

Chapter 1

Invariant Manifold Techniques for Control of Underactuated Mechanical Systems

We propose an algorithm for constructing static, quasi-continuous, feedback control laws guaranteeing exponential convergence for a class of underactuated mechanical systems. The methodology uses the concept of invariant manifolds, i.e., invariant smooth hypersurfaces in the state space. These manifolds are constructed by direct integration of the closed-loop dynamics subject to a linear feedback control law. The methodology is applied to n -dimensional nonholonomic systems in power form. A recursive algorithm is presented which uses these manifolds to design feedback controllers which achieve global exponential convergence of the system to the origin.

1.1 Introduction

Control of underactuated systems, i.e, with less control inputs than coordinates is emerging as an important topic in control theory and applications. The potentials of a complete understanding of this problem are enormous. For example, the possibility of building mechanisms that can perform complex tasks using a small number of actuators will reduce cost, weight as well as occurrences of failures. One of the main obstacles in the development of a comprehensive methodology for controlling such systems is, however, the fact that linear control techniques cannot be used for these problems. These are “inherently nonlinear” systems which require new approaches taking into consideration their nonlinear character. One challenging aspect of these systems is that they are controllable but not stabilizable by smooth static or dynamic state feedback control laws [1]. Control of underactuated systems often fall under the area of control of nonholonomic systems. Nonholonomic control systems commonly arise from mechanical systems when non-integrable constraints are imposed on the motion, i.e., velocity constraints, which can not be integrated to generate constraints on the configuration space. Examples include mobile robots [2], underactuated symmetric rigid spacecraft [3, 4], etc. An additional motivating aspect of these systems is that their

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treatment requires knowledge from two different but complementary disciplines: *Dynamical Systems Theory* and *Control Theory*. A number of approaches have been proposed for solving the stabilization and the path-planning problems for nonholonomic systems. These methodologies can be broadly classified as discontinuous, time-invariant control laws [3, 4, 5, 6, 7, 8] or time-varying (usually smooth or continuous) control laws [9, 10, 11, 12, 13]. Time-varying, nonsmooth exponentially convergent controllers have been reported in [14]. Hybrid controllers have also been proposed [15, 16]. A discussion of the advantages and the disadvantages of these different schemes can be found in the survey article [17].

The use of invariant manifold methods for controlling nonlinear systems has probably its origins in singular optimal control [18, 19], where a “singular surface” is constructed and the optimal control is chosen to lie on this surface. Sliding control has also its origins in the same idea [20]. Invariant manifold techniques have been recently used in conjunction with problems in nonholonomic systems [4, 8, 21, 22]. In this work we extend the results of [4, 8, 22] to derive feedback control laws for nonholonomic systems in power form with two inputs. We first show that a set of invariant manifolds can be constructed for n -dimensional nonholonomic systems in power form, using the exact closed-loop system solution subject to a linear feedback law. If the initial conditions lie on these invariant manifolds, the states of the closed-loop system converge exponentially to zero. The proposed control laws are thus composed of two parts. The first part is the linear control on the manifold while the the second part forces the trajectories to asymptotically converge to these manifolds. The resulting controllers are discontinuous at the origin and are similar *in form* to the ones recently proposed by Astolfi [5, 6] for nonholonomic systems in chained form. The approach in [5, 6], however, uses a singular transformation for resolution of the singularities (σ process).

The paper is organized as follows. To motivate the discussion, Section 1.2 presents two examples where the construction of an invariant manifold for the closed-loop system leads to the rather straightforward design of exponentially convergent control laws. These examples also demonstrate the potential benefits of introducing these manifolds as new coordinates for describing the system dynamics. Section 1.3 extends these results to a general class of nonholonomic systems without drift, i.e., n -dimensional nonholonomic systems in power form with two inputs. First we present a recursive algorithm which uses the manifolds as new coordinates to generate a series of systems in power form of reduced dimension. The end result of this recursive process is a 3-dimensional system in power form. The main result is given in Section 1.5 (Theorem 1) which basically shows that the feedback control law for the 3-dimensional system constructed through this recursive process can be used for the original n -dimensional system with a proper choice of the control gains.

1.2 Two Motivating Examples

1.2.1 Mobile Robot

Consider the kinematic model of a mobile robot subject to a nonholonomic constraint.

$$\begin{aligned}\dot{x} &= v \cos \gamma \\ \dot{y} &= v \sin \gamma \\ \dot{\gamma} &= \omega\end{aligned}\tag{1.1}$$

where (x, y) denotes the position of the wheel axis center, γ denotes the cart orientation with respect to the x -axis, and v and ω denote the control inputs which are the tangential and the angular velocities, respectively. Introducing the state and input transformations [9, 23]

$$x_2 = \gamma, \quad x_1 = x \cos \gamma + y \sin \gamma, \quad x_3 = x \sin \gamma - y \cos \gamma \quad (1.2a)$$

$$u_1 = v - x_3 \omega, \quad u_2 = \omega \quad (1.2b)$$

system (1.1) can be written as a 3-dimensional system in power form

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1 u_2 \end{aligned} \quad (1.3)$$

Consider now the linear feedback control

$$u_1 = -k x_1, \quad u_2 = -k x_2, \quad (k > 0) \quad (1.4)$$

which gives the closed-loop system

$$\begin{aligned} \dot{x}_1 &= -k x_1 \\ \dot{x}_2 &= -k x_2 \\ \dot{x}_3 &= -k x_1 x_2 \end{aligned} \quad (1.5)$$

which can be analytically integrated to obtain

$$\begin{aligned} x_1(t) &= x_{10} e^{-kt} \\ x_2(t) &= x_{20} e^{-kt} \\ x_3(t) &= s(x_0) + \frac{1}{2} x_{01} x_{02} e^{-2kt} \end{aligned} \quad (1.6)$$

where $x_0 = (x_{10}, x_{20}, x_{30})^T \in \mathbb{R}^3$ is the initial condition at time $t = 0$ and $s(x)$ is defined by

$$s(x) = x_3 - \frac{1}{2} x_1 x_2 \quad (1.7)$$

Clearly, from Eq. (1.6) x_1 and x_2 decay exponentially to zero with rate of decay k , whereas $\lim_{t \rightarrow \infty} x_3(t) = s(x_0)$. If the initial conditions are chosen such that $s(x_0) = 0$, then x_3 will also decay exponentially to zero with rate $2k$.

The smooth function $s(x)$ can thus be used to define a set of initial conditions

$$\mathcal{M}_1 = \{x \in \mathbb{R}^3 : s(x) = 0\} \quad (1.8)$$

such that for all $x_0 \in \mathcal{M}_1$ the system (1.3) decays exponentially to zero with rate k . Since $s(x)$ is a smooth mapping of constant rank, the set $s^{-1}(0) = \mathcal{M}_1$ is a 2-dimensional smooth manifold [24]. It is also an *invariant manifold* for the system (1.5) since

$$\begin{aligned} \dot{s} &= \dot{x}_3 - \frac{1}{2} (x_1 \dot{x}_2 + \dot{x}_1 x_2) \\ &= -k x_1 x_2 + \frac{k}{2} (x_1 x_2 + x_1 x_2) = 0 \end{aligned} \quad (1.9)$$

The manifold $\mathcal{M}_1$ is shown in Fig. 1.1(a).

The control law (1.4) will work for all initial conditions $x_0 \in \mathcal{M}_1$ but it needs modification if $x_0 \notin \mathcal{M}_1$. Specifically, one can choose a control law of the form

$$u_1 = -k x_1 + \beta_1(x, s) \quad u_2 = -k x_2 + \beta_2(x, s) \quad (1.10)$$

where $\beta_i(x, s)$ are functions which vanish on $\mathcal{M}_1$, i.e., $\beta_i(x, 0) = 0$ ($i = 1, 2$). If one chooses, for example [8]

$$\beta_1 = -\mu \frac{s(x)}{x_1^2 + x_2^2} x_2, \quad \beta_2 = \mu \frac{s(x)}{x_1^2 + x_2^2} x_1, \quad (\mu > 2k) \quad (1.11)$$

one obtains the following result

Proposition 1 *Consider the system (1.3) and the control law in equations (1.10)-(1.11). For all initial conditions such that $x_{10}^2 + x_{20}^2 \neq 0$, the closed-loop system is globally exponentially convergent to the origin with rate k . Moreover, the control history is bounded along closed-loop trajectories.*

Proof. The complete proof of this result is shown in [8] and will not be repeated here. However, a sketch of the proof can be easily obtained by noticing that the function $V(x) = x_1^2 + x_2^2$ decreases exponentially along closed-loop trajectories since

$$\dot{V} = 2(x_1 \dot{x}_1 + x_2 \dot{x}_2) = -2k(x_1^2 + x_2^2) = -2kV \quad (1.12)$$

whereas the time derivative of $s(x)$ along closed-loop trajectories gives

$$\dot{s} = -\frac{\mu}{2}s \quad (1.13)$$

Thus, x_1 and x_2 decay exponentially with rate k , whereas s (and thus also x_3) decays to zero with rate $\mu/2$. Moreover, The control law is bounded since $\lim_{t \rightarrow \infty} \beta_i(t) = 0$ exponentially with rate $\mu/2 - k > 0$. $\triangleleft$

1.2.2 Underactuated Axi-symmetric Spacecraft

The control methodology described previously required the analytical integration of the closed loop equations subject to a linear control law. This was easily accomplished for the mobile robot example, once the system is transformed in the 3-dimensional normal form. In this section we present a less trivial example arising in the reorientation of a rigid, axi-symmetric body (i.e., spacecraft) when there is no available torque along the symmetry axis. We again consider the kinematic model only, assuming that the angular velocity component ω_3 along the symmetry axis (the 3-axis) is initially zero. If $\omega_3(t) \neq 0$, the system is not stabilizable at the origin [3, 4].

Using a 3-2-1 Eulerian angle sequence (ψ, θ, ϕ) for attitude orientation one obtains the following kinematic equations

$$\begin{aligned} \dot{\phi} &= \omega_1 + \omega_2 \sin \phi \tan \theta \\ \dot{\theta} &= \omega_2 \cos \phi \\ \dot{\psi} &= \omega_2 \sin \phi \sec \theta \end{aligned} \quad (1.14)$$

where ω_1, ω_2 are the available control inputs.

The analysis of this problem is simplified by introducing stereographic coordinates w_1 and w_2 to describe the location of the 3-axis in lieu of θ and ϕ [25]

$$w_1 = \frac{\sin \phi \cos \theta}{1 + \cos \phi \cos \theta}, \quad w_2 = \frac{\sin \theta}{1 + \cos \phi \cos \theta}, \quad \zeta = \psi \quad (1.15)$$

In these coordinates, Eqs. (1.14) take the form

$$\dot{w} = \frac{\omega}{2} + \frac{\bar{\omega}}{2} w^2 \quad (1.16a)$$

$$\dot{\zeta} = \frac{i}{2}(\omega - \bar{\omega}) \frac{(w + \bar{w})(1 + |w|^2)}{(1 + w^2)(1 + \bar{w}^2)} \quad (1.16b)$$

where w and ω denote the complex quantities $w = w_1 + i w_2$ and $\omega = \omega_1 + i \omega_2$ ($i = \sqrt{-1}$), bar denotes complex conjugate, and $|\cdot|$ denotes the magnitude of a complex number ($z\bar{z} = |z|^2$).

Consider now the following linear control law in terms of w

$$\omega = -k w \quad (1.17)$$

which results in the following closed-loop dynamics

$$\dot{w} = -\frac{k}{2}(1 + |w|^2) w \quad (1.18a)$$

$$\dot{\zeta} = -i \frac{k}{2} \frac{(w^2 - \bar{w}^2)(1 + |w|^2)}{(1 + w^2)(1 + \bar{w}^2)} \quad (1.18b)$$

Note that w decays monotonically (in fact, exponentially) to zero. Also, the time history of the state variable ζ is completely determined by the time history of w . Using the change in the independent variable

$$\tau = \frac{k}{2}(1 + |w|^2) t \quad (1.19)$$

Eq. (1.18a) can be immediately integrated to obtain

$$w(\tau) = w_0 e^{-\tau} \quad (1.20)$$

while the differential equation for ζ yields

$$\zeta' = i \frac{(\bar{w}_0^2 - w_0^2) e^{-2\tau}}{(1 + w_0^2 e^{-2\tau})(1 + \bar{w}_0^2 e^{-2\tau})} \quad (1.21)$$

where prime denotes derivative with respect to τ . Integrating both sides from 0 to T and letting $T \rightarrow \infty$ one obtains that

$$\zeta_\infty - \zeta_0 = \frac{i}{2} \ln \left(\frac{1 + \bar{w}_0^2}{1 + w_0^2} \right) \quad (1.22)$$

where $\lim_{T \rightarrow \infty} \zeta(T) = \zeta_\infty$. Equation (1.22) allows one to calculate the final value (as $\tau \rightarrow \infty$) of ζ . Using standard results from complex analysis [26] one obtains that if the initial conditions (w_0, ζ_0) satisfy the equation

$$\zeta_0 = -\arctan \left(\frac{2w_{10}w_{20}}{1 + w_{10}^2 - w_{20}^2} \right) \quad (1.23)$$

then $\zeta_\infty = 0$. That is, one can define a manifold of initial conditions in $\mathbb{R}^3$ from

$$\mathcal{M}_2 = \{(w_1, w_2, \zeta) \in \mathbb{R}^3 : \zeta + \arctan\left(\frac{2w_1 w_2}{1 + w_1^2 - w_2^2}\right) = 0\} \quad (1.24)$$

such that the *linear* feedback control law $\omega = -k w$ will drive all trajectories exponentially to the origin. The manifold $\mathcal{M}_2$ is shown in Fig. 1.1(b).

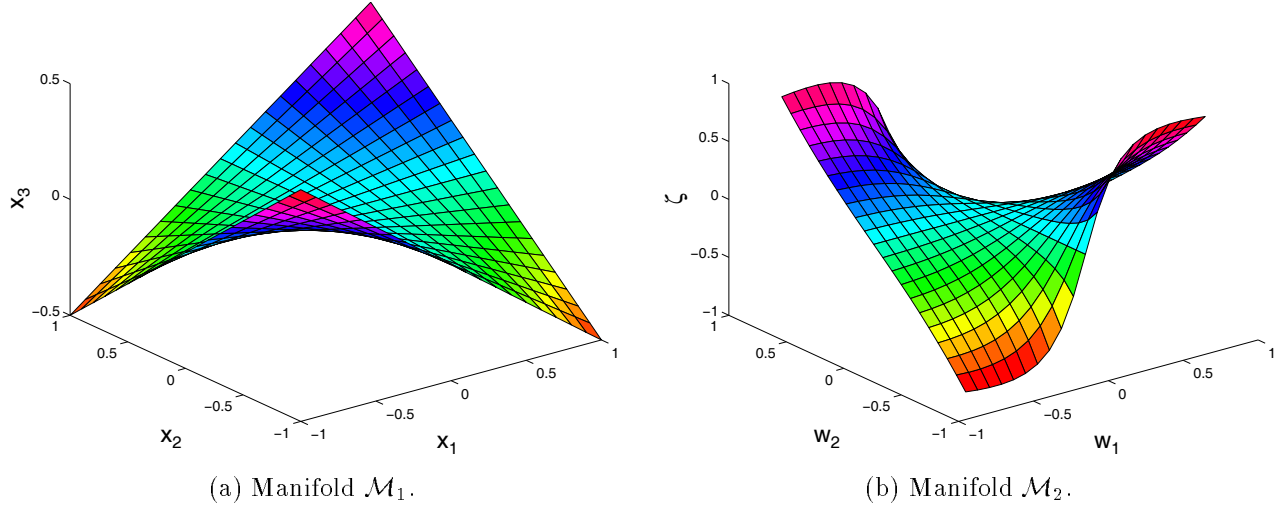


Figure 1.1 Invariant Manifolds for the Two Examples.

To finish the controller construction, one suffices to make the manifold $\mathcal{M}_2$ attractive. To this end, introduce the variable z defined by

$$z = \zeta + \arctan\left(\frac{2w_1 w_2}{1 + w_1^2 - w_2^2}\right) \quad (1.25)$$

Notice that z provides a measure of the distance from the manifold $\mathcal{M}_2$. Moreover, it can be shown that z satisfies the differential equation

$$\dot{z} = \text{Im}(\omega \bar{w}) \quad (1.26)$$

Proposition 2 Consider the system described by Eqs. (1.16a) and (1.26) and the feedback control law

$$\omega = -k w - i\mu \frac{z}{\bar{w}}, \quad (2\mu > k) \quad (1.27)$$

Then for all initial conditions $w_0 \neq 0$ we have that the closed-loop trajectory is bounded and

$$\lim_{t \rightarrow \infty} (w(t), z(t)) = 0 \quad (1.28)$$

with exponential rate of convergence. Moreover, the control history $\omega(t)$ is bounded.

Proof. The proof is obtained by noticing that for the closed-loop system one has

$$\frac{d|w|^2}{dt} = -k(1 + |w|^2)|w|^2 \quad (1.29a)$$

$$\dot{z} = -\mu z \quad (1.29b)$$

so both w and z decay to zero exponentially.

The control law in Eq. (1.27) remains bounded for all $t \geq 0$ since z decays faster than w if $2\mu > k$. $\triangleleft$

The control law (1.27) was first derived in [4]. The control law in Eqs. (1.10)-(1.11) for the 3-dimensional system in power form was given in [8] using the same arguments as in [4]. In fact, the similarity between these two examples is more than superficial. The system (1.14) can be transformed to (1.3) by the following state and input transformations [27]

$$x_1 = \ln\left(\frac{1+\sin\phi}{\cos\theta}\right) \cos\phi + \psi \sin\phi, \quad x_2 = \phi, \quad x_3 = \ln\left(\frac{1+\sin\phi}{\cos\theta}\right) \cos\phi + \psi \sin\phi \quad (1.30a)$$

$$u_1 = \omega_2 \sec\theta - x_3(\omega_2 \sin\phi \tan\theta + \omega_1), \quad u_2 = \omega_2 \sin\phi \tan\theta + \omega_1 \quad (1.30b)$$

The alternative form of the kinematics in terms of w and z seems to be, nevertheless, more appealing because of the physical significance of these coordinates [28, 25, 4].

1.3 Systems in Power Form and their Invariant Manifolds

The previous methodology can be extended to a class of underactuated mechanical systems without drift. Two important subclasses of such systems are systems in *chained* and *power* normal forms. Although not all underactuated mechanical systems can be transformed into chained or power form, a great number of systems encountered in practice can be converted into one of these forms. Wheeled robots, multi-trailers, underactuated symmetric rigid spacecraft are only but a few examples [2]. Moreover, there are transformations from one form to the other [29].

The mathematical model for an underactuated, nonholonomic system in power form with two inputs can be described by [8, 17]

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_j &= \frac{1}{(j-2)!} x_1^{j-2} u_2, \quad j = 2, \dots, n \end{aligned} \quad (1.31)$$

The construction of the invariant manifolds for a system in the form (1.31) can be easily performed using the methodology of Section 1.2.1; see also [8]. To this end, consider the system in (1.31) and the following linear feedback

$$u_1 = -k x_1, \quad u_2 = -k_1 x_2, \quad (k > 0, \quad k_1 > 0) \quad (1.32)$$

With this linear control law, the closed-loop equations are

$$\begin{aligned} \dot{x}_1 &= -k x_1 \\ \dot{x}_j &= -\frac{1}{(j-2)!} k_1 x_1^{j-2} x_2, \quad j = 2, \dots, n \end{aligned} \quad (1.33)$$

These equations can be explicitly integrated to obtain

$$\begin{aligned} x_1(t) &= x_{10} e^{-kt} \\ x_2(t) &= x_{20} e^{-k_1 t} \\ x_j(t) &= s_{1,j-2}(x_0) + \frac{k_1}{(j-2)!((j-2)k + k_1)} x_{10}^{j-2} x_{20} e^{-((j-2)k + k_1)t}, \quad j = 3, \dots, n \end{aligned} \quad (1.34)$$

where $x_0 = (x_{10}, x_{20}, x_{30}, \dots, x_{n0})^T \in \mathbb{R}^n$ is the initial state of the system, and where

$$s_{1,j} = x_{j+2} - \frac{k_1}{j!(j k + k_1)} x_1^j x_2, \quad j = 1, \dots, n-2 \quad (1.35)$$

Each equation in (1.35) defines a smooth function in terms of $x_1, x_2, \dots, x_n$. Therefore, Eqs. (1.35) define a series of smooth manifolds by

$$\Pi_{1,j} = \{x \in \mathbb{R}^n : s_{1,j}(x) = 0\}, \quad j = 1, \dots, n-2 \quad (1.36)$$

each of dimension $(n-1)$. Moreover, since

$$\text{rank} \left[\frac{\partial s_{1,j}}{\partial x_i} \right] = n-2 \quad (1.37)$$

the manifold

$$\Pi_1 = \bigcap_{j=1}^{n-2} \Pi_{1,j} = \{x \in \mathbb{R}^n : s_{1,j}(x) = 0, \quad j = 1, \dots, n-2\} \quad (1.38)$$

defined by the intersection of the manifolds $\Pi_{1,j}$ is also a (2-dimensional) smooth manifold [24].

Lemma 1 *Consider the system (1.31) under the feedback control (1.32). Then, for all initial conditions $x_0 \in \Pi$ the closed-loop trajectories of the system converge exponentially to the origin.*

Proof. First we show that each manifold $\Pi_{1,j}$ is invariant for the closed-loop system. Indeed,

$$\begin{aligned} \dot{s}_{1,j} &= \dot{x}_{j+2} - \frac{k_1}{j!(j k + k_1)} (j x_1^{j-1} \dot{x}_1 x_2 + x_1^j \dot{x}_2) \\ &= \frac{k k_1}{(j-1)!(j k + k_1)} x_1^j x_2 + \frac{k}{(j-1)!(j k + k_1)} x_1^j u_2 = 0 \end{aligned} \quad (1.39)$$

Subsequently, the manifold Π is also invariant for Eqs. (1.33). For $x_0 \in \Pi$ the solutions of the closed-loop system are given by (1.34) with $s_{1,j}(x_0) = 0$ ($j = 1, \dots, n-2$). The assertion of the lemma follows immediately. $\triangleleft$

1.4 A Recursive Algorithm for Systems in Power Form

In this section we present a recursive algorithm to create a series of systems which will be used to construct an exponentially convergent controller for the system in Eqs. (1.31). All

the systems generated by this recursive process (herein called the *generated systems*) can be put in power form through a linear transformation. These generated systems are, however, of reduced dimension. The methodology is based on the idea that by constructing a set of $(n-2)$ manifolds for the n -dimensional system, the problem of designing a feedback controller for the initial system becomes one of designing a feedback controller for a similar system in power form but of dimension $(n-1)$. By repeating this process, one ends up with a system in power form of dimension three, for which is relatively easy to design static, nonsmooth controllers.

1.4.1 The Recursive Process

Consider the n -dimensional system given in Eqs. (1.31), and construct a set of $n-2$ invariant manifolds for this system under the linear feedback $u_1 = -k x_1$, $u_2 = -k_1 x_2$ as in Eqs. (1.35). Let the following linear transformation

$$\begin{aligned} x_{2,1} &= x_1 \\ x_{2,j} &= ((j-1)k + k_1) s_{1,j-1} \quad j = 2, \dots, n-1 \end{aligned} \quad (1.40)$$

One can then define the following system in terms of $x_{2,j}$ for $1 \leq j \leq n-1$

$$\begin{aligned} \dot{x}_{2,1} &= u_{2,1} \\ \dot{x}_{2,j} &= \frac{1}{(j-2)!} x_{2,1}^{j-2} u_{2,2} \quad j = 2, \dots, n-1 \end{aligned} \quad (1.41)$$

where

$$u_{2,1} = u_1 \quad u_{2,2} = k x_1 u_2 - k_1 x_2 u_1 \quad (1.42)$$

The system in Eqs. (1.41) will be called the second *generated system* and we use the first index in the subscript of the state elements to denote this. For consistency, we define the first generated system to be simply the original system in Eqs. (1.31), that is, we let $x_{1,j} = x_j$ for $1 \leq j \leq n$. Notice that Eqs. (1.41) represent a system in power form of dimension $(n-1)$. The same process can be therefore repeated for this system.

After repeating this process $(i-1)$ times one obtains the i th generated system (of dimension $n-i+1$)

$$\begin{aligned} \dot{x}_{i,1} &= u_{i,1} \\ \dot{x}_{i,j} &= \frac{1}{(j-2)!} x_{i,1}^{j-2} u_{i,2} \quad j = 2, \dots, n-i+1 \end{aligned} \quad (1.43a)$$

For the i th generated system, one can construct $(n-i-1)$ invariant manifolds using the linear control law

$$u_{i,1} = -k x_{i,1} \quad u_{i,2} = -k_i x_{i,2} \quad (k_i > 0) \quad (1.44)$$

and the methodology described earlier. The corresponding manifolds are defined by

$$\Pi_{i,j} = \{x \in \mathbb{R}^{n-i+1} : s_{i,j}(x) = 0\} \quad 1 \leq j \leq n-i-1 \quad (1.45)$$

where

$$s_{i,j} = x_{i,j+2} - \frac{k_i}{j!(j k + k_i)} x_{i,1}^j x_{i,2} \quad j = 1, \dots, n-i-1 \quad (1.46)$$

Defining now the transformation

$$\begin{aligned} x_{i+1,1} &= x_{i,1} \\ x_{i+1,j} &= (j k + k_i) s_{i,j-1} \quad j = 2, \dots, n-i \end{aligned} \quad (1.47)$$

the $(i+1)$ th generated system can be described as follows

$$\begin{aligned} \dot{x}_{i+1,1} &= u_{i+1,1} \\ \dot{x}_{i+1,j} &= \frac{1}{(j-2)!} x_{i+1,1}^{j-2} u_{i+1,2} \quad j = 2, \dots, n-i \end{aligned} \quad (1.48)$$

where

$$u_{i+1,1} = u_{i,1} \quad u_{i+1,2} = k x_{i,1} u_{i,2} - k_i u_{i,1} x_{i,2} \quad (1.49)$$

This process can be continued until the $(n-2)$ th generated system, which is the 3-dimensional system in power form

$$\begin{aligned} \dot{x}_{n-2,1} &= u_{n-2,1} \\ \dot{x}_{n-2,2} &= u_{n-2,2} \\ \dot{x}_{n-2,3} &= x_{n-2,1} u_{n-2,2} \end{aligned} \quad (1.50)$$

1.5 The Feedback Controller

Our objective is to use the manifolds $\Pi_{i,j}$ in Eq. (1.45) to design exponentially convergent controllers for the system (1.31). By construction, it is immediate that if for the $(i+1)$ th generated system we have that

$$x_{i+1,1} = x_{i+1,2} = x_{i+1,3} = \dots = x_{i+1,n-i} = 0 \quad (1.51)$$

then for the i th generated system we have that

$$x_{i,1} = x_{i,3} = \dots = x_{i,n-i+1} = 0 \quad (1.52)$$

Thus, any convergent controller about the origin for the $(i+1)$ th generated system, will also make the i th generated system converge to the one-dimensional manifold

$$\mathcal{M}_i = \{x \in \mathbb{R}^{n-i+1} : x_{i,1} = x_{i,3} = \dots = x_{i,n-i+1} = 0\} \quad (1.53)$$

In particular, if the controller for the $(i+1)$ th generated system is chosen such that, in addition, satisfies the property

$$\lim_{t \rightarrow \infty} x_{i,2} = 0 \quad (1.54)$$

then the same control law will drive the i th generated system to the *origin*.

Theorem 1 Consider the system in Eqs. (1.31) ($n \geq 3$) and the feedback controller

$$u_1 = u_{n-2,1} \quad (1.55a)$$

$$u_2 = \frac{u_{n-2,2}}{k^{n-3} x_1^{n-3}} - \sum_{\ell=1}^{n-3} k_\ell \frac{x_{\ell,2}}{k^{\ell-1} x_1^{\ell-1}} \quad (1.55b)$$

where

$$u_{n-2,1} = -k x_1 \quad u_{n-2,2} = -k_{n-2} x_{n-2,2} - \mu \frac{s}{x_1} \quad (1.56)$$

where

$$k > 0, \quad k_1 > 0, \quad k_j > k + k_{j-1}, \quad (j = 2, \dots, n-2), \quad \mu > (k + k_{n-2})^2/k \quad (1.57)$$

and

$$s = s_{n-2,1} = x_{n-2,3} - \frac{k_{n-2}}{k_{n-2} + k} x_1 x_{n-2,2} \quad (1.58)$$

and where $x_{n-2,3}$ and $x_{\ell,2}$ ($\ell = 1, \dots, n-2$) are derived through the recursive process described in Section 1.4. This control law is bounded along closed loop trajectories and has the property that

$$\lim_{t \rightarrow \infty} (x_1(t), x_2(t), \dots, x_n(t)) = 0 \quad (1.59)$$

exponentially, for all initial conditions such that $x_1(0) \neq 0$.

Proof. First, notice that the recursive algorithm guarantees that the following equations hold

$$x_1 = x_{2,1} = x_{3,1} = \dots = x_{n-2,1} \quad (1.60a)$$

$$u_1 = u_{2,1} = u_{3,1} = \dots = u_{n-2,1} \quad (1.60b)$$

From Eqs. (1.50) and (1.56) we have that the $(n-2)$ th generated closed-loop system is given by

$$\begin{aligned} \dot{x}_{n-2,1} &= -k x_1 \\ \dot{x}_{n-2,2} &= -k_{n-2} x_{n-2,2} - \mu \frac{s}{x_1} \\ \dot{x}_{n-2,3} &= -k_{n-2} x_{n-2,2} x_1 - \mu s \end{aligned} \quad (1.61)$$

From these equations one has immediately that $x_1(t) = x_{10} e^{-kt}$ and x_1 decreases exponentially with rate of decay k . The variable s in Eqs. (1.58) is the invariant manifold for the system in Eq. (1.50) under the linear feedback

$$u_{n-2,1} = -k x_1 \quad u_{n-2,2} = -k_{n-2} x_{n-2,2} \quad (1.62)$$

The differential equation for s is

$$\begin{aligned} \dot{s} &= \frac{k}{k_{n-2} + k} x_1 u_{n-2,2} - \frac{k_{n-2}}{k_{n-2} + k} x_{n-2,2} u_{n-2,1} \\ &= -\frac{k\mu}{k_{n-2} + k} s \end{aligned} \quad (1.63)$$

and s decreases exponentially with rate of decay $k\mu/(k_{n-2} + k)$. Moreover, by definition, $\lim_{t \rightarrow \infty} s(t) = 0$ implies that $\lim_{t \rightarrow \infty} x_{n-2,3}(t) = 0$.

The differential equation for $x_{n-2,2}$ can be written as follows

$$\dot{x}_{n-2,2} = -k_{n-2} x_{n-2,2} - \mu \gamma_1(t) \quad (1.64)$$

where the function

$$\gamma_1(t) = \frac{s(t)}{x_1(t)} \quad (1.65)$$

is exponentially decaying with rate greater than k_{n-2} . Thus, from Eq. (1.64) $\lim_{t \rightarrow \infty} x_{n-2,2}(t) = 0$ exponentially, with rate of decay k_{n-2} . Thus, the control law in Eqs. (1.56) achieves $\lim_{t \rightarrow \infty} x_{n-2,j}(t) = 0$ for $j = 1, 2, 3$.

The rest of the proof follows by induction. To this end, let us assume that for the $(i+1)$ th generated system we have that $\lim_{t \rightarrow \infty} x_{i+1,j}(t) = 0$, for $j = 1, 2, 3, \dots, n-i$, which implies that $\lim_{t \rightarrow \infty} x_{i,j}(t) = 0$, for $j = 1, 3, \dots, n-i+1$. It has been shown previously that with the control law in Eq. (1.56) $x_{n-2,2}$ decays exponentially with rate greater than k_{n-2} . Assume also that the functions $x_{i+\ell,2}$, where $\ell = n-2-i, n-3-i, \dots, 1$, decay exponentially, each with corresponding rate $k_{i+\ell}$. We would like to show that $\lim_{t \rightarrow \infty} x_{i,2}(t) = 0$ with rate k_i .

To this end, consider the differential equation for $x_{i,2}$, given by

$$\dot{x}_{i,2} = u_{i,2} = - \sum_{\ell=0}^{n-3-i} \frac{k_{i+\ell}}{k^\ell} \frac{x_{i+\ell,2}}{x_1^\ell} + \frac{u_{n-2,2}}{k^{n-2-i} x_1^{n-2-i}} \quad (1.66)$$

and using the expression for $u_{n-2,2}$ from (1.56)

$$\dot{x}_{i,2} = -k_i x_{i,2} - \sum_{\ell=1}^{n-2-i} \frac{k_{i+\ell}}{k^\ell} \frac{x_{i+\ell,2}}{x_1^\ell} - \frac{\mu}{k^{n-2-i}} \frac{s}{x_1^{n-1-i}}, \quad i = 1, \dots, n-3 \quad (1.67)$$

From (1.57) it is easy to see that the following equations are satisfied

$$\begin{aligned} k_{n-2} &> k_i + (n-2-i)k \\ k_{n-3} &> k_i + (n-3-i)k \\ &\vdots \\ k_{i+1} &> k_i + k \end{aligned}$$

According to Eq. (1.63), s decays exponentially with rate greater than $k_{n-2} + k$ thus the functions

$$\gamma_\nu = \frac{s}{x_1^\nu}, \quad \nu = 2, \dots, n-2 \quad (1.69)$$

decay exponentially with rate greater than k_i . By assumption, $x_{i+\ell,2}$ ($\ell = n-2-i, n-3-i, \dots, 1$) decay exponentially, each with corresponding rate $k_{i+\ell}$ whereas x_1 decays with rate k . Thus, the functions

$$\rho_\ell = \frac{x_{i+\ell,2}}{x_1^\ell}, \quad \ell = n-2-i, n-3-i, \dots, 1 \quad (1.70)$$

decay exponentially, each with rate greater than k_i .

Equation (1.67) is thus a linear differential equation in terms of $x_{i,2}$ perturbed by exponentially decaying terms of rate greater than k_i . Thus, we have that

$$\lim_{t \rightarrow \infty} x_{i,2}(t) = 0, \quad (1.71)$$

with rate of k_i .

The fact the the control law is bounded follows immediately from the fact that x_1 reaches the origin only asymptotically (not in finite time), and the fact that the functions $\rho_\ell(t)$ in Eq. (1.70) and $\gamma_\nu(t)$ in Eq. (1.69) are bounded along trajectories of the closed-loop system $\triangleleft$

Remark 1 The control law in Eq. (1.55) makes the constructed invariant manifolds in each step attractive, such that the trajectories of the closed-loop system tend (asymptotically) to these manifolds. On these manifolds the control laws reduce to the linear control laws $u_1 = -k x_1$ and $u_{i,2} = -k_i x_{i,2}$ which force the the trajectories to the origin. It is clear that for this procedure to work the attraction of the trajectories to the corresponding manifolds at each step should take place on different times scales. This is achieved by making the gains of the control laws $u_{i,2}$ bigger than the one in the previous step. •

Remark 2 The control law in Eqs. (1.55) will work as long as $x_1(0) \neq 0$. In fact, the proposed controllers are discontinuous at the origin of the state space. They are, nevertheless, differentiable functions of time along closed-loop trajectories. If initially $x_1(0) = 0$ one can use any (open-loop) control law such that x_1 becomes nonzero and then switch to the control law (1.55) [4, 8]. The term “quasi-smooth” or “quasi-continuous” has been proposed to describe such control laws [8, 22]. •

Remark 3 The boundedness of the control laws is guaranteed by imposing specific convergence rates between the different states. In addition, the manifolds have been constructed using exact (i.e., analytic) integration of the closed-loop dynamics. Since convergence rates and closed-loop dynamics are affected by disturbances, these control laws may require robustification [30]. •

Simulation results show that the previously described algorithm provides an effective way to design exponentially convergent controllers for an n -dimensional nonholonomic control system in power form. Figure 1.5 shows the trajectories of a 5th order system with initial state $x(0) = (4, 1, 3, 2, 1)$ subject to the feedback control in Eq. (1.55). The gains were chosen as

$$k = k_1 = 1, \quad k_2 = 3, \quad k_3 = 5, \quad \mu = 37 \quad (1.72)$$

Notice that the control law drives the states to the origin at different time scales. This is more evident in Fig. 1.2(b) where it is shown the time history of the logarithm of the euclidean norm of the state. The linear slopes of the states indicate the exponential rate of convergence. The corresponding control history is shown in Fig. 1.3.

1.6 Conclusions

We present a methodology for constructing exponentially convergent controllers for nonholonomic systems in power form. The construction of the proposed control laws is based on a recursive algorithm which uses a series of invariant manifolds in order to construct a sequence of generated systems in power form of reduced dimension. Using this process, one ends up with a 3-dimensional system in power form. The proposed control law for the n -dimensional system is an exponentially convergent control law for this 3-dimensional system

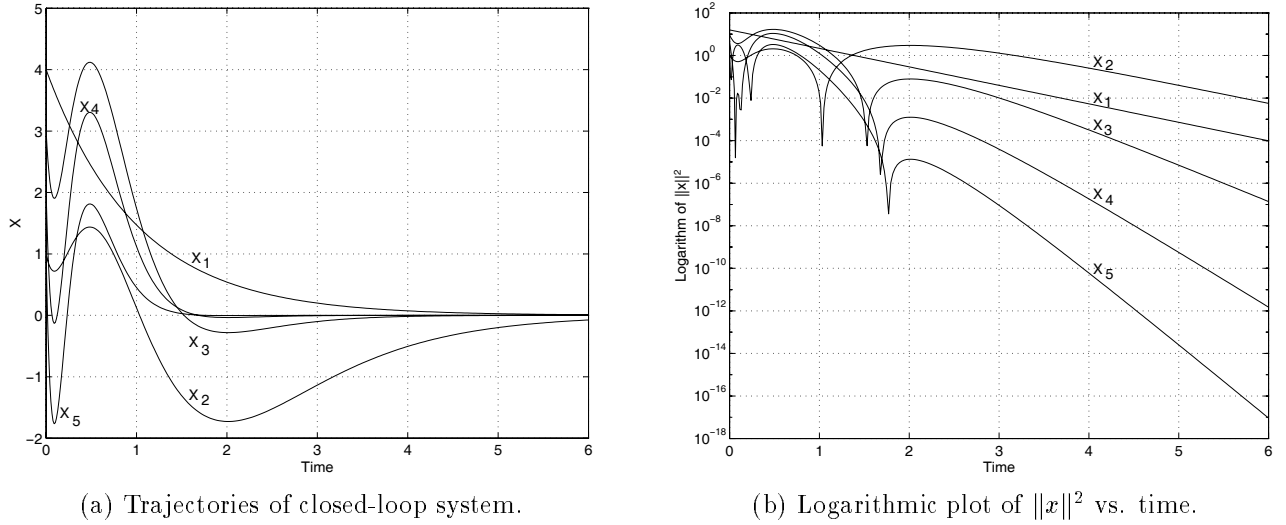


Figure 1.2 State Histories for the Numerical Example.

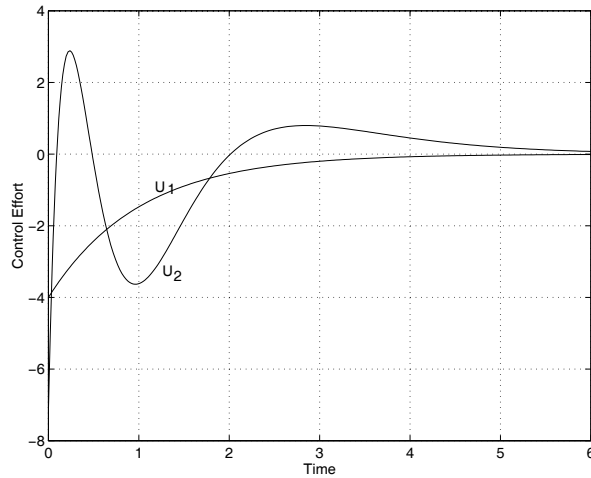


Figure 1.3 Control input history.

by proper choice of the gains. Finally, because of the equivalence between chained and power form systems, the control laws proposed here can also be used for nonholonomic systems in chained form.

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