

Friction Tire/Road Modeling, Estimation and Optimal Braking Control

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SUMMARY

In this paper we present a series of results concerning the problem of modeling, estimation of contact road/tire friction. We also present some aspects related to the problem of optimal braking control.

The modeling part of the paper we discuss new dynamic friction force model for the longitudinal road/tire interaction for wheeled ground vehicles. The model is based on a dynamic friction model developed previously for contact-point friction problems, called the LuGre model [7]. By assuming a contact patch between the tire and the ground the model is indeed given by a PDE. It is however possible to approximate the PDE by a ODE (the lumped model) for the friction force is developed based on the patch boundary conditions and the normal force distribution along the contact patch. We also discuss possible extension of longitudinal models to high dimensional models including lateral force and self alignment torque. Finally, we discuss how these models may be affected by other external factors such as: road conditions (roughness), road/tire interfaces (wet road), temperature variation, etc.

In the estimation section we discuss the problem of tire-road friction estimation using only angular wheel velocity which can not always be computed from actual sensors. Tire forces information is relevant to problems like: optimization of Anti-lock brake systems (ABS), traction system, diagnostic of the road friction conditions, etc.

These results may suggest alternative traction control methodologies, other than the current ones based on the use of tracking of the “optimal” slip coefficient using, for example, sliding mode control. These aspects are presented in the last two section of the paper. First, we investigate optimal braking strategies for a simplified model of a wheeled vehicle. We derive the *exact* optimality conditions for the minimum braking distance problem by solving the associated optimal control problem. It is shown that the optimal control is singular and can be written in a state feedback form. Then, we present a control scheme for emergency braking of vehicles. The controller utilizes estimated state feedback control to achieve a near maximum deceleration.

Keywords: Tire/road contact friction models, ABS and traction systems.

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1 INTRODUCTION

The problem of predicting the friction force between the tire and the ground for wheeled vehicles is of enormous importance to automotive industry. Since friction is the major mechanism for generating forces on the vehicle, it is extremely important to have an accurate characterization of the magnitude (and direction) of the friction force generated at the ground/tire interface. However, accurate tire/ground friction models are difficult to obtain analytically. Subsequently, in the past several years, the problem of modeling and predicting tire friction has become an area of intense research in the automotive community. In particular, ABS and traction control systems rely on knowledge of the friction characteristics. Such systems have enhanced safety and maneuverability to such an extent, that they have become almost mandatory for all future passenger vehicles.

Traction control systems reduce or eliminate excessive slipping or sliding during vehicle acceleration and thus enhance the controllability and maneuverability of the vehicle. Proper traction control design has a paramount effect on safety and handling qualities for passenger vehicles. Traction control aims to achieve maximum torque transfer from the wheel axle to forward acceleration. Similarly, anti-lock braking systems (ABS) prohibit wheel lock and skidding during braking by regulating the pressure applied on the brakes, thus increasing lateral stability and steerability, especially during wet and icy road conditions. As with the case of traction control, the main difficulty in designing ABS systems is the nonlinearity and uncertainty of the tire/road models. In either case, the friction force at the tire/road interface is the main mechanism for converting wheel angular acceleration or deceleration (due to the motor torque or braking) to forward acceleration or deceleration (longitudinal force). Therefore, the study of the friction force characteristics at the road/tire interface is of paramount importance for the design of ABS and/or traction control systems. Moreover, tire friction models are also indispensable for accurately reproducing friction forces for simulation purposes. Active control mechanisms, such as ESP, TCS, ABS, steering control, active suspension, etc. may be tested and optimized using vehicle mechanical 3D simulators with suitable tire/road friction models.

A common assumption in most tire friction models is that the normalized tire friction μ

$$\mu = \frac{F}{F_n} = \frac{\text{Friction force}}{\text{Normal force}}$$

is a nonlinear function of the normalized relative velocity between the road and the tire (slip coefficient s) with a distinct maximum; see Fig. 1. In addition, it is understood that μ also depends on the velocity of the vehicle and road surface conditions, among other factors (see [6] and [16]). The curves shown in Fig. 1 illustrate how these factors influence the shape of μ .

The curves shown in Fig. 1 are derived empirically, based solely on steady-state (i.e., constant linear and angular velocity) experimental data [16, 3] in a highly controlled laboratory environment or using specially designed test vehicles. Under such steady-state conditions, experimental data seem to support the force vs. slip curves of Fig. 1. In reality, the linear and angular velocities can never be controlled independently and hence, such idealized steady-state conditions are not reached except during the rather uninteresting case of cruising with constant speed. The development of the friction force at the tire/road interface is very much a dynamic phenomenon. In other words, the friction force does not reach its steady-state value shown in Fig. 1 instantaneously, but rather exhibits transient behavior which may differ significantly from its steady-state value. Experiments performed in commercial vehicles, have shown that the tire/road forces do not necessarily vary along the curves shown Fig. 1, but rather “jump” from one value to another when these forces are displayed in the $\mu - s$ plane [26]. In addition, in realistic situations, these variations are most likely to exhibit hysteresis loops, clearly indicating the dynamic nature of friction.

In this paper we present a series of results concerning the problem of modeling, estimation of contact road/tire friction. We also present some aspects related to the problem of optimal braking control.

The modeling part of the paper we discuss new dynamic friction force model for the longitudinal road/tire interaction for wheeled ground vehicles. The model is based on a dynamic friction model developed previously for contact-point friction problems, called the LuGre model [7]. By assuming a contact patch between the tire and the ground the model is indeed given by a PDE. It is however possible to approximate the PDE by a ODE (the lumped model) for the friction force is developed based on the patch boundary conditions and the normal force distribution along the contact patch. We also discuss possible extension of longitudinal

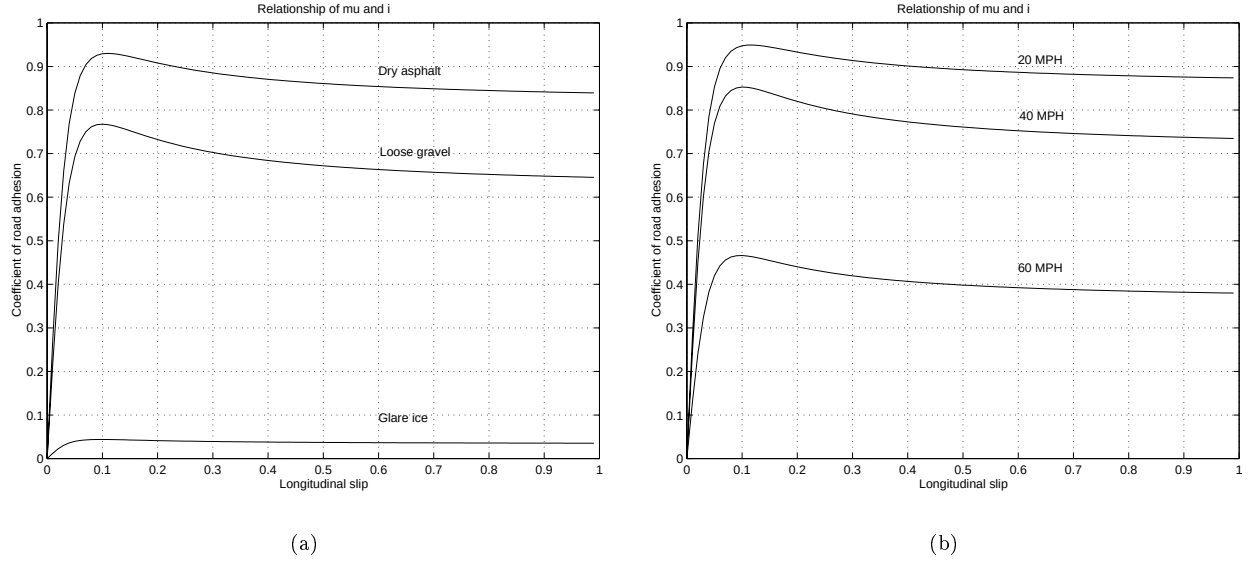


Figure 1: Typical variations of the tire/road friction profiles for different road surface conditions (a), and different vehicle velocities (b). Curves given by Harned et al. [16].

models to high dimensional models including lateral force and self alignment torque. Finally, we discuss how these models may be affected by other external factors such as: road conditions (roughness, wet road), road/tire interfaces (wet road), temperature variation, etc.

In the estimation section we discuss the problem of tire-road friction estimation using only angular wheel velocity which can not always be computed from actual sensors. Tire forces information is relevant to problems like: optimization of Anti-lock brake systems (ABS), traction system, diagnostic of the road friction conditions, etc.

These results may suggest alternative traction control methodologies, other than the current ones based on the use of tracking of the “optimal” slip coefficient using, for example, sliding mode control. These aspects are presented in the last two section of the paper. First, we investigate optimal braking strategies for a simplified model of a wheeled vehicle. We derive the *exact* optimality conditions for the minimum braking distance problem by solving the associated optimal control problem. It is shown that the optimal control is singular and can be written in a state feedback form. Then, we present a control scheme for emergency braking of vehicles. The controller utilizes estimated state feedback control to achieve a near maximum deceleration.

2 ROAD/TIRE CONTANT FRICTION MODELS

In this section we consider the simplified motion dynamics of a quarter-vehicle model. The system is then of the form

$$m\dot{v} = F \quad (1)$$

$$J\dot{\omega} = -rF + u, \quad (2)$$

where m is 1/4 of the vehicle mass and J , r are the inertia and radius of the wheel, respectively. v is the linear velocity of the vehicle, ω is the angular velocity of the wheel, u is the accelerating (or braking) torque, and F is the tire/road friction force. For the sake of simplicity, only longitudinal motion will be considered in this section. The lateral motion and well as combined longitudinal/lateral dynamics are discussed in

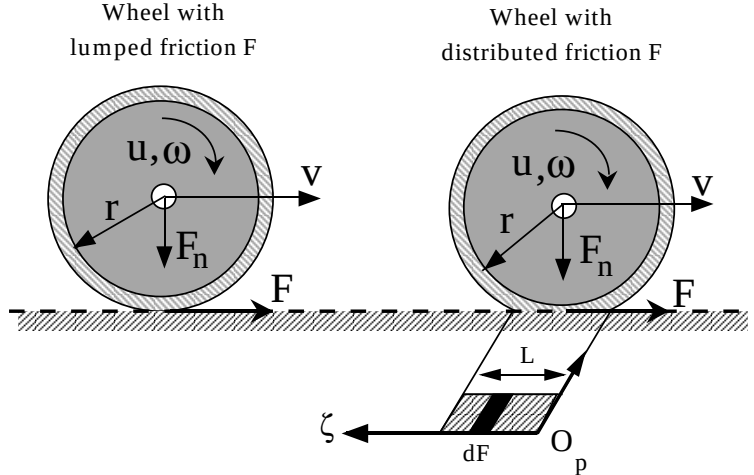


Figure 2: One-wheel system with lumped friction (left), and distributed friction (right).

subsequent sections. The dynamics of the braking and driving actuators, suspension dynamics, etc. are also neglected.

2.1 Static slip/force models

The most common tire friction models used in the literature are those of algebraic slip/force relationships. They are defined as one-to-one (memoryless) maps between the friction F , and the longitudinal slip rate s , which defined as

$$s = \begin{cases} s_b = \frac{r\omega}{v} - 1 & \text{if } v > r\omega, \ v \neq 0 \quad \text{for braking} \\ s_d = 1 - \frac{v}{r\omega} & \text{if } v < r\omega, \ \omega \neq 0 \quad \text{for driving} \end{cases} \quad (3)$$

The slip rate results from the reduction of the effective circumference of the tire as a consequence of the tread deformation due to the elasticity of the tire rubber [20]. This, in turn, implies that the ground velocity v will not be equal to $r\omega$. The absolute value of the slip rate is defined in the interval $[0, 1]$. When $s = 0$ there is no sliding (pure rolling), whereas $|s| = 1$ indicates full sliding/skidding. It should be pointed out that in this paper we always define the relative velocity as $v_r = r\omega - v$. As a result, the slip coefficient in (3) is positive for driving and negative for braking. This is somewhat different than what is done normally in the literature, where the relative velocity (and hence the slip s) is kept always positive by redefining $v_r = v - r\omega$ in case of braking. Since we wish our results to hold for both driving and braking, we feel that it is more natural to keep the same definition for the relative velocity for both cases. This also avoids any inconsistencies and allows for easy comparison between the braking and driving regimes.

The slip/force models aim at describing the shapes shown in Fig. 1 via static maps $F(s) : s \mapsto F$. They may also depend on the vehicle velocity v , i.e. $F(s, v)$, and vary when the road characteristics change.

One of the most well-known models of this type is Pacejka's model (see, Pacejka and Sharp [22]), also known as the "Magic Formula." This model has been shown to suitably match experimental data, obtained under particular conditions of constant linear and angular velocity. The Pacejka model has the form¹

$$F(s) = c_1 \sin(c_2 \arctan(c_3 s - c_4(c_3 s - \arctan(c_3 s)))) ,$$

where the c'_i 's are the parameters characterizing this model. These parameters can be identified by matching experimental data, as shown in Bakker et al. [3].

¹In the formulas that follow, it is assumed that $s \in [0, 1]$. Hence these formulas give the magnitude of the friction force. The sign of F is then determined from the sign of $v_r = r\omega - v$.

Another static model is the one proposed by Burckhardt [6]. The tire/road friction characteristic is of the form

$$F(s, v) = (c_1(1 - e^{-c_2 s}) - c_3 s) e^{-c_4 v}, \quad (4)$$

where $c_1, \dots, c_4$ are some constants. This model has a velocity dependency, seeking to match variations like the ones shown in Fig. 1(b).

Kiencke and Daiss [17] neglect the velocity-dependent term in equation (4) and after approximating the exponential function in (4), they obtain the following expression for the friction/slip curve

$$F(s) = k_s \frac{s}{c_1 s^2 + c_2 s + 1}, \quad (5)$$

where k_s is the slope of the $F(s)$ vs. s curve at $s = 0$, and c_1 and c_2 are properly chosen parameters.

Alternatively, Burckhardt [5] proposes a simple, velocity-independent three-parameter model as follows

$$F(s) = c_1(1 - e^{-c_2 s}) - c_3 s. \quad (6)$$

All the previous friction models are highly nonlinear in the unknown parameters, and thus they are not well-adapted to be used for on-line identification. For this reason, simplified models like

$$F(s) = c_1 \sqrt{s} - c_2 s \quad (7)$$

have been proposed in the literature.

It is also well understood that the “constant” $c'_i s$ in the above models, are not really invariant, but they may strongly depend on the tire characteristics (e.g., compound, tread type, tread depth, inflation pressure, temperature), on the road conditions (e.g., type of surface, texture, drainage, capacity, temperature, lubricant, etc.), and on the vehicle operational conditions (velocity, load); see, for instance the discussion in [21].

2.2 Lumped Dynamic friction models

The static friction models of the previous section are appropriate when we have steady-state conditions for the linear and angular velocities. In fact, the experimental data used to validate the friction/slip curves are obtained using specialized equipment that allow independent linear and angular velocity modulation so as to transverse the whole slip range. This steady-state point of view is rarely true in reality, especially when the vehicle goes through continuous successive phases between acceleration and braking.

As an alternative to the static $F(s)$ maps, different forms of dynamic models can be adopted. The so-called “dynamic friction models” attempt to capture the transient behaviour of the tire-road contact forces under time-varying velocity conditions. Generally speaking, dynamic models can be formulated either as lumped or as distributed models, as shown in Fig. 2. A *lumped* friction model assumes a point tire-road friction contact. As a result, the mathematical model describing such a model is an ordinary differential equations that can be easily solved by time integration. *Distributed* friction models, on the other hand, assume the existence of a *contact patch* between the tire and the ground with an associated normal pressure distribution. This formulation results in a partial differential equation, that needs to be solved both in time and space.

A number of dynamic models have been proposed in the literature that can be classified under the term “dynamic friction models.” One such model, for example, has been proposed by Bliman et al. in [4]. In that reference the friction is calculated by solving a differential equation of the following form

$$\begin{aligned} \dot{z} &= |v_r| A z + B v_r \\ F(z, v_r) &= C z + \text{sgn}(v_r) D \end{aligned} \quad (8)$$

The matrix A is required to be Hurwitz of dimension either one or two, with the latter case being more accurate. Another lumped dynamic model that can be used to accurately predict the friction forces during transients is the LuGre friction model [9]. The derivation of this model is discussed in great detail [10], and summarized in Section 2.3. Before doing that, we next present an example of a frequently used dynamic

friction model, the so-called “kinematic model”. This will allow us to motivate our introduction in Section 2.3 of a new distributed (and later, an average lumped) friction model. In addition, it will be shown that these two commonly used lumped models are not able to reproduce the steady-state characteristics, similar to those of Pacejka’s “Magic Formula.”

Kinematic-Based Models

These models are derived from the idealization of a contact point deformation and from kinematic considerations (velocity relations between the points that concern the tire deformations). Their derivation follows semi-empirical considerations and assumes that the contact forces result from the product of the tire deformation and the tire stiffness.

An example of a two-dimensional model characterizing the lateral force and the aligning moment, can be found in [19]. A brush model for the longitudinal tire dynamics has been derived in [11, 2].

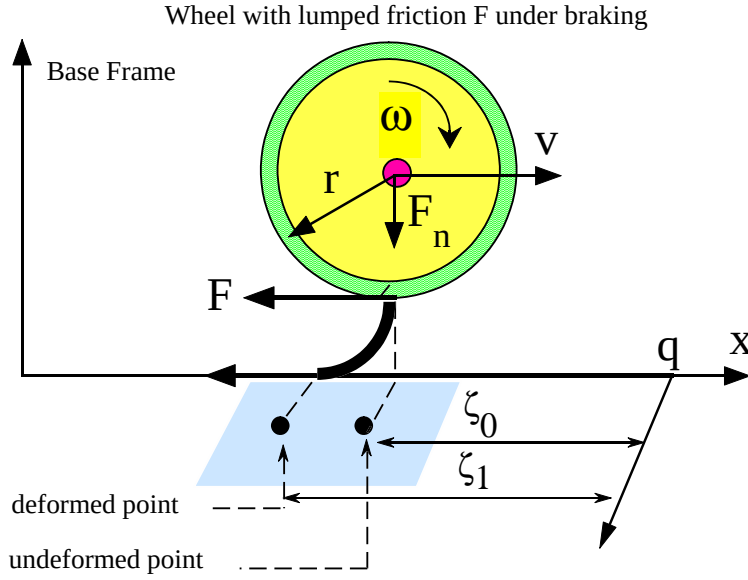


Figure 3: View of the contact area with the position of the undeformed contact point ζ_0 , and the point ζ_1 that deforms under longitudinal shear forces during a brake phase.

For simplicity, next we only discuss the breaking case. The traction case follows a similar development, with an appropriate definition of z . In that case, the relaxation length may be defined as the wheel arc length required to build friction.

A model for the longitudinal dynamics in [11] is derived by defining the *normalized* longitudinal slip z as

$$z = \frac{\zeta_1 - \zeta_0}{\zeta_0} \quad (9)$$

where ζ_1 locates a hypothetical element which follows the road, and it defines the distance from the *deformed* wheel point to a forward point q . ζ_0 locates a hypothetical element which is undeformed under longitudinal shear forces, and defines the distance from the *undeformed* wheel point (i.e. center of the wheel’s rotational axis) to a forward point q as shown in Fig. 3. The undeformed point ζ_0 and the forward point q are moving at the same ground speed v , thus the distance ζ_0 is constant with respect to the moving point q . The length ζ_0 is known as the longitudinal relaxation length.

Differentiating z in (9) with respect to time, and noticing that $\dot{\zeta}_0 = v$, $\dot{\zeta}_1 = r\omega$, we get²

$$\frac{1}{\sigma} \frac{dz}{dt} = v_r \text{sgn}(v) - |v|z \quad (10)$$

$$F = h(z) \quad (11)$$

²In this expression both positive and negative v are considered.

where v is the linear velocity, $v_r = r\omega - v$ is the relative velocity, and the friction force F is defined by the function $h(z)$ that describes the stationary slip characteristics. In the simplest case, $h(z)$ is given by a linear relationship between the longitudinal slip and the tire (linear) stiffness k ,

$$h(z) = kz$$

The constant $1/\sigma = \zeta_0$ is called the relaxation length, and can be defined as the distance required to reach the steady-state value of F

$$F_{ss} = h(z_{ss}) = kz_{ss} = k \frac{r\omega - v}{v} = ks$$

after a step change of the slip longitudinal velocity, $s = s_b = v_r/v = (r\omega - v)/v$. The role of the relaxation length $1/\sigma$ in equation (10), can be better understood by rewriting this equation in terms of the spatial coordinate η ,

$$\eta(t) = \int_0^t |v(\tau)| d\tau$$

rather than as a time-differential equation, i.e.,

$$\frac{1}{\sigma} \frac{dz}{d\eta} = \frac{1}{\sigma} \frac{dz}{d\eta} \frac{d\eta}{dt} = v_r \text{sgn}(v) - |v|z \quad (12)$$

$$\frac{1}{\sigma} \frac{dz}{d\eta} = -z + \frac{v_r}{v} = -z + s \quad (13)$$

Equation (13) can thus be seen as a first order spatial equation with the sliding velocity s as its input. It thus becomes clear that σ represents the *spatial* constant of this equation.

As pointed out in [11], this model works well for high speeds, but it generates lightly damped oscillations at low speeds. The reason for this is that at quasi-steady-state regimes, z is close to its steady-state value ($z \approx s$), and the friction F is dominated by its spring-like behaviour ($F \approx kz$), resulting in a lightly damped mechanical system. Additional considerations are necessary to make this model consistent for all possible changes in the velocity sign. This restricts the usefulness of this model for tire friction analysis and control development. Reference [11] actually provides a twelve-step algorithm for implementing this model in simulations.

Note that the kinematic model do not exhibit a maximum for values $|s| < 1$, as suggested by experimental data and also by Pacejka's formula. However, the kinematic model can be modified by redefining the function $h(z)$ in (11) so as to produce a steady-state behaviour similar to the one predicted by the "Magic Formula" [2].

The Lumped LuGre Model

The LuGre model is an extension of the Dahl model that includes the Stribeck effect (see, [7]). This model will be used as a basis for further developments for the final model proposed in this paper. The lumped, LuGre model as proposed in [9], and [8] is given as,

$$\dot{z} = v_r - \frac{\sigma_0 |v_r|}{g(v_r)} z \quad (14)$$

$$F = (\sigma_0 z + \sigma_1 \dot{z} + \sigma_2 v_r) F_n \quad (15)$$

with

$$g(v_r) = \mu_c + (\mu_s - \mu_c) e^{-|v_r/v_s|^\alpha} \quad (16)$$

where σ_0 is the rubber longitudinal lumped stiffness, σ_1 the rubber longitudinal lumped damping, σ_2 the viscous relative damping, μ_c the normalized Coulomb friction, μ_s the normalized static friction, ($\mu_c \leq \mu_s$, $\mu_c + \mu_s$), v_s the Stribeck relative velocity, F_n the normal force, $v_r = r\omega - v$ the relative velocity, and z the internal friction state. The constant parameter α is used to capture the steady-steady friction/slip characteristic³.

³The model in (15) differs from the point-contact LuGre model in [7] in the way that the function $g(v)$ is defined. Here we propose to use $\alpha = 1/2$ instead of $\alpha = 2$ as in the LuGre point-contact model in order to better match the pseudo-stationary characteristic of this model (map $s \mapsto F(s)$) with the shape of the Pacejka's model.

In contrast to the Dahl model, the lumped LuGre model does exhibit a maximum friction for $|s| \leq [0, 1]$, but it still displays a discontinuous steady-state characteristic, at zero relative velocity. Nevertheless, by letting the internal bristle deflection z depend on both the time and the contact position ζ along the contact patch, it is possible to show that this model will have the appropriate steady-state properties. This is discussed next.

2.3 The LuGre Distributed model

Distributed models assume the existence of an area of contact (or patch) between the tire and the road, as shown in Fig. 2. This patch represents the projection of the part of the tire that is in contact with the road. With the contact patch is associated a frame O_p , with ζ -axis along the length of the patch in the direction of the tire rotation. The patch length is L .

Brief Review of Existing Models

Distributed dynamical models, have been studied previously, for example, in the works of Bliman et al. [4]. In these models, the contact patch area is discretized to a series of elements, and the microscopic deformation effects are studied in detail. In particular, Bliman et al. [4] characterize the elastic and Coulomb friction forces at each point of the contact patch, and they give the aggregate effect of these distributed forces by integrating over the whole patch area. They propose a second order rate-independent model (similar to Dahl's model), by applying the point friction model (8) to a rubber element situated at ζ at time t . By letting $z(\zeta, t)$ denote the corresponding friction state, they obtain the partial differential equation

$$\frac{\partial z}{\partial t} + r\omega \frac{\partial z}{\partial \zeta} = |v_r|Az + Bv_r, \quad z(\zeta, 0) = z(0, t) = 0 \quad (17)$$

$$F(t) = \frac{F_p}{L} \int_0^L Cz(\zeta, t) d\zeta \quad (18)$$

They also show that, under constant v and ω , there exists a choice of parameters A , B and C that closely match a curve similar to the one characterizing the "Magic Formula."

In [26] van Zanten et al. use a distributed brush model. The contact patch is described by a brush-type model where the displacement of each bristle is characterized by the state z_i , $i = 1, \dots, N$. The discretized version of this model, in its simplest form, is given by

$$\frac{dz_i}{dt} = v - \omega r - \frac{z_i - z_{i-1}}{L/N} \omega r \quad (19)$$

$$F = \sum_{i=1}^N c_i z_i \quad (20)$$

where N is the number of discrete elements (bristles) and c_i is the stiffness of the bristle. The imposed boundary condition $dz_1/dt = 0$ implies that the bristle at the beginning of the contact area has no displacement.

Distributed LuGre Model

One can also extend the point friction model (14)-(15) to a distributed friction model along the patch by letting $z(\zeta, t)$ denote the friction state (deflection) of the bristle/patch element located at the point ζ along the patch at a certain time t . At every time instant $z(\zeta, t)$ provides the deflection distribution along the contact patch. The model (14)-(15) can now be written as

$$\frac{dz}{dt}(\zeta, t) = v_r - \frac{\sigma_0 |v_r|}{g(v_r)} z \quad (21)$$

$$F = \int_0^L dF(\zeta, t), \quad (22)$$

with $g(v_r)$ defined as in (16) and where

$$dF(\zeta, t) = \left(\sigma_0 z(\zeta, t) + \sigma_1 \frac{\partial z}{\partial t}(\zeta, t) + \sigma_2 v_r \right) dF_n(\zeta, t),$$

where $dF(\zeta, t)$ is the differential friction force developed in the element $d\zeta$ and $dF_n(\zeta, t)$ is the differential normal force applied in the element $d\zeta$ at time t . This model assumes that the contact velocity of each differential state element is equal to v_r .

Assuming a steady-state normal force distribution $dF_n(\zeta, t) = dF_n(\zeta)$ and introducing a normal force density function $f_n(\zeta)$ (force per unit length) along the patch, i.e.,

$$dF_n(\zeta) = f_n(\zeta)d\zeta$$

one obtains the total friction force as

$$F(t) = \int_0^L (\sigma_0 z(\zeta, t) + \sigma_1 \frac{\partial z}{\partial t}(\zeta, t) + \sigma_2 v_r) f_n(\zeta) d\zeta \quad (23)$$

Noting that⁴ $\dot{\zeta} = |r\omega|$, and that

$$\frac{dz}{dt}(\zeta, t) = \frac{\partial z}{\partial \zeta} \frac{\partial \zeta}{\partial t} + \frac{\partial z}{\partial t},$$

we have that equation (21) describes a partial differential equation, i.e.

$$\frac{\partial z}{\partial \zeta}(\zeta, t) |r\omega| + \frac{\partial z}{\partial t}(\zeta, t) = v_r - \frac{\sigma_0 |v_r|}{g(v_r)} z(\zeta, t) \quad (24)$$

that should be solved in both in time and space. More details on the derivation of this distributed friction model are given in [10].

Steady-State Characteristics

The time steady-state characteristics of the model (21)-(22) are obtained by setting $\frac{\partial z}{\partial \zeta}(\zeta, t) \equiv 0$ and by imposing that the velocities v and ω are constant. Enforcing these conditions in (24) results in

$$\frac{\partial z(\zeta, t)}{\partial \zeta} = \frac{1}{|\omega r|} \left(v_r - \frac{\sigma_0 |v_r|}{g(v_r)} z(\zeta, t) \right) \quad (25)$$

At steady-state, v, ω (and hence v_r) are constant, and (25) can be integrated along the patch with the boundary condition $z(0, t) = 0$. A simple calculation shows that

$$z_{ss}(\zeta) = \text{sgn}(v_r) \frac{g(v_r)}{\sigma_0} \left(1 - e^{-\frac{\sigma_0}{g(v_r)} \left| \frac{v_r}{\omega r} \right| \zeta} \right) = c_2 (1 - e^{c_1 \zeta}) \quad (26)$$

where

$$c_1 = -\frac{\sigma_0}{g(v_r)} \left| \frac{v_r}{\omega r} \right|, \quad c_2 = \text{sgn}(v_r) \frac{g(v_r)}{\sigma_0} \quad (27)$$

Notice that when $\omega = 0$ (locked wheel case) the distributed model, and hence the steady-state expression (26) collapses into the one predicted by the standard point-contact LuGre model. This agrees with the expectation that for a locked wheel the friction force is only due to pure sliding.

The steady-state value of the total friction force is calculated from (23)

$$F_{ss} = \int_0^L (\sigma_0 z_{ss}(\zeta) + \sigma_2 v_r) f_n(\zeta) d\zeta \quad (28)$$

To proceed with the calculation of F_{ss} we need to postulate a distribution for the normal force $f_n(\zeta)$. The typical form of the normal force distribution reported in the literature [25, 20, 15, 14], is shown in Fig. 4. However, for the sake of simplicity, other forms can be adopted. Some examples are given next.

⁴It is assumed here that the origin of the ζ -frame changes location when the wheel velocity reverses direction, such that $\dot{\zeta} = r\omega$, for $\omega > 0$, and $\dot{\zeta} = -r\omega$, for $\omega < 0$.

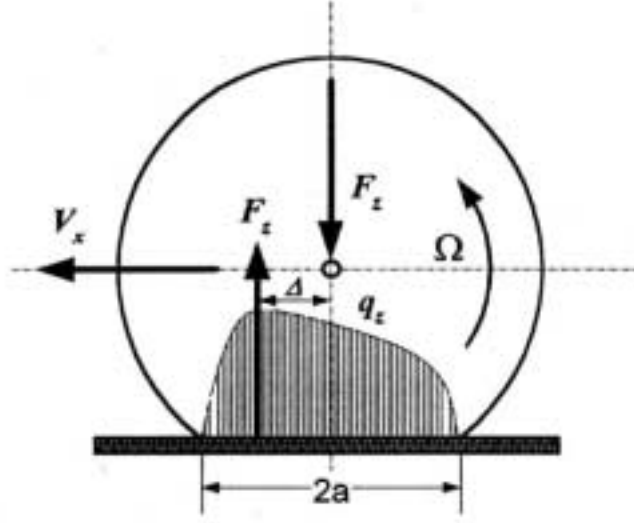


Figure 4: Typical normal load distribution along the patch; from [15].

- *Constant norm distribution.* A simple result can be derived if we assume uniform load distribution, as done in [9] and [13]. For uniform normal load

$$f_n(\zeta) = \frac{F_n}{L}, \quad 0 \leq \zeta \leq L \quad (29)$$

and one obtains,

$$F_{ss} = \left(\text{sgn}(v_r)g(v_r) \left[1 - \frac{Z}{L}(1 - e^{-L/Z}) \right] + \sigma_2 v_r \right) F_n \quad (30)$$

where

$$Z = \left| \frac{\omega r}{v_r} \right| \frac{g(v_r)}{\sigma_0} \quad (31)$$

- *Exponentially decreasing distribution.* In this case, the decrease of the normal load along the patch shown in Fig. 4 is approximated with an exponentially decreasing function

$$f_n(\zeta) = e^{-\lambda(\frac{\zeta}{L})} f_{n0}, \quad 0 \leq \lambda, \quad 0 \leq \zeta \leq L \quad (32)$$

where $f_n(0) = f_{n0}$ denotes the distributed normal load at $\zeta = 0$. This particular choice will become clear later on, when we reduce the infinite dimension distributed model to a simple lumped one having only one state variable. Moreover, for $\lambda > 0$ we have a strictly decreasing function of f_n . With the choice (32) one obtains

$$F_{ss} = \sigma_0 c_2 k_1 \left(1 - e^{-\lambda} + k_2 e^{(-\lambda a + CL)} + k_2 \right) + \sigma_2 v_r k_1 (1 - e^{-\lambda}) \quad (33)$$

where

$$k_1 = \frac{f_{n0} L}{\lambda} \quad \text{and} \quad k_2 = \frac{\lambda}{c_1 L - \lambda}$$

The details of these calculations are given [10]. The value of f_{n0} can be computed from λ, L and the total normal load F_n acting on the wheel shaft. That is,

$$\begin{aligned} F_n &= \int_0^L f_n(\zeta) d\zeta = f_{n0} \int_0^L e^{-\frac{\lambda}{L}\zeta} d\zeta \\ &= -\frac{L}{\lambda} f_{n0} \left[e^{-\frac{\lambda}{L}\zeta} \right]_0^L = -\frac{L}{\lambda} f_{n0} (e^{-\lambda} - 1) \\ &= \frac{L}{\lambda} f_{n0} (1 - e^{-\lambda}) \end{aligned} \quad (34)$$

which yields,

$$f_{n0} = F_n \frac{\lambda}{(1 - e^{-\lambda})L} \quad (35)$$

- *Distributions with zero boundary conditions.* As shown in Fig. 4, a realistic force distribution has, by continuity, zero values for the normal load at the boundaries of the patch. Several forms satisfy this constraint. Some possible examples proposed herein are given below:

$$f_n(\zeta) = \frac{3F_n}{2L} \left[1 - \left(\frac{\zeta - L/2}{L/2} \right)^2 \right] \quad \text{parabolic} \quad (36)$$

or,

$$f_n(\zeta) = \frac{\pi F_n}{2L} \sin(\pi\zeta/L) \quad \text{sinusoidal} \quad (37)$$

or,

$$f_n(\zeta) = \frac{\gamma^2 L^2 + \pi^2}{\pi L (e^{-\gamma L} + 1)} \exp^{-\gamma\zeta} \sin(\pi\zeta/L) \quad \text{sinusoidal/exponential} \quad (38)$$

where F_n denotes the total normal load.

Relation with the Magic Formula

The previously derived steady-state expressions, depend on both v and ω . They can also be expressed as a function of s and either v or ω . For example, for the constant distribution case, we have that $F_{ss}(s)$, can be rewritten as:

- *Driving case.* In this case $v < r\omega$, see also (3), and the force at steady-state is given by

$$F_d(s) = \text{sgn}(v_r) F_n g(s) \left(1 + \frac{g(s)}{\sigma_0 L |s|} (e^{-\frac{\sigma_0 L |s|}{g(s)}} - 1) \right) + F_n \sigma_2 r \omega s \quad (39)$$

with $g(s) = \mu_c + (\mu_s - \mu_c) e^{-|r\omega s/v_s|^\alpha}$, for some constant ω , and $s = s_d$.

- *Braking case.* Noticing that the following relations hold between the braking s_b and the driving s_d slip definitions,

$$r\omega s_d = v s_b, \quad s_d = \frac{s_b}{s_b + 1}$$

the steady-state friction force for the braking case can be written as

$$F_b(s) = \text{sgn}(v_r) F_n g(s) \left(1 + \frac{g(s)|1+s|}{\sigma_0 L |s|} (e^{-\frac{\sigma_0 L |s|}{g(s)|1+s|}} - 1) \right) + F_n \sigma_2 v s \quad (40)$$

where $g(s) = \mu_c + (\mu_s - \mu_c) e^{-|vs/v_s|^\alpha}$, for constant v , and $s = s_b$; see also (3).

Remark: Note that the above expressions depend not only on the slip s , but also on either the vehicle velocity v or the wheel velocity ω , depending on the case considered (driving or braking). Therefore static plots of F vs. s can only be obtained for a specified (constant) velocity. This dependence of the steady-state force/slip curves on vehicle velocity is evident in experimental data found in the literature. Nonetheless, it should be stressed here that it is impossible to reproduce such a curves form experimental data obtained from standard vehicles during normal driving conditions, since v and ω cannot be independently controlled. For that, specially design equipment is needed. Figure 5(a) shows the steady-state dependence on the vehicle velocity for the braking case.

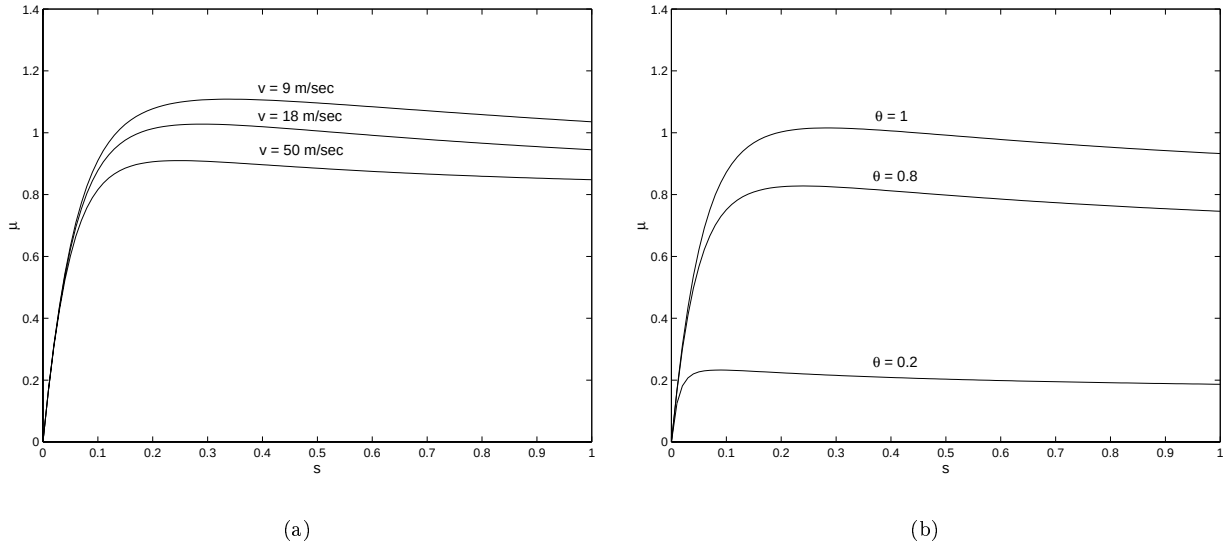


Figure 5: Static view of the distributed LuGre model with uniform force distribution (braking case) under: (a) different values for v , (b) different values for θ with $v = 20$ m/s = 72 Km/h. These curves show the normalized friction $\mu = F(s)/F_n$, as a function of the slip rate s .

Dependency on Road Conditions

The level of tire/road adhesion, can be modeled by introducing a multiplicative parameter θ in the function $g(v_r)$. To this aim, we substitute $g(v_r)$ by

$$\tilde{g}(v_r) = \theta g(v_r),$$

where $g(v_r)$ is the nominal known function given in (16). Computation of the function $F(s, \theta)$, from equation (40) as a function of θ , gives the curves shown in Fig. 5(b). These curves match reasonably well the experimental data shown in Fig. 1(a) for several coefficients of road adhesion.

We also note that the steady-state representation of equations (39) or (40) can be used to identify most of the model parameters by fitting this model to experimental data. These parameters can also be used in a simple one-dimensional lumped model, which can be shown to suitably approximate the (average) solution of the partial differential equation (21) and (22). This approximation is discussed next.

2.4 Average Lumped Model

It is clear that the distributed model captures reality better than the lumped, point contact model. It is also clear that in order to use the distributed model for control purposes it is necessary to choose a discrete number of states to describe the dynamics for each tire. This has the disadvantage that a possibly large number of states is required to describe the friction generated at each tire. Alternatively, one could define a

mean friction state $\bar{z}$ for each tire and then derive an *ordinary differential equation* for $\bar{z}$. This will simplify the analysis and can also lead to much simpler control design synthesis procedures for tire friction problems.

To this end, let us define

$$\bar{z}(t) \equiv \frac{1}{F_n} \int_0^L z(\zeta, t) f_n(\zeta) d\zeta \quad (41)$$

where F_n is the total normal force, given by

$$F_n = \int_0^L f_n(\zeta) d\zeta$$

Thus,

$$\dot{\bar{z}}(t) = \frac{1}{F_n} \int_0^L \frac{\partial z}{\partial t}(\zeta, t) f_n(\zeta) d\zeta \quad (42)$$

Using (24) we get

$$\begin{aligned} \dot{\bar{z}}(t) &= \frac{1}{F_n} \int_0^L \left(v_r - \frac{\sigma_0 |v_r|}{g} z(\zeta, t) - \frac{\partial z(\zeta, t)}{\partial \zeta} |\omega r| \right) f_n(\zeta) d\zeta \\ &= v_r - \frac{\sigma_0 |v_r|}{g} \bar{z}(t) - \frac{|\omega r|}{F_n} \int_0^L \frac{\partial z(\zeta, t)}{\partial \zeta} f_n(\zeta) d\zeta \\ &= v_r - \frac{\sigma_0 |v_r|}{g} \bar{z}(t) - \frac{|\omega r|}{F_n} \left[z(\zeta, t) f_n(\zeta) \right]_0^L + \frac{|\omega r|}{F_n} \int_0^L z(\zeta, t) \frac{\partial f_n(\zeta)}{\partial \zeta} d\zeta \end{aligned}$$

The term in the square brackets describes the influence of the boundary conditions, whereas the integral term accounts for the particular form of the force distribution.

From (23) the friction force is

$$\begin{aligned} F(t) &= \int_0^L \left(\sigma_0 z(\zeta, t) + \sigma_1 \frac{\partial z}{\partial t}(\zeta, t) + \sigma_2 v_r \right) f_n(\zeta) d\zeta \\ &= (\sigma_0 \bar{z}(t) + \sigma_1 \dot{\bar{z}}(t) + \sigma_2 v_r) F_n \end{aligned}$$

As a general goal, one wishes to introduce normal force distributions, that leads to the following form for the lumped LuGre model,

$$\dot{\bar{z}}(t) = v_r - \frac{\sigma_0 |v_r|}{g} \bar{z}(t) - \kappa(\cdot) |\omega r| \bar{z}(t) \quad (43)$$

$$F(t) = (\sigma_0 \bar{z}(t) + \sigma_1 \dot{\bar{z}}(t) + \sigma_2 v_r) F_n \quad (44)$$

where $\kappa(\cdot)$ is defined as:

$$\kappa(\cdot) \bar{z} = \frac{1}{F_n} \left\{ \left[z(\zeta, t) f_n(\zeta) \right]_0^L - \int_0^L z(\zeta, t) \frac{\partial f_n(\zeta)}{\partial \zeta} d\zeta \right\} \quad (45)$$

and F_n as above. When comparing this model with the point contact LuGre model (14)-(15), it is clear that κ captures the distributed nature of the former model. It is also expected that $\kappa > 0$, so that the map $v_r(t) \mapsto F(t)$ preserves the passivity properties of the point contact LuGre model [7, 1].

2.5 Influence of the Force Distribution on $\kappa(\cdot)$

Depending on the postulated normal force distribution density function, several expressions for the average lumped model can be developed. For instance, κ may be a constant, an explicit or an implicit function of the mean friction state $\bar{z}$. We study some of these forms next.

Exponentially Decreasing Distribution

Assuming (32) along with $z(0, t) = 0$ one obtains

$$\begin{aligned}\kappa(\cdot)\bar{z} &= \frac{1}{F_n} \left[z(\zeta, t) e^{-\lambda(\zeta/L)} f_{n0} \right]_0^L + \frac{1}{F_n} \int_0^L z(\zeta, t) \frac{\lambda}{L} e^{-\lambda(\zeta/L)} f_{n0} d\zeta \\ &= \frac{1}{F_n} z(L, t) e^{-\lambda} f_{n0} + \frac{\lambda}{L} \bar{z}(t)\end{aligned}\quad (46)$$

Next, recall that we require $\lambda \geq 0$. For large values of λ it is possible to ignore the term containing $z(L, t)$ in the equation above, and approximate $\kappa(\cdot)$ by a constant

$$\kappa = \frac{\lambda}{L}, \quad \text{with } 0 \leq \lambda \quad (47)$$

Uniform Normal Distribution

The case of the uniform normal distribution can be viewed as a special case of (32) with $\lambda = 0$. In this case $f_n(\zeta) = f_{n0} = F_n/L$ and we obtain the following expression

$$\kappa(\cdot)\bar{z} = \frac{1}{F_n} z(L, t) f_{n0} = \frac{1}{L} z(L, t) \quad (48)$$

Deur [13] proposed that the boundary condition for the last element $z(L, t)$ be approximated by a linear expression of the average deflection $\bar{z}$,

$$z(L, t) \approx \kappa_0(\cdot)\bar{z} \quad (49)$$

resulting in the relation,

$$\kappa(\cdot) = \frac{\kappa_0(\cdot)}{L} \quad (50)$$

The function $\kappa_0(\cdot)$ in (49) is chosen in [13] so that the steady state solutions of the total friction force for the average/lumped model in (43)-(44), and the one of the distributed model (30) are the same. This approximation results in the following expression for κ_0

$$\kappa_0 = \kappa_0(Z) = \frac{1 - e^{-L/Z}}{1 - \frac{Z}{L}(1 - e^{-L/Z})} \quad (51)$$

In [13] it is also shown that, κ_0 belongs to the range $[1, 2]$. Often, a constant value for $\kappa_0 \in [1, 2]$ can be chosen, without significantly changing the steady states of the distributed and lumped models.

Measurements of the longitudinal slip s , friction coefficient μ , and linear velocity v collected during brakings of a specially equipped test vehicle, have been used to identify the parameters of the average/lumped LuGre tire friction model [10]. These parameters can be used as a basis to validate the dynamic friction model by comparing the time histories of the friction force predicted by our model with the friction force from the three experiments.

The 3-D (μ, v, s) steady-state solution of the distributed dynamical LuGre model at the mean velocity, is compared to the friction coefficient μ given by the experiments, as shown in Fig. 6. This allows, via a optimization procedure, to identify the parameters $(\sigma_0, \sigma_2, \mu_s, \mu_c \text{ and } v_s)$. By comparing the time histories of the friction force given by our the model with the ones given by the experiments we can determine the rest of the parameters (e.g., σ_1). An example of such comparison is shown in Fig. 7, for time-varying κ .

3 HIGH DIMENSIONAL MODELS

In this section we recall the definitions of slip that are commonly used in the tire/road friction modeling. We use $V = [V_x, V_y]$ to denote the translational longitudinal velocity at the center of the contact patch O (see

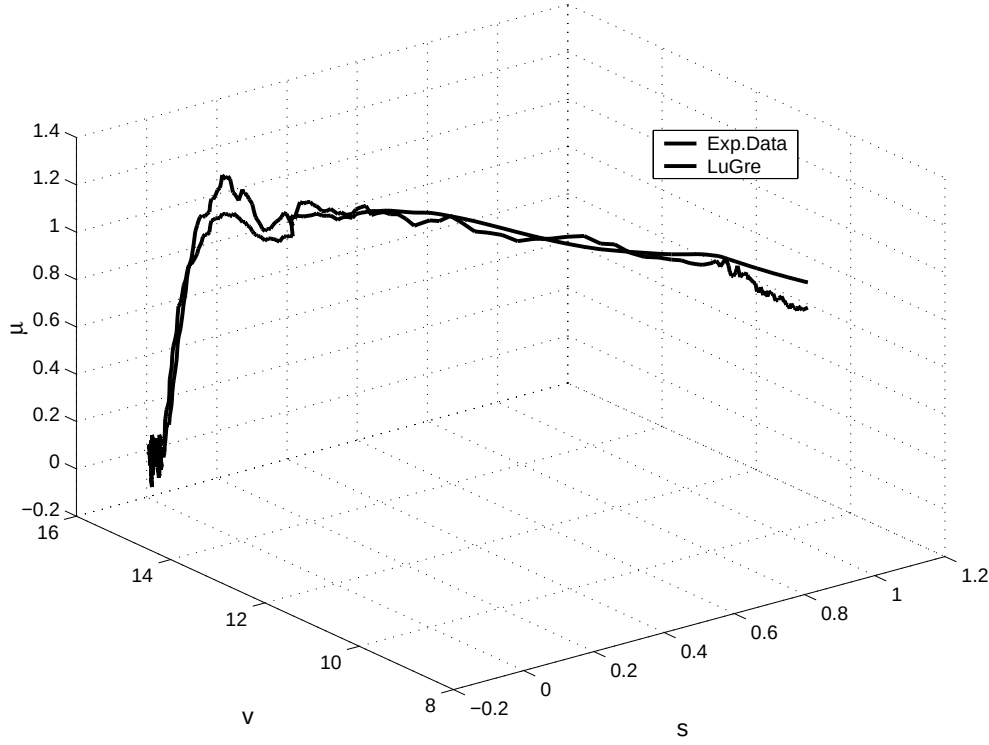


Figure 6: 3-D Plots of the corresponding (μ, s, v) curves for the collected data and the estimated predicted steady-state LuGre average lumped model, with $\alpha = 2$.

Fig. 8) in the wheel plane XOY . The angular velocity of the wheel is denoted by w , V_c as the equivalent wheel translational velocity at the point O and $V_c = \rho w$ where ρ is the free radius of the tire. When w is positive (the vehicle is moving forward), the wheel equivalent longitudinal velocity V_c is positive in the coordinate system R_0 , where R_0 is defined as a moving frame with velocity V_x with the origin O in the center contact patch.

We define the slip velocity of the point O as $V_s = [V_{sx}, V_{sy}]$ in the wheel plane XOY . The slip angle is denoted by α . The slip ratios used to parameterize the friction model are defined as S_s and S_α , for longitudinal and lateral directions, respectively. Two conventions will be used to separate the braking and traction cases (see Fig 8). The pseudo-static braking curves are performed under constant velocity whereas pseudo-static traction curves are given for constant angular velocity. These conventions also prevent the slip from becoming undefined when either the wheel speed or the longitudinal speed reach zero.

- In the braking case, longitudinal slip S_s and lateral slip S_α are given by

$$S_s = \frac{V_x - V_c}{V_x} = \frac{V_{sx}}{V_x} \quad (52)$$

$$S_\alpha = \frac{V_{sy}}{V_x} = |\tan \alpha| \quad (53)$$

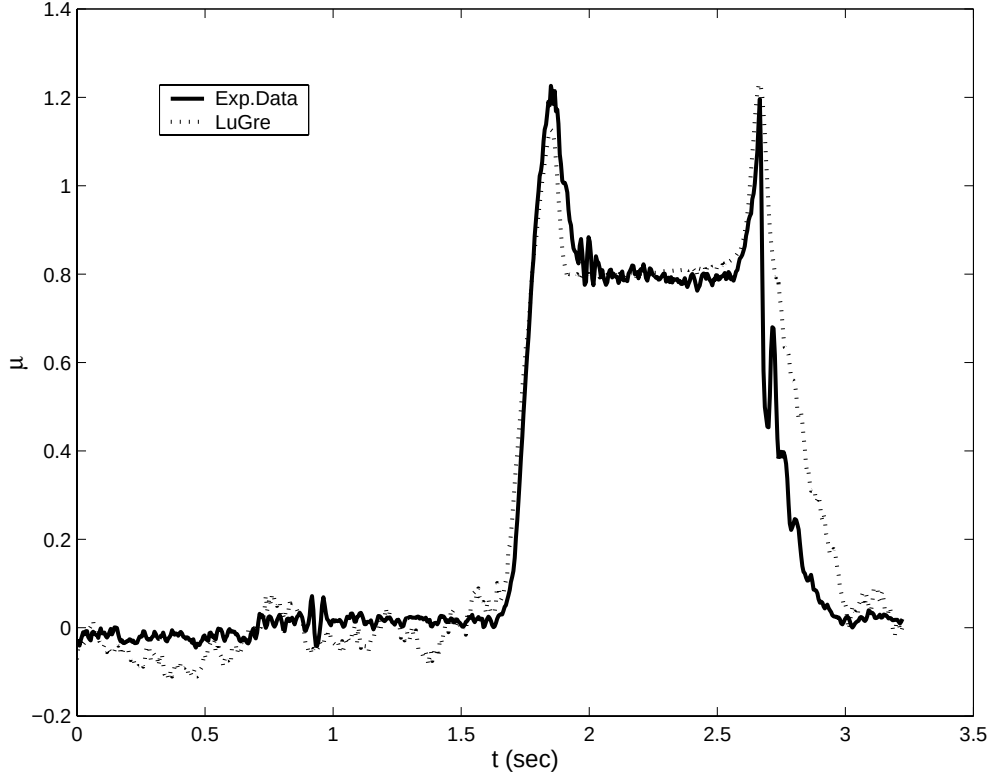
In braking $V_x - V_c > 0$, $V_x \neq 0$ then $1 \geq S_s > 0$.

- In the traction case, longitudinal slip S_s and lateral slip S_α are given by

$$S_s = \frac{V_c - V_x}{V_c} = \frac{V_{sx}}{V_c} \quad (54)$$

$$S_\alpha = \frac{V_{sy}}{V_c} = (1 - S_s)|\tan \alpha| \quad (55)$$

In traction, if $V_x - V_c < 0$, $w \neq 0$ then $1 \geq S_s > 0$

Figure 7: case (ii): varying κ .

Other conventions, like those used for the “magic formula” (see [3]), can easily be employed and do not change the final results. In this particular case, a specific definition needs to be considered when the longitudinal speed V_x or V_c tend to zero, in order to prevent a singularity in the definition of the slip.

In this section we consider both longitudinal and lateral motions, and investigate the resulting forces and torque at the center of the contact patch. Some of these ideas can be found in [41].

3.1 Two dimensional model

Let $\delta\xi$ represent a small slice of the deformed belt crossing the contact patch at position ξ in coordinate R_0 (see Fig. 8). The slice $\delta\xi$ is moving at the speed $V_{\delta\xi} = [V_x, V_y + \xi\dot{\varphi}]$ with $\dot{\varphi}$ the yaw speed of the rim [41]. We can model the dry friction occurring in each slice using the LuGre dynamic friction model. The contact between the two surfaces can be represented by microscopic bristle deflections with the coordinates $\delta z(\xi, t) = [\delta z_x(\xi, t), \delta z_y(\xi, t)]$ and the relative velocity of each slice at ξ with respect to O is given as $V_r(\xi, t) = [V_{rx}(\xi, t), V_{ry}(\xi, t)] = [-V_{sx}(t), -V_{sy}(t) - \xi\dot{\varphi}]$ (the direction of the total force is opposite to the slip vector). However, in this paper we consider the tire belt as a rigid body. The model might be extended in order to include dynamic properties of the rubber belt. In this case, the longitudinal velocity of each slide $V_{\delta\xi}$ of the contact patch would have a more complex expression that might include camber angle dependencies or other factors. These notions have already been explained in the literature [41, 52].

For the rigid tire belt model, the extended two dimensional distributed tire/road friction model are given by

$$\delta\dot{z}_x(\xi, t) = V_{rx} - \frac{\sigma_{0x}}{g_x(V_{rx})} \delta z_x(\xi, t) |V_{rx}| \quad (56a)$$

$$\delta\dot{z}_y(\xi, t) = V_{ry} - \frac{\sigma_{0y}}{g_y(V_{ry})} \delta z_y(\xi, t) |V_{ry}| \quad (56b)$$

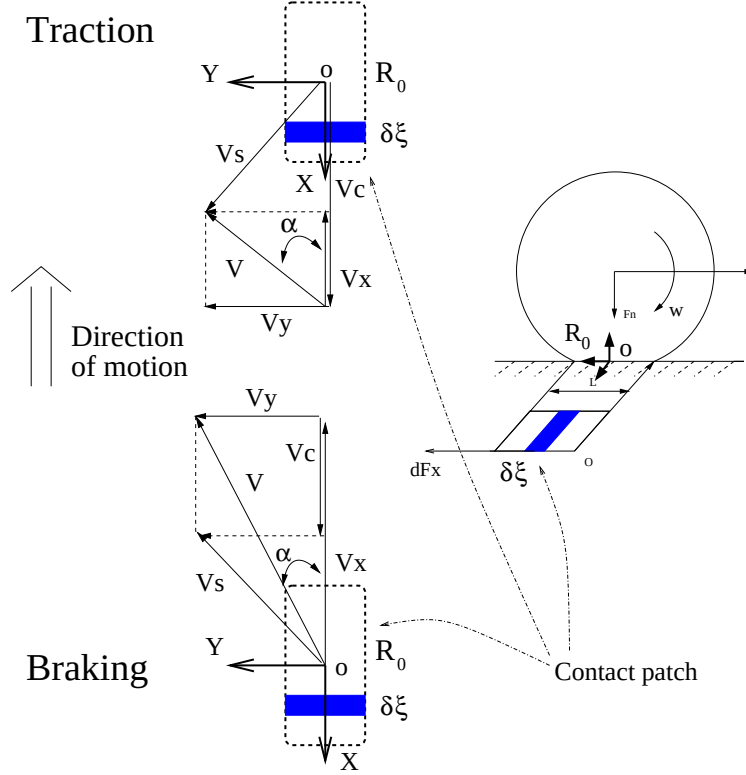


Figure 8: Convention for the slip definition for both braking and traction cases

and the friction forces

$$\delta F_x = \{\sigma_{0_x} \delta z_x(\xi, t) + \sigma_{1_x} \delta \dot{z}_x(\xi, t) + \sigma_{2_x} V_{rx}\} \delta F_n \quad (57a)$$

$$\delta F_y = \{\sigma_{0_y} \delta z_y(\xi, t) + \sigma_{1_y} \delta \dot{z}_y(\xi, t) + \sigma_{2_y} V_{ry}\} \delta F_n \quad (57b)$$

with σ_{j_i} are the dynamic coefficients of the LuGre friction model for lateral and longitudinal direction ($i = x, y$; $j = 1, 2, 3$) known as the normalize rubber stiffness (σ_{0_i}), the normalized rubber damping (σ_{1_i}), and the normalized viscous relative damping (σ_{2_i}). The normal load δF_n is considered uniformly distributed over the patch along ξ direction, thus $\delta F_n = F_n/L$, and

$$g_x(V_{rx}) = \mu_{c_x} + (\mu_{s_x} - \mu_{c_x}) e^{-|V_{rx}|^{1/2}} \quad (58a)$$

$$g_y(V_{ry}) = \mu_{c_y} + (\mu_{s_y} - \mu_{c_y}) e^{-|V_{ry}|^{1/2}} \quad (58b)$$

are two functions that characterize the steady state properties of the friction, where μ_{c_i} , μ_{s_i} , v_{s_i} are, respectively, the Coulomb friction coefficient, the normalized static friction and the Stribeck relative velocity. The two LuGre models for longitudinal and lateral motions are using different parameters since the friction properties of the contact tire/road are different in longitudinal and lateral directions. The fact that the tire has non isotropic properties is well known in this area and has already been introduced in most of the current models (see [3, 5]).

The system given in the Eqs. (56) and (57) is both time and space dependent, as a consequence the derivative of $\delta z_i(\xi, t)$, $i = x, y$, for longitudinal and lateral directions is a full derivative given by

$$\delta \dot{z}_i(\xi, t) = \frac{\partial \delta z_i(\xi, t)}{\partial \xi} \frac{\partial \xi}{\partial t} + \frac{\partial \delta z_i(\xi, t)}{\partial t} \quad (59)$$

3.2 Self-aligning torque

The self-aligning torque is an important part of the tire model because the reaction force applied to the vehicle is strongly dependent upon it (steering wheel feedback force). The self-aligning torque consists of two important effects, the yaw motion of the tire that creates a friction torque $\delta M_{z_1}(\xi, t)$ and the transfer of the forces in the center of wheel frame where forces apply. Both this two effects generate the torque at the center of patch, known as the self-aligning torque. In this presentation we neglect the yaw motion of the tire $\delta M_{z_1}(\xi, t)$, due to the fact that the yaw motion effect is small (see [41]).

The tire/road forces and torque are always calculated at the center of the patch O in the Fig. 8. Consequently, a self-aligning torque is produced resulting from a non-symmetry of the contact patch deformation $\delta z(\xi, t)$, or forces $\delta F_y(\xi, t)$, over the contact patch length L . The equivalent forces and torque produced by a slice $\delta \xi$ at position ξ with respect to the center of the patch O is given by $[\delta F_x, \delta F_y, \delta M_z = \xi \delta F_y]$ and the total tire/road interaction in the patch center is now expressed by two forces F_x, F_y and the self-aligning torque M_z :

$$F_x = \int_{-\frac{L}{2}}^{\frac{L}{2}} \delta F_x(\xi, t) d\xi \quad (60a)$$

$$F_y = \int_{-\frac{L}{2}}^{\frac{L}{2}} \delta F_y(\xi, t) d\xi \quad (60b)$$

$$M_z = \int_{-\frac{L}{2}}^{\frac{L}{2}} \left[\underbrace{\delta M_{z_1}(\xi, t)}_{\text{neglected}} + \delta M_{z_2}(\xi, t) \right] d\xi = \int_{-\frac{L}{2}}^{\frac{L}{2}} \xi \delta F_y(\xi, t) d\xi \quad (60c)$$

3.3 Stationary properties

The stationary characteristics of the tire are widely used in this research area. To produce this characteristics, a complex experiment setup is usually utilized. These conditions are hardly obtainable on a real vehicle since the maneuvers would then be very severe for the passengers. Each point on the stationary curve is given for a constant slip and a constant wheel velocity or wheel angular velocity, therefore V_c and V remain constant and a defined slip angle α is obtained. The yaw motion of the rim is also not considered i.e. $\dot{\varphi} = 0$. Therefore, during the stationary conditions, ξ and t are no longer independent because of constant velocity, we have $\delta z_i(\xi, t) = \delta z_i(t)$ if we like to find a time varying solution, or $\delta z_i(\xi, t) = \delta z_i(\xi)$ if we look for the spatial solution. We choose to calculate the spatial solution in the frame R_0 defined previously. Notice that, if $\dot{\xi} = V_c$ is constant during stationary conditions, we have

$$\frac{d}{dt} \delta z_i(\xi, t) = \frac{d \delta z_i(\xi)}{d \xi} \frac{d \xi}{dt} = \frac{d \delta z_i(\xi)}{d \xi} V_c \quad (61)$$

with spatial coordinates. As a consequence the stationary bristle model using spatial coordinates becomes

$$\frac{d \delta z_i(\xi)}{d \xi} V_c = -V_{si} - \frac{\sigma_{0i}}{g_i(V_{si})} \delta z_i(\xi) |V_{si}| \quad (62)$$

with $\delta z_i(-\frac{L}{2}) = 0$ the boundary condition. The spatial solution is given by

$$\delta z_i(\xi) = \text{sign}(-V_{si}) \frac{g_i(V_{si})}{\sigma_{0i}} \left[1 - e^{-\frac{\sigma_{0i} |V_{si}| / V_c}{g_i(V_{si})} (\xi + \frac{L}{2})} \right] \quad (63)$$

Remark. $\delta z_i(\xi)$ does not represent the behaviour of a slide δz_i at a given time instant t and a given position ξ , both t and ξ are varying since $\dot{\xi} = V_c$. The stationary solution gives the history of one slide $\delta z_i(\xi, t)$ during the interval $\Delta t = \frac{\Delta \xi}{V_c}$.

Integrating the forces and torque along the contact patch, we obtain three components for the stationary tire model F_x, F_y, M_z for traction case. As discussed in the previous remark, these expressions do not

represent the instantaneous forces and torques at the contact interval (like given in Eqs. (60)), but the average force produce by one slide during the time interval $\Delta t = \frac{L}{V_c}$ necessary to cross the contact patch L .

$$\frac{F_x}{F_n} = -\gamma_x(V_{sx})\text{sign}(V_{sx})g_x(V_{sx}) \left[1 + \frac{g_x(V_{sx})}{L\sigma_0 S_s} \left(e^{-\frac{\sigma_{0x} L S_s}{g_x(V_{sx})}} - 1 \right) \right] - (\sigma_{1x} + \sigma_{2x})V_{sx} \quad (64a)$$

$$\frac{F_y}{F_n} = -\gamma_y(V_{sy})\text{sign}(V_{sy})g_y(V_{sy}) \left[1 + \frac{g_y(V_{sy})}{L\sigma_{0y} S_\alpha} \left(e^{-\frac{\sigma_{0y} L S_\alpha}{g_y(V_{sy})}} - 1 \right) \right] - (\sigma_{1y} + \sigma_{2y})V_{sy} \quad (64b)$$

$$\frac{M_z}{F_n} = -\frac{\gamma_y(V_{sy})\text{sign}(V_{sy})g_y^2(V_{sy})}{2\sigma_{0y}^2 S_\alpha} \left[\left(e^{-\frac{\sigma_{0y} L S_\alpha}{g_y(V_{sy})}} - 1 \right) + \left(\frac{2g_y(V_{sy})}{L\sigma_{0y} S_\alpha} \right) \left(e^{-\frac{\sigma_{0y} L S_\alpha}{g_y(V_{sy})}} - 1 \right) \right] \quad (64c)$$

The function γ_i defined as $\gamma_i(V_{si}) = 1 - \frac{\sigma_{1i}|V_{si}|}{g_i(V_{si})}$, for $i = x, y$. Parameter identification, and comparison with “magic formula” will be presented later. For braking case, we can find the similar formula for F_x , F_y , and M_z .

3.4 Lumped dynamic tire/road friction model

Distributed model are difficult to use for estimation and control purposes. Thus, we will now developed a simplified lumped parametric representation. In this paper, we obtained the lumped model by defining lumped variables $\bar{z}_i$ as follow,

$$\bar{z}_i(t) = \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \delta z_i(\xi, t) d\xi, \quad i = x, y \quad (65)$$

where L is defined as an “elementary surface length”, which could be a *tread block element* or the *full contact patch length* between the tire and the road. Neglecting the yaw motion of the rim, i.e. $\dot{\varphi} = 0$, the distributed friction model changes into

$$\dot{\bar{z}}_i + \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{\partial \xi}{\partial t} \frac{\partial \delta z_i(\xi, t)}{\partial \xi} d\xi = -V_{si} - \frac{\sigma_{0i}}{g(V_{si})} \bar{z}_i |V_{si}| \quad (66)$$

$\frac{\partial \xi}{\partial t} = V_c$ is assumed uniformly along the patch.

- **Internal states**

$$\dot{\bar{z}}_i = -V_{si} - \frac{\sigma_{0i}}{g(V_{si})} \bar{z}_i |V_{si}| \quad (67)$$

with $\bar{z}_i(0) = 0$. Assuming $\delta z(-\frac{L}{2}, 0) = \delta z(\frac{L}{2}, 0) = 0$, because the patch is at free when $t = 0$.

- **Lumped forces**

$$F_i = \{\sigma_{0i} \bar{z}_i + \sigma_{1i} \dot{\bar{z}}_i + \sigma_{2i} V_{ri}\} F_n \quad (68)$$

Using the new lumped internal states $\bar{z}_i$, we can obtain

- **Lumped self-aligning torque**

$$M_z = \int_{-\frac{L}{2}}^{\frac{L}{2}} \xi \delta F_y(\xi, t) d\xi \quad (69)$$

we have (see [42])

$$\begin{aligned} \dot{M}_z &= \int_{-\frac{L}{2}}^{\frac{L}{2}} \left[\dot{\xi} \delta F_y(\xi, t) + \xi \frac{\partial \delta F_y(\xi, t)}{\partial t} \right] d\xi \\ &= V_c F_y + \left(\frac{\dot{\gamma}_y(V_{sy})}{\gamma_y(V_{sy})} - \frac{\sigma_{0y} |V_{sy}|}{g_y(V_{sy})} \right) M_z \end{aligned} \quad (70)$$

3.5 Influence on the External factors.

Friction characteristics can change dramatically with external conditions like humidity or different type of road, etc. In this subsection, we provide some insight on the way the friction model is affected by typical perturbations. We first discuss some variations that can be parametrized using the actual model parameters (elementary effects), then we discuss external perturbation that may affect the accuracy of the used model (i.e. road conditions, temperature variation, etc.).

Elementary effects

- **Dry friction characteristic** is represented by the function g_i of Eqs. (58) and several dynamic coefficients σ_i . In the following we will consider that a change in friction will affect primarily the static characteristics of the model uniformly (i.e. the Coulomb friction and Stribeck effect). Hence, we may use a parameter θ_1 to scale the function g_i .
- **Normal load** variations can easily be introduced in the model using a second parameter θ_2 scaling F_N in Eqs. (57). Eventually, as reported in [7], normal load may affect the static parameters of the LuGre model and then change θ_1 .
- **Contact surface length.** As shown in Eqs. (64), the variation of L could be introduced using a third coefficient θ_3 scaling the dynamic parameter σ_{0_i} .

And we propose the following structure for the tire friction model

$$\dot{\bar{z}}_i = -V_{si} - \theta_1 \frac{\sigma_{0_i} \theta_3}{g(V_{si})} \bar{z}_i |V_{si}| \quad (71)$$

$$\dot{M}_z = V_c F_y + \left(\frac{\dot{\gamma}_y(V_{sy})}{\gamma_y(V_{sy})} - \frac{\theta_3 \sigma_{0_y} |V_{sy}|}{g_y(V_{sy})} \right) M_z \quad (72)$$

$$F_i = \theta_2 \{ \theta_3 \sigma_{0_i} \bar{z}_i + \sigma_{1_i} \dot{\bar{z}}_i + \sigma_{2_i} V_{ri} \} F_n \quad (73)$$

with $\bar{z}_i(0) = 0$ and $i = x, y$.

External perturbations

- **Different road surface** result in variations of the dry friction characteristic of the model. This effect could easily be introduced using the parameter θ_1 described in the previous section.
- **Contamination of the interface** has a tremendous effect on the tire/road contact behavior. The depth of water on the road greatly influences tire/road friction. These phenomena are complex and related to the different aspects of the contact patch between the tire and the road. Previous analysis made use of different water level categories such as thin or thick film of water on the road [43, 44, 45]. Even if the distinction between these two situations sounds artificial, it helps in understanding the contamination of the tire/road contact patch by water. In the limiting situation of a thin water layer, the contact between tire and road is completely lost due to full contamination of the interface⁵. This problem is called viscous hydro-planing [47, 48]. On the other hand, for a thick water layer, an extra force is generated in front of the tire due to the accumulation of water (hydro-dynamic forces). In this situation, the contact between the two surfaces could be lost without necessarily full contamination of the interface but mostly because of the hydro-dynamic force. This phenomenon is named dynamic hydro-planing. The last case is more general than viscous hydro-planing due to the fact that other hydro-dynamical effects take place during the contact process.

Consequently, we can summarize these remarks as follows: for every set of experimental conditions in wet weather, there exists a definite velocity called the hydro-planing limit, beyond which hydro-planing occurs (see [48]). At this velocity, the wheel is in general supported by (see Fig. 9):

⁵When water has invaded the complete contact area between the tire and the road

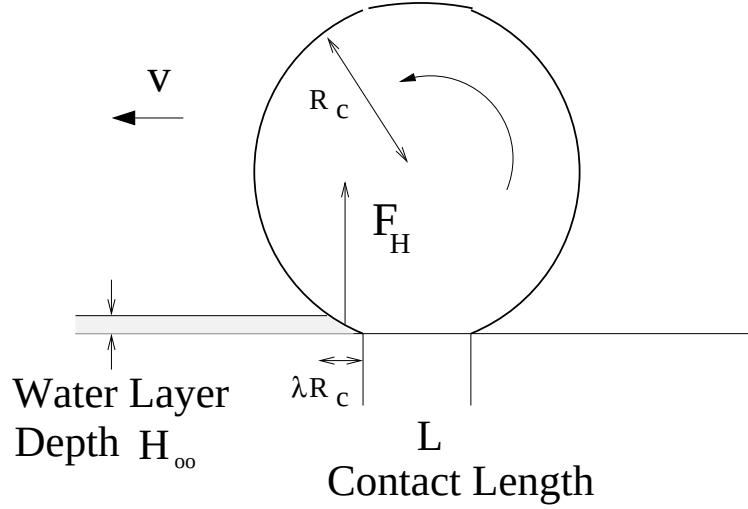


Figure 9: Schematic representation of tire under wet road

- A hydro-dynamic upward thrust, just ahead of the contact area of the tire. The magnitude of this force is determined by the water layer depth h_∞ (for thick films) and vehicle velocity v . An expression for the upward thrust, F_H , can be found in [48]. As the hydro-dynamic upward thrust result in a decrease of the effective normal load, we propose in this paper to include this effect using the parameter θ_2 scaling the load F_N .
- A squeezed film and wiping effect in the contact area. The squeezed film effect is due to the inability of the tread elements traversing the contact area to remove the last few tenths of a millimeter of water in the available time. The wiping effect is due to the longitudinal and lateral relative motion of the tire elements in the contact area.

Remark. F_H approaches the load of the wheel at high velocities (over 200 km/h, which is not realistic) depending of the amount of water h_∞ on the road. Thus, as stated before, the hydro-dynamic thrust F_H is not the only factor contributing to hydro-planning. This discrepancy is due to variations in the contact area due to a thin film of water squeezed at the tire/road interface: the thin water layer supports a fraction of the load of the wheel, and hence the effective contact area and the effective normal load decreases. The inclusion of all of these effects in the tire model is possible if we can develop dynamic expressions for the contact patch and the effective load of the tire, based on physic laws. Determining the contact patch is a difficult problem, and several empirical or semi empirical approaches have been proposed in the literature [49, 47]. In these references, the road/tire interface (contact area) is divided into three areas, where the major effects are different: dry contact, micro-water layer region, and macro-water layer region. A simple model can then be obtained to scale the effective contact area, given certain drainage characteristics of the ground and the tire. A more sophisticated interpretation of the contact patch dynamics based on fluid mechanics has also been proposed in [50, 48, 51]. These discontinuous interpretations are not satisfactory for simulation or control purposes, because there require a significant amount of characterizations and experimental measurements.

To simplify the problem, variations of the contact patch length could be included using a third parameter θ_3 scaling σ_{0_i} as mentioned previously.

- **Pressure and temperature variations** can also be decomposed in terms of the elementary effects given in section ???. Hence, for example temperature variation changes the dry friction characteristic therefore θ_1 and pressure variations affect the contact patch length, therefore affects θ_3 . However, these two perturbations may be more complex to include in the model since they may modify more than one parameter. Further investigation will be necessary on this problem.

A simple theory is discussed to include different road condition. Effects of perturbations could be investigated in terms of three varying coefficient of the friction model. This method facilitates the design of new scheme for estimation and control based on the dynamic friction model (i.e. braking control, friction estimator). Since a single perturbation may have complex effect on the friction model, a multi-parameter analysis might be necessary and experimental verification has to be done to validate this theory.

Figure Fig. 10 shows a comparison of the identified stationary tire/road model, with the curves obtained from the “magic formula” calibrated on dry surface condition for a typical tire model 165-65R14, given from Renault SA.

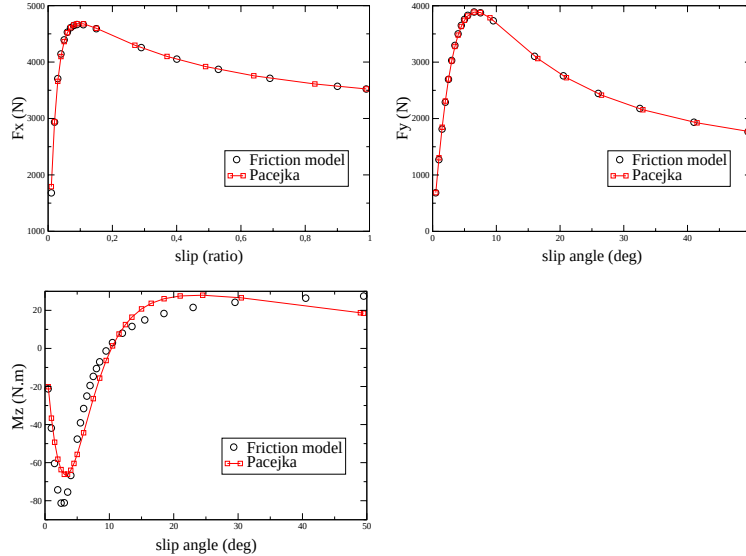


Figure 10: Comparison of the stationary tire/road friction model and the “magic formula” (constant velocity during braking $v = 15m/s$)

The influence of different water films and various texture amplitudes are illustrated in Figs. 11 and 12. Fig. 11 illustrates the hydro-dynamic properties of the tire under wet road conditions. As predicted, increasing the thickness of the water on the road h_∞ for a given travel velocity v dramatically reduces the resulting friction. Furthermore, Fig. 12 shows the effect of varying the texture amplitude ϵ on the friction characteristics using relation given in [51]. In this case, the water layer h_∞ is given and the road pattern is specified as triangular. As ϵ is varied the setting time is slightly affected. We observe that friction increases as texture amplitude increases.

4 ROAD/TIRE FRICTION OBSERVERS

This section is devoted to the problem of tire-road friction estimation using only angular wheel velocity which can not always been computed from actual sensors. Tire forces information is relevant to problems like: optimization of Anti-lock brake systems (ABS), traction system, diagnostic of the road friction conditions, etc.

In applications like ABS system the linear velocity cannot be computed from existing sensors, it is important to design observers that also estimates the linear wheel (or car) velocity. In this section we present the nonlinear observer that estimate on-line both the tire/road characteristics and the slip ratio, using only rotational wheel velocity, see [8], and [35].

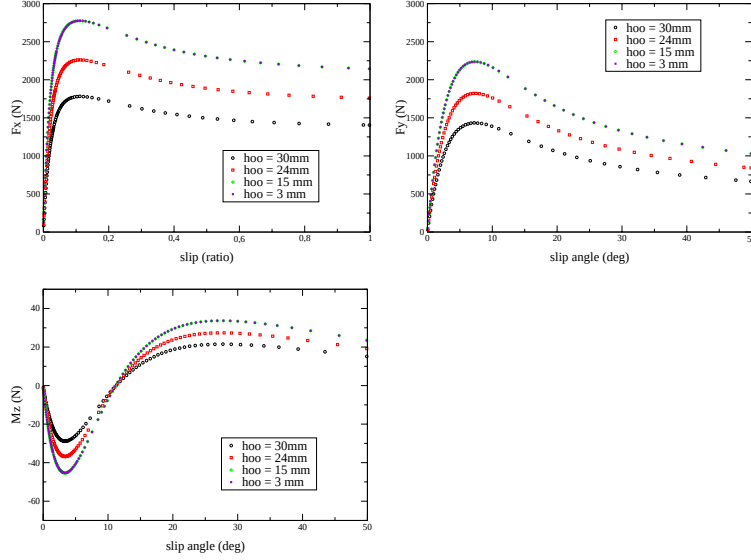


Figure 11: Effect of water depth h_∞ on a stationary tire model ($v = 25m/s$ and triangular texture of amplitude $\epsilon = 0.06mm$)

4.1 The estimation problem

We consider the one-wheel model with the lumped tire/road friction model, i.e.

$$m\dot{v} = F_n(\sigma_0 z + \sigma_1 \dot{z}) + F_n \sigma_2 v_r \quad (74)$$

$$J\dot{\omega} = -rF_n(\sigma_0 z + \sigma_1 \dot{z}) - \sigma_\omega \omega + u_\tau \quad (75)$$

$$\dot{z} = v_r - \theta \frac{\sigma_0 |v_r|}{g(v_r)} z \quad (76)$$

with,

$$g(v_r) = \mu_C + (\mu_S - \mu_C) e^{-|v_r/v_s|^{1/2}},$$

where we have neglected the term σ_2 in the equation (75), σ_ω is the viscous rotational friction, and θ is the parameter related to changes in the road conditions, that is to be estimated. The observation problem can be now formulated as follows.

Problem formulation: Assume that the lumped friction parameters with $\theta = 1$, has been identified off-line, and assume that only the variable ω , is measurable from some sensor. The problem is to design an on-line observer for θ , that allows the controller to monitor the eventual changes in the road conditions, by using only the measure of the rotational wheel velocity ω .

5 GENERAL OBSERVER DESIGN

Consider the following system:

$$\dot{x} = Ax + B[\theta\varphi(y, u, x)] + Ru + Ey \quad (77)$$

$$\dot{\theta} = 0 \quad (78)$$

$$y = C^T x \quad (79)$$

with $y, \theta, \varphi(y, u, x) \in R, x \in R^n$, and $u \in R^m$.

We assume that system states are bounded, and that the following holds:

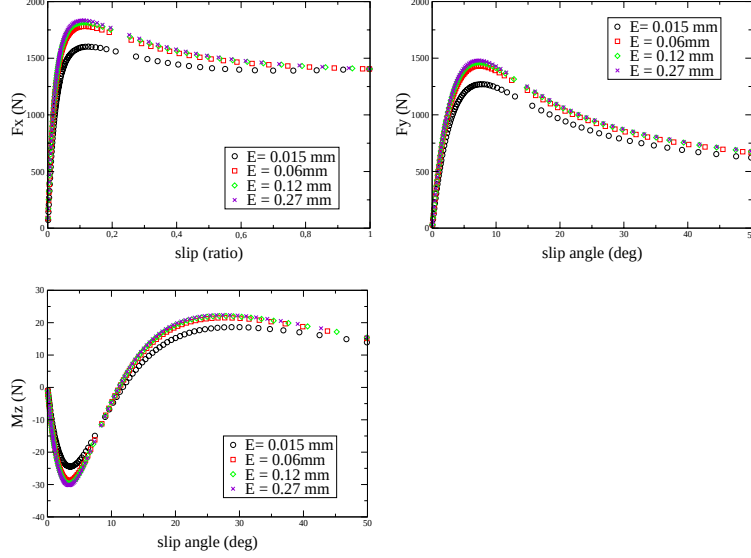


Figure 12: Effect of road texture amplitude E on a stationary tire model (texture: triangular motif, $v = 25m/s$ and $h_\infty = 30mm$)

A1) (A, C^T) is an observable pair,

A2) There exist a known function $\infty > \rho_0 \geq \rho(y, u) \geq 0$, and a known upper bound f_{max} such that either

- (i) $|\varphi(y, u, x_1) - \varphi(y, u, x_2)| \leq \rho(y, u) \|x_1 - x_2\|, \quad \forall x_1, x_2,$
- (ii) $|\varphi(y, u, x)| \leq f(\|x\|) \leq f_{max}, \quad \forall |x| < \infty, \forall |y| < \infty, |u| < \infty$

holds.

A3) The map $G(s) : \psi \mapsto \tilde{y}$ of the system

$$\dot{\tilde{x}} = [A - KC^T]\tilde{x} + B\psi \quad (80)$$

$$\tilde{y} = C^T \tilde{x} \quad (81)$$

with $[A - KC^T]$ Hurwitz, is strictly passive, i.e. $\exists Q > 0, P = P^T > 0$, and K , such that

$$P[A - KC^T] + [A - KC^T]^T P = -Q \quad (82)$$

$$PB = C. \quad (83)$$

A4) The trajectories of the system $(y(t), u(t), x(t))$, satisfy

$$\lim_{t \rightarrow \infty} \varphi(y(t), u(t), x(t)) \neq 0.$$

A5) The following condition holds

$$\lambda_{min} Q - 2\|C^T\|\theta_{max}\rho_0 = q_0 > 0.$$

with $\theta_{max} \geq |\theta|$.

Under this hypothesis, we propose the following observer structure:

$$\dot{\hat{x}} = A\hat{x} + B \left[\hat{\theta} \varphi(y, u, \hat{x}) \right] + Ru + Ey + K(y - \hat{y}) + B\nu_1 \quad (84)$$

$$\dot{\hat{\theta}} = \nu_2 \quad (85)$$

$$\hat{y} = C^T \hat{x} \quad (86)$$

where ν_1 , and ν_2 , are design variables, which will be defined subsequently.

Introducing the error variables:

$$\tilde{x} = x - \hat{x} \quad (87)$$

$$\tilde{\theta} = \theta - \hat{\theta} \quad (88)$$

$$\tilde{y} = y - \hat{y} = C^T \tilde{x}, \quad (89)$$

The error equation becomes

$$\dot{\tilde{x}} = [A - KC^T]\tilde{x} + B [\theta\varphi(y, u, x) - \hat{\theta}\varphi(y, u, \hat{x})] - B\nu_1 \quad (90)$$

$$\dot{\tilde{\theta}} = -\nu_2 \quad (91)$$

$$\dot{\tilde{y}} = C^T \tilde{x}, \quad (92)$$

where,

$$\theta\varphi(y, u, x) - \hat{\theta}\varphi(y, u, \hat{x}) = \tilde{\theta}\varphi(y, u, \hat{x}) + \theta[\varphi(y, u, x) - \varphi(y, u, \hat{x})]$$

Now, defining the Lyapunov function

$$V = \tilde{x}^T P \tilde{x} + \frac{1}{\gamma} \tilde{\theta}^2$$

and using properties A1, and A3, we have

$$\dot{V} = -\tilde{x}^T Q \tilde{x} + 2\tilde{\theta} [\tilde{y}\varphi(y, u, \hat{x}) - \gamma^{-1}\nu_2] + 2\tilde{y}\theta [\varphi(y, u, x) - \varphi(y, u, \hat{x})] - \tilde{y}\nu_1. \quad (93)$$

Defining the adaptation law ν_2 as

$$\nu_2 = \gamma\varphi(y, u, \hat{x})\tilde{y} \quad (94)$$

we obtain

$$\dot{V} \leq -\tilde{x}^T Q \tilde{x} + 2|\tilde{y}||\theta| |\varphi(y, u, x) - \varphi(y, u, \hat{x})| - \tilde{y}\nu_1.$$

If A2 – (i) holds, then we have,

$$\dot{V} \leq -\tilde{x}^T Q \tilde{x} + 2|\tilde{y}||\theta|\rho(y, u)||\tilde{x}| - \tilde{y}\nu_1 \quad (95)$$

$$\leq -q||\tilde{x}||^2 + 2||C^T|||\theta|\rho(y, u)||\tilde{x}|^2 - \tilde{y}\nu_1 \quad (96)$$

$$\leq -||\tilde{x}||^2(\lambda_{min}Q - 2||C^T|||\theta|\rho_0) - \tilde{y}\nu_1 \quad (97)$$

$$\leq -q_0||\tilde{x}||^2 - \tilde{y}\nu_1. \quad (98)$$

where the last inequality follows from A4. In this case, we can simple set $\nu_1 = 0$, to get $\dot{V} \leq -q_0||\tilde{x}||^2$.

In the second case, when only A2 – (ii) holds, we have

$$\begin{aligned} \dot{V} &\leq -\tilde{x}^T Q \tilde{x} + 2|\tilde{y}||\theta|(f(|x|) + f(|\hat{x}|)) - \tilde{y}\nu_1 \\ &\leq -\tilde{x}^T Q \tilde{x} - |\tilde{y}| [-2\theta_{max}(f_{max} + f(|\hat{x}|)) + \text{sgn}(\tilde{y})\nu_1], \end{aligned}$$

which suggests that ν_1 should be defined to have a high-gain component, i.e.

$$\nu_1 = 2\theta_{max}(f_{max} + f(|\hat{x}|))\text{sgn}(\tilde{y}),$$

where θ_{max} , is an upper bound of the parameter θ .

With this choice of ν_1 we have that

$$\dot{V} \leq -q||\tilde{x}||^2.$$

Thus, in both of the cases considered by assumption A2 $\tilde{x}$, and $\tilde{\theta}$ are bounded, and $\tilde{x} \rightarrow 0$. Finally, from the error equation (90), we have that

$$\lim_{t \rightarrow \infty} \{B\tilde{\theta}\varphi(y, u, x)\} = 0,$$

which together with assumption A4 leads us to conclude that

$$\lim_{t \rightarrow \infty} \hat{\theta} = \theta.$$

We have thus proved the following theorem:

Theorem 1 *Consider the following system*

$$\begin{aligned}\dot{x} &= Ax + B[\theta\varphi(y, u, x)] + Ru + Ey \\ \dot{\theta} &= 0 \\ y &= C^T x\end{aligned}$$

under the assumptions A1 – A5, with $y, \theta, \varphi(y, u, x) \in R$, $x \in R^n$, and $u \in R^m$. Then the following observer

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + B[\hat{\theta}\varphi(y, u, \hat{x})] + Ru + Ey + K(y - \hat{y}) + B\nu_1 \\ \dot{\hat{\theta}} &= \gamma\varphi(y, u, \hat{x})\tilde{y} \\ \hat{y} &= C^T \hat{x},\end{aligned}$$

with

$$\nu_1 = \begin{cases} 0 & \text{if } A2\text{-(i) holds} \\ 2\theta_{max}(f_{max} + f(|\hat{x}|))\text{sgn}(\tilde{y}) & \text{if } A2\text{-(ii) holds} \end{cases}$$

ensures (under verification of A4), that

$$\lim_{t \rightarrow \infty} \hat{\theta} = \theta.$$

5.1 Application to the one-wheel model

We consider the one-wheel model with lumped friction as described by the equations (74)-(76). As formulated previously, we assume that only ω is measurable. To set our system in the same framework as the structure (77)-(79), we introduce the following change of coordinates:

$$\eta = rmv + J\omega \tag{100}$$

$$\chi = J\omega + rF_n\sigma_1 z, \tag{101}$$

from which we get:

$$\dot{\eta} = -\frac{F_n\sigma_2}{m}\eta + \left(\frac{JF_n\sigma_2}{m} + r^2F_n\sigma_2 - \sigma_\omega\right)\omega + u_\tau \tag{102}$$

$$\dot{\chi} = -\frac{\sigma_0}{\sigma_1}\chi + \left(J\frac{\sigma_0}{\sigma_1} - \sigma_\omega\right)\omega + u_\tau \tag{103}$$

$$\dot{z} = (r\omega - v) - \theta\sigma_0 \frac{|r\omega - v|}{g(v_r)} z \tag{104}$$

$$y = \frac{1}{J}(\chi - rF_n\sigma_1 z) = \omega. \tag{105}$$

Defining x , u and y respectively as

$$x = \begin{bmatrix} \eta \\ \chi \\ z \end{bmatrix}, \quad u = u_\tau, \quad y = \omega,$$

we can rewrite the above system as

$$\begin{aligned}\dot{x} &= \begin{bmatrix} -\frac{F_n\sigma_2}{m} & 0 & 0 \\ 0 & -\frac{\sigma_0}{\sigma_1} & 0 \\ -\frac{1}{rm} & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \theta\varphi(y, u, x) \\ &+ \begin{bmatrix} \left(\frac{JF_n\sigma_2}{m} + r^2F_n\sigma_2 - \sigma_\omega\right) \\ \left(J\frac{\sigma_0}{\sigma_1} - \sigma_\omega\right) \\ r \end{bmatrix} y + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u,\end{aligned}$$

where $\varphi(y, u, x)$, is defined as

$$\varphi(y, x) = \frac{\sigma_0 |ry - v|}{g(ry - v)} z \quad v = (\eta - J\omega)/rm.$$

With this representation we shall now verify condition under which the assumptions A1 – A3 hold. The condition A4 depends on the operational conditions, in particular on the applied torque u_τ , and condition A5 depends on the level of friction involved.

Condition A1 (linear observability). With A , and C defined as above, i.e

$$A = \begin{bmatrix} -\frac{F_n \sigma_2}{m} & 0 & 0 \\ 0 & -\frac{\sigma_0}{\sigma_1} & 0 \\ -\frac{1}{rm} & 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 \\ \frac{1}{J} \\ \frac{-r F_n \sigma_1}{J} \end{bmatrix}$$

iff $\frac{F_n \sigma_2}{m} \neq \frac{\sigma_0}{\sigma_1}$

$$\text{rank } \mathcal{O} = \text{rank} \begin{bmatrix} 0 & \frac{F_n \sigma_1}{m} & \frac{F_n^2 \sigma_1 \sigma_2}{m^2} \\ 1 & -\frac{\sigma_0}{\sigma_1} & \left(\frac{\sigma_0}{\sigma_1}\right)^2 \\ -r F_n \sigma_1 & 0 & 0 \end{bmatrix} = 3$$

with $\mathcal{O} = [C, A^T C (AA)^T C]$. Hence, condition A1 holds. This rank condition clearly shows that the existence a non-zero normal force F_n is necessary to design the friction observer.

Condition A – (ii) (bounds). With $\varphi(y, x) = \frac{\sigma_0 |ry - v|}{g(ry - v)} z$, we have that

$$\begin{aligned} |\varphi(y, x)| &\leq \frac{\sigma_0 |ry - v|}{g(ry - v)} |z| \leq \frac{\sigma_0}{\mu_C} |ry - v| |z| \\ &\leq \frac{\sigma_0}{\mu_C} (|ry| + |v|) |z| \\ &\leq \frac{\sigma_0}{\mu_C} \left(\left(r + \frac{J}{rm}\right) y_{sup} + \frac{|\eta|}{rm} \right) |z| \\ &\leq \frac{\sigma_0}{\mu_C} \left(\left(r + \frac{J}{rm}\right) y_{sup} + \frac{|\eta|}{rm} \right) |z| = f(|x|) \end{aligned}$$

and that

$$|\varphi(y, \hat{x})| \leq \frac{\sigma_0}{\mu_C} \left(\left(r + \frac{J}{rm}\right) y_{sup} + \frac{|\hat{\eta}|}{rm} \right) |\hat{z}| = f(|\hat{x}|)$$

Condition A3 (Passivity). Finding a vector K , so that the map $\psi \mapsto \tilde{y}$ of the system description (80)-(81), is strictly passive, is equivalent to searching for a vector $K = [k_1, k_2, k_3]^T$, such that the I/O-map $G(s)$, defined as

$$G(s) = C^T [Is - A + KC^T]^{-1} B, \quad (106)$$

is strictly positive real (SPR), i.e. $\text{Re}\{G(j\omega)\} > 0, \forall \omega \in [0, \infty]$

Computation of $G(s)$ with the corresponding values for A, B, C , and with $k_2 = 0$, gives the map

$$G(s) = k_G \frac{(s + \beta)(s + \rho)}{(s^2 + \alpha_1 s + \alpha_2)(s + \rho)} \quad (107)$$

with

$$\begin{aligned}
k_G &= \frac{rF_n\sigma_1}{J} \\
\rho &= \frac{\sigma_0}{\sigma_1} \\
\beta &= \frac{F_n\sigma_2}{m} \\
\alpha_1 &= \beta - k_3 r F_n \sigma_1 \\
\alpha_2 &= -k_3 \beta r F_n \sigma_1 + k_1 \frac{F_n \sigma_1}{m}
\end{aligned}$$

Since ρ is positive, a zero/pole cancellation can be done in the map $G(s)$. A sufficient condition for this function be SPR is thus that $\alpha_1 > \beta$, and that all the coefficients $\beta, \alpha_1, \alpha_2$ be positive. This condition can be achieved if

$$\begin{aligned}
k_1 &> 0 \\
k_3 &< 0.
\end{aligned}$$

From the Kalman-Yakubovich-Popov lemma, we thus ensure with this choice of K that there exist a $P > 0$ satisfying the Lyapunov equation with $PB = C$.

Condition A4 (Persistence of excitation). To ensure parameter convergence we need to guarantee that

$$\lim_{t \rightarrow \infty} \varphi(y(t), u(t), x(t)) = \lim_{t \rightarrow \infty} \frac{\sigma_0 |ry(t) - v(t)|}{g(ry - v)} z(t) \neq 0.$$

This implies that the relative velocity should not tend to zero in order for the estimated parameter to converge. This in turn implies that the internal friction state $z(t)$ will not asymptotically converge to zero.

Condition A5 is not required for this application. Summarizing, we have,

Theorem 2 *Consider the one-wheel model with lumped dynamic friction (103)-(105), then the following observer:*

$$\begin{aligned}
\dot{\hat{\eta}} &= -\frac{F_n\sigma_2}{m}\hat{\eta} + \left(\frac{F_n\sigma_2}{m} + rF_n\sigma_2 - \sigma_\omega\right)\omega + u_\tau + k_1(\omega - \hat{\eta}) \\
\dot{\hat{\chi}} &= -\frac{\sigma_0}{\sigma_1}\hat{\chi} + \left(J\frac{\sigma_0}{\sigma_1} - \sigma_\omega\right)\omega + u_\tau \\
\dot{\hat{z}} &= \hat{v}_r - \hat{\theta} \frac{\sigma_0 |\hat{v}_r|}{g(\hat{v}_r)} \hat{z} - k_3(\omega - \hat{\eta}) \\
&\quad - 2\theta_{max}(f_{max} + f(|\hat{x}|)) \operatorname{sgn}(\hat{y}) \\
\dot{\hat{\theta}} &= \gamma \frac{\sigma_0 |\hat{v}_r|}{g(\hat{v}_r)} \hat{z}(\omega - \hat{\eta}) \\
\dot{\hat{y}} &= \frac{1}{J}(\hat{\chi} - rF_n\sigma_1 \hat{z}),
\end{aligned}$$

where $\hat{v}_r = r\omega - \hat{v}$ with positive nonzero k_1, k_3 and γ , ensures that all the estimated states are bounded, and that:

$$\begin{aligned}
\lim_{t \rightarrow \infty} \hat{\eta} &= \eta \\
\lim_{t \rightarrow \infty} \hat{\chi} &= \chi \\
\lim_{t \rightarrow \infty} \hat{z} &= z.
\end{aligned}$$

If in addition, the relative contact velocity does not vanishes, then we also have that

$$\lim_{t \rightarrow \infty} \hat{\theta} = \theta.$$

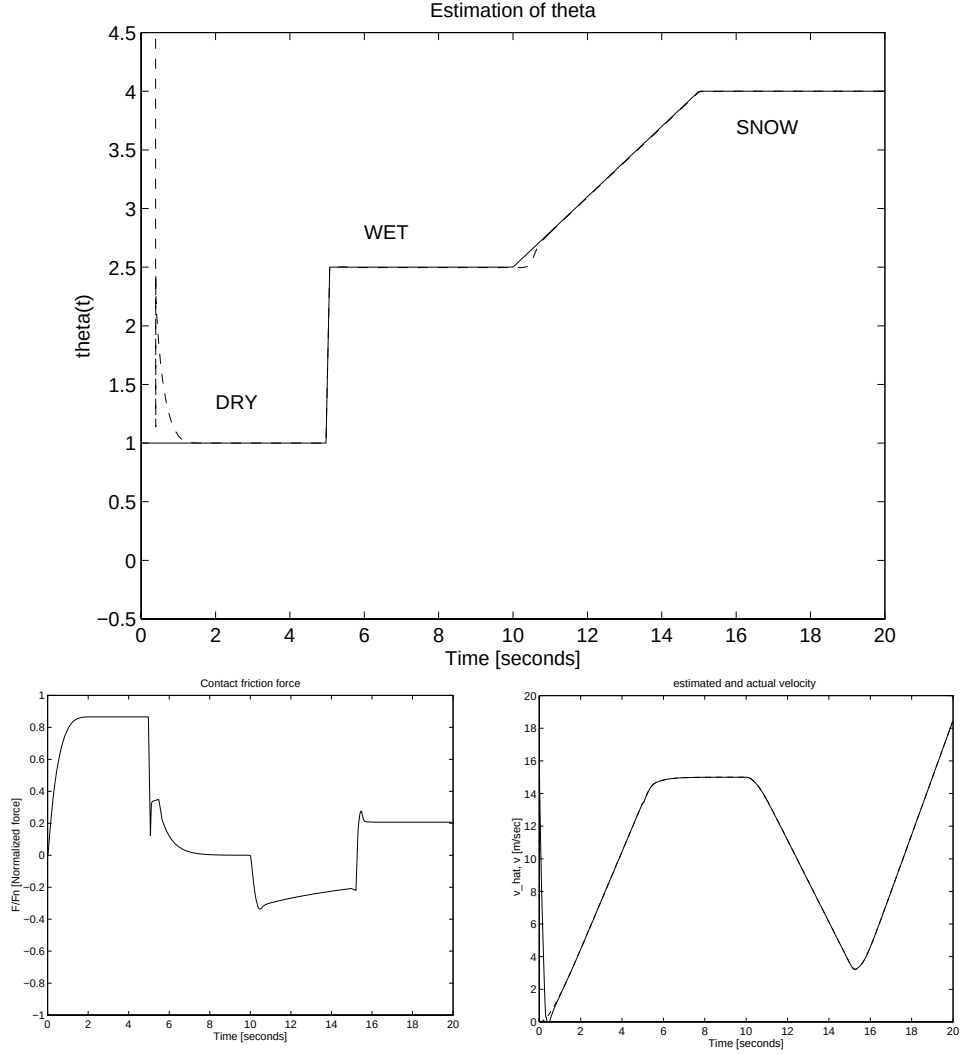


Figure 13: a) Estimated parameter $\hat{\theta}(t)$, and evolution of θ (upper figure). b) Contact torque friction (normalized) $F(t)/F_n$ (low left). c) Actual and estimated wheel linear velocity $v(t)$ (low right).

Simulations have been performed with the one-wheel system and the lumped LuGre model. The simulation has first an acceleration phase, and then a braking phase. Fig. 13-(b) shows the time-profile of the contact friction force. The estimation of linear velocity v is shown in Fig. 13-(c).

Simulation were done under variations of the parameter θ , representing the road variation conditions. Fig. 13-(a) shows in bold lines the value of θ , which evolves within fourth different conditions: the first quarter of the simulation corresponds to dry asphalt conditions. The second quarter corresponds to a sudden change from dry to wet. During the third quarter, there is a smooth variation from wet to snow. The last quarter is keep constant at the snow conditions. In dotted lines we can see the evolution of the estimate $\hat{\theta}(t)$. As we can observe, a good parameter tracking is obtained, as long as the relative contact velocity is different from zero.

6 OPTIMAL BRAKING

In this section we investigate optimal braking strategies for a simplified model of a wheeled vehicle. We derive the optimality conditions for the minimum braking distance problem by solving the associated optimal control problem. For the case of a static tire friction model, it is shown that the optimal control is singular and can be written in a state feedback form. In the most general case, the optimal solution involves both subarcs of singular and bang-bang controls. In accordance with classical results, the bang-bang subarcs are included in order to satisfy the initial and final boundary conditions. In addition, we also provide a formal proof of the optimality of the “maximum friction” strategy used in standard ABS control system design. Thus, under the hypothesis that the friction force is an algebraic (memoryless) function of the slip coefficient, it is shown that the optimal singular control is the one that keeps the slip rate constant at the maximum friction value. Such static friction models may not be realistic, however. Dynamic friction models are more accurate in representing the actual friction generation mechanism. We thus also provide the corresponding optimality conditions for the case of the LuGre tire dynamic friction model.

6.1 State Equations

Consider the equations (74)-(75) and let $\rho = mr^2/J$, $u = ru_r/J$ and $f(s) = F_r(v_r, v)/m$, where F_r is the friction force, assumed to be a static map of the slip coefficient s . Here $v_r = \omega r - v$ is the relative velocity. For braking, recall that $s = r\omega/v - 1 = v_r/v$. Let now the new variables

$$z_1 = v_r + (1 + \rho)v, \quad z_2 = v \quad (108)$$

Then the state equations (74)-(75) can be written in the form

$$\dot{z}_1 = u \quad (109a)$$

$$\dot{z}_2 = f(s) \quad (109b)$$

where $s = (z_1 - (1 + \rho)z_2)/z_2 = z_1/z_2 - (1 + \rho)$.

Given the equations (109) the objective is to minimize the performance index

$$\mathcal{J} = \int_0^{t_f} z_2(\tau) d\tau \quad (110)$$

subject to the following initial and final conditions

$$z_1(0) = z_{10}, \quad z_2(0) = z_{20}, \quad z_1(t_f) = z_2(t_f) = 0 \quad (111)$$

and the final time t_f is free. To solve the previous optimization problem we introduce the associated Hamiltonian

$$\mathcal{H} = z_2 + \lambda_1 u + \lambda_2 f(z) \quad (112)$$

The adjoint system is the given by

$$\dot{\lambda}_1 = -\lambda_2 \frac{\partial f}{\partial z_1} \quad (113a)$$

$$\dot{\lambda}_2 = -\lambda_2 \frac{\partial f}{\partial z_2} \quad (113b)$$

Since the final time is not specified, the transversality condition gives

$$\mathcal{H}(t_f) = 0 \quad (114)$$

Along with the fact that the Hamiltonian is not an explicitly function of time, the last equation implies that $\mathcal{H}(t) = 0 \quad \forall t \in [0, t_f]$ along the optimal trajectory. From Pontryagin's Maximum Principle, the optimal control is given by

$$u_{\text{opt}} = \text{argmin} \mathcal{H}(z, \lambda, u) \quad (115)$$

It is assumed here that the control input u is bounded, that is, any allowable control must satisfy the constraint

$$u_{\min} \leq u \leq 0 \quad (116)$$

From (112) the switching function is

$$\mathcal{H}_1 = \lambda_1 \quad (117)$$

and using Eqs. (115)-(116), one obtains the following optimal control strategy

$$u_{\text{opt}} = \begin{cases} u_{\min} & \text{for } \mathcal{H}_1 > 0 \\ 0 & \text{for } \mathcal{H}_1 < 0 \\ u_{\text{sing}} & \text{for } \mathcal{H}_1 \equiv 0 \end{cases} \quad (118)$$

Note that the singular control u_{sing} is used when the switching function remains zero over a *finite* time interval, i.e., $\mathcal{H}_1(t) = 0$ for $t \in [t_1, t_2] \subset [0, t_f]$. The singular control is given by [32]

$$u_{\text{sing}} = \frac{\alpha(z)}{\beta(z)} \quad (119)$$

where

$$\alpha(z) = -\frac{\partial^2 f}{\partial z_1 \partial z_2} f + \frac{\partial f}{\partial z_1} \frac{\partial f}{\partial z_2} \quad (120a)$$

$$\beta(z) = \frac{\partial^2 f}{\partial z_1^2} \quad (120b)$$

For optimality, the control in Eq. (119) must also satisfy a local second-order convexity condition, known as the generalized Legendre-Clebsch condition or the Kelley-Contensou test [30]

$$\frac{\partial}{\partial u} \left(\frac{d^{2q} \mathcal{H}_1}{dt^{2q}} \right) \geq 0 \quad (121)$$

where q is the order of the singular arc. In our case, $q = 1$ and (121) reduces to testing whether

$$\beta(z) \leq 0 \quad (122)$$

along the singular part of the trajectory.

6.2 Computation of Singular Control

The computation of the singular control from Eq. (119) requires the calculation of the partial derivatives in (120). To proceed with the calculation of the singular control, we need to consider the explicit expression of the friction force as a function of z_1 and z_2 . Given the assumption that f is a function of only the slip coefficient $s = z_1/z_2 - (1 + \rho)$, one obtains for the partial derivatives in (120)

$$\frac{\partial f}{\partial z_1} = \frac{\partial f}{\partial s} \frac{\partial s}{\partial z_1}, \quad \frac{\partial f}{\partial z_2} = \frac{\partial f}{\partial s} \frac{\partial s}{\partial z_2} \quad (123)$$

and hence

$$\frac{\partial^2 f}{\partial z_1 \partial z_2} = \frac{\partial^2 f}{\partial s^2} \frac{\partial s}{\partial z_1} \frac{\partial s}{\partial z_2} + \frac{\partial f}{\partial s} \frac{\partial^2 s}{\partial z_1 \partial z_2} \quad (124a)$$

$$\frac{\partial^2 f}{\partial z_1^2} = \frac{\partial^2 f}{\partial s^2} \left(\frac{\partial s}{\partial z_1} \right)^2 + \frac{\partial f}{\partial s} \frac{\partial^2 s}{\partial z_1^2} \quad (124b)$$

Theorem 3 ([32]) *On a singular subarc, necessarily $\frac{\partial f}{\partial s} = 0$.*

It should be pointed out that the mere appearance of the singular control in the composite optimal trajectory is not ensured *a priori*, even in the case the local optimality of the control law is guaranteed by the satisfaction of Kelley's condition (121). For most automotive applications, however, the maximum braking torque is large enough, such that a singular subarc is always part of the optimal trajectory.

Notice that the expression of the singular control can also be written in the form

$$u_{\text{sing}} = \frac{z_1}{z_2} f(z_1, z_2) \quad (125)$$

and the control is in a purely *feedback* form (independent of co-states). This is the expression of the singular control during braking. The only assumption used in the derivation of (125) is that the friction force is a function of the slip coefficient s .

6.3 Maximum Friction Control

Assuming that the friction force is given as an algebraic map of the slip coefficient s , the control that keeps the friction force to its maximum value of the f vs. s curve can be calculated by imposing $\dot{s} = 0$, along with the conditions

$$\frac{\partial f}{\partial s} = 0, \quad \frac{\partial^2 f}{\partial s^2} < 0 \quad (126)$$

Using the definition for the slip coefficient (assume $z_2 \neq 0$), one obtains

$$\dot{s} = \frac{z_2 \dot{z}_1 - z_1 \dot{z}_2}{z_2^2} \quad (127)$$

Thus, $\dot{s} = 0$ if and only if

$$z_2 \dot{z}_1 - z_1 \dot{z}_2 = z_2 u - z_1 f(z_1, z_2) = 0 \quad (128)$$

or that

$$u_{f_{\max}} = \frac{z_1}{z_2} f(z_1, z_2) \quad (129)$$

which is the same as the expression for the singular control in Eq. (125), under the additional assumption that $\partial f / \partial s = 0$. Therefore, the singular control u_{sing} achieves $\dot{s} = 0$ and $\partial f / \partial s = 0$, i.e., it forces operation at the maximum of the friction vs. slip curve.

From (124b) the second condition in Eq. (126) implies that

$$\frac{\partial^2 f}{\partial z_1^2} < 0$$

which is exactly Kelley's condition for optimality of the singular control.

Remark. It is clear that the singular control is constant. It can also be re-written as

$$u_{\text{sing}} = s^* f_{\max} + (1 + \rho) f_{\max} \quad (130)$$

The last expression has the advantage that it does not require the calculation of the partial derivatives in Eq. (124). It needs only the maximum value of the friction force $f_{\max}$, and the corresponding value of the slip coefficient s^* .

6.4 Dynamic Friction Model

The previous derivation was based on the assumption that the friction force between the tire and the ground can be accurately described as a static relation in terms of the slip coefficient. In fact, it is well-known that transient effects can be important. This drawback of static friction models can be addressed by using dynamic friction models like the ones discussed here. Next, we use the average lumped LuGre model from (43)-(44) in the optimal braking problem. The average lumped model is preferable over the distributed LuGre model since the latter one consists of an infinite number of states that makes it very difficult to use it for control design.

The system dynamics are given by

$$\dot{z}_1 = \dot{v}_r + (1 + \rho)\dot{v} = u \quad (131)$$

$$\dot{z}_2 = \dot{v} = f(v_r, z_3) \quad (132)$$

$$\dot{z}_3 = \dot{\bar{z}} = v_r - \frac{\sigma_0 |v_r|}{g(v_r)} z_3 - \frac{\kappa}{L} |v_r + v| z_3 \quad (133)$$

where z_1 and z_2 as in (108) and where $z_3 = \bar{z}$, the internal friction state which characterizes the mean displacement of the bristles at the contact patch. The total friction force is given by

$$f(v_r, z_3) = \frac{F_n}{m} (\sigma_0 \bar{z} + \sigma_1 \dot{\bar{z}} + \sigma_2 v_r) \quad (134)$$

In (133) $g(v_r)$ is assumed to be given by

$$g(v_r) = \mu_c + (\mu_s - \mu_c) e^{-(v_r/v_s)^2} \quad (135)$$

The parameters of the friction model are the normalized longitudinal stiffness coefficient σ_0 [1/m], the normalized longitudinal damping coefficient σ_1 [sec/m], the normalized viscous damping factor σ_2 [sec/m], the normalized Coulomb and static friction μ_c and μ_s and the Stribech characteristic velocity v_s [m/sec]. When uniform load distribution is assumed, κ can be taken as a dimensionless constant in the interval [1, 2]; see the discussion in Section (2.5).

The performance index to be minimized is given by (110). The final time t_f is free and the boundary conditions are

$$z_1(0) = z_{10}, \quad z_2(0) = z_{20}, \quad z_1(t_f) = 0, \quad z_2(t_f) = 0, \quad z_3 = \text{free} \quad (136)$$

In order to solve this minimization problem we use an “optimal backstepping” approach [33]. To this end, we consider the reduced system

$$\dot{z}_2 = f(v_r, z_3) \quad (137)$$

$$\dot{z}_3 = v_r - \frac{\sigma_0 |v_r|}{g(v_r)} z_3 - \frac{\kappa}{L} |v_r + z_2| z_3 \quad (138)$$

$$f(v_r, z_3) = \frac{F_n}{m} (\sigma_0 z_3 + \sigma_1 \dot{z}_3 + \sigma_2 v_r) \quad (139)$$

where v_r is now the control input. If the optimal “control” v_r^* for the system (137)-(139) is known, then the optimal braking torque can be computed from (131). The full-system optimization problem requires knowledge of the boundary conditions for z_1 . Since the boundary conditions of z_1 of the reduced-system will be in general different than the ones for the complete system, bang-bang subarcs will be required to satisfy these boundary conditions at $t = 0$ and $t = t_f$. We avoid the calculation of such bang-bang subarcs by making the realistic assumption that for large enough braking torque (typically the case), z_1 can be changed almost instantaneously via (131). Therefore, without loss of generality one can assume that the actual value of v_r matches the optimal ones from the reduced optimization system at $t = 0$ and $t = t_f$. This approach has the benefit of avoiding the numerically ill-conditioned calculation of the boundary subarcs which, at any rate, have an extremely small duration.

In order to proceed with the solution of the optimal control problem (137)-(139) subject to (110) we need to consider the natural control constraint $-v \leq v_r \leq 0$. From the definition of the longitudinal slip rate for braking $s = \frac{\omega r}{v} - 1$ this constraint implies $-1 \leq s \leq 0$ or $0 \leq \omega r \leq v$. The constraint $v_r \in U = [-v, 0]$ is imposed by the typical braking systems, that is, the brakes can only apply a torque opposite to the angular velocity of the wheel.

6.5 Singular Perturbation Approach

As mentioned in [27] the LuGre friction model together with the equations of motion can be regarded as a singular perturbed system [28, 29]. In the LuGre tire friction model we have that the parameter σ_0 is

typically very big compared to the other terms of the internal state $\bar{z}$ equation. Moreover, the patch length L which is small. Now define $\ell = \sigma_0 L$ and let

$$\epsilon = \frac{1}{\sigma_0}, \quad \hat{z} = \frac{1}{\epsilon} \bar{z}, \quad \hat{z}_3 = \hat{z} \quad (140)$$

We can re-write the system equations (137)-(139) in a singular perturbed form as follows:

$$\dot{z}_2 = \hat{f}(v_r, \hat{z}_3, \epsilon) \quad (141)$$

$$\epsilon \dot{\hat{z}}_3 = v_r - \frac{|v_r|}{g(v_r)} \hat{z}_3 - \frac{\kappa}{\ell} |v_r + z_2| \hat{z}_3 \quad (142)$$

$$\hat{f}(v_r, \hat{z}_3, \epsilon) = \frac{F_n}{m} (\hat{z}_3 + \epsilon \sigma_1 \dot{\hat{z}}_3 + \sigma_2 v_r) \quad (143)$$

We observe that the fast state of the system is the internal state of the friction model $\bar{z}_3$ and the slow state is the velocity of the vehicle z_2 . This means that the friction develops faster than the vehicle changes its speed. Setting $\epsilon = 0$ we derive the reduced model

$$\dot{z}_2 = \hat{f}(v_r, \hat{z}_{3ss}) \quad (144)$$

$$\hat{z}_{3ss} = \frac{v_r}{\frac{|v_r|}{g(v_r)} + \frac{\kappa |v_r + z_2|}{\ell}} \quad (145)$$

$$\hat{f}(v_r, \hat{z}_{3ss}) = \frac{F_n}{m} (\hat{z}_{3ss} + \sigma_2 v_r) \quad (146)$$

Observe that $\hat{z}_{3ss} = \sigma_0 \bar{z}_{ss}$ where $\bar{z}_{ss}$ is the steady state of the mean deflection of the bristles at the LuGre model. That is, $\bar{z}$ will reach $\bar{z}_{ss}$ very quickly. This allow us to use the reduced model (144)-(146) for our analysis.

The Hamiltonian for the optimization problem using the reduced system (144)-(146) is given by

$$\hat{\mathcal{H}} = z_2 + \lambda \hat{f}(v_r, \hat{z}_{3ss}) \quad (147)$$

The optimal controller must satisfy

$$\begin{aligned} v_r = \operatorname{argmin}_{v_r \in U} (\hat{\mathcal{H}}) &= \operatorname{argmin}_{v_r \in U} (\hat{f}(v_r, \hat{z}_{3ss})) \\ &= \operatorname{argmin}_{v_r \in U} \left(\frac{F_n}{m} (\hat{z}_{3ss} + \sigma_2 v_r) \right) \end{aligned} \quad (148)$$

Observe that this equation implies that for optimality we require minimum (maximum negative) friction, a result that was also seen in the previous section where the Pacejka model was used. Recall that $\hat{f}(v_r, \hat{z}_{3ss}) = F_n (\hat{z}_{3ss} + \sigma_2 v_r) / m$. Figure 14 shows how $\hat{z}_{3ss} + \sigma_2 v_r$ changes with $v_r \in U$ for different v (or z_2). We observe that the minimizing v_r is either the one that makes $\frac{\partial(\hat{z}_{3ss} + \sigma_2 v_r)}{\partial v_r} = 0$, or, when the velocity is small enough, $v_r = -v$ (when $s = -1$).

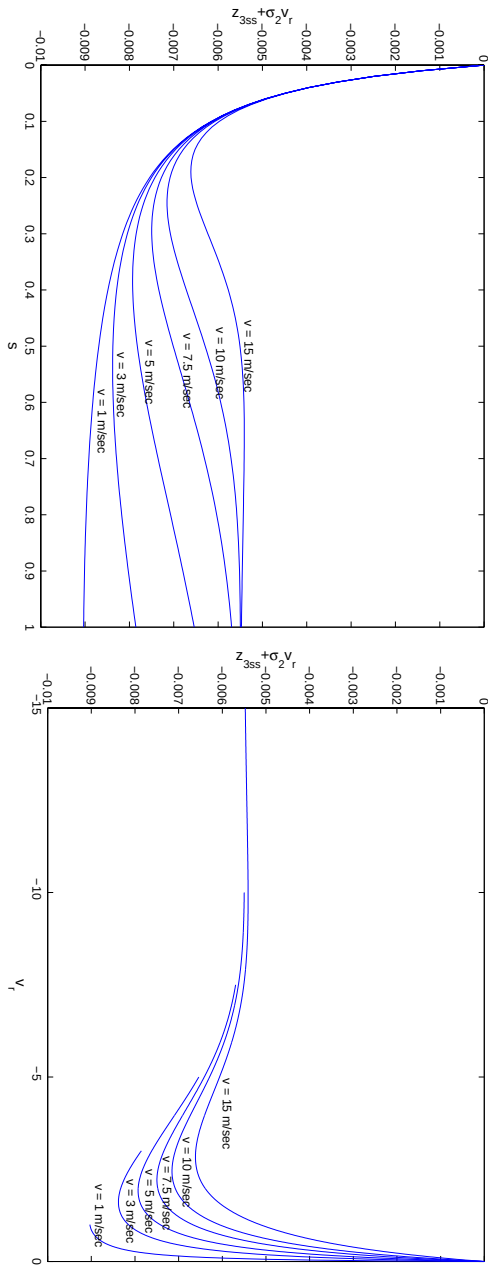
6.6 Numerical Example

To illustrate the previous analysis we consider a numerical example of minimum braking distance for a vehicle with mass 1000 kg ($m = 250$ kg). The wheel radius is $r = 0.25$ m and the wheel and drivetrain moment of inertia is $J = 1$ kg m². These values correspond to a value of $\rho = 15.625$. We first consider the case using Pacejkas' static friction model.

$$F_r / F_n = D \sin(C \arctan(Bs)) \quad (149)$$

with values $D = 0.7, B = 7, C = 1.6$. The maximum normalized friction force can be computed directly from Eq. (149) and is $F_r / F_n = 0.7$. The corresponding slip coefficient is $s^* = -0.2138$.

The initial conditions are given by $v(0) = r\omega(0) = 15$ m/s. To avoid the singularity of the friction model at $v = 0$ and $s = 0$, the final conditions are given by $v(t_f) = 0.1$ m/s and $r\omega(t_f) = 0.099$ m/s. The initial

Figure 14: $\hat{z}_{3ss} + \sigma_2 v_r$ with longitudinal slip s and relative velocity v_r

conditions correspond to a value of the slip coefficient $s(0) = 0$ and the final conditions correspond to a value of slip $s(t_f) = -0.01$. The maximum value of the braking torque is given by 1,500 Nm which corresponds to $u_{\min} = -375$.

To numerically solve the associated two-boundary value optimization problem, a special FORTRAN code was written based on a root-solving Newton method using the subroutine hybrd of the MINPACK library [31]. All calculations were performed using double precision arithmetic to avoid numerical ill-conditioning. The results are shown in Figs. 15-16. The optimal solution has three subarcs in this case. As expected from our analysis, during most of the trajectory a singular control law is used to achieve maximum friction force. The initial and final bang subarcs take care of the required boundary conditions (slip not at the maximum friction force). The distance traveled before complete stop is $\mathcal{J} = 16.382$ m. The total time to stop the vehicle under this applied braking torque profile (shown on the bottom of Fig. 16(a)) is $t_f = 2.17$ sec. The initial an final bang subarcs take place in a very short time interval. The duration of the initial and final subarcs are approximately 0.0123 and 3×10^{-4} seconds, respectively. That is, for all practical purposes, the initial and final bang subarcs can be replaced by two impulses at the initial and final parts of the trajectory. This justifies the reduced-system approach followed for the dynamic model case.

Figure 15 shows the time histories of the states and the co-states. The angular velocity of the wheel and the corresponding co-state λ_1 exhibit a very fast transient during the initial part of the trajectory, as expected from the optimal torque profile. The optimality of the trajectory is verified by the time history of the switching function λ_1 in Fig. 15. A more detailed depiction of the switching function just before the entry to the singular subarc is shown on the top plot of Fig. 16(a). Finally, Fig. 16(b) shows the friction force and the slip coefficient history.

Next we compare the optimal controller designed using Pacejka's model (149) with the one designed using the LuGre model. The first step is to find the parameters of the LuGre model (σ_0, μ_s, μ_c etc.) such that the two models are comparable. To do so we fit to Pacejka's $s - \mu$ plot, a $s - \mu$ plot constructed using (145)-(146) together with the definition of longitudinal slip, at an intermediate velocity of the simulation (7.5 m/sec). We keep the same values for the Pacejka model coefficients. The fitting of the two curves is shown in figure 17. The parameters identified are shown in Table 1. We have used patch length $L = 0.2$ m, constant parameter $\kappa = 1.2$ and $\alpha = 2$. The optimal controller for the case of the LuGre friction model is given by (148). The simulations that follow are made using the same parameters as for the static case: $m = 250$ kg, $J = 1$ kg m², $\rho = 15.625$ and initial velocity $v(0) = 15$ m/sec. Observe the way the controllers work. The optimal controller based on the static model, keeps the slip constant to its maximizing value, while in the LuGre model-based optimal controller, the slip $s = -v_r/v$ is changing to provide the maximum friction at all times. We observe that in the LuGre model case, v_r at some point hits the boundary of the constraint $-v \leq v_r \leq 0$ and we have $v_r = -v$. Then v_r stays on the boundary of the constraint until the

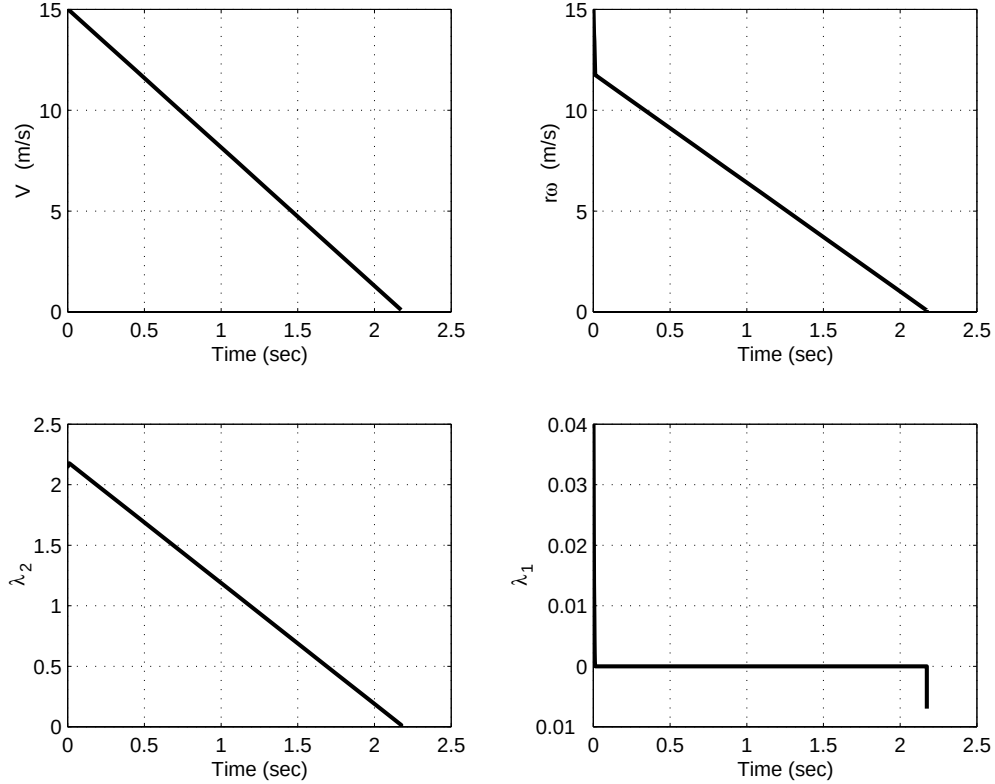


Figure 15: State and co-state time histories.

Table 1: Identified Parameters

Parameter	Value
σ_0	96.19 m^{-1}
σ_2	0 sec / m
μ_c	0.52
μ_s	0.89
v_s	3.46 m / sec

end. This is expected since, as seen previously, for small velocities the maximum friction appears at $s = 1$ i.e. $v_r = -v$.

Recall that the optimal braking controller maximizes friction at all times. Together with the optimal controller for the LuGre friction model case we present a simulation using the optimal controller designed for the Pacejka model (constant slip) but this time applied to the equations using the LuGre model. The constant slip selected is the one maximizing friction at the velocity that the steady state of the LuGre model was matched with the Pacejka model ($v = 7.5 \text{ m/sec}$). This slip is $s \approx 0.25$. As expected, it takes longer to stop the vehicle compared to using the optimal controller. The stopping distance using the optimal controller was 17.34m while using the constant slip controller it was 17.81m.

7 OBSERVED-BASED EMERGENCY BRAKING CONTROL

In this section a control scheme for emergency braking of vehicles is designed. The scheme is based on the LuGre dynamic friction model and estimation of the tire/road friction. The control system output is the pressure in the braking system, and is calculated using only the wheel angular speed information.

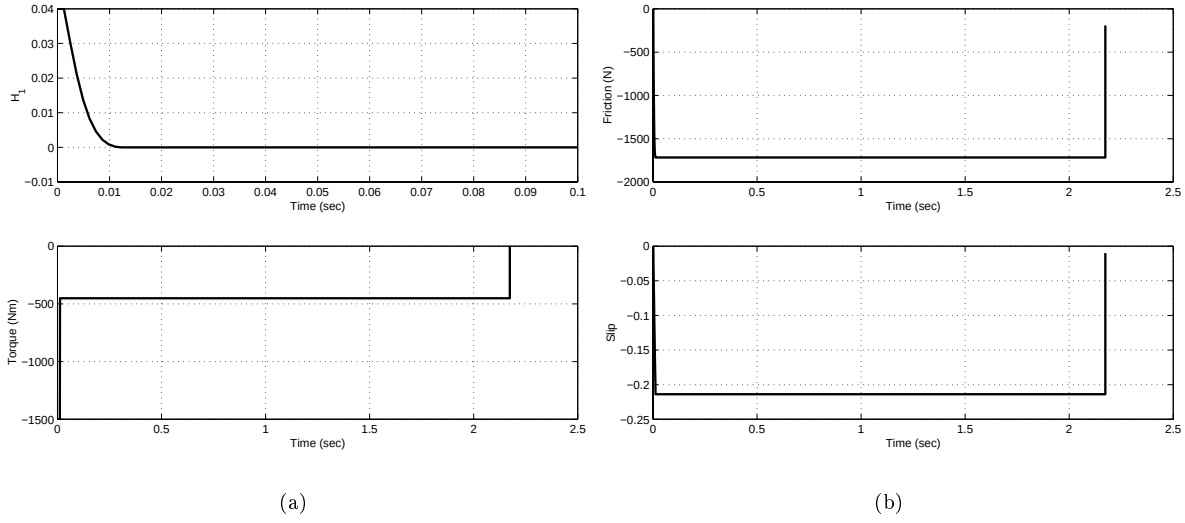


Figure 16: (a): Switching function and braking torque history. (b) Friction force and slip coefficient time histories.

The controller utilizes estimated state feedback control to achieve a near maximum deceleration. The state observer gain is calculated by using linear matrix inequality (LMI) techniques. This system has two advantages when compared with an antilock braking system (ABS): it generates less chattering during braking and produces a source of a priori information regarding safe spacing.

7.1 System Dynamics

In this section only the longitudinal dynamics of the vehicle is considered. We assume that the four wheels of the vehicle apply the same braking force. For simplicity, we also assume that the road has no slope and that the weight of the vehicle is distributed evenly among the four wheels. A quarter vehicle model is used and we consider a modified lumped LuGre friction model as follows:

$$\begin{cases} \dot{z} = -v_r - \theta \frac{\sigma_0 |v_r|}{h(v_r)} z \\ J\dot{\omega} = -rF_x - u_\tau \\ m\dot{v} = 4F_x - F_r \end{cases} \quad (150)$$

where z is the friction internal state, $v_r = v - r\omega$ is the relative velocity, $h(v_r) = \mu_c + (\mu_s - \mu_c)e^{-|\frac{v_r}{v_s}|^{1/2}}$, u_τ is the traction/braking torque, F_x the traction/braking force given by the tire/road contacting, F_r the rolling resistance, m the vehicle mass, J the tire rotational inertia, and the parameter θ is used to model the effect of different tire/road Coulomb friction coefficients. The braking force F_x is given by

$$F_x = F_n(\sigma_0 z + \sigma_1 \dot{z} - \sigma_2 v_r)$$

where F_x is a negative number in the case of braking and $F_n = mg/4$. By [39], the rolling resistance can be modeled as

$$F_r = \sigma_v mgv$$

where σ_v is rolling resistance coefficient and g is the gravity constant.

Defining the state variables as

$$x_1 := \sigma_0 z, \quad x_2 := v, \quad x_3 := v_r = v - r\omega$$

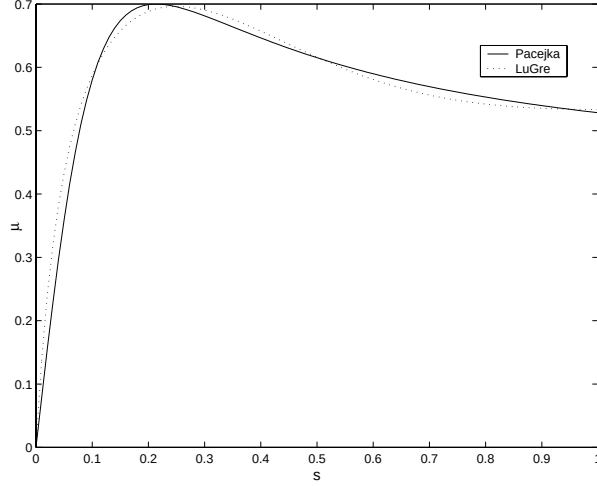


Figure 17: Pacejka and LuGre models fitting

the system dynamics Eq.(150) are rewritten as:

$$\dot{x}_1 = \sigma_0 \dot{z} = -\sigma_0 x_3 - \theta \sigma_0 f(x_3) x_1 \quad (151)$$

where $f(x_3) = \frac{x_3}{h(x_3)}$, here we use the fact that $x_3 = v_r = v - r\omega \geq 0$ during braking and

$$\dot{x}_2 = g[x_1 + \sigma_1(-x_3 - \theta f(x_3)x_1) - \sigma_2 x_3] - g\sigma_v x_2 \quad (152)$$

For the state variable x_3 we have

$$\dot{x}_3 = \dot{v} - r\dot{\omega} = \alpha[x_1 + \sigma_1(-x_3 - \theta f(x_3)x_1) - \sigma_2 x_3] - g\sigma_v x_2 + \frac{r}{J}K_b P_b, \quad (153)$$

where $\alpha = g(1 + \frac{mr^2}{4J})$. In the above equation we use the formula $u_r = K_b P_b$, where K_b is the braking system gain and P_b the brake pressure, which is the controlled variable.

For most vehicles, we can measure the angular velocity of each wheel.

$$y = \omega = \frac{1}{r}(x_2 - x_3) \quad (154)$$

7.2 Observer-based Braking Controller Design

In this section we formulate the design of a controller based on the available angular velocity output. Arrange the system dynamics (151), (152), (153) and output measurements (154) as

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + B_1\theta\psi(\mathbf{x}) + B_2u \\ y = C\mathbf{x} \end{cases} \quad (155)$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & -\sigma_0 \\ g & -g\sigma_v & -g(\sigma_2 + \sigma_1) \\ \alpha & -g\sigma_v & -\alpha(\sigma_2 + \sigma_1) \end{bmatrix}, \quad B_1 = \begin{bmatrix} -\sigma_0 \\ -g\sigma_1 \\ -\alpha\sigma_1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ \frac{r}{J}K_b \end{bmatrix}, \quad C = \begin{bmatrix} 0 & \frac{1}{r} & -\frac{1}{r} \end{bmatrix},$$

$$\psi(\mathbf{x}) = x_1 f(x_3), \quad u = P_b$$

Since the internal state z , given by LuGre model, and the vehicle longitudinal velocity are unavailable, we must design an observer to estimate these states. Construct the following model-based nonlinear observer

$$\dot{\hat{\mathbf{x}}} = A\hat{\mathbf{x}} + B_1\hat{\theta}\psi(\hat{\mathbf{x}}) + B_2u + L(y - C\hat{\mathbf{x}}) + B_1\mathcal{G} \quad (156)$$

where $\mathcal{G}$ is a tuning function to be determined subsequently.

The following assumptions are made for the system (155) and observer (156):

- (i) (A, B_1) is controllable and (A, C) is observable;
- (ii) $f(x_3)$ is positive and bounded and $f'(x_3)$ is bounded, i.e.

$$0 \leq f(x_3) \leq f_{max} \leq \rho_2 < \infty, \quad |f'(x_3)| \leq \rho_3 < \infty, \quad \forall \mathbf{x} \in \mathcal{D}_1 \subset \mathcal{R} \quad (157)$$

- (iii) The unknown parameter θ is bounded, i.e.

$$0 < \theta \leq \theta_{max} \quad (158)$$

- (iv) The map $w \mapsto \xi$ of the system

$$\begin{cases} \dot{\zeta} = (A - LC)\zeta + B_1 w \\ \xi = C\zeta \end{cases} \quad (159)$$

with $(A - LC)$ Hurwitz, is strictly passive; moreover, $\exists \rho_1 > 0$ a constant, and $\exists P = P^T > 0$ such that

$$(A - LC)^T P + P(A - LC) + (\rho_1^2 + \rho_4)I < 0 \quad (160)$$

as well as

$$PB_1 = C^T \quad (161)$$

where $\rho_4 = \frac{2\theta_{max}\rho_2}{r} > 0$.

Theorem 4 Under assumptions (i) – (iv) there exists an adaptive emergency braking controller that achieves

$$\hat{\lambda} \rightarrow \hat{\lambda}_{max}$$

asymptotically for the system (155) using measured angular velocity ω , where the estimated slip $\hat{\lambda} := \frac{\hat{x}_3}{\hat{x}_2} = \frac{\hat{v} - r\hat{\omega}}{\hat{v}}$ and $\hat{\lambda}_{max} := \hat{\lambda}_{max}(\hat{v}_r, \hat{v})$ is the longitudinal slip corresponding to the estimated maximum friction coefficient $\hat{m}_{u_{max}}$ in the pseudo-static relationship between μ and λ .

Proof: Define $\tilde{\mathbf{x}} := \mathbf{x} - \hat{\mathbf{x}}$, $\tilde{y} := y - \hat{y} = C\tilde{\mathbf{x}}$ and $\tilde{\theta} := \theta - \hat{\theta}$, then the error dynamics for the system is

$$\dot{\tilde{\mathbf{x}}} = (A - LC)\tilde{\mathbf{x}} + B_1 \left[\theta\psi(\mathbf{x}) - \hat{\theta}\psi(\hat{\mathbf{x}}) \right] - B_1 \mathcal{G} \quad (162)$$

define the dynamic surface $\tilde{s}$ as

$$\tilde{s} := \hat{x}_3 - \hat{\lambda}_{max} \hat{x}_2$$

differentiate $\tilde{s}$

$$\begin{aligned} \dot{\tilde{s}} &= \dot{\hat{x}}_3 - \dot{\hat{x}}_2 \hat{\lambda}_{max} - \dot{\hat{\lambda}}_{max} \hat{x}_2 \\ &= \frac{r}{J} K_b P_b - \sigma_1 (\alpha - g \hat{\lambda}_{max}) f(\hat{x}_3) \hat{\theta} + \left\{ (\alpha - g \hat{\lambda}_{max}) [\hat{x}_1 - (\sigma_2 + \sigma_1) \hat{x}_3 - (1 - \hat{\lambda}_{max}) g \sigma_v \hat{x}_2] \right. \\ &\quad \left. + (l_3 - l_2 \hat{\lambda}_{max}) \tilde{y} \right\} - \sigma_1 (\alpha - g \hat{\lambda}_{max}) \mathcal{G} - \dot{\hat{\lambda}}_{max} \hat{x}_2 \\ &= d K_b P_b + \beta_1(\hat{\mathbf{x}}) \hat{\theta} + \beta_2(\hat{\mathbf{x}}) + \beta_3(\hat{\mathbf{x}}) \mathcal{G} \end{aligned} \quad (163)$$

where

$$\begin{aligned} d &= \frac{r}{J}, \quad \beta_1(\hat{\mathbf{x}}) = -\sigma_1 (\alpha - g \hat{\lambda}_{max}) f(\hat{x}_3) \hat{x}_1, \\ \beta_2(\hat{\mathbf{x}}) &= (\alpha - g \hat{\lambda}_{max}) [\hat{x}_1 - (\sigma_2 + \sigma_1) \hat{x}_3] + (l_3 - l_2 \hat{\lambda}_{max}) \tilde{y} - \dot{\hat{\lambda}}_{max} \hat{x}_2 - (1 - \hat{\lambda}_{max}) g \sigma_v \hat{x}_2, \\ \beta_3(\hat{\mathbf{x}}) &= -\sigma_1 (\alpha - g \hat{\lambda}_{max}), \end{aligned}$$

and l_2, l_3 are the second and third elements of the gain vector $L \in \mathcal{R}^3$.

Consider the following Lyapunov function candidate

$$V = \frac{1}{2}\tilde{s}^2 + \frac{1}{2\gamma}\tilde{\theta}^2 + \tilde{\mathbf{x}}^T P \tilde{\mathbf{x}}$$

where $\gamma > 0$. Then

$$\begin{aligned} \dot{V} &= \dot{\tilde{\mathbf{x}}}^T P \tilde{\mathbf{x}} + \tilde{\mathbf{x}}^T P \dot{\tilde{\mathbf{x}}} + \tilde{s}\dot{\tilde{s}} + \frac{1}{\gamma}\tilde{\theta}\dot{\tilde{\theta}} \\ &= \tilde{\mathbf{x}}^T [(A - LC)^T P + P(A - LC)] \tilde{\mathbf{x}} + 2\tilde{\mathbf{x}}^T P B_1 [\theta\psi(\mathbf{x}) - \hat{\theta}\psi(\hat{\mathbf{x}})] + \frac{1}{\gamma}\tilde{\theta}\dot{\tilde{\theta}} + \tilde{s}\dot{\tilde{s}} - 2\tilde{\mathbf{x}}^T P B_1 \mathcal{G} \end{aligned}$$

Notice that

$$\theta\psi(\mathbf{x}) - \hat{\theta}\psi(\hat{\mathbf{x}}) = \tilde{\theta}\psi(\hat{\mathbf{x}}) + \theta[\psi(\mathbf{x}) - \psi(\hat{\mathbf{x}})]$$

and use the fact (161) to obtain

$$\begin{aligned} \dot{V} &= \tilde{\mathbf{x}}^T [(A - LC)^T P + P(A - LC)] \tilde{\mathbf{x}} + 2\tilde{y}\tilde{\theta}\psi(\hat{\mathbf{x}}) + 2\tilde{\mathbf{x}}^T P B_1 \theta[\psi(\mathbf{x}) - \psi(\hat{\mathbf{x}})] + \\ &\quad \frac{1}{\gamma}\tilde{\theta}\dot{\tilde{\theta}} + \tilde{s} [dK_b P_b + \beta_1(\hat{\mathbf{x}})\hat{\theta} + \beta_2(\hat{\mathbf{x}}) + \beta_3(\hat{\mathbf{x}})\mathcal{G}] - 2\tilde{\mathbf{x}}^T P B_1 \mathcal{G}. \end{aligned} \quad (164)$$

Letting the control input be

$$u = P_b = \frac{1}{dK_b} [-\beta_1(\hat{\mathbf{x}})\hat{\theta} - \beta_2(\hat{\mathbf{x}}) - \beta_3(\hat{\mathbf{x}})\mathcal{G} - \eta\tilde{s}]$$

Eq.(163) becomes

$$\dot{\tilde{s}} = -\eta\tilde{s}. \quad (165)$$

Using (161) and (165) and letting

$$\dot{\tilde{\theta}} = 2\gamma\tilde{y}\psi(\hat{\mathbf{x}}) \quad (166)$$

we obtain from Eq.(164)

$$\begin{aligned} \dot{V} &= \tilde{\mathbf{x}}^T [(A - LC)^T P + P(A - LC)] \tilde{\mathbf{x}} + 2\tilde{y}\tilde{\theta}\psi(\hat{\mathbf{x}}) + 2\tilde{\mathbf{x}}^T P B_1 \theta[\psi(\mathbf{x}) - \psi(\hat{\mathbf{x}})] + \\ &\quad \frac{1}{\gamma}\tilde{\theta}\dot{\tilde{\theta}} - \eta\tilde{s}^2 - 2\tilde{y}\mathcal{G}. \end{aligned}$$

Note that

$$\psi(\mathbf{x}) - \psi(\hat{\mathbf{x}}) = x_1 f(x_3) - \hat{x}_1 f(\hat{x}_3) = f(x_3)\tilde{x}_1 + \hat{x}_1 f'(x_3^*)(x_3 - \hat{x}_3)$$

where x_3^* is a value between x_3 and $\hat{x}_3$ derived by using the Mean Value Theorem for the smooth function $f(x) = \frac{x}{h(x)}$. Moreover, by (157), (158) and (166)

$$\begin{aligned} \dot{V} &\leq \tilde{\mathbf{x}}^T [(A - LC)^T P + P(A - LC)] \tilde{\mathbf{x}} + 2\tilde{\mathbf{x}}^T C^T \theta_{max} \rho_2 \tilde{x}_1 + 2\rho_3 \theta_{max} |\tilde{y}| |\hat{x}_1| |\tilde{x}_3| - \eta\tilde{s}^2 - 2\tilde{y}\mathcal{G} \\ &= \tilde{\mathbf{x}}^T [(A - LC)^T P + P(A - LC)] \tilde{\mathbf{x}} + \frac{1}{r} \theta_{max} \rho_2 (2\tilde{x}_1 \tilde{x}_2 + 2\tilde{x}_1 \tilde{x}_3) + 2\rho_3 \theta_{max} |\tilde{y}| |\hat{x}_1| |\tilde{x}_3| - \eta\tilde{s}^2 - 2\tilde{y}\mathcal{G} \\ &\leq \tilde{\mathbf{x}}^T [(A - LC)^T P + P(A - LC)] \tilde{\mathbf{x}} + \rho_4 \left(\tilde{x}_1^2 + \frac{1}{2}\tilde{x}_2^2 + \frac{1}{2}\tilde{x}_3^2 \right) - \frac{\rho_4}{2} \left(|\tilde{x}_3| - \frac{\rho_3 r}{\rho_2} |\tilde{y}\hat{x}_1| \right)^2 \\ &\quad + \frac{\rho_4}{2} \tilde{x}_3^2 + \frac{\rho_4}{2} \left(\frac{\rho_3 r}{\rho_2} \right)^2 \hat{x}_1^2 \tilde{y}^2 - \eta\tilde{s}^2 - 2\tilde{y}\mathcal{G} \\ &\leq \tilde{\mathbf{x}}^T [(A - LC)^T P + P(A - LC) + \rho_4 I] \tilde{\mathbf{x}} - \eta\tilde{s}^2 - \frac{\rho_4}{2} \left(|\tilde{x}_3| - \frac{\rho_3 r}{\rho_2} |\tilde{y}\hat{x}_1| \right)^2 + \\ &\quad \tilde{y} \left[\frac{\rho_4}{2} \left(\frac{\rho_3 r}{\rho_2} \right)^2 \hat{x}_1^2 \tilde{y} - 2\mathcal{G} \right] \\ &\leq -\rho_1^2 \|\tilde{\mathbf{x}}\|^2 - \eta\tilde{s}^2 - \frac{\rho_4}{2} \left(|\tilde{x}_3| - \frac{\rho_3 r}{\rho_2} |\tilde{y}\hat{x}_1| \right)^2 + \tilde{y} \left[\frac{\rho_4}{2} \left(\frac{\rho_3 r}{\rho_2} \right)^2 \hat{x}_1^2 \tilde{y} - 2\mathcal{G} \right]. \end{aligned}$$

If we choose $\mathcal{G}$ such that

$$\mathcal{G} = \frac{\rho_4}{4} \left(\frac{\rho_3 r}{\rho_2} \right)^2 \hat{x}_1^2 \tilde{y} \quad (167)$$

then

$$\dot{V} \leq -\rho_1^2 \|\tilde{\mathbf{x}}\|^2 - \eta \tilde{s}^2 - \frac{\rho_4}{2} \left(|\tilde{x}_3| - \frac{\rho_3 r}{\rho_2} |\tilde{y} \hat{x}_1| \right)^2 \leq 0$$

using Barbalat's Lemma, we can conclude that

$$\tilde{s} \rightarrow 0, \quad \tilde{\mathbf{x}} \rightarrow 0, \quad \text{as } t \rightarrow \infty.$$

Thus, by definition of $\tilde{s}$ and λ we have

$$\hat{\lambda} \rightarrow \hat{\lambda}_{max} \quad \text{as } t \rightarrow \infty$$

■

Remark 1 The adaptive nonlinear observer structure presented in this paper is similar to the scheme presented in [36]. [36] presented results that require an additional Lipschitz assumption on the function $\psi(x)$, and condition (161) has been replaced by $B_1^T P C^\perp = 0$ where $C^\perp$ is the projection on to $\text{null}(C)$.

Remark 2 The tuning function $\mathcal{G}$ given by (167) is a linear function of $\tilde{y}$ and appears both in the observer and the control input. Compared with the tuning function in [35], Eq. (167) does not require switching in the control input, and therefore produces a smoother control.

Remark 3 Assumptions (i) through (iv) must be satisfied in order for the system dynamics described by Eq. (155) and, thus for the theorem to hold:

(i) Regarding assumption (i), we can calculate the observability matrix

$$\mathcal{O} = \begin{bmatrix} 0 & \frac{1}{r} & -\frac{1}{r} \\ \frac{g-\alpha}{r} & 0 & \frac{a}{r} \\ \frac{\alpha a}{r} & -\frac{g\sigma_v a}{r} & \frac{(g-\alpha)[\alpha(\sigma_2+\sigma_1)^2+\sigma_0]}{r} \end{bmatrix}$$

where $a = (\sigma_2 + \sigma_1)(\alpha - g)$, thus $\text{rank}(\mathcal{O}) = 3$, and (A, C) is an observable pair. Hence, assumption (i) always holds;

(ii) To see that assumption (ii) is always satisfied, we have

$$0 \leq f(x_3) = \frac{x_3}{h(x_3)} \leq \frac{x_3}{\mu_c} \leq \frac{\lambda_{max} v_{max}}{\mu_c} = \rho_2$$

and

$$|f'(x_3)| \leq \frac{1}{\mu_c} \left\{ 1 + \left(\frac{\mu_s}{\mu_c} - 1 \right) \left[1 + \frac{1}{2} \left(\frac{v_{max}}{v_s} \right)^{1/2} \right] \right\} = \rho_3$$

(iii) As for assumption (iv), we have to pick an observer gain L and a positive symmetric matrix P such that following optimization problem is feasible

$$\begin{cases} \max & \rho_1 \\ \text{s.t.} & (A - LC)^T P + P(A - LC) + \rho_1^2 I + \rho_4 I < 0 \\ & P B_1 = C^T, \quad P = P^T > 0 \text{ and } \rho_1 > 0 \end{cases}$$

This can be calculated by linear matrix inequality (LMI) algorithms, such as those presented in [37].

7.3 Simulation Results

In the following simulation example we use the parameters from the LeSabre cars used by the California PATH program. These parameters are: $M = 1701.0 \text{ Kg}$, $Ca = 0.3693 \text{ N} \cdot \text{s}^2/\text{m}^2$, $J = 2.603 \text{ Kg} \cdot \text{m}^2$, $R = 0.323 \text{ m}$. We also take the road LuGre friction parameter in Eq.(150) parameter to be $\theta = 1$ and the braking gain $K_b = 0.9$. The nominal values of the parameters in the dynamic LuGre friction model are the same as those in [40].

We simulate an emergency braking maneuver with a vehicle initial velocity of $v = 30 \text{ m/s}$ and the designed observed-based controller. The initial condition for observer dynamics is $\hat{\mathbf{x}}(0) = [0 \ 29.5 \ 0]^T$ and the true state is $\mathbf{x}(0) = [0 \ 30 \ 0.5]^T$, namely we use the measurement $R\omega (=29.5 \text{ m/s})$ as the initial condition for $\hat{v}$. Fig. 19 shows the time responses of the real state vector $\mathbf{x}$ and estimated state vector $\hat{\mathbf{x}}$, while Fig. 20(c) shows the time responses of the estimated friction parameter $\hat{\theta}$. Fig. 20(a) shows the time responses of the controlled pressure P while Fig. 20(b) shows the controlled sliding surface $\tilde{s}$. Fig. 20(d) illustrates the difference, $\tilde{y}$, of measurement y and output of observer $\hat{y}$. From these figures we can see that the estimated state $\hat{\mathbf{z}}$ and parameter $\hat{\theta}$ converge to their respective true values quickly, and that the controlled input (pressure P) remains within its feasible domain, enabling the vehicle to come to a quick halt (decelerating at around 10 m/s^2). This example verifies the results of the previous section. However, the simulation results also reveal that the estimated states $\hat{v}$ and $\hat{v}_r$ do not converge to their true states during the braking process, even though the vehicle achieved its maximum estimated deceleration level, which is based on estimated states, as shown by Fig. 20(b). We will discuss this result in more detail in the following section.

7.4 Discussions

In the previous section we presented a simulation example in which the vehicle achieves its maximum estimated deceleration level in spite of the slow convergence rate of the control systems's estimated linear velocity $\hat{v}$ and relative velocity $\hat{v}_r$. In this section we produce an approximate analytical explanations of this observation.

From the state error dynamics (162) we find

$$\dot{\tilde{y}} = -\frac{1}{r}[l_2 - l_3 + \sigma_1(g - \alpha)\mathbf{g}]\tilde{y} + f_1(\tilde{\mathbf{x}}) \quad (168)$$

where $f_1(\tilde{\mathbf{x}}) = (g - \alpha)[(1 - \sigma_1\theta f(x_3))\tilde{x}_1 - \sigma_v\tilde{x}_2 - (\sigma_1 + \sigma_2)\tilde{x}_3 - \sigma_1[\theta f(x_3) - \hat{\theta}f(\hat{x}_3)]\hat{x}_1]$ and $\mathbf{g} = \frac{\rho_4}{4} \left(\frac{\rho_3 r}{\rho_2} \right)^2 \hat{x}_1^2$.

In our example, we chose a relative large value for the gain L with $l_2 > l_3$ ($L = [-400 \ -60 \ -500]^T$). As a consequence, $\tilde{y} \rightarrow 0$ quickly. Similarly, we can also assume that $\tilde{\theta} \rightarrow 0$ quickly due to our choice of a high adaptation gain $\gamma (=200)$ and the presence of persistence of excitation, which we observed in the numerical example. The analysis for absence of persistence of excitation is in progress.

Using the approximation $\tilde{y} \approx 0$ and $\tilde{\theta} \approx 0$, we now analyze the dynamics of the state errors (162) and obtain

$$\dot{\tilde{\mathbf{x}}} = \bar{A}(x_3)\tilde{\mathbf{x}} \quad (169)$$

where

$$\bar{A}(x_3) = \begin{bmatrix} -\sigma_0\theta f(x_3) & 0 & 0 \\ g[1 - \sigma_1\theta f(x_3)] & -g\sigma_v & -g\sigma_2 \\ \alpha[1 - \sigma_1\theta f(x_3)] & -g\sigma_v & -\alpha\sigma_2 \end{bmatrix}$$

Notice that $\sigma_0 = 280$, $\hat{\theta} \approx 1$ and $\frac{v}{\mu_c} \geq f(x_3) > \frac{v_r}{\mu_s}$. We can therefore conclude that $\tilde{x}_1 \rightarrow 0$ quickly with a decaying rate of around $\sigma_0\theta f(x_3)$ during the beginning of braking process, due to the fact that v_r is large. This explains why the estimated state $\hat{x}_1$ converges quickly to the real state x_1 . In the case of the state estimates $\hat{x}_2$ and $\hat{x}_3$, from Eq.(169) we find that the eigenvalues of matrix $\bar{A}(x_3)$ associated with these two states are

$$s_{2,3} = \frac{-(g\sigma_v + \alpha\sigma_2) \pm \sqrt{(g\sigma_v)^2 + (\alpha\sigma_2)^2 + 4g^2\sigma_v\sigma_2}}{2}$$

Since σ_v and σ_2 are very small,

$$-1 \ll s_{2,3} < 0, \quad \forall t \geq 0$$

The rate of decay for $\tilde{x}_2$ and $\tilde{x}_3$ is small and the eigenvector associated with s_2 is around $w_2 \approx [0 \ 1 \ 1]^T$. Moreover, the eigenvector w_2 is independent of state variables, as shown in Fig. 21. In section 7.2 we proved that $\tilde{\mathbf{x}} \rightarrow \mathbf{0}$ and, from the analysis in this section, we found that the convergence of states $\hat{x}_2$ and $\hat{x}_3$ is very slow, which is illustrated by the numerical example of the previous section.

Fig. 21 shows a sketch of the trajectory portrait of the approximate nonlinear system (169). For any initial condition $P_0 = (\tilde{x}_1(0), \tilde{x}_2(0), \tilde{x}_3(0)) \in \mathcal{R}^3$, the flow trajectory will quickly approach to the $\tilde{x}_2 \times \tilde{x}_3$ plane because of the rapid convergence of $\tilde{x}_1$ (s_1 is large). Moreover, the trajectory will converge to w_2 on the $\tilde{x}_2 \times \tilde{x}_3$ plane if $\tilde{x}_2(0) > 0$ and $\tilde{x}_3(0) > 0$, as shown in Fig. 21. Thus, if we pick

$$\tilde{x}_2(0) \geq 0, \quad \tilde{x}_3(0) \geq 0, \quad (170)$$

then

$$\max\{\tilde{x}_2(0), \tilde{x}_3(0)\} \geq \tilde{x}_2(t) \approx \tilde{x}_3(t) \geq 0, \quad \forall t \geq t_0$$

where t_0 is fairly small and depends on the convergence rate and initial conditions of $\tilde{x}_1(t)$.

Remark 4 Note that by Theorem 1 we obtained $\hat{\lambda} \rightarrow \hat{\lambda}_{max}$ and $\lambda \rightarrow \lambda_{max}$ due to the fact $\tilde{\mathbf{x}} \rightarrow \mathbf{x}$. However, since the states $\hat{x}_3$ and $\hat{x}_2$ converge slowly, there will be some error between λ and $\hat{\lambda}$. This error can be estimated as follows.

$$\begin{aligned} \lambda(t) - \hat{\lambda}(t) &= \frac{x_3}{x_2} - \frac{\hat{x}_3}{\hat{x}_2} = \frac{x_2\tilde{x}_3 - x_3\tilde{x}_2}{x_2\hat{x}_2} \\ &\approx \frac{(x_2 - x_3)\tilde{x}_2}{x_2\hat{x}_2} = \left(1 - \frac{x_3}{x_2}\right) \frac{\tilde{x}_2}{\hat{x}_2} \\ &\leq (1 - \lambda(t))\lambda(0), \quad \forall t \geq t_0 \end{aligned}$$

Note that in general $\lambda(0) \leq 3\%$ during normal driving conditions before braking. As a consequence, the slip estimate error is small. Similarly, it can be shown that $\lambda_{max} - \hat{\lambda}_{max}$ will also be small after the state $\tilde{x}_1 \rightarrow 0$. Therefore, the proposed control system will achieve a near maximum deceleration level, in spite of the fact that state estimation errors $\tilde{x}_2$ and $\tilde{x}_3$ converge slowly.

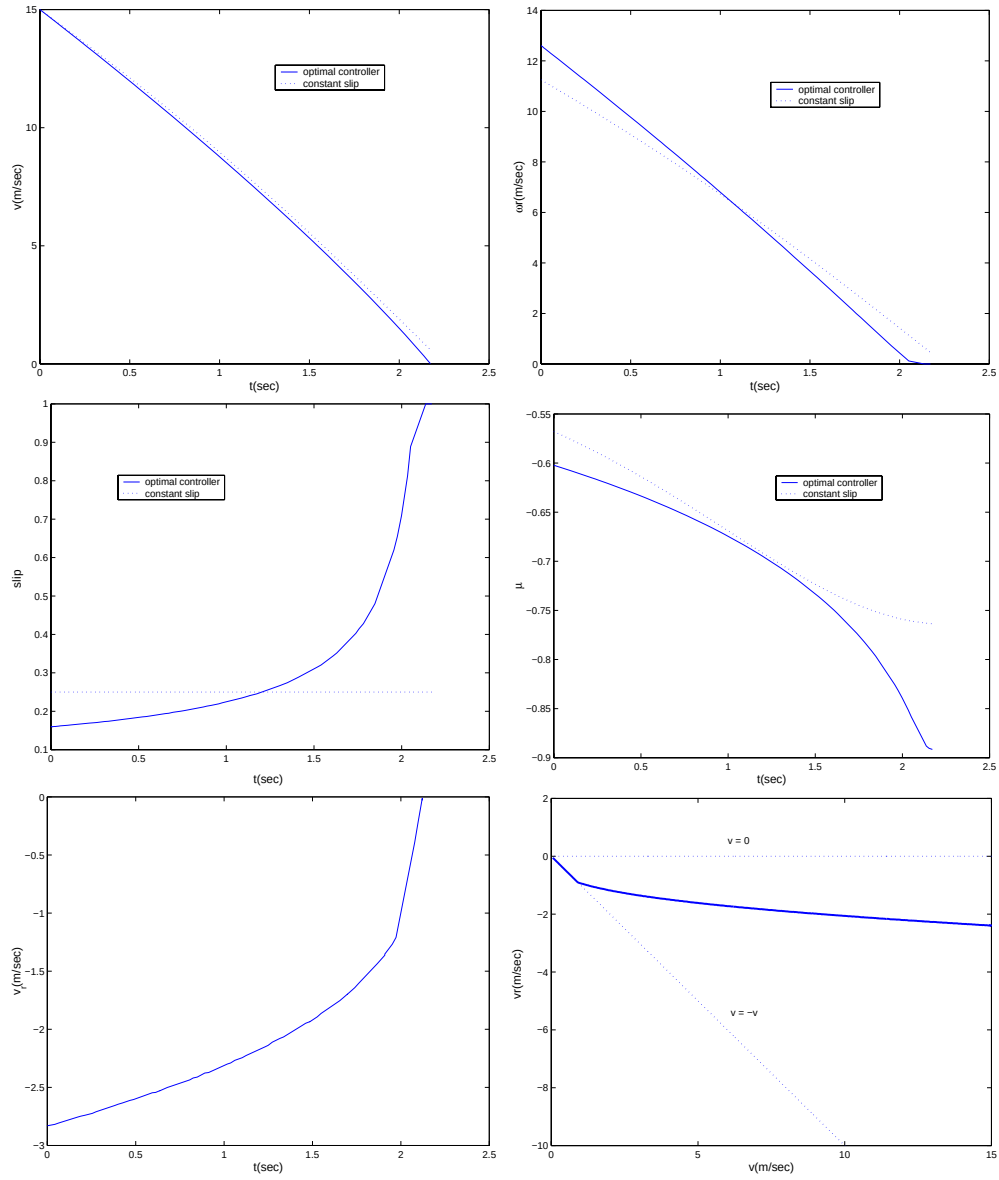


Figure 18: Braking simulation using the LuGre friction model: Optimal and constant slip controllers

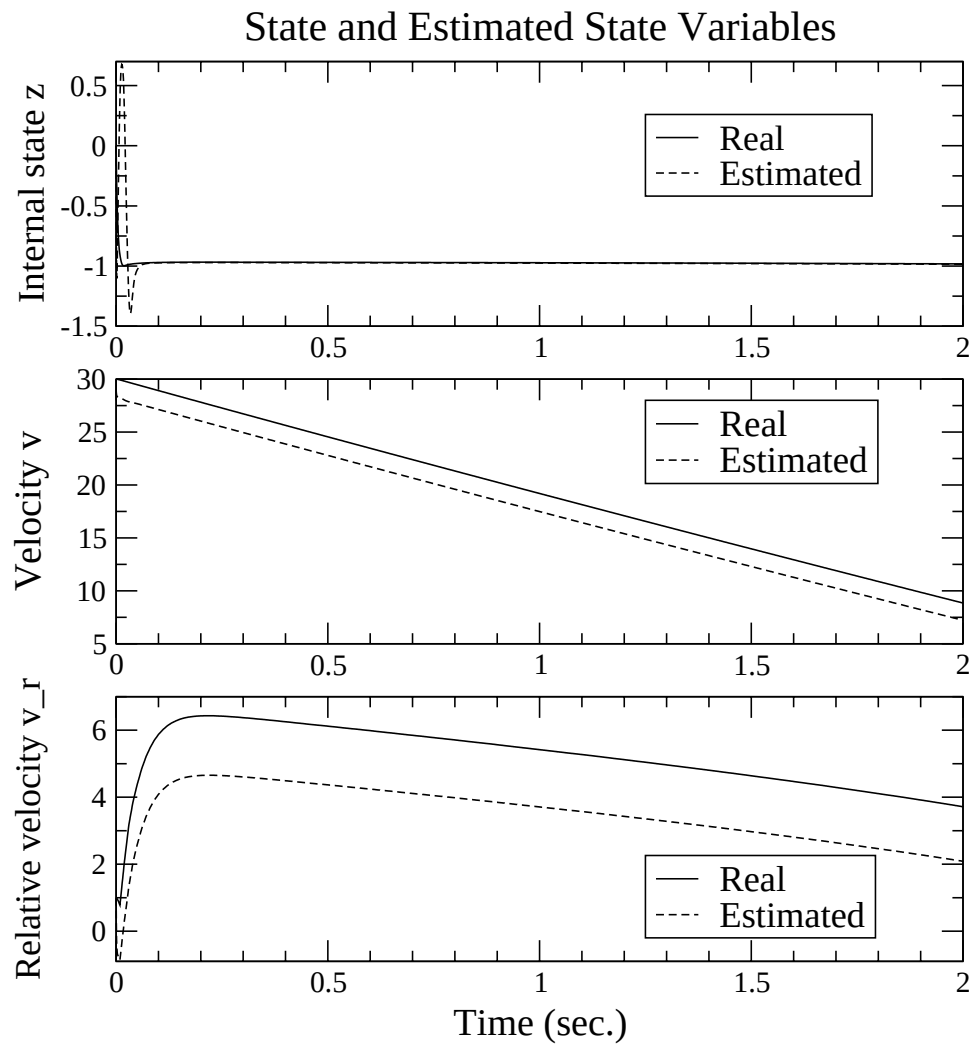


Figure 19: Estimated and real state variables.

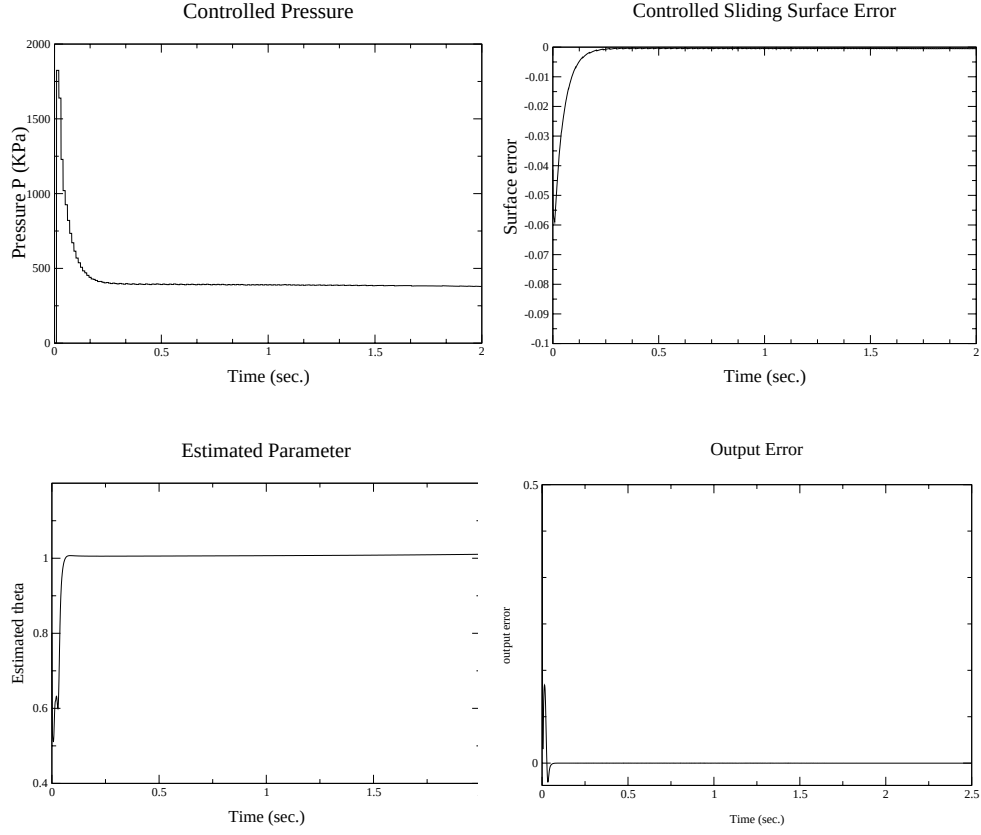


Figure 20: (a) Controlled braking pressure P and (b) sliding surface $\tilde{s}$ (c) Estimated friction parameter $\hat{\theta}$ and (d) measurement error $\tilde{y}$ (rad/s).

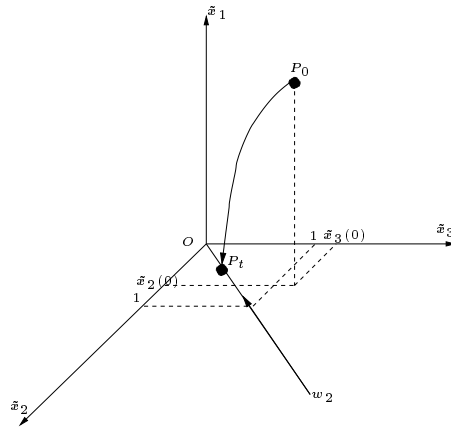


Figure 21: A schematic trajectory plot for nonlinear system $\dot{\tilde{\mathbf{x}}} = \bar{A}(\tilde{x}_3)\tilde{\mathbf{x}}$.

8 CONCLUSIONS

In this paper we have revisited the problem of characterizing the friction at the tire/surface interface for wheeled vehicles. We have reviewed the major models used in the literature, namely, static, dynamic, lumped and distributed models. We have shown that static friction models are inadequate for describing the transient nature of friction build-up. Dynamic friction models are necessary to capture such transients during abrupt braking and acceleration phases. We propose a new dynamic friction model that accurately captures friction transients, as well as velocity-dependent characteristics and tire/road properties. The model is developed by extending the well-known LuGre point friction model to the case of a contact patch at the tire/surface interface. We have also show how the friction force for lateral/cornering or combined longitudinal/lateral motion can also be modeled.

We have presented a method to estimate on-line the changes in road condition using only wheel rotational velocity information. To achieve this goal we have used a lumped friction model with internal dynamics that, one hand provide a more accurate description of the contact friction, and on the other hand, allow us to characterize road condition variations via a single parameter. We have presented a model-base observer that ensure asymptotic tracking of road condition, under mild conditions implying a non-vanishing evolution of the slip rate. This condition are quite natural in this context (it imply that the vehicle should operate away to the ideal pure rolling condition). The observer presented here has been derived in a general framework allowing to also study to the case where the vehicle velocity is not measurable. In particular, assumption A2 – (ii) allows for this possibility.

We have shown that the optimal braking strategy for a wheeled vehicle is to operate at the maximum point of the friction/slip curve. This strategy has been used extensively in the past in the design of ABS systems. One of the interesting results of our analysis is that the optimal “max-friction” control is singular. Maximum torque control is used to match the boundary conditions. The total optimal trajectory is therefore composed of a sequence of bang-bang and singular arcs. For typical automotive applications, where the applied torque is high, the bang-bang subarcs can be safely approximated by impulses to achieve the specified boundary conditions of the slip.

We have discussed emergency braking control under unknown tire/road conditions and system states, based on a dynamical friction model and the assumption that the only available measurable signal to the controller is the wheel angular velocity. The braking pressure controller is determined based on the estimation of system state variables and the unknown friction parameter. The simulation results show that the vehicle can be stopped quickly with near maximum deceleration by applying this controller. The asymptotic convergence of the estimated states and parameters estimates has been proven. Moreover, it was also shown that the friction properties can be estimated and near maximum deceleration achieved, in spite of the slow convergence rate of the vehicle velocity and wheel relative velocity error estimates.

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REFERENCES

1. Barahanov, N., and Ortega, R., “Necessary and Sufficient Conditions for Passivity of the LuGre Friction Model,” *IEEE Transactions on Automatic Control*, Vol. 45, No. 4, pp. 830–832, 2000.
2. Bernard, J., and Clover, C. L., “Tire Modeling for Low-Speed and High-Speed Calculations,” Society of Automotive Engineers, Paper 950311, 1995.

3. Bakker, E., Nyborg, L. and H. Pacejka, H., "Tyre Modelling for Use in Vehicle Dynamic Studies," Society of Automotive Engineers, Paper 870421, 1987.
4. Bliman, P.A., Bonaldi, T. and Sorine, M., "Hysteresis Operators and Tire Friction Models: Application to Vehicle Dynamic Simulations," *Proc. of ICIAM'95*, Hamburg, Germany, 3-7 July, 1995.
5. Burckhardt, M., "ABS und ASR, Sicherheitsrelevantes, Radschlupf-Regel System," *Lecture Scriptum*, University of Braunschweig, Germany, 1987.
6. Burckhardt, M., *Fahrwerktechnik: Radschlupfregelsysteme*, Vogel-Verlag, Germany, 1993.
7. Canudas de Wit, C., Olsson, H., Åström, K.J., and Lischinsky, P., "A New Model for Control of Systems with Friction," *IEEE Transactions on Automatic Control*, Vol. 40, No. 3, pp. 419-425, 1995.
8. Canudas de Wit, C., Horowitz, R. and Tsotras, P., "Model-Based Observers for Tire/Road Contact Friction Prediction," In *New Directions in Nonlinear Observer Design*, Nijmeijer, H. and T.I Fossen (Eds), Springer Verlag, Lectures Notes in Control and Information Science, May 1999.
9. Canudas de Wit, C. and Tsotras, P., "Dynamic Tire Friction Models for Vehicle Traction Control," In *Proceedings of the IEEE Conference on Decision and Control*, Phoenix, AZ, pp. 3746-3751, 1999.
10. Canudas de Wit, C., Tsotras, P., E. Velenis, M. Basset, and G. Gissinger, "Dynamic Friction Models for Road/Tire Longitudinal Interaction," *Submitted to Vehicle System Dynamics*, May 2001.
11. Clover, C.L., and Bernard, J.E., "Longitudinal Tire Dynamics," *Vehicle System Dynamics*, Vol. 29, pp. 231-259, 1998.
12. Dahl, P.R., "Solid Friction Damping of Mechanical Vibrations," *AIAA Journal*, Vol. 14, No. 12, pp. 1675-1682, 1976.
13. Deur, J., "Modeling and Analysis of Longitudinal Tire Dynamics Based on the LuGre Friction Model," In *Proceedings of the IFAC Conference on Advances in Automotive Control*, Kalsruhe, Germany, pp. 101-106, 2001.
14. Faria, L.O., Oden, J.T., Yavari, B.T., Tworzydło, W.W., Bass, J.M., and Becker, E.B., "Tire Modeling by Finite Elements," *Tire Science and Technology*, Vol. 20, No. 1, pp. 33-56, 1992.
15. Gim, G. and Nikravesh, P.E., "A Unified Semi-Empirical Tire Model with Higher Accuracy and Less Parameters," *SAE International Congress and Exposition*, Detroit, MI, 1999.
16. Harned, J., Johnston, L. and Scharpf, G., "Measurement of Tire Brake Force Characteristics as Related to Wheel Slip (Antilock) Control System Design," *SAE Transactions*, Vol. 78, Paper 690214, pp. 909-925, 1969.
17. Kiencke, U. and Daiss, A., "Estimation of Tyre Friction for Enhanced ABS-Systems," In *Proceedings of the AVEG'94*, 1994.
18. Liu, Y. and Sun, J., "Target Slip Tracking Using Gain-Scheduling for Antilock Braking Systems," In *Proceedings of the American Control Conference*, pp. 1178-1182, Seattle, WA, 1995.
19. Maurice, J.P., Berzeri, M., and Pacejka, H.B., "Pragmatic Tyre Model for Short Wavelength Side Slip Variations," In *Vehicle System Dynamics*, Vol. 31, pp. 65-94, 1999.
20. Moore, D.F., *The Friction of Pneumatic Tyres*, Elsevier Scientific Publishing Co., New York, 1975.
21. Pasterkamp, W.R. and Pacejka, H.B., "The Tire as a Sensor to Estimate Friction," *Vehicle Systems Dynamics*, Vol. 29, pp. 409-422, 1997.
22. Pacejka, H.B. and Sharp, R.S., "Shear Force Developments by Pneumatic Tires in Steady-State Conditions: A Review of Modeling Aspects," *Vehicle Systems Dynamics*, Vol. 20, pp. 121-176, 1991.
23. Ramberg W., and Osgood W.R., "Description of Stress-Strain Curves by Three Parameters," *Technical Note 902*, National Advisory Committee for Aeronautics, Washington, DC, 1943.
24. Sargin, M., "Stress-Strain Relationship for Concrete and the Analysis of Structural Concrete Sections," *SM Study 4*, Solid Mechanics Division, University of Waterloo, Canada, 1971.
25. Wong J.Y., *Theory of Ground Vehicles*, John Wiley & Sons, Inc., New York, 1993.
26. van Zanten, A., Ruf, W.D., and Lutz, A., "Measurement and Simulation of Transient Tire Forces," In *International Congress and Exposition*, Detroit, MI, SAE Technical Paper Series, Paper 890640, 1989.
27. F. Altpeter, F. Ghorbel, and R. Longchamp, "Relationship between two friction models: A singular perturbation approach," in *Proceedings of 37th IEEE Conference of Decision and Control*, (Tampa, FL).
28. P. Kokotovic, R. O'Malley, and P. Sannuti, "Singular perturbations and order reduction in control theory - an overview," *Automatica*, vol. 12, pp. 123-132, 1976.
29. H. Chow and P. Kokotovic, "Near-optimal feedback stabilization of a class of nonlinear singularly perturbed systems," *SIAM Journal on Control and Optimization*, vol. 16, no. 5, September 1978.
30. H. Kelley, R. Kopp, and H. Moyer, "Singular extremals," in *Topics in Optimization*, New York: Academic Press, 1967.
31. J. J. Moré, B. Garbow, and K. E. Hillstom, "User Guide for MINPACK-1," Technical Report ANL-80-74, Argonne National Laboratory, Applied Mathematics Division, Argonne, IL, 1980.
32. P. Tsotras and C. Canudas de Wit, "On the Optimal Braking of Wheeled Vehicles," in *American Control Conference*, June 28-30 2000. Chicago, IL.
33. P. Tsotras and H. J. Kelley, "Drag-Law Effects in the Goddard Problem," *Automatica*, Vol. 27, pp. 481-490, 1991.
34. Alvarez, L., Yi, J., and Horowitz, R. (2000). Adaptive Emergency Brake Control in AHS with Underestimation of Friction Coefficient. In *Proceedings of American Control Conference*, pages 574-579, Chicago, IL.
35. Canudas de Wit, C. and Horowitz, R. (1999). Observers for Tire/Road Contact Friction using only wheel angular velocity information. In *Proceedings of 38th IEEE Conference of Decision and Control*, Phoenix, AZ.
36. Cho, Y. M. and Rajamani, R. (1997). A Systematic Approach to Adaptive Observer Synthesis for Nonlinear Systems. *IEEE Transactions on Automatic Control*, 42(4):534-537.

37. El Ghaoui, L., Nikoukhan, R., and Delebecque, F. (1995). LMIT00L: a Package for LMI Optimization. In *Proceedings of 34th IEEE Conference of Decision and Control*, New Orleans, LA.
38. Kiencke, U. (1993). Realtime Estimation of Adhesion Characteristic Between Tyres and Road. In *Proceedings of the IFAC World Congress*, volume 1.
39. Wong, J. Y. (1993). *Theory of Ground Vehicles*. John Wiley & Sons, New York, NY, 2nd edition.
40. Yi, J., Alvarez, L., Horowitz, R., and Canudas de Wit, C. (2000). Adaptive Emergency Braking Control in Automated Highway System Using Dynamic Tire/Road Friction Model. In *Proceedings of 39th IEEE Conference of Decision and Control*, Sydney, Australia.
41. Sorine, M. and Szymanski, J., "A new all-vehicle-speed dynamic tire model", *IFAC Symposium on Control in Transportation Systems 2000*, Braunschweig, Germany, June 2000.
42. X. Claeys and J. Yi and L. Alvarez and R. Horowitz and C. Canudas de Wit and L. Richard, "A Simple 3D Parametric Dynamic Tire/Road Friction Model for Vehicle Simulation and Control", *Submitted to the 2001 ASME-IMECE*, New-York, USA, November 2001.
43. A. Browne and D. Whicker, "Design of Tire Tread Elements for Optimum Thin Film Wet Traction", *SAE paper*, Nb. 770278, Detroit, MI, USA, 1977.
44. S. Rohde, "On the Combined Effects of Tread Element Flexibility and Pavement Microtexture on Thin Wet Traction", *SAE paper*, Nb. 770277, Detroit, MI, USA, 1977.
45. S. Rohde, "On the Effect of Pavement Microtexture on Thin Film Traction", *International Journal of Mechanical Science*, 1975, Pages 95-101, volume 18.
46. Meyer, W. and Walter, J., "Frictional Interaction of Tire and Pavement", *ASTM Publication*, ASTM Publication Code Number 04-793000-37, 1983.
47. Pottinger, M. and Yager, T., "The Tire Pavement Interface", *ASTM Publication*, ASTM Publication Code Number 04-929000-27, 1985.
48. Moore, D., "Drainage Criteria for Runway Surface Roughness", *Royal Aeronautical Society*, 1964.
49. Horne, W. and Buhlmann, F., "A Method for Rating the Skid Resistance of Micro/Macrotexture Characteristics of Wet Pavements, Frictional Interaction of Tire and Pavement", *American Society for Testing and Materials*, STP 793, pp. 191-218, 1983.
50. Moore, D., "The Friction and Lubrification of Elastomers", *Pergamon Press*, 1972.
51. X. Claeys and J. Yi and L. Alvarez and R. Horowitz and C. Canudas de Wit and L. Richard, "Tire Friction Modeling under Wet Road Conditions", *Submitted to the 2001 American Control Conference*, Washington DC, USA, June 2001.
52. Gwanghun Gim and P.E. Nikraves, "An Analytical Model of Pneumatic Tyres for Vehicle Dynamic Simulations. Part I: Pure Slips", *International Journal of Vehicle Design*, Vol. 11, pp. 589-618, 1990.