



Minimum-Fuel Closed-Loop Powered Descent Guidance with Stochastically Derived Throttle Margins

Jack Ridderhof* and Panagiotis Tsiotras[†]
Georgia Institute of Technology, Atlanta, Georgia 30332

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If during a guided powered descent maneuver the descent vehicle deviates from the planned trajectory, then the vehicle may need to adjust the commanded thrust in order to still reach the target landing site or to avoid violating mission constraints. However, often the nominal thrust command at any time along a minimum-fuel powered descent trajectory is either at the maximum or the minimum throttle, and as a result the corrective thrust command may be outside the allowable throttle range. A margin must therefore be added between the planned throttle command and the engine throttle limits, but this margin may be overly conservative to the detriment of performance. In this paper, the powered descent trajectory is modeled as a stochastic process in order to nonconservatively adjust the bounds on the feed-forward optimal thrust magnitude command to allow for sufficient feedback authority. The margin on the nominal throttle is computed as a function of the covariance of the closed-loop thrust commands so that if the nominal throttle is within the limits, plus the margin, then the closed-loop throttle is within the allowable limits with high probability. The proposed method can be solved onboard without iteration.

I. Introduction

DURING the powered descent phase of a planetary landing mission, a descent vehicle uses the thrust from the rocket engines to remove the excess kinetic energy and steer the vehicle to the desired landing site. The powered descent guidance (PDG) controller is responsible for determining and executing the sequence of thrust vectors for the vehicle to follow in order to reach the landing site, while satisfying certain constraints, such as avoiding subsurface flight or not exceeding a maximum velocity.

The majority of current PDG strategies still follow the powered descent architecture developed for the Apollo program. This PDG scheme represents trajectories as polynomials parameterized in time. The Apollo PDG algebraically solved for the polynomial coefficients in order to meet prescribed boundary conditions [1]. The thrust commands were then set to track the acceleration profile obtained by twice differentiating the polynomial representing the descent path. The Apollo polynomial guidance was later adapted for the Mars Science Laboratory PDG [2–4]. Although this polynomial guidance is computationally efficient, it is not fuel-optimal [5]; that is, the guidance solution requires more fuel than necessary in order to complete the maneuver.

Minimum-fuel PDG, on the other hand, has, until recently, been primarily the focus of academic research [6,7]. In the special case when the descent trajectory is strictly vertical, the minimum-fuel control sequence is such that the vehicle first descends in free-fall, and then, at a particular time that can be computed algebraically, the maximum throttle is applied until touchdown [8]. Hence, the optimal control strategy has a bang-bang structure. For the general case of a three-dimensional descent trajectory, however, there is no known algebraic representation of the minimum-fuel solution, although properties of optimal solutions can be derived from the minimum principle. Specifically, the minimum-fuel optimal thrust profile is nonsingular with a max-min-max structure [9].

Fortunately, due to recent theoretical developments and technological advances in onboard computational capabilities, it is now possible to compute minimum-fuel PDG trajectories onboard as the solution to a numerical optimization problem [10]. One such method is due to the theory of lossless convexification, which allows for the minimum-fuel PDG to be posed as a convex optimization problem. As a result, the globally optimal powered descent trajectory can be computed by convex optimization with guaranteed convergence [11,12]. The G-FOLD algorithm, which is based on the theory of lossless convexification, has been demonstrated in test flights [13]. Furthermore, when the descent trajectory is solved as a convex optimization problem, it is possible to enforce convex path constraints, such as minimum glide slope or maximum off-vertical thrust direction [11]. Recent works have included more complex state-triggered and line-of-sight constraints [14,15]. Iterative methods extending lossless convexification theory have been applied to handle nonlinear drag forces [16] and vehicle attitude dynamics [17]. The convergence guarantees stemming from the problem convexity make these previous methods ideally suited for onboard closed-loop trajectory generation, although to-date the approach has only been used to autonomously generate a reference trajectory onboard, which is then tracked using a linear controller [13].

On a different research direction, an indirect method has been developed to solve onboard for minimum-fuel descent trajectories, which includes optimization over the time-of-flight and the maneuver start time [18]. This indirect method determines the optimal thrust profile from the solution to a multivariate root-finding problem, which can be computed quickly onboard. This approach thus enables an implicit form of closed-loop control wherein the optimal thrust profile is continuously resolved during the descent maneuver. On the other hand, this indirect method does not have a guarantee of convergence and does not include constraints on the glide slope or thrust-pointing angle. See also Refs. [19–23].

Both the lossless convexification method as well as the indirect method can enable closed-loop control during powered descent by either continuously resolving for a nominal trajectory or by using closed-loop tracking; however, neither approach accounts for stochasticity. As has been shown in [24,25], accounting for stochasticity yields better performance than simply applying deterministic closed-loop guidance. This initially may seem surprising, but it is easy to see why this is the case. While closing the loop, a purely deterministic framework can be overly optimistic about the future of disturbances (i.e., there will be none), and hence it will be more aggressive in its effort to suppress the current disturbances. A stochastic treatment, on the other hand, pays equal attention to both the current and future disturbances when closing the loop. For this reason, in this paper we follow a stochastic approach with the primary objective to fill this gap

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*Graduate Student, School of Aerospace Engineering.

[†]David and Andrew Lewis Chair and Professor, School of Aerospace Engineering, Institute for Robotics and Intelligent Machines.

between deterministic open-loop (or even closed-loop) guidance and stochastic feedback control. In other words, by introducing stochasticity we are able to obtain less conservative solutions that save fuel.

If the fuel-optimal PDG trajectory is given before the mission starts or even if it solved once before the initiation of the powered descent maneuver, then the computed open-loop descent trajectory must be tracked using feedback control in order to reject external disturbances and to minimize the effect of unmodeled dynamics [13]. The engine throttle along a fuel-optimal powered descent trajectory, however, will always be either at the maximum or at the minimum allowable value [9]. On the other hand, and because the magnitude of the closed-loop thrust commands is not known when designing the open-loop trajectory, conservative margins must be included on the open-loop throttle commands. Fuel-optimality must then be traded against the control authority allocated for closed-loop thrust corrections by setting margins on the open-loop throttle. The open-loop PDG solution therefore implicitly depends on the intensity of the external disturbances. Indeed, even if closed-loop control is achieved by continuously resolving for the optimal, open-loop trajectory, the robustness of the trajectory to disturbances can be improved by including control margins [26,27].

In this paper, we address this interdependency between open-loop minimum-fuel PDG and closed-loop tracking control by studying the powered descent trajectory as a stochastic process. The descent vehicle states, and by extension the closed-loop thrust commands, are then described by random vectors with known distributions. In this setting, the nominal (i.e., open-loop) thrust steers the mean of the state distribution while the feedback gain steers the dispersion (i.e., the covariance) of the state distribution [28–30]. The margins imposed on the nominal throttle commands thus have a natural interpretation: If the nominal control satisfies the throttle margin, then the closed-loop control satisfies the maximum and minimum throttle constraints with high probability. This definition of the control margin can still be applied for the case that the open-loop trajectory is resolved during the maneuver, in the sense that the control margin provides a probabilistic guarantee on resolvability.

Furthermore, the closed-loop thrust corrections are provided as the minimum-variance control solution such that the covariance of the descent vehicle state at the end of the maneuver is no greater than a given maximum value [29,31]. This formulation has several implications for minimizing the fuel-cost of the nominal trajectory. First, the variance of the feedback control influences the throttle margin, since if the uncertainty of the future closed-loop thrust increases, then a larger margin must be imposed on the nominal thrust. This, in turn, increases the fuel cost. Second, there is a tradeoff between the closed-loop control variance and the final state variance. For a given intensity of external disturbances, the uncertainty in the vehicle states at the end of the maneuver can be decreased by taking more aggressive corrective control actions. In the standard linear quadratic (LQ) optimal control formulation, this tradeoff is achieved by manually tuning the weight matrices for the final state error and for the running control cost [7]. For the present problem, in contrast, we apply a recent result from constrained stochastic optimal control in order to solve for the minimum-variance control while meeting a final state covariance constraint [31–33], which removes the need to hand-tune weight matrices and is thus better suited for onboard implementation.

The main contributions of this paper are summarized below. First, we extend the aforementioned existing theory on covariance control [31–33] to allow for nonconstant, bounded mass values, as is the case for the vehicle mass after thrusting. This result is then applied to solve for the feedback gains such that the covariance of the closed-loop trajectory at a final time is constrained by a user-defined upper bound. Second, we provide a method for determining the throttle margins as a function of the closed-loop control covariance that can be computed onboard with convergence guarantees. The throttle margins are then provided as the input to a deterministic minimum-fuel PDG problem; this approach contrasts the classical method of closed-loop guidance for which the feedback gains are computed after the determination of the open-loop trajectory [7]. Third, we provide a link between minimum-variance feedback control and minimum-fuel powered descent trajectories by showing that decreasing the variance of future

feedback controls allows for smaller throttle margins, which in turn decreases the fuel cost.

The paper is organized as follows. In Sec. II, minimum-fuel PDG is introduced as a stochastic optimal control problem. In Sec. III.A, we separate the system dynamics into a deterministic mean part and a stochastic deviation from the mean part. The feedback control is derived in Sec. III.B, the throttle margin is constructed in Sec. III.C, the mean control is posed as a deterministic PDG problem in Sec. III.D, and an algorithm for the proposed closed-loop PDG is presented in Sec. III.E. The organization of Sec. III is shown graphically in Fig. 1. Finally, we present a numerical example for a powered descent maneuver at Mars in Sec. IV and we compare the results with a traditional approach that does not account for the closed-loop control variance when determining the throttle margins.

II. Problem Formulation

Consider a spacecraft in powered descent modeled as a point-mass with mass m and with position and velocity $x = (r, v) \in \mathbb{R}^3 \times \mathbb{R}^3$ in a surface-fixed inertial frame. At the initial time $t = 0$, the mass $m(0) = m_0 > 0$ is assumed to be fixed and known. The initial position and velocity $x(0) = x_0 = (r_0, v_0)$, which we refer to as the state, is assumed to be a random vector distributed as

$$x_0 \sim \mathcal{N}(\bar{x}_0, P_{x_0}) \quad (1)$$

where the positive semidefinite matrix P_{x_0} and the vector $\bar{x}_0$ are both fixed and known. Random force disturbances acting on the vehicle are modeled as a three-dimensional Brownian motion (w_t) scaled by a positive constant γ . The spacecraft motion is thus described by the stochastic differential equation (SDE)

$$d \begin{bmatrix} r \\ v \\ m \end{bmatrix} = \begin{bmatrix} v \\ u/m + g \\ -\alpha \|u\| \end{bmatrix} dt + \begin{bmatrix} 0 \\ \gamma/m \\ 0 \end{bmatrix} dw_t \quad (2)$$

where $g \in \mathbb{R}^3$ is the gravitational acceleration, α is a positive mass-flow parameter, and $u \in \mathbb{R}^3$ is the control thrust given at each time t by the affine feedback law

$$u(t) = \bar{u}(t) + K(t)\tilde{x}(t) \quad (3)$$

where $\tilde{x} = x - \mathbb{E}(x)$ is the state deviation from the mean, $\bar{u} = \mathbb{E}(u)$ is the mean control, and K is a feedback gain matrix to be determined. We assume that the mean control is sufficiently larger than the feedback control so that we may approximate

$$\mathbb{E}(\|u\|) \approx \|\bar{u}\| \quad (4)$$

From this assumption, it follows that the mass is an approximate deterministic function of the mean control, given as the solution to the ordinary differential equation (ODE)

$$\dot{m} = -\alpha \|\bar{u}\| \quad (5)$$

with the initial value $m(0) = m_0$. After a still to-be-determined time-of-flight $t_f > 0$, we require that the mean and the covariance of the spacecraft state satisfy the terminal constraints

$$\mathbb{E}(x_{t_f}) = \bar{x}_f \quad (6a)$$

$$\text{Cov}(x_{t_f}) \leq P_{x_f} \quad (6b)$$

where the final state mean $\bar{x}_f$ is a given vector and the maximum final state covariance P_{x_f} is a given positive-definite matrix.[‡] The spacecraft's engines have a minimum and maximum throttle that cannot be

[‡]For any two matrices A and B , the notation $A \leq B$ means that the matrix $B - A$ is positive semidefinite.

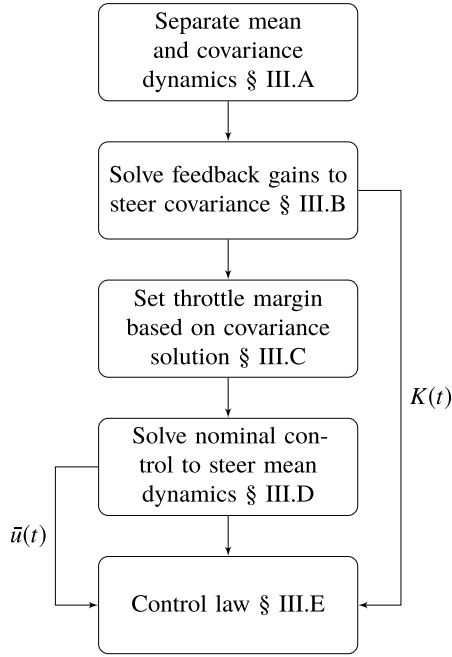


Fig. 1 Organization of solution procedure.

exceeded, and therefore the control is constrained to lie inside the set

$$\Omega = \{u: \rho_1 \leq \|u\| \leq \rho_2\} \quad (7)$$

where $\rho_2 > \rho_1 > 0$ are the maximum and minimum throttle limits. Since the state is initially Gaussian distributed, there will be a nonzero probability that the throttle constraint is violated when the feedback gain K is nonzero. We therefore require that the control remains within the set Ω with high probability. During flight, the thrust commands are saturated in the event that the throttle constraint is violated. That is, we require that

$$\mathbb{P}(u(t) \in \Omega) \geq 1 - \beta \quad \text{for all } t \in [0, t_f] \quad (8)$$

where $1 \geq \beta > 0$ is a prescribed constant specifying the maximum probability of control saturation. The parameter β allows for the system designer to specify a balance between robustness to disturbances and fuel cost: as will be shown analytically in Sec. III.D and numerically in Sec. IV, lower values of β will result in a larger throttle margin and a correspondingly higher fuel cost. The mean state $\bar{x} = \mathbb{E}(x)$ and mean control $\bar{u}$ are further constrained by

$$(\bar{x}(t), \bar{u}(t)) \in \mathcal{X} \times \mathcal{U} \quad \text{for all } t \in [0, t_f] \quad (9)$$

where $\mathcal{X}$ is the convex set restricting the glide slope angle θ to be less than the maximum glide slope angle $\theta_0 \leq \pi/2$, defined as

$$\mathcal{X} = \{(\bar{r}, \bar{v}): \theta(\bar{r}) \leq \theta_0 \leq \pi/2\} \quad (10)$$

where $\theta(\bar{r})$ is the angle between the mean position vector $\bar{r}$ and the local vertical, and where $\mathcal{U}$ is the cone of thrust vectors with off-vertical angle no more than a given maximum angle ϕ_0 (see Fig. 2), that is,

$$\mathcal{U} = \{\bar{u}: \|\bar{u}\| \cos \phi_0 + g^\top \bar{u} / \|g\| \leq 0\} \quad (11)$$

We are interested in solving the following optimal control problem to safely land the spacecraft while minimizing fuel consumption.

Problem 1: Find the time-of-flight $t_f > 0$, the mean control $\bar{u}(t)$ for $t \in [0, t_f]$, and the feedback gain $K(t)$ for $t \in [0, t_f]$ that minimize the expected fuel cost

$$J(\bar{u}, t_f) = \int_0^{t_f} \|\bar{u}(t)\| dt \quad (12)$$

subject to the dynamics (2) with the control law (3), the initial state distribution (1), the throttle constraint (8), the mean path constraints (9), and the final state distribution constraint (6).

III. Problem Analysis

A. Separation of Deterministic and Stochastic Dynamics

Since the mass is deterministic due to the assumption (4), we can take the expectation of the dynamics (2) to obtain an ODE for the mean dynamics given by

$$\begin{bmatrix} \dot{\bar{r}} \\ \dot{\bar{v}} \\ \dot{m} \end{bmatrix} = \begin{bmatrix} \bar{v} \\ \bar{u}/m + g \\ -\alpha \|\bar{u}\| \end{bmatrix} \quad (13)$$

with the initial conditions $(\bar{r}(0), \bar{v}(0)) = \bar{x}_0$ and $m(0) = m_0$. Then, by subtracting the mean dynamics (13) from the original dynamics (2), and substituting the control law (3), we obtain the SDE

$$d\tilde{x} = (A + BK/m)\tilde{x}dt + (\gamma/m)Bdw_t \quad (14)$$

for the state deviation dynamics with the initial condition $\tilde{x}(0) = x_0 - \bar{x}_0$, and where

$$A = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I \end{bmatrix} \quad (15)$$

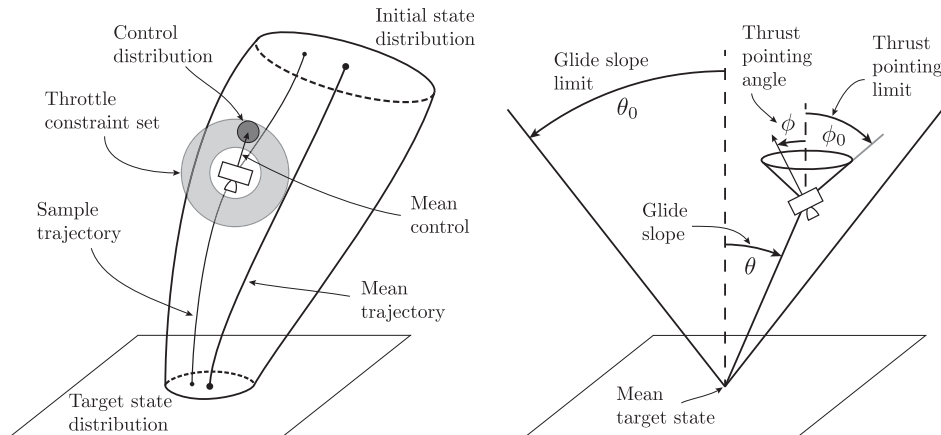


Fig. 2 Left: Schematic of powered descent with the state modeled as a random vector. Right: Definitions of glide slope and thrust pointing limits.

The mass dynamics do not depend on the deviation state $\tilde{x}$, and therefore the deviation state dynamics are linear. It follows that the state x remains normally distributed over the entire problem horizon with mean $\bar{x}$ and covariance $P_x = \mathbb{E}(\tilde{x}\tilde{x}^\top)$, which comes as the solution to the differential Riccati equation

$$\dot{P}_x = (A + BK/m)P_x + P_x(A + BK/m)^\top + (\gamma/m)^2 BB^\top \quad (16)$$

with the initial condition $P_x(0) = P_{x_0}$. Furthermore, since the feedback control $\tilde{u} = K\tilde{x}$ is also a zero-mean normal random vector with covariance $P_u = KP_xK^\top$, we have that the control is distributed as

$$u \sim \mathcal{N}(\bar{u}, KP_xK^\top) \quad (17)$$

B. Optimal Covariance Control

The covariance dynamics (16) depend on the spacecraft mass m , which is a function on the mean control $\bar{u}$. Therefore, the deviation state dynamics, and hence the state and the control covariance evolution, depend on the choice of the mean control. On the other hand, the set of allowable values for the mean control that satisfy the throttle constraint (8) depends on the control covariance, and hence, by Eq. (16), on the feedback gain K . To break this circular dependency, we will first solve for a feedback gain that will steer the covariance dynamics (16) to a final state covariance that satisfies the terminal constraint (6b) for any value of the mass that is bounded by

$$m_\ell(t) \leq m(t) \leq m_u(t) \quad \text{for all } t \in [0, t_f] \quad (18)$$

where $m_\ell(t)$ and $m_u(t)$ are given functions of time. For instance, since the throttle is bounded by ρ_1 and ρ_2 , it follows from Eq. (5) that the mass will always be bounded as in Eq. (18) with the bounds given by

$$m_\ell(t) = m_0 - \rho_2 \alpha t \quad (19)$$

$$m_u(t) = m_0 - \rho_1 \alpha t \quad (20)$$

for $t \geq 0$. We first consider the following problem to find the feedback gain K^ℓ , which steers the covariance dynamics in the case the mass m is equal to its lower bound m_ℓ over the entire problem horizon.

Problem 2: For a fixed time-of-flight $t_f > 0$ and for the mass given as $m(t) = m_\ell(t)$ for $t \in [0, t_f]$, find the feedback gain $K(t)$ for $t \in [0, t_f]$ that steers the state covariance dynamics (16) to the final state covariance $P_x(t_f) = P_{x_f}$, while minimizing the cost

$$\tilde{J}(K) = \int_0^{t_f} \text{tr}(RK P_x K^\top + Q P_x) dt \quad (21)$$

where $R = R(t) > 0$ and $Q = Q(t) \geq 0$ are given matrix-valued functions for $t \in [0, t_f]$. We denote the solution to this problem by the pair (K^ℓ, P_x^ℓ) , where $P_x^\ell = P_x^\ell(t)$ is the solution to Eq. (16) with the gain $K^\ell = K^\ell(t)$ and mass $m_\ell = m_\ell(t)$.

A solution to Problem 2 is guaranteed to exist provided that the pair (A, B) is controllable [28], which, by inspection of Eq. (15), holds for the present problem. Furthermore, the feedback gain K^ℓ that solves Problem 2 is given in terms of the solution to a backward matrix Riccati equation with the initial condition provided as a closed-form function of P_{x_0} , P_{x_f} , and γ ; see Ref. [33] (Theorem 1). Thus, Problem 2 can be solved onboard. The cost (21) penalizes the state and control deviation the nominal trajectory, which, as we will show in Sec. III.C, corresponds to how much throttle margin must be allowed to allow for closed-loop controls.

Next, we use the solution (K^ℓ, P_x^ℓ) of Problem 2 to construct a new feedback gain K such that the terminal covariance constraint (6b) will be satisfied for any mass function satisfying Eq. (18). By definition, P_x^ℓ solves the equation

$$\dot{P}_x^\ell = (A + BK^\ell/m_\ell)P_x^\ell + P_x^\ell(A + BK^\ell/m_\ell)^\top + (\gamma/m_\ell)^2 BB^\top \quad (22)$$

with boundary conditions $P_x^\ell(0) = P_{x_0}$ and $P_x^\ell(t_f) = P_{x_f}$. Therefore, if we set K as a function of m , m_ℓ , and K^ℓ in such a way that the state covariance P_x is always less than or equal to P_x^ℓ , then we will have that

$$P_x(t_f) \leq P_x^\ell(t_f) = P_{x_f} \quad (23)$$

Before we state our main result we will need a comparison formula for Riccati equations, which is taken from Ref. [34] with minor modifications.

Lemma III.1 ([34], Theorem 2.1): Let P_i , $i = 1, 2$, be a solution of

$$\dot{P}_i = \mathcal{A}_i^\top(t)P_i + P_i\mathcal{A}_i(t) + \mathcal{Q}_i(t) \quad (24)$$

on the interval $[0, t_f]$. If $P_1(0) = P_2(0)$, and if

$$\begin{bmatrix} \mathcal{Q}_1 - \mathcal{Q}_2 & \mathcal{A}_1 - \mathcal{A}_2 \\ (\mathcal{A}_1 - \mathcal{A}_2)^\top & 0 \end{bmatrix} \geq 0, \quad t \in [0, t_f] \quad (25)$$

then $P_2(t) \leq P_1(t)$ for all $t \in [0, t_f]$.

Theorem III.2: Let P_x be the solution to Eq. (16) with feedback gain K given by

$$K(t, m) = \frac{m}{m_\ell(t)} K^\ell(t), \quad t \in [0, t_f] \quad (26)$$

where $m \geq m_\ell(t)$ for all $t \in [0, t_f]$. Then $P_x(t) \leq P_x^\ell(t)$ for all $t \in [0, t_f]$.

Proof: First, we rewrite Eq. (22) as

$$\dot{P}_x^\ell = m_\ell^{-2}[(m_\ell^2 A - m_\ell B K^\ell)P_x^\ell + P_x^\ell(m_\ell^2 A - m_\ell B K^\ell)^\top + \gamma^2 BB^\top] \quad (27)$$

Substituting the feedback gain (26) into Eq. (16) yields

$$\dot{P}_x = m_\ell^{-2}[(m_\ell^2 A - m_\ell B K^\ell)P_x + P_x(m_\ell^2 A - m_\ell B K^\ell)^\top + (\gamma m_\ell/m)^2 BB^\top] \quad (28)$$

Then, using

$$\mathcal{A}_1 = \mathcal{A}_2 = m_\ell^2 A - m_\ell B K^\ell \quad (29)$$

$$\mathcal{Q}_1 = \gamma^2 BB^\top, \quad \mathcal{Q}_2 = \gamma^2 \frac{m_\ell^2}{m^2} BB^\top \quad (30)$$

we may rewrite Eqs. (27) and (28) as

$$\dot{P}_x^\ell = m_\ell^{-2}(\mathcal{A}_1 P_x^\ell + P_x^\ell \mathcal{A}_1^\top + \mathcal{Q}_1) \quad (31)$$

$$\dot{P}_x = m_\ell^{-2}(\mathcal{A}_2 P_x + P_x \mathcal{A}_2^\top + \mathcal{Q}_2) \quad (32)$$

Since $m \geq m_\ell$ and $BB^\top \geq 0$, it follows that

$$\mathcal{Q}_1 - \mathcal{Q}_2 = \gamma^2 BB^\top \left(1 - \frac{m_\ell^2}{m^2}\right) \geq 0 \quad (33)$$

and therefore, for all $t \in [0, t_f]$,

$$\begin{bmatrix} \mathcal{Q}_1 - \mathcal{Q}_2 & 0 \\ 0 & 0 \end{bmatrix} \geq 0 \quad (34)$$

Finally, we apply Lemma III.1 with $P_1 = P_x^\ell$ and $P_2 = P_x$ to conclude that $P_x(t) \leq P_x^\ell(t)$ for all $t \in [0, t_f]$. $\square$

Corollary III.3: Let P_x be the solution to Eq. (16) with the feedback gain K as in Eq. (26), and assume that the mass is bounded by Eq. (18). Then, the control covariance matrix $P_u = KP_xK^\top$ is bounded from above by

$$P_{u,\max} = \frac{m_u^2}{m_\ell^2} K^\ell P_x^\ell K^{\ell\top} \quad (35)$$

where (K^ℓ, P_x^ℓ) is the solution to Problem 2.

Proof: Substituting the feedback gain K from Eq. (26) into $P_u = KP_xK^\top$, and using the fact that the mass m is bounded from above by m_u , along with the fact that $P_x \leq P_x^\ell$, we obtain

$$P_u = KP_xK^\top = \frac{m^2}{m_\ell^2} K^\ell P_x K^{\ell\top} \leq \frac{m_u^2}{m_\ell^2} K^\ell P_x^\ell K^{\ell\top} = P_{u,\max} \quad (36)$$

□

We note that in the special case in which the mass is known precisely when solving Problem 2, then we can set a prior bound on the mass such that $m_u = m = m_\ell$, from which it follows that $P_u = P_{u,\max}$. More generally, the difference between P_u and $P_{u,\max}$ decreases monotonically as the ratio of the upper to the lower bound on mass decreases.

C. Mean Throttle Margin

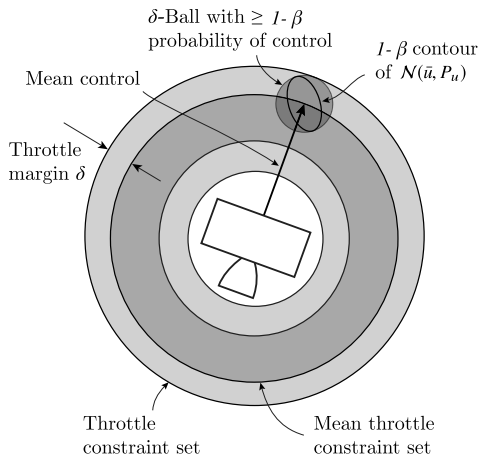
In general, solving for the optimal control with respect to the probabilistic throttle constraint (8) is intractable. Therefore, we develop the following relaxation. Given a control covariance P_u and maximum saturation probability β as in Eq. (8), we wish to find the largest subset $\Omega_\delta = \Omega_\delta(P_u, \beta)$ of admissible controls given by

$$\Omega_\delta = \{\bar{u}: \rho_1 + \delta \leq \|\bar{u}\| \leq \rho_2 - \delta\} \quad (37)$$

with the property that

$$u \sim \mathcal{N}(\bar{u}, P_u) \text{ and } \bar{u} \in \Omega_\delta \Rightarrow \mathbb{P}(u \in \Omega) \geq 1 - \beta \quad (38)$$

We refer to the parameter δ as the *throttle margin*. In general, finding the smallest throttle margin such that Eq. (38) holds is a nonlinear optimization problem with constraints given in terms of three-dimensional integrals, and thus is not computationally feasible for onboard applications. We therefore solve the following problem: find the smallest value of the radius for a three-dimensional ball centered at $\bar{u}$ that has probability of containing u of at least equal to $1 - \beta$. This radius will, in turn, provide a conservative value for the throttle margin δ in Eq. (37), as shown geometrically in Fig. 3.



Lemma III.4: Let $\tilde{u} \sim \mathcal{N}(0, P_u)$ and let σ_u^2 be the maximum eigenvalue of P_u . Then, for all $\delta \geq 0$,

$$\mathbb{P}(\|\tilde{u}\| \leq \delta) \geq \operatorname{erf}\left(\frac{\delta}{\sigma_u \sqrt{2}}\right) - \frac{2\delta}{\sigma_u \sqrt{2\pi}} e^{-\delta^2/2\sigma_u^2} \quad (39)$$

Proof: It can be shown [35] that for any $a \geq 0$,

$$\mathbb{P}(\tilde{u}^\top P_u^{-1} \tilde{u} \leq a^2) = \operatorname{erf}\left(\frac{a}{\sqrt{2}}\right) - \frac{2a}{\sqrt{2\pi}} e^{-a^2/2} \quad (40)$$

To obtain the desired result we find the radius δ of a ball that contains the ellipsoid $\{\tilde{u}: \tilde{u}^\top P_u^{-1} \tilde{u} \leq a^2\}$. First, we diagonalize the control covariance matrix by $P_u = UDU^\top$, where U is orthogonal and D is a diagonal matrix of the eigenvalues λ_i of P_u . Since $\sigma_u^2 \geq \lambda_i$, it follows that

$$D^{-1} = \frac{1}{\sigma_u^2} \operatorname{diag}(\sigma_u^2/\lambda_i) \geq \frac{1}{\sigma_u^2} I \quad (41)$$

From the previous expression it follows that

$$\tilde{u}^\top P_u^{-1} \tilde{u} = \tilde{u}^\top U D^{-1} U^\top \tilde{u} \geq \|\tilde{u}\|^2 / \sigma_u^2 \quad (42)$$

The inequality $\tilde{u}^\top P_u^{-1} \tilde{u} \leq a^2$ implies that

$$\|\tilde{u}\|^2 / \sigma_u^2 \leq \tilde{u}^\top P_u^{-1} \tilde{u} \leq a^2 \quad (43)$$

For $\delta^2 = a^2 \sigma_u^2$, this implies $\|\tilde{u}\|^2 \leq \delta^2$ and thus $\|\tilde{u}\| \leq \delta$. Therefore $\{\tilde{u}: \tilde{u}^\top P_u^{-1} \tilde{u} \leq a^2\} \subseteq \{\tilde{u}: \|\tilde{u}\| \leq \delta\}$, and hence $\mathbb{P}(\tilde{u}^\top P_u^{-1} \tilde{u} \leq a^2) \leq \mathbb{P}(\|\tilde{u}\| \leq \delta)$. The result follows by substituting $a = \delta/\sigma_u$ into Eq. (40):

$$\mathbb{P}(\|\tilde{u}\| \leq \delta) \geq \mathbb{P}(\tilde{u}^\top P_u^{-1} \tilde{u} \leq \delta^2 / \sigma_u^2) \geq \operatorname{erf}\left(\frac{\delta}{\sigma_u \sqrt{2}}\right) - \frac{2\delta}{\sigma_u \sqrt{2\pi}} e^{-\delta^2/2\sigma_u^2} \quad (44)$$

□

Theorem III.5: Let $u \sim \mathcal{N}(\bar{u}, P_u)$, and let Ω_δ be defined as in Eq. (37), where the throttle margin δ satisfies

$$\operatorname{erf}\left(\frac{\delta}{\sigma_u \sqrt{2}}\right) - \frac{2\delta}{\sigma_u \sqrt{2\pi}} e^{-\delta^2/2\sigma_u^2} \geq 1 - \beta \quad (45)$$

where σ_u^2 be the maximum eigenvalue of P_u . Then Eq. (38) holds.

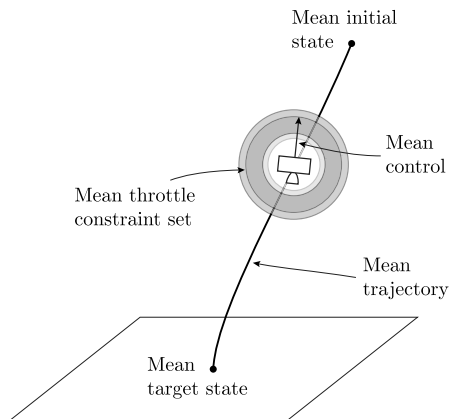


Fig. 3 Conversion from stochastic to deterministic problem with throttle margin.

Proof: If $\bar{u} \in \Omega_\delta$ and $\|\bar{u}\| \leq \delta$, then

$$\|u\| = \|\bar{u} + \tilde{u}\| \leq \|\bar{u}\| + \|\tilde{u}\| \leq \rho_2 - \delta + \delta = \rho_2 \quad (46)$$

$$\|u\| = \|\bar{u} + \tilde{u}\| \geq \|\bar{u}\| - \|\tilde{u}\| \geq \rho_1 + \delta - \delta = \rho_1 \quad (47)$$

Hence $\{u: \bar{u} \in \Omega_\delta, \|\bar{u}\| \leq \delta\} \subseteq \Omega$, which implies that $\mathbb{P}(u \in \Omega) \geq \mathbb{P}(\|\bar{u}\| \geq \delta)$ when $\bar{u} \in \Omega_\delta$. Lastly, if δ satisfies inequality (45), we conclude from Lemma III.4 that $\mathbb{P}(u \in \Omega) \geq 1 - \beta$. $\square$

Proposition III.6: If there exists a value of δ that satisfies inequality (45), then the minimum value of δ satisfying Eq. (45) is the unique solution to

$$\operatorname{erf}\left(\frac{\delta}{\sigma_u \sqrt{2}}\right) - \frac{2\delta}{\sigma_u \sqrt{2\pi}} e^{-\delta^2/2\sigma_u^2} - 1 + \beta = 0 \quad (48)$$

Proof: Take $\xi(\delta)$ to be the left-hand side of inequality (45). We have that $\xi(0) = 0$. Then, taking the derivative, we obtain

$$\frac{d\xi}{d\delta}(\delta) = \frac{2\delta^2}{\sigma_u^3 \sqrt{2\pi}} e^{-\delta^2/2\sigma_u^2} \quad (49)$$

which is strictly positive for $\delta > 0$. Hence $\xi(\delta)$ is monotonically increasing for positive δ , and so the minimum value of δ that satisfies Eq. (45) is the unique solution to Eq. (48). $\square$

Therefore, given a control covariance P_u and maximum saturation threshold β , we can obtain the set Ω_δ by solving a one-dimensional zero-finding problem for Eq. (48), which can be performed onboard. We henceforth denote dependence of variables on P_u and β by a subscript δ , since the thrust margin δ is a function of P_u and β , that is, $\delta = \delta(t; \beta) = \delta(P_u(t); \beta)$.

D. Optimal Mean Control

Suppose that, for a given time-of-flight $t_f > 0$, we have solved for both the feedback gain K from Eq. (26) and the corresponding maximum control covariance $P_{u,\max}$ from Eq. (35). The throttle margin $\delta(t)$ is then computed from Eq. (48) to obtain the mean control constraint set $\Omega_{\delta(t)}$. We are now prepared to solve for the mean control by substituting the probabilistic thrust constraint Eq. (8) in Problem 1 with the deterministic constraint

$$\bar{u}(t) \in \Omega_{\delta(t)} \quad \text{for all } t \in [0, t_f] \quad (50)$$

The mean control problem is formalized as follows.

Problem 3: For a fixed time-of-flight $t_f > 0$ and given throttle margin $\delta(t)$ for $t \in [0, t_f]$, find the mean control $\bar{u}(t)$ for $t \in [0, t_f]$ that minimizes the expected fuel cost (12) subject to the mean dynamics (13), the mean throttle constraint (50), the mean path constraints (9), and the final mean state constraint (6).

Remark III.7: Problem 3 is a standard, deterministic PDG problem, and therefore may be solved by existing methods, such as Ref. [11].

The mean control solving Problem 3 satisfies the mean throttle constraint (50), and therefore, by Theorem III.5, the original probabilistic thrust constraint (8) is satisfied. Furthermore, we obtain the optimal time-of-flight as

$$t_f^* = \arg \min J_\delta^*(t_f) \quad (51)$$

where $J_\delta^*(t_f)$ denotes the optimal cost for Problem 3 with the throttle margin δ and time-of-flight t_f ; that is, for a given throttle margin $\delta = \delta(t)$ and time-of-flight $t_f > 0$, the resulting optimal cost for Problem 3 is $J_\delta^*(t_f)$. The function $J_\delta^*(t_f)$ is unimodal [11], and therefore, for a fixed throttle margin, the optimal time-of-flight can be found by a simple line search. It is well known that the control solving Problem 3 is bang-bang, which we formally state in the following theorem.

Theorem III.8: The optimal thrust profile $\bar{u}^*$ solving Problem 3 has a max-min-max structure with no singular arcs. That is, for any time-of-flight $t_f > 0$ (including $t_f = t_f^*$),

Lower bound on
 $\mathbb{P}(\text{throttle in bounds})$

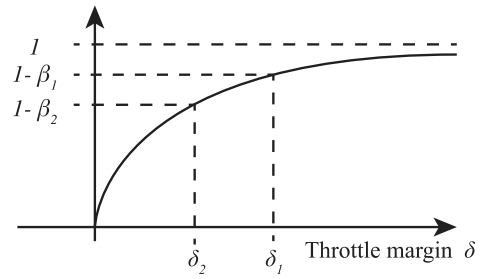


Fig. 4 Diagram of $\xi(\delta)$ as in Eqs. (48) and (49).

$$\|\bar{u}^*(t)\| = \begin{cases} \rho_2 - \delta(t), & 0 \leq t \leq t_1, \\ \rho_1 + \delta(t), & t_1 < t \leq t_2, \\ \rho_2 - \delta(t), & t_2 < t \leq t_f \end{cases} \quad (52)$$

for all $t \in [0, t_f]$.

Proof: See Refs. [6,9,11]. $\square$

Corollary III.9: Let δ_1 and δ_2 be piecewise continuous, positive functions defined over the interval $[0, t_f]$ such that $\delta_1(t) \leq \delta_2(t)$ for all $t \in [0, t_f]$. If, in addition, $\delta_1(t) < \delta_2(t)$ for any nonzero time interval, then $J_{\delta_1}^*(t_f) < J_{\delta_2}^*(t_f)$.

Proof: Let $t_f > 0$ be fixed, and let $\bar{u}_2^*$ be the control solving Problem 3 with the throttle margin δ_2 . First, we observe that the control $\bar{u}_2^*$ satisfies the throttle constraint with the margin δ_1 , since $\rho_1 + \delta_1 \leq \|\bar{u}_2^*\| \leq \rho_2 - \delta_1$ for all time. Since $J_{\delta_1}^*(t_f)$ is the minimum cost for all controls that satisfy this constraint, it follows that

$$J_{\delta_1}^*(t_f) \leq J(\bar{u}_2, t_f) = J_{\delta_2}^*(t_f) \quad (53)$$

It remains to show that equality does not hold. Indeed, if the throttle margin is given by δ_1 , then $\rho_1 + \delta_1 < \|\bar{u}_2^*\| < \rho_2 - \delta_1$ on a nonzero interval implies that the control $\bar{u}_2^*$ is singular on a nonzero interval. This contradicts Theorem III.8, and thus $\bar{u}_2^*$ is not an optimal control for Problem 3 with margin δ_1 , which implies that the inequality in Eq. (53) is strict. $\square$

Corollary III.10: Increasing the maximum probability of throttle saturation β decreases the fuel cost.

Proof: As β increases, then, by Eqs. (48) and (49), the throttle margin δ decreases; see Fig. 4. Suppose first that the time-of-flight is held constant while β is increased. Then the throttle margin will decrease at every time instant, and thus, by Corollary III.9, the fuel cost will decrease. It follows that the fuel will also decrease when the time-of-flight is left as an optimization variable. $\square$

Corollary III.9 is a formal statement of the intuitive result that decreasing the throttle margin at any point along the trajectory will decrease the fuel cost. Furthermore, since our definition (48) of the throttle margin depends on the control covariance, Corollary III.9 implies a connection between minimum-variance feedback control and constrained minimum-fuel control: if the control variance increases, then the fuel cost increases. We also observe, as a consequence of Corollary III.9, that the noise intensity γ and maximum probability of control saturation β influence the fuel cost. Indeed, either increasing γ or decreasing β will increase the throttle margin, and hence will increase the fuel cost. These observations were confirmed by our numerical experiments reported in Sec. IV.

E. Closed-Loop Powered Descent Guidance

The results of the preceding sections are summarized in the following algorithm.

Proposition III.11: The control law $u(t, x)$ returned by Algorithm 1.0 provides a feasible solution to Problem 1.

Proof: Since the mean control is the solution to Problem 3, the mean path constraints (9) and the final mean state (6a) constraint are satisfied. Furthermore, by Theorem III.2, we have that $P_x(t) \leq P_x^\ell(t)$

Algorithm 1: Closed-loop powered descent guidance**Input:** $t_f, m_0, \beta, \gamma > 0, P_{x_0}, P_{x_f} > 0, \bar{x}_0$, and $m_e(t), m_u(t)$ for $t \in [0, t_f]$ **Output:** Control law $u(t, x)$

- 1 Solve Problem 2 to obtain $P_x^e(t)$ and $K^e(t)$ for $t \in [0, t_f]$
- 2 Set $K(t, m) = mK^e(t)/m_e(t)$;
- 3 Solve (48) with $\sigma_u = \sqrt{\lambda_{\max}(P_{u,\max}(t))}$, where $P_{u,\max}$ given as in (35), to obtain $\delta(t)$ for $t \in [0, t_f]$
- 4 Solve Problem 3 to obtain $\bar{x}(t)$, $\bar{u}(t)$, and $m(t)$ for $t \in [0, t_f]$;
- 5 Return control law $u(t, x) = \bar{u}(t) + K(t, m(t))(x - \bar{x}(t))$.

for all $t \in [0, t_f]$. Taking $t = t_f$ we have that the final state covariance constraint (6b) is satisfied.

Next, we show that the throttle constraint (8) is satisfied as well. For fixed $\delta > 0$, the function

$$g(\sigma_u) = \operatorname{erf}\left(\frac{\delta}{\sigma_u \sqrt{2}}\right) - \frac{2\delta}{\sigma_u \sqrt{2\pi}} e^{-\delta^2/2\sigma_u^2} \quad (54)$$

is strictly decreasing since

$$\frac{dg}{d\sigma_u}(\sigma_u) = -\frac{2\delta^3}{\sqrt{2\pi}\sigma_u^2} e^{-\delta^2/2\sigma_u^2} < 0 \quad (55)$$

It follows that $g(\sqrt{\lambda_{\max}(P_u)}) \geq g(\sqrt{\lambda_{\max}(P_{u,\max})})$ since $P_u \leq P_{u,\max}$. Therefore, the inequality (45) holds for u distributed as $\mathcal{N}(\bar{u}, P_u)$ and δ as given by Algorithm 1, and thus, by Theorem III.5, we have that $\mathbb{P}(u \in \Omega) \leq 1 - \beta$. Thus, we have shown that all of the constraints in Problem 1 are satisfied. $\square$

IV. Numerical Example

Consider a Mars Science Laboratory-like descent vehicle, with parameters as shown in Table 1, performing a powered descent maneuver at Mars. We use a surface-fixed frame with the first coordinate as the altitude, and we construct the maneuver so that the mean trajectory remains in the plane defined by the first and third coordinates. The initial mean position, velocity, and mass of the vehicle are given as

$$\bar{r}_0 = (1.9, 0, 1.5) \text{ km}, \quad \bar{v}_0 = (-80, 0, 40) \text{ m/s}, \quad m_0 = 1905 \text{ kg} \quad (56)$$

The initial position and velocity have covariance

$$\operatorname{Cov}\left(\begin{bmatrix} r_0 \\ v_0 \end{bmatrix}\right) = \begin{bmatrix} (15 \text{ m/3})^2 I & 0 \\ 0 & (5 \text{ m} \cdot \text{s}^{-1}/3)^2 I \end{bmatrix} \quad (57)$$

The target mean position and velocity are

$$\bar{r}_f = (100, 0, 100) \text{ m}, \quad \bar{v}_f = (0, 0, 0) \text{ m/s} \quad (58)$$

and the final covariance of the position and velocity is constrained per Eq. (6b) by

Table 1 Vehicle properties

Property	Value	Unit
Specific impulse	210	S
Maximum thrust	16,573	N
Minimum throttle	30	%
Maximum throttle	80	%
Pointing angle θ_{pc0}	75	deg
Glide slope angle θ_{gs0}	70	deg
Wet mass m_0	1,905	kg
Propellant mass	400	kg

$$\operatorname{Cov}\left(\begin{bmatrix} r(t_f) \\ v(t_f) \end{bmatrix}\right) \leq P_{x_f} = \begin{bmatrix} (5 \text{ m/3})^2 I & 0 \\ 0 & (3 \text{ m} \cdot \text{s}^{-1}/3)^2 I \end{bmatrix} \quad (59)$$

that is, the standard deviation of each position and velocity coordinate at the final time should be at most $5/3 \approx 1.67$ m and 1 m/s, respectively. The disturbance intensity γ is set to be 5% of the maximum thrust given in Table 1. The constant weighting matrices

$$Q = \begin{bmatrix} 75I & 0 \\ 0 & 0.5I \end{bmatrix}, \quad R = I \quad (60)$$

where used when solving Problem 2. These values for Q and R were chosen qualitatively in order to balance the state covariance and the control covariance along the closed-loop trajectory.

The optimal mean control was solved as a convex program, as in Ref. [11], for a time-discretization of $\Delta t = 1$ s, in MATLAB with CVX and Mosek [36–38]. The optimal times-of-flight were computed by golden section line search. We solved for the closed-loop PDG following Algorithm 3.5 for probabilities of throttle saturation $\beta = 10\%, 30\%, 50\%$, and 70% . For comparison, we also solved for the optimal mean control for a range of fixed (not time-varying) throttle margins with the feedback controls given as the solution to Problem 2. The simulation cases are listed in Table 2. We compare the targeting performance for each solution by the root-norm error metric given by

$$\text{error} = \sqrt{\lambda_{\max}(P_x(t_f) - P_{x_f})} \quad (61)$$

The fuel cost and the position error, the velocity error, and the targeting error (61) are shown in Figures 5–7 for each of the solution cases listed in Table 2. We see that, for a comparable targeting error, the fuel cost is decreased by setting dynamic throttle margins per

Table 2 Simulation results

Case number	β (%)	Throttle limits (%)		Max std. dev. ^a		
		Lower	Upper	Fuel cost (kg)	Pos. (m)	Vel. (m/s)
1	10	—	—	260.7	1.70	1.08
2	30	—	—	256.3	1.82	1.08
3	50	—	—	254.0	2.30	1.17
4	70	—	—	252.4	3.20	1.26
5	—	35	75	266.0	2.28	1.09
6	—	33	77	254.7	4.12	1.23
7	—	32	78	251.8	5.28	1.34
8	—	30	80	248.3	8.57	2.40

^aSquare root of the maximum eigenvalue of the covariance matrix.

Pos. = Position.

Vel. = Velocity.

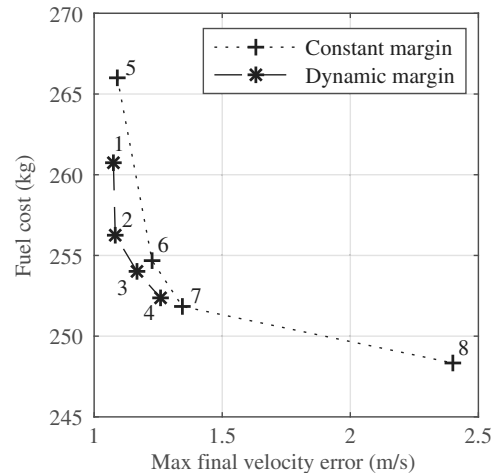


Fig. 5 Maximum velocity covariance standard deviation for cases listed in Table 2.

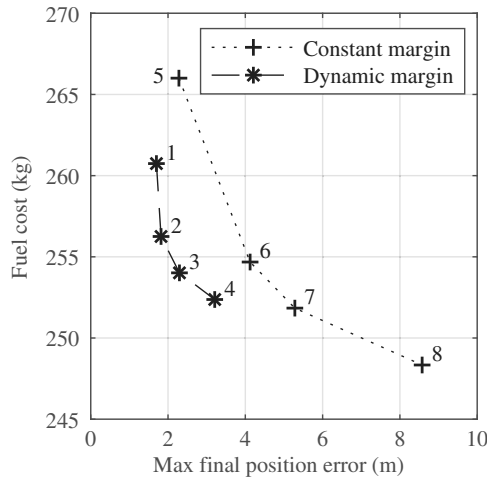


Fig. 6 Maximum position covariance standard deviation for cases listed in Table 2.

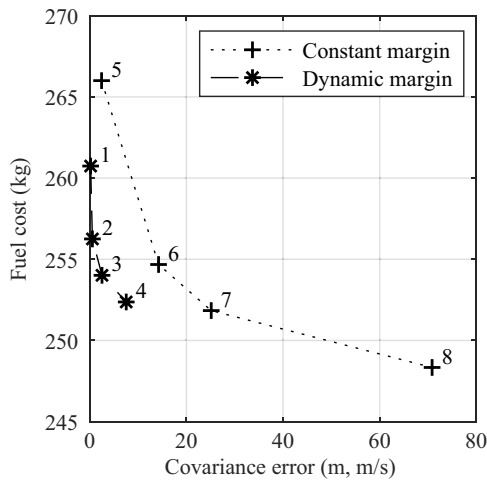
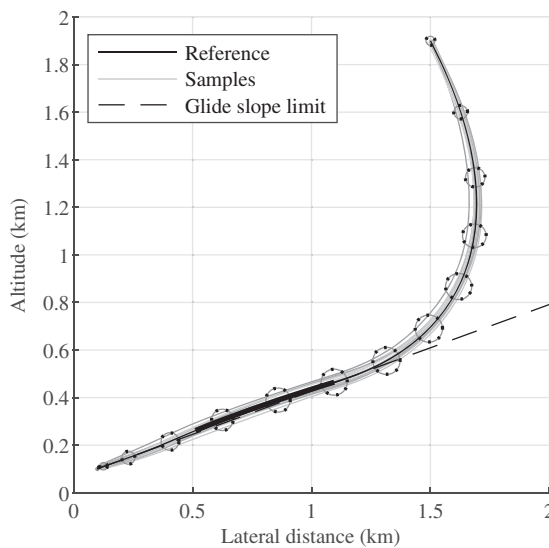


Fig. 7 Comparison of covariance targeting performance for cases listed in Table 2. The error is given by the square root of the maximum eigenvalue of the matrix $P_x(t_f) - P_{x_f}$ as in Eq. (61).



Algorithm 1. Intuitively, the fixed throttle margins must be large enough to allow for the most extreme corrective control actions, which may only be required for a short segment of the trajectory.

Consider, for example, the solution for $\beta = 30\%$, which is shown in Fig. 8. The optimal time-of-flight was 53 s. The 3σ position uncertainty is shown every 4 s, where the covariance was computed by both the analytical solution per Eq. (16), which is shown in gray, and by a 2000-trial Monte Carlo, which is shown as dotted lines. The trajectory segment during which the thrust is at the minimum value is emphasized. We see that, despite some throttle saturation events, the covariance of the closed-loop trajectory closely tracks the predicted covariance solution. Near the end of the trajectory, the feedback control actions become more aggressive in order to meet the final state-covariance constraint. Therefore, the throttle variance is largest near the end of the trajectory, and the throttle margin increases accordingly; that is, the nominal trajectory is adjusted in response to predicted future disturbances. A planar view of the control sample path trajectories in polar coordinates, with zero degrees corresponding to the vertical, for the PDG solution with $\beta = 30\%$ is shown in Fig. 9. Here we can see two-dimensional slices of the δ -balls corresponding to the throttle margin as in Fig. 3, which are shown every 4 s, and samples of the closed-loop control.

The solutions for $\beta = 10\%$ and 70% are shown in Fig. 10. When $\beta = 10\%$, the throttle margins are large enough so that the state covariance computed from Monte Carlo closely tracks the predicted solution. On the other hand, when $\beta = 70\%$, the throttle frequently saturates and the state covariance increases. However, as we see in Figs. 5–7, and as given by Corollary III.10, there is a corresponding decrease in the fuel cost.

Next, we consider the PDG solutions for fixed throttle margins. The trajectories for the two cases with numbers six and eight from Table 2 are compared in Fig. 11. The throttle limits for the nominal trajectory are shown in gray and the absolute throttle limits are shown in black. We see that in the absence of any throttle margin, the state covariance increases and the final targeting error is significant. As expected, increasing the constant throttle margin improves targeting performance. However, for the nonzero fixed margin case shown in Fig. 11, there is a high probability of control saturation in the final 10 s of the trajectory. Consequently, the targeting error is larger than for any of the cases with dynamic margins, including the case that $\beta = 70\%$. For comparison, we increased the fixed throttle margin until the targeting error was comparable to the case when $\beta = 50\%$; the fuel cost was approximately 12 kg, or 4.7%, greater than the fuel cost for the dynamic margin with $\beta = 50\%$.

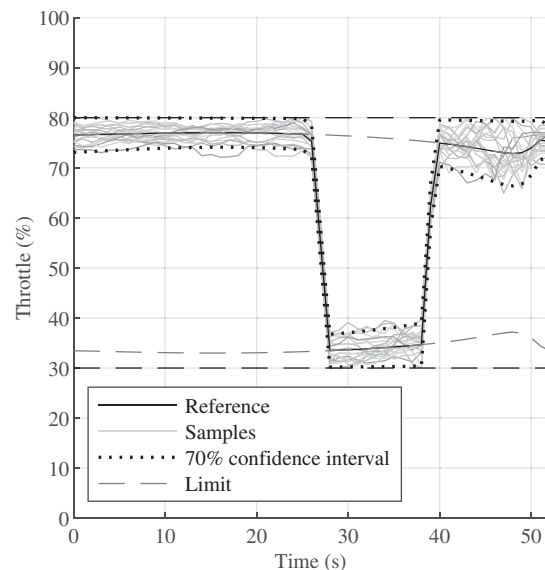


Fig. 8 Solution with $\beta = 30\%$. Left: Powered descent trajectories with 3σ confidence ellipses. Right: Closed-loop throttle trajectories with $1 - \beta$ confidence intervals.

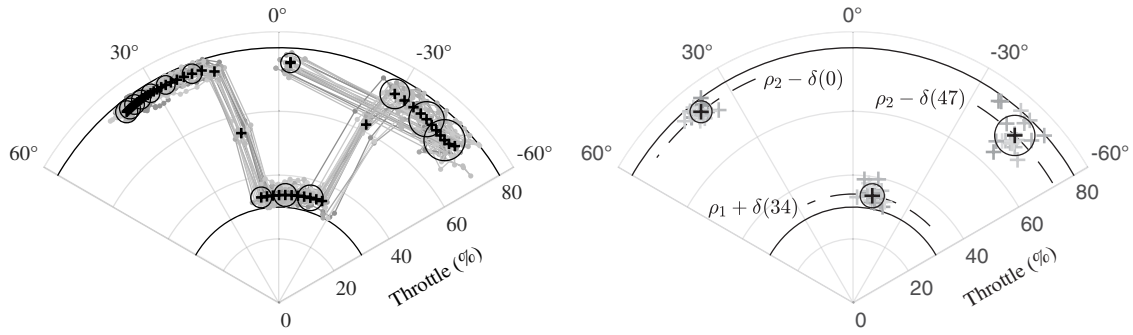


Fig. 9 Solution with $\beta = 30\%$. Left: Planar view of sample control trajectories. Right: Control samples at $t = 0, 34, \text{ and } 47$ s, and the mean constrained throttle (dashed) at these times.

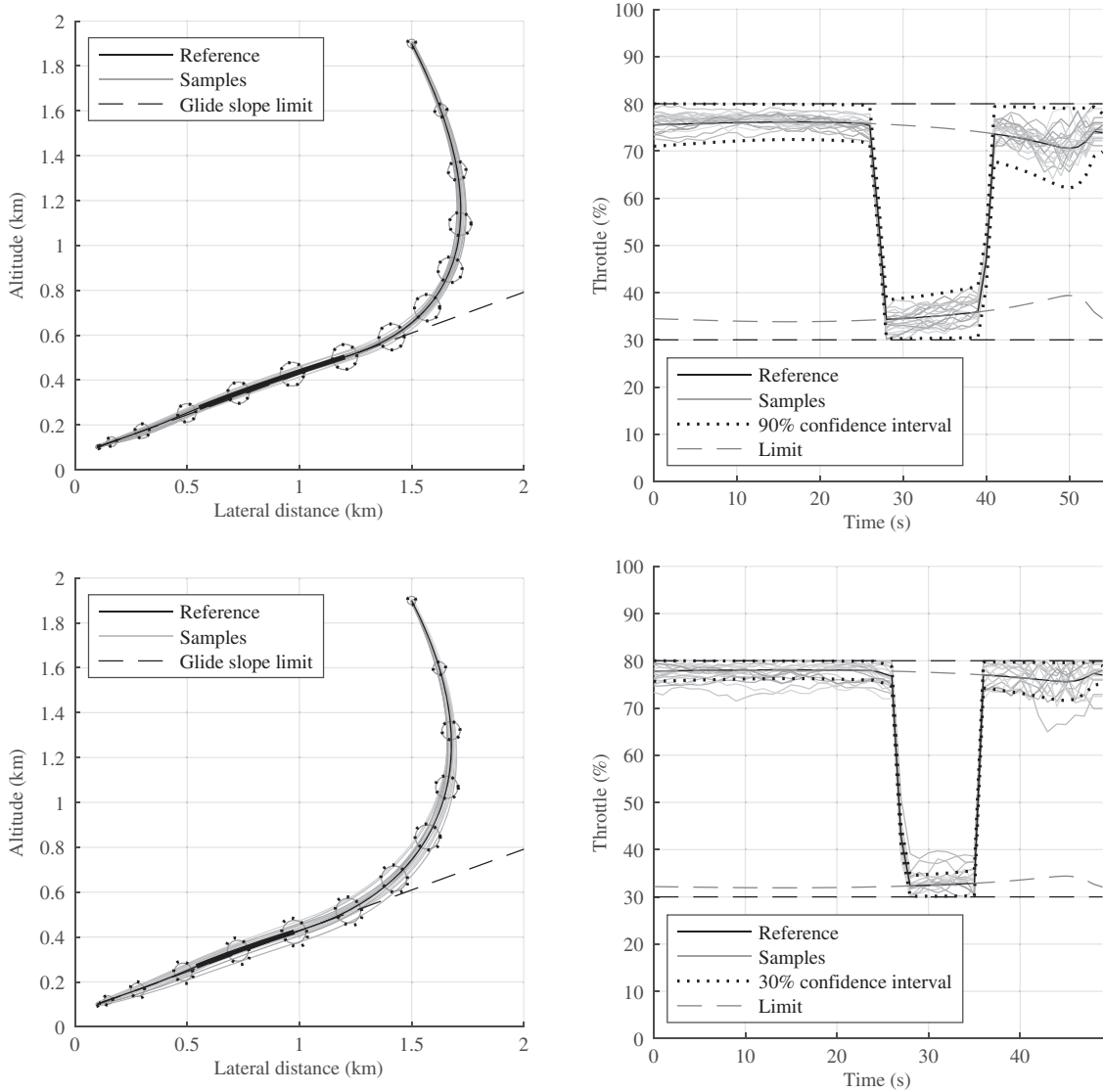


Fig. 10 Solutions with $\beta = 10\%$ (top) and $\beta = 70\%$ (bottom). Left: Powered descent trajectories with 3σ confidence ellipses. Right: Closed-loop throttle trajectories with $1 - \beta$ confidence intervals.

Finally, recall that as a consequence of the assumption (4), the mass dynamics were modeled as a deterministic function of the mean throttle. This assumption is supported by the Monte Carlo results: For the solution with $\beta = 30\%$, for example, the 3σ error in the final mass as computed from the Monte Carlo was approximately

10 kg, which amounts to a 0.6% deviation from the deterministic predicted final mass value of 1648 kg. The small error between the targeted final state covariance and the final state covariance computed from Monte Carlo, as reported in Table 2, further supports this assumption.

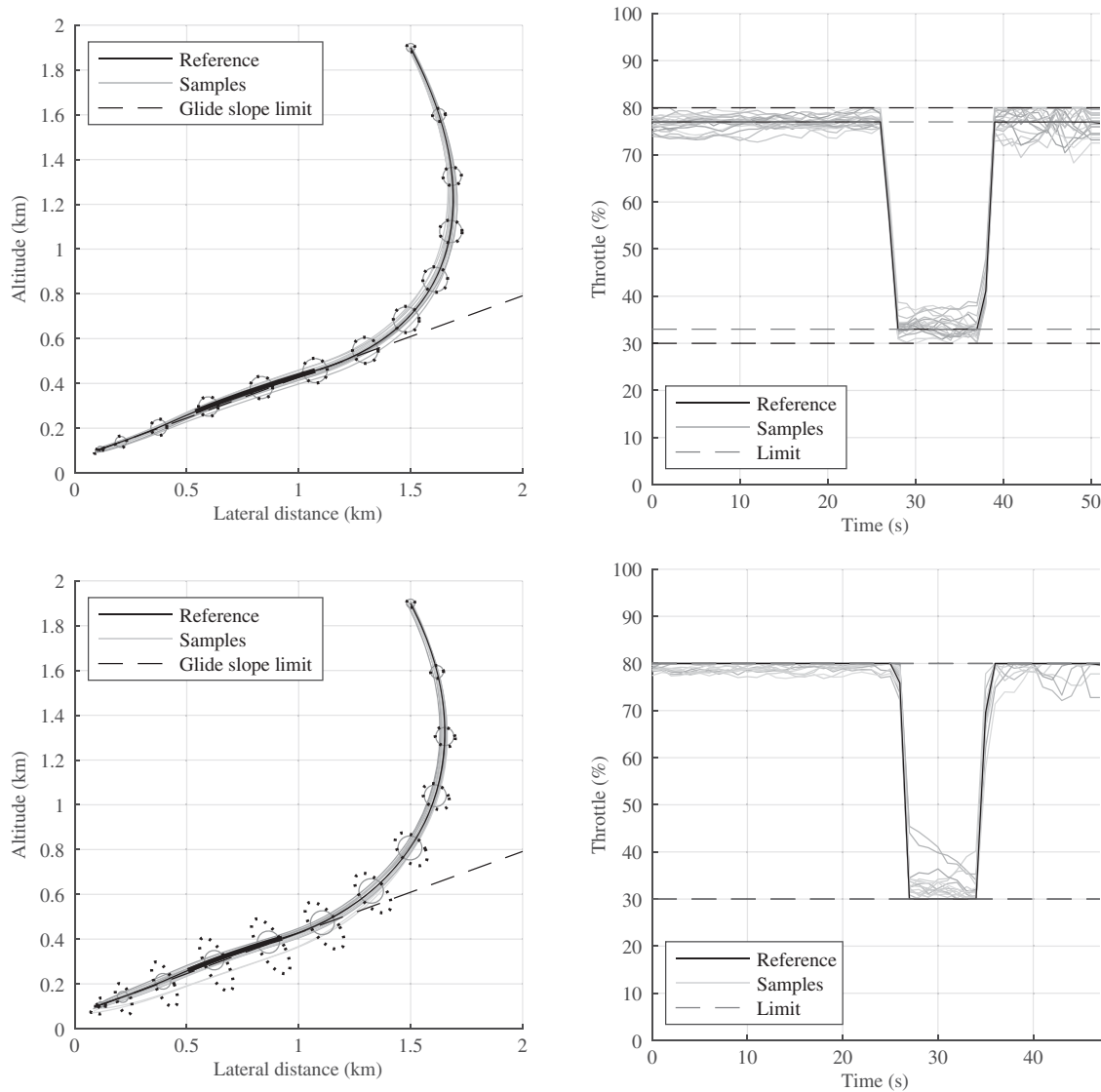


Fig. 11 Solutions no. 6 (top) and no. 8 (bottom) with fixed margins as given in Table 2. Left: Powered descent trajectories with 3σ confidence ellipses. Right: Closed-loop throttle trajectories.

V. Conclusions

In this paper, a method has been presented for adding closed-loop control to existing open-loop PDG solutions. An important property of the closed-loop control law is that the feedback gains are computed before the feed-forward controls, and therefore iteratively solving for the feedback gains and the feed-forward controls can be avoided. Furthermore, the feedback gains are computed by integrating a system of ordinary differential equations with the initial condition given algebraically. The throttle margin is computed by solving for the root of a monotonic function, and the nominal trajectory is solved by any existing method, such as convex programming, using the provided throttle margin. Therefore, the proposed closed-loop PDG can be computed onboard. In numerical simulations, it is demonstrated that dynamically setting the throttle margins, per the proposed method, results in lower fuel costs required to meet particular terminal position and velocity covariance requirements when compared with just scaling constant throttle margins.

The proposed method has several limitations that may be addressed in future work. First, aerodynamic forces are not included in the powered descent model, and the disturbance process is assumed to be Brownian with a constant intensity. Furthermore, in order to enable quick onboard computation and to provide performance guarantees, the proposed method solves for the feedback gains before solving for the nominal control; however, in principle, performance may be

improved by jointly optimizing the feedback gains and the nominal control.

Looking forward, the framework presented in this paper can be applied to other nonsingular optimal control problems with external disturbances, since, for general optimal control problems with constrained control inputs, the nominal control is coupled to the statistics of the closed-loop system. The proposed method constitutes a computationally straightforward method for computing a closed-loop control law by means of setting open-loop control margins while avoiding iterative methods.

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