

Optimal Stochastic Vehicle Path Planning Using Covariance Steering

Kazuhide Okamoto¹ and Panagiotis Tsiotras²

Abstract—This work addresses the problem of vehicle path planning in the presence of obstacles and uncertainties, a fundamental robotics problem. While several path planning algorithms have been proposed over the years, many of them have dealt with only deterministic environments or with only *open-loop* uncertainty, i.e., the uncertainty of the system state is not controlled and, typically, increases with time due to exogenous disturbances. This may lead to potentially conservative nominal paths. The typical approach to deal with disturbances and reduce uncertainty is to use a lower-level feedback controller. We advocate the premise that, if a path planner can consider the *closed-loop* evolution of the system uncertainty, it can lead to less conservative, but still feasible, paths. To this end, in this work we develop an approach that is based on optimal covariance steering, which explicitly steers the state covariance for stochastic linear systems. We verify the proposed framework using extensive numerical simulations.

Index Terms—Motion and Path Planning, Robust/Adaptive Control of Robotic Systems, Probability and Statistical Methods, Formal Methods in Robotics and Automation.

I. INTRODUCTION

VEHICLE path planning problems have been an active research topic for many years [1]. Among the existing literature, sampling-based algorithms, such as the rapidly-exploring random trees (RRT) [2], have been popular for solving motion planning problems subject to constraints. Variants of RRT ([3], [4]) have been proposed that offer asymptotic optimality guarantees. However, in general, these approaches deal with deterministic dynamics and cannot directly address uncertain systems.

In order to solve path planning problems under uncertainty, several approaches have been proposed, such as chance-constrained RRTs [5], [6] and mixed-integer programming approaches [7], [8]. These approaches consider only the *open-loop* dynamics of the covariance, i.e., the covariance of the system state is not controlled and, typically, increases with time due to the exogenous disturbances. In order to add robustness, state-feedback *closed-loop* controllers are applied

after the fact. Alternatively, it is more natural to consider the closed-loop evolution of the covariance for path planning.

Recently, path-planning problems with closed-loop covariance dynamics have been addressed in the literature, such as [9], [10], [11], [12], [13]. Among these, the work of Vitus and Tomlin [12] is most closely related to our work, as it also uses mixed-integer programming (MIP) to deal with non-convex domains. MIP approaches have been actively investigated for path-planning problems in non-convex state constraint environments formulated as mixed-integer linear programming (MILP) [14], [15], mixed-integer quadratic programming (MIQP) [16], and mixed-integer semi-definite programming (MISDP) problems [17]. MIP approaches that cope with system uncertainty have been discussed by several researchers [7], [8]. However, these works only consider the *open-loop* dynamics of the covariance, which may lead to unnecessarily conservative solutions. Vitus et al. [18] dealt with a path-planning problem with *closed-loop* dynamics of the covariance using the so-called Tunnel-MILP approach, which decomposes a non-convex environment into convex polygons and solves the optimal control problem through these convex polygons. In this formulation, integer variables indicate in which polygon the state variable belongs to at each time step. This is computationally more efficient than other MIP approaches, which typically use separate integer variables for each face of every obstacle. While the original Tunnel-MILP approach could not cope with constraint violations between time steps, Deits and Tedrake [19] proposed a new constraint such that two consecutive system states have to belong to the same convex polygon in order for the system state to remain in the same polygon between consecutive time steps, implying no collision with obstacles.

These previous MIP approaches do not consider the terminal state distribution. Steering the covariance to a pre-specified value at a given time needs to be formulated as a numerical optimization problem [20]. The approach proposed in this work extends these previous works by adding the element of actively controlling the state covariance to achieve less conservative paths. Specifically, in this paper we utilize the closed-loop dynamics of the covariance to compute a collision-free path under non-convex state chance constraints and steer the state to a pre-specified Gaussian distribution, and at the same time, minimize a quadratic cost that depends on the expectation of the state and control.

In this work, we design a stochastic optimal path planner that steers the mean and the covariance of a linear time-varying system from some initial values to some pre-specified terminal values at a given time in the presence of obstacles and uncer-

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ainties. We assume that the initial-state Gaussian distribution and the white-noise diffusion are given and independent of the state and input with statistics known a-priori. We minimize a state and control expectation-dependent cost while steering the state distribution to the target Gaussian distribution.

Controlling the state covariance of a linear system has been researched since the late '80s [21]. Most of the literature have worked on the *infinite* horizon problem, for both discrete and continuous time systems [22], [23]. On the other hand, until recently [24], [25], [26], the *finite*-horizon problem, in which the controller steers the system covariance to a pre-specified value at a given time step, had not been investigated. In [27], [28], the authors converted the finite optimal covariance steering problem into a convex programming problem.

We consider the planning task with obstacles in the environment. Chance constraints are used for stochastic control problems and impose a maximum probability of constraint violation instead of imposing a hard constraint. Chance-constrained optimization has been extensively studied [29]. As shown in [30], using Boole's inequality [31], in the case of Gaussian additive disturbance and a linear system, the optimal control problem can be converted into a convex programming problem with little conservatism. In our previous work [20], we converted the finite-horizon optimal covariance steering problem into a convex programming problem when convex state chance constraints are present. In this work, we extend the approach in [20] in order to cope with arbitrary (non-convex) state chance-constraints.

The contributions of this work are twofold. The first is the solution of the optimal covariance steering problem under *non-convex* state chance constraints, which, to the best of our knowledge, has not been addressed in the literature. The proposed approach facilitates path planning by computing optimal trajectories in the presence of obstacles and uncertainties. To achieve this, we use mixed-integer convex programming to efficiently decompose the admissible state space to a union of overlapping convex sets. Similarly to the Tunnel-MILP [18], the state trajectory needs to go through these convex sets. Since mixed-integer problems are in general NP-hard, numerically efficient approaches are required. Thus, as the second contribution of this work, we introduce a new optimal covariance steering approach that is faster than the one proposed in [20]. Together, these two contributions allow us to solve path-planning problems more efficiently in more realistic domains.

II. PROBLEM STATEMENT

We assume a system with dynamics consisting of the following stochastic discrete-time linear time-varying (LTV) system with additive noise:

$$x_{k+1} = A_k x_k + B_k u_k + D_k w_k, \quad (1)$$

where $k = 0, \dots, N-1$ is the time index, $x_k \in \mathbb{R}^{n_x}$ is the state, $u_k \in \mathbb{R}^{n_u}$ is the control input, and $w_k \in \mathbb{R}^{n_w}$ is a Gaussian noise with zero mean and unit covariance, i.e., $\mathbb{E}[w_k] = 0$, $\mathbb{E}[w_{k_1} w_{k_2}^\top] = I_{n_w}$ if $k_1 = k_2$, $\mathbb{E}[w_{k_1} w_{k_2}^\top] = 0$ otherwise. In addition, we assume that $\mathbb{E}[x_{k_1} w_{k_2}^\top] = 0$, $0 \leq$

$k_1 \leq k_2 \leq N$. The random vector x_0 is an initial state and is drawn from the following Gaussian distribution

$$x_0 \sim \mathcal{N}(\mu_0, \Sigma_0), \quad (2)$$

where $\mu_0 \in \mathbb{R}^{n_x}$ is the mean and $\Sigma_0 \in \mathbb{R}^{n_x \times n_x}$ is the covariance of the initial state, respectively. This work assumes that $\Sigma_0 \succeq 0$. We wish to guide the system trajectories of (1) from (2) to the following target Gaussian distribution

$$x_N \sim \mathcal{N}(\mu_N, \Sigma_N), \quad (3)$$

at a given time N , where $\mu_N \in \mathbb{R}^{n_x}$ and $\Sigma_N \in \mathbb{R}^{n_x \times n_x}$ with $\Sigma_N \succ 0$. We, at the same time, wish to minimize the following quadratic cost function

$$J(x_{0:N-1}, u_{0:N-1}) = \mathbb{E} \left[\sum_{k=0}^{N-1} x_k^\top Q_k x_k + u_k^\top R_k u_k \right], \quad (4)$$

where $x_{0:N-1}$ is the state sequence, $u_{0:N-1}$ is the control sequence, and $Q_k \succeq 0$ and $R_k \succ 0$ for all $k = 0, \dots, N-1$. Owing to (3), (4) does not include a terminal cost.

Furthermore, we consider state constraints. As the noise in (1) is possibly unbounded, the system state is also possibly unbounded. Thus, instead of using hard constraints, we consider state chance constraints, which enforces the probability of the constraint violation being less than a threshold, i.e.,

$$\Pr(x_k \notin \chi) \leq P_{\text{fail}}, \quad (5)$$

for all $k = 0, \dots, N-1$, where $\Pr(\cdot)$ is the probability, $\chi \subset \mathbb{R}^{n_x}$ denotes the feasible state set, and $P_{\text{fail}} \in (0, 0.5)$ is maximum allowed probability of failure. Note that, unlike the problem setup in [20], the set χ in the current work may be *non-convex* owing to the presence of several obstacles, i.e.,

$$\chi = \chi_\Omega \setminus \left(\bigcup_{j=1}^{N_{\text{obs}}} \chi_j \right), \quad (6)$$

where $\chi_\Omega, \chi_1, \dots, \chi_{N_{\text{obs}}} \subset \mathbb{R}^{n_x}$ are (typically) polytopes and N_{obs} is the number of obstacles.

Remark 1. We assume the controllability of (1) if (6) does not exist. Namely, any $x_N \in \mathbb{R}^{n_x}$ is reachable from x_0 , if $w_k = 0$ for $k = 0, \dots, N-1$.

We provide an alternative description of (1), which will be instrumental to solve our problem. The discussion below is borrowed from our previous papers [20], [32]. We start by reformulating (1) as

$$X = \mathcal{A}x_0 + \mathcal{B}U + \mathcal{D}W, \quad (7)$$

where X , U , and W denote the state, input, disturbance sequences, respectively, e.g., $X = [x_0^\top \ x_1^\top \ \dots \ x_N^\top]^\top$ and $U = [u_0^\top \ u_1^\top \ \dots \ u_{N-1}^\top]^\top$, and the matrices $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{D}$ are defined accordingly. Notice that $\mathbb{E}[x_0 x_0^\top] = \Sigma_0 + \mu_0 \mu_0^\top$, $\mathbb{E}[x_0 W^\top] = 0$, and $\mathbb{E}[W W^\top] = I_{N n_w}$. Using this new notation, we may rewrite the cost (4) as

$$J(X, U) = \mathbb{E} [X^\top \bar{Q} X + U^\top \bar{R} U], \quad (8)$$

It follows from $Q_k \succeq 0$ and $R_k \succ 0$ that $\bar{Q} \succeq 0$ and $\bar{R} \succ 0$. The initial and terminal conditions (2) and (3) can be written as

$$\mu_0 = E_0 \mathbb{E}[X], \quad \Sigma_0 = E_0 \Sigma_X E_0^\top, \quad (9)$$

and

$$\mu_N = E_N \mathbb{E}[X], \quad \Sigma_N = E_N \Sigma_X E_N^\top, \quad (10)$$

where $\Sigma_X = \mathbb{E}[XX^\top] - \mathbb{E}[X]\mathbb{E}[X]^\top$ and $E_k \triangleq [0_{n_x, kn_x} \ I_{n_x} \ 0_{n_x, (N-k)n_x}]$. Last, but not least, we can rewrite the state chance constraints (5) as

$$\Pr(E_k X \notin \chi) \leq P_{\text{fail}}, \quad k = 0, \dots, N-1. \quad (11)$$

In summary, we wish to solve the following stochastic optimal control problem.

Problem 1. *Given the system state sequence (7), find the optimal control sequence $U^* = U_{N-1}^*$ that minimizes the objective function (8), subject to the initial state (9), the terminal state (10), and the state chance constraints (11).*

III. OPTIMAL COVARIANCE CONTROL WITH OBSTACLES

This section introduces the proposed approach for solving Problem 1. First, we introduce a computationally more efficient approach than the one in [20] to solve the problem for the case of *convex* state chance constraints. We then show how to deal with *non-convex* state chance constraints.

A. Proposed Covariance Steering Approach

In [20] the control used state values of all previous time steps, which did not utilize the Markov property of the system dynamics. Thus, if the time horizon is long, large memory may be required to implement the control policy. In this section, we propose a computationally more efficient approach. The main result is given in the following theorem.

Theorem 1. The following control policy

$$u_k = v_k + K_k y_k, \quad (12)$$

where $v_k \in \mathbb{R}^{n_u}$, $K_k \in \mathbb{R}^{n_u \times n_x}$, and $y_k \in \mathbb{R}^{n_x}$ is given by

$$y_{k+1} = A_k y_k + D_k w_k, \quad (13a)$$

$$y_0 = x_0 - \mu_0, \quad (13b)$$

results in a convex programming formulation of Problem 1.

Proof. The control sequence U can be represented as $U = V + KY$, where $V = [v_0^\top, \dots, v_{N-1}^\top]^\top \in \mathbb{R}^{Nn_u}$, $Y = [y_0^\top, \dots, y_{N-1}^\top]^\top \in \mathbb{R}^{(N+1)n_x}$, and $K \in \mathbb{R}^{Nn_u \times (N+1)n_x}$ is defined accordingly. It is straightforward from (13) to derive $Y = Ay_0 + DW$. Thus,

$$U = V + K(Ay_0 + DW). \quad (14)$$

Because $\mathbb{E}[y_0] = 0$ and $\mathbb{E}[W] = 0$, $\mathbb{E}[U] = V$. Thus, it follows from (7) that

$$\bar{X} = \mathbb{E}[X] = A\mu_0 + BV, \quad (15)$$

$$\tilde{X} = X - \mathbb{E}[X] = (I + BK)(Ay_0 + DW). \quad (16)$$

According to [20], the cost function (8) may be converted into $J(\bar{X}, \tilde{X}, V, \tilde{U}) = \text{tr}(\bar{Q}\mathbb{E}[\tilde{X}\tilde{X}^\top]) + \text{tr}(\bar{R}\mathbb{E}[\tilde{U}\tilde{U}^\top]) + \bar{X}^\top \bar{Q} \bar{X} + V^\top \bar{R} V$, where $\tilde{U} = U - V$. It follows from (15)

and (16) and $\mathbb{E}[y_0 y_0^\top] = \Sigma_0$ that, defining $\Sigma_Y = A\Sigma_0 A^\top + DD^\top$, one can further convert the cost function into

$$J(V, K) = (A\mu_0 + BV)^\top \bar{Q} (A\mu_0 + BV) + V^\top \bar{R} V + \text{tr}[(I + BK)^\top \bar{Q} (I + BK) + K^\top \bar{R} K] \Sigma_Y. \quad (17)$$

Note that we used the following facts: $\mathbb{E}[y_0 W^\top] = 0$, and $\mathbb{E}[WW^\top] = I_{Nn_w}$. In addition, we reformulate the terminal constraint (10) as

$$\mu_N = E_N (A\mu_0 + BV), \quad (18a)$$

$$\Sigma_N = E_N (I + BK) \Sigma_Y (I + BK)^\top E_N^\top. \quad (18b)$$

Note that V steers the mean and K steers the covariance to the pre-specified values μ_N and Σ_N , respectively. As discussed in [20], we relax the covariance equality constraint (18b) to an inequality constraint. Thus, $\Sigma_N \succeq E_N (I + BK) \Sigma_Y (I + BK)^\top E_N^\top$, which leads to

$$1 - \|\Sigma_Y^{1/2} (I + BK)^\top E_N^\top \Sigma_N^{-1/2}\|_2 \geq 0. \quad (19)$$

Finally, the chance constraint (11) can be formulated as follows. We assume that the feasible state space χ can be represented as a convex polytope: $\chi \triangleq \bigcap_{j=1}^M \{x : \alpha_j^\top x \leq \beta_j\}$ where $\alpha_j \in \mathbb{R}^{n_x}$ and $\beta_j \in \mathbb{R}$. Under this assumption, we convert the chance constraint (11) to $\Pr(\alpha_j^\top E_k X > \beta_j) \leq p_j$ and $\sum_{j=1}^M p_j \leq P_{\text{fail}}$. It follows from (15) and (16) that $\alpha_j^\top E_k X$ is a scalar random variable sampled from univariate Gaussian distribution. Following the discussion in [20], it is straightforward to derive the inequality constraint

$$\alpha_j^\top E_k (A\mu_0 + BV) + \|\Sigma_Y^{1/2} (I + BK)^\top E_k^\top \alpha_j\| \Phi^{-1}(1 - p_{j,\text{fail}}) - \beta_j \leq 0, \quad (20)$$

where $p_{j,\text{fail}}$ is a pre-specified value, and Φ^{-1} is the inverse function of the cumulative distribution function of the standard normal distribution. $\square$

In summary, we have converted Problem 1 into the following convex programming problem.

Problem 2. *Given the system state sequence (15) and (16), find V and K that minimizes the cost function (17) subject to the terminal state constraints (18a) and (19), and the chance constraint (20) with pre-specified probability thresholds $p_{j,\text{fail}}$.*

Remark 2. The new control strategy in Theorem 1 is computationally more efficient than the one in [20]. Specifically, the vector V and the matrix K in (14) are a full vector and a block diagonal matrix. Thus, the number of entries one needs to compute is $\mathcal{O}(Nn_x n_u)$. In contrast, the matrix K in [20] has $\mathcal{O}(N^2 n_x n_u)$ entries.

Remark 3. Instead of (17), it is possible to separately design the cost functions for the mean and covariance steering as

$$J(V, K) = J_m(V) + J_v(K), \quad (21)$$

where $J_m(V) = (A\mu_0 + BV)^\top \bar{Q}_m (A\mu_0 + BV) + V^\top \bar{R}_m V$ and $J_v(K) = \text{tr}[(I + BK)^\top \bar{Q}_v (I + BK) + K^\top \bar{R}_v K] \Sigma_Y$.

B. Optimal Covariance Control with Obstacles

In this section, we propose a new approach to efficiently deal with non-convex state chance constraints in a covariance steering framework. By steering the state covariance, we mitigate the conservativeness of the ensuing path.

We start by representing the non-convex set of obstacle-free states as the union of a finite number of overlapping *convex* sub-regions $\mathcal{R}_r$. Specifically, we represent χ in (6) as

$$\chi = \bigcup_{r=0}^{N_R-1} \mathcal{R}_r, \quad (22)$$

where N_R is the number of sub-regions. We assume that each $\mathcal{R}_r$ is a polytope which can be represented as

$$\mathcal{R}_r \triangleq \bigcap_{q=0}^{M_r-1} \{x : \alpha_{r,q}^\top x \leq \beta_{r,q}\}, \quad (23)$$

where $\alpha_{r,q} \in \mathbb{R}^{n_x}$ and $\beta_{r,q} \in \mathbb{R}$. Let now the Boolean matrix $\mathcal{M} \in \{0,1\}^{N_R \times (N-1)}$, where $\mathcal{M}_{r,k} = 1$ implies that the state at time steps k and $k+1$ belongs to $\mathcal{R}_r$ with high probability. Note that, because of the noise, the state constraints need to be probabilistically formulated, i.e., using chance constraints. Namely, $\mathcal{M}_{r,k} = 1$ implies that $\Pr(x_k \notin \mathcal{R}_r) < \epsilon$ and $\Pr(x_{k+1} \notin \mathcal{R}_r) < \epsilon$, where $0 < \epsilon \ll 1$. In order to ensure that, with high probability, the state is collision-free at time step k , we use the following form

$$\sum_{r=0}^{N_R-1} \mathcal{M}_{r,k} = 1. \quad (24)$$

As there can be overlaps between sub-regions, the state variables at steps k and $k+1$ can belong to multiple regions. Thus, the implication is only one directional [17].

Next, we prove the following lemma.

Lemma 1. Given $\mathcal{R}_r$ in (23), the conditions $\Pr(x_k \notin \mathcal{R}_r) < \epsilon$ and $\Pr(x_{k+1} \notin \mathcal{R}_r) < \epsilon$, are converted to second-order cone constraints in terms of V and K .

Proof. It is straightforward from Section III-A to convert the conditions of the lemma to

$$\alpha_{r,q}^\top E_\kappa (\mathcal{A}\mu_0 + \mathcal{B}V) + \|\Sigma_Y^{1/2} (I + \mathcal{B}K)^\top E_\kappa^\top \alpha_{r,q}\| \Phi^{-1}(1 - \epsilon) - \beta_{r,q} \leq 0, \quad (25)$$

where $\kappa \in \{k, k+1\}$ and $q = 0, \dots, M_r - 1$. $\square$

Finally, we reformulate Problem 1 into the following mixed-integer convex programming problem summarized in Algorithm 1. The element-wise constraints on V and K in line 4 were introduced to expedite the solution. Note that, although MIP problems are, in general, NP-hard, many tools have been recently developed in the literature to efficiently solve such problems. In the following section, using simple numerical examples, we demonstrate that our problem setup can be efficiently solved with current MIP solvers.

Algorithm 1: Optimal Covariance Control

Input : $\mu_0, \Sigma_0, \mu_N, \Sigma_N, \epsilon, A_k, B_k, D_k, \alpha_{r,q}, \beta_{r,q}$.
Output: V^* and K^*

- 1 Find V^* and K^* in (14) that minimizes (17) or (21)
- 2 subject to
- 3 (18a) and (19) $\triangleright$ Terminal Condition
- 4 $V_{\min} \leq V \leq V_{\max}, K_{\min} \leq K \leq K_{\max}$
- 5 (24) $\triangleright$ Condition on $\mathcal{M}$
- 6 $\mathcal{M}_{r,k} = 1 \Rightarrow (25) \triangleright$ Chance Constraints

IV. NUMERICAL SIMULATIONS

A. Runtime Performance Comparison

We compared the performance proposed in Section III-A to the approach in [20]. To this end, we conducted the same numerical simulation as the one in [20]. Figure 1 depicts the time to solve the problem. We checked the horizon settings of $N = 5, 10, 15, 20, 25, 30$, and with each horizon length, we conducted 50 trials. The mean and the one standard deviation is illustrated in Fig. 1. It is worth noticing that the optimal cost function value found by the proposed approach is smaller than the one found by [20] as shown in Table I although the proposed approach has more restrictive structure in K matrix.

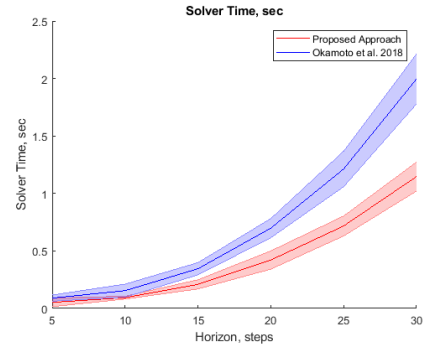


Fig. 1. Time comparison between proposed approach and the approach introduced in [20].

TABLE I
COST FUNCTION VALUES, $\times 10^5$.

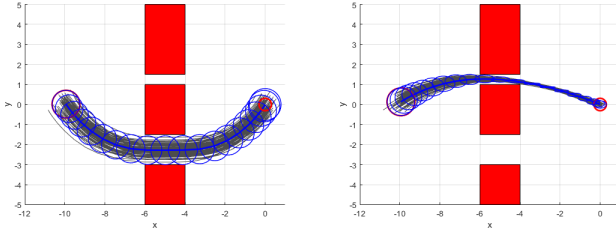
Horizon	Proposed Approach	Okamoto et al. 2018 [20]
5	1.2645	1.2647
10	0.1556	0.1556
15	0.0485	0.0485
20	0.0234	0.0234
25	0.0150	0.0174
30	0.0120	0.0257

B. Double Slit

This example considers the vehicle path-planning problem under the double-integrator time-invariant system dynamics with $x_k \in \mathbb{R}^4$, $u_k \in \mathbb{R}^2$, $w_k \in \mathbb{R}^4$ with $D = 10^{-2}I_4$ and $\Delta t = 0.2$ time-step interval. We set the horizon to $N = 20$ and the probability threshold to $\epsilon = 1e - 3$. We used YALMIP [33] and MOSEK [34]. In this example, and in order to reduce the search space and the computational

time, we restricted the control vector and gain matrix as $-10 \leq K \leq 10$, $-100 \leq V \leq 100$. We consider the case illustrated in Fig. 2, where we wish to find the trajectory to go from left to right through one of the two “slits.” Figure 2a shows the trajectories when we solve only the mean steering problem ($K \equiv 0$ in (14)). In this case, the covariance is not controlled, and thus, the terminal covariance constraint cannot be satisfied. Therefore, (19) is not imposed. Since the initial covariance is large, and as the covariance grows, the mean steering controller has difficulty in guiding the trajectory through the top slit, and the optimal path has to go through the larger but further away slit. Figure 2b illustrates the result of the proposed approach. Although the top slit is narrower than the bottom one, the controller shrinks the covariance and successfully computes an optimal path. In order to compute this path, we used the rectangular-shaped sub-regions shown in Fig. 3. Although this division was conducted manually, algorithms are available to automatically represent the entire feasible region as a union of convex polytopes [35].

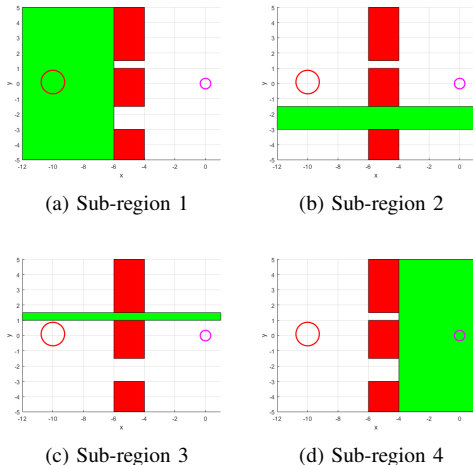
We also conducted a similar numerical simulation with slightly different setup, where the top slit is much narrower as illustrated in Fig. 4a. The algorithm found that the cost of shrinking the error ellipses and going through the top slit is higher than going through the lower slit as shown in Fig. 4b.



(a) Mean steering.

(b) Proposed approach.

Fig. 2. Double slit example.



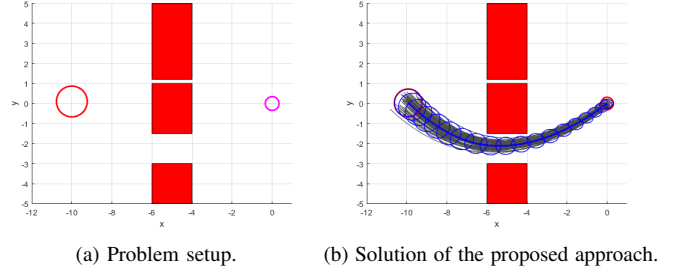
(a) Sub-region 1

(b) Sub-region 2

(c) Sub-region 3

(d) Sub-region 4

Fig. 3. Representation of the feasible sub-regions.



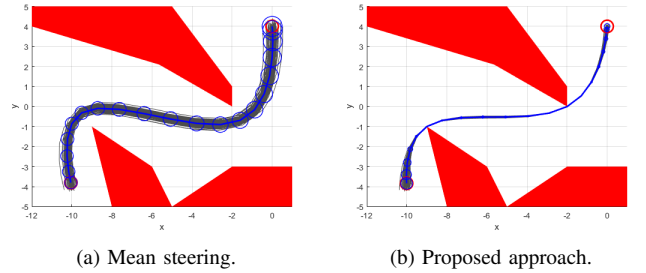
(a) Problem setup.

(b) Solution of the proposed approach.

Fig. 4. Double slit example with narrower top slit.

C. Non-Convex Obstacles

We consider the case illustrated in Fig. 5, where we computed the optimal collision-free trajectory in a more general environment with obstacles arranged randomly having different orientations. Figure 5b illustrates the trajectory generated by the proposed approach, which, unlike a mean steering controller in Fig. 5a, successfully shrinks the error ellipse and takes the shortest path.



(a) Mean steering.

(b) Proposed approach.

Fig. 5. Vehicle path planning with non-convex obstacles.

V. DISCUSSION AND CONCLUSION

This work has addressed the optimal stochastic vehicle path planning problem, and has solved the problem as an optimal covariance steering problem under non-convex state chance constraints. The proposed approach is a non-trivial extension of our prior work [20] to the problem of optimal covariance steering under *non-convex* state chance constraints. In order to apply covariance steering to path planning problems, we followed an approach similar to the Tunnel-MILP [18] to determine which subregion the state variable belongs to at each time step. In the process, we also proposed a computationally more efficient approach than the one in [20].

Since we represent the feasible state space as the union of feasible convex sub-regions (22), the solution may become somewhat conservative, as illustrated in Fig. 6, which shows the solution in Fig. 4b around the lower left corner of the middle obstacle. The green areas are the feasible overlapping regions, and the red area is the obstacle. The black lines indicate the edges of the sub-regions. The error ellipse touches these boundaries, but it does not touch the corner of the obstacle, which indicates that the path has unnecessarily large safety margin from the obstacle. This extra margin is owing to the requirement that this error ellipse needs to belong to

both sub-regions 1 and 2 (see Fig. 3). Furthermore, note that

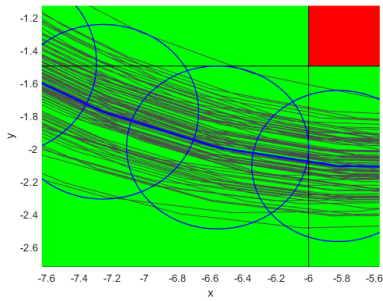


Fig. 6. Magnification of the solution illustrated in Fig. 4b.

the decomposition in (22) is not unique. Thus, an interesting question would be to find the “best” decomposition to a union of convex sets for our problem. In the scenarios we tested we observed that the performance depends not so much on the number of obstacles, but rather on the way the state space is represented as the union of convex sub-regions. The clearance between obstacles and length of the optimal path does not seem to affect the performance significantly.

To our knowledge, this work is the first to solve the optimal covariance steering problem under non-convex state chance constraints. Future work will include the investigation of an effective decomposition approach of the feasible state space.

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