

Optimal Covariance Steering for Continuous-Time Linear Stochastic Systems With Martingale Additive Noise

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Abstract—In this paper, we study the problem of how to optimally steer the state covariance of a general continuous-time linear stochastic system over a finite time interval subject to additive noise. Optimality here means reaching a target state covariance with minimal control energy. The additive noise may include a combination of white Gaussian noise and abrupt “jump noise” that is discontinuous in time. We first establish the controllability of the state covariance for linear time-varying stochastic systems. We then derive the optimal control, which entails solving two coupled matrix ordinary differential equations (ODEs) with split boundary conditions. We show the existence and uniqueness of the solution to these coupled matrix ODEs, and thus those of the optimal control.

I. INTRODUCTION

All dynamical systems are prone to disturbances whose effects persist with time. Controlling uncertainty is critical for the robustness and overall performance of all such systems. Among the various approaches, covariance control theory, which has been developed since the mid-1980s, provides a *direct* way to regulate the state covariance of a stochastic system subject to noise [1], [2]. By quantifying uncertainty in a direct manner, covariance control theory has found applications in many real-life engineering and physics problems, such as spacecraft soft landing [3] and trajectory optimization [4], vehicle path planning [5]–[7], drone delivery, multi-agent systems [8], active cooling of stochastic oscillators [9], Schrödinger bridge [10], and optimal mass transport [11].

Much of the earlier work dealt exclusively with controlling the stationary, or asymptotic, state covariance over an infinite time horizon [12]–[16]. Most recent work has focused on the finite-horizon optimal covariance steering problem in both continuous-time [17]–[20] and discrete-time settings [21]–[26]. It is shown that when the noise channel coincides with the control channel, and the noise is modeled by a Wiener process, there exists a unique optimal control that steers the state covariance from any initial positive definite matrix $\Sigma_0 \succ 0$ to any final $\Sigma_T \succ 0$ over a finite time interval $[0, T]$

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[17], [19]. When the noise channel differs from the control channel, and the coefficient matrices A , B , and C are constant, controllability of the state covariance via state feedback is established first in [27] and then in [18]. However, the question of the existence and uniqueness of an optimal control policy is still open for different noise and control channels and more comprehensive noise models.

Contributions: In this paper, we first extend the noise model used in covariance steering problems to include jumps of any size. Next, we establish a controllability result for the covariance equation of a linear time-varying stochastic system under mild assumptions. Lastly, and most importantly, we show the existence and uniqueness of the optimal control law when the noise and control channels are different. In particular, the technique for proving the existence and uniqueness of the controller is novel to the optimal covariance steering problems.

The rest of the paper is organized as follows. The main problem of interest is formulated in Section II. The comprehensive noise model is discussed in Section III. The controllability of the state covariance for a linear time-varying stochastic system is shown in Section IV. The existence and uniqueness of the optimal control are shown in Section V. Finally, one example is presented in Section VI to illustrate the results of this paper. Some of the longer proofs are provided in the Appendix.

II. PROBLEM FORMULATION

Consider the following time-varying linear stochastic system subject to additive martingale noise

$$dx(t) = A(t)x(t)dt + B(t)u(t)dt + C(t)dm(t), \quad (1)$$

$$\mathbb{E}[x(0)] = 0, \quad \mathbb{E}[x(0)x^T(0)] = \Sigma_0 \succ 0, \quad (2)$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^p$ is the control input, $m(t) \in \mathbb{R}^q$ is a martingale [28] independent of $x(0)$ with $d\mathbb{E}[m(t)m^T(t)]/dt = D(t) \succeq 0^1$, and $A(t) \in \mathbb{R}^{n \times n}$, $B(t) \in \mathbb{R}^{n \times p}$, and $C(t) \in \mathbb{R}^{n \times q}$ are the coefficient matrices. Let $\mathcal{C}^k$ denote the class of k -times continuously differentiable functions. We assume that $A(t) \in \mathcal{C}^{n-1}$, $B(t) \in \mathcal{C}^n$, and $C(t), D(t) \in \mathcal{C}^0$. Without loss of generality, assume that (1) is defined on the time interval $[0, 1]$, and that the desired terminal

¹The assumption that $d\mathbb{E}[m(t)m^T(t)]/dt = D(t) \succeq 0$ can be justified by Itô calculus [29, Sections II].

state $x(1)$ is characterized by its mean² and covariance matrix given by

$$\mathbb{E}[x(1)] = 0, \quad \mathbb{E}[x(1)x^\top(1)] = \Sigma_1 \succ 0. \quad (3)$$

A control input $u \in \mathcal{U}$ is *admissible* if, for each $t \in [0, 1]$, it depends only on t and on the past history of the states $\{x(s) : 0 \leq s \leq t\}$, and satisfies

$$J(u) \triangleq \mathbb{E} \left[\int_0^1 u^\top(t) R(t) u(t) dt \right] < \infty, \quad (4)$$

for some given continuous $R(t) \succ 0$ of dimension $p \times p$ for all $t \in [0, 1]$, such that (1) with the initial condition (2) has a strong solution [30], and the desired terminal state mean and covariance given by (3) are achieved.

This paper addresses the following two problems:

Problem P1: Determine whether $\mathcal{U}$ is nonempty.

Problem P2: Find the optimal control $u^* \in \mathcal{U}$ that minimizes the quadratic cost functional $J(u)$ defined in (4) subject to the boundary constraints (2) and (3).

III. MARTINGALE NOISE MODEL

In this section, we give some examples of the martingale noise used in this work, along with a brief discussion on how the martingale noise affects the optimal control of the system.

The martingale noise $dm(t)$ has a mean value of zero. Generally speaking, the noise may not be white, since a martingale may not have stationary and independent increments. Some examples of noise modeled by dm are as follows.

- 1) **White Gaussian noise:** $dm_1(t) = dw(t)$, where $w(t) \in \mathbb{R}^q$ is a Wiener process.
- 2) **Noise with jumps:** $dm_2(t) = dh(t) - \lambda(t)dt$, where $h(t) \in \mathbb{R}^q$ is a nonhomogeneous Poisson process with (deterministic) arrival rate $\lambda(t) = [\lambda_1(t) \ \lambda_2(t) \ \dots \ \lambda_q(t)]^\top$, since the compensated nonhomogeneous Poisson process $m_2(t) = h(t) - \int_0^t \lambda(\tau) d\tau$ is a martingale [28]. In general, the jump size may not be fixed but it is a random variable following a certain distribution. We may take m_2 to be a nonhomogeneous compound Poisson process subtracting its compensator [28]. Thus, dm_2 can model random noise of any jump size.
- 3) **Combinations:** We can model continuous noise and noise with jumps, such as $dm_3(t) = \kappa dw(t) + dm_2(t)$, where $\kappa > 0$, or $dm_4(t) = d\ell(t) - de(t)$, where ℓ is a Lévy process and e is its compensator [28].

As the martingale noise is characterized solely by its mean and covariance, it is not surprising that the optimal control for martingale noise has the same structure as that for Gaussian noise. In both cases, the control boils down to solving two coupled matrix ordinary differential equations (ODEs),

²Assuming zero mean is just for convenience, otherwise the control will have an additional term; see Remark 9 in [17].

one Riccati equation and one Lyapunov equation, with split boundary conditions on the latter. In Section V, we provide a new, unifying approach to analyze these coupled matrix ODEs.

IV. CONTROLLABILITY OF THE STATE COVARIANCE

In this section, the controllability of the state covariance for (1) is established. The state covariance of (1), denoted by

$$\Sigma(t) \triangleq \mathbb{E} \left[(x(t) - \mathbb{E}[x(t)]) (x(t) - \mathbb{E}[x(t)])^\top \right],$$

is said to be *controllable* on the time interval $[0, 1]$ if, for any given $\Sigma_0, \Sigma_1 \succ 0$, there exists an admissible control $u \in \mathcal{U}$ that steers the state covariance $\Sigma(t)$ of (1) from $\Sigma(0) = \Sigma_0$ to $\Sigma(1) = \Sigma_1$, while maintaining $\Sigma(t) \succ 0$ for all $t \in [0, 1]$.

Consider the state feedback control law of the form

$$u(t) = K(t)x(t), \quad t \in [0, 1], \quad (5)$$

where $K(t) \in \mathbb{R}^{p \times n}$ is a feedback matrix to be determined. Since $\mathbb{E}[x(0)] = 0$, we have $\mathbb{E}[x(t)] = 0$ for all $t \in [0, 1]$. Therefore, the state covariance $\Sigma(t) = \mathbb{E}[x(t)x^\top(t)]$ satisfies

$$\begin{aligned} \dot{\Sigma} = & (A(t) + B(t)K(t))\Sigma + \Sigma(A(t) + B(t)K(t))^\top \\ & + C(t)D(t)C^\top(t). \end{aligned} \quad (6)$$

Note that with the control (5), controlling the covariance of the stochastic system (1) amounts to controlling the deterministic system (6) via $K(t)$. When the matrices A , B , C , and D are constant and the pair (A, B) is controllable, the state covariance dynamics (6) is controllable via $K(t)$ [27], since (6) can be reduced to the form of (27) in [18, Theorem 3]. It is easy to check that this statement still holds when $C(t)$ and $D(t)$ are time-varying. It will be shown below that when $A(t)$ and $B(t)$ are time-varying, and under some mild assumptions, (6) is also controllable and therefore $\mathcal{U}$ is non-empty.

Given a time-varying matrix pair $(A(t), B(t))$ of dimensions $n \times n$ and $n \times p$, respectively, define

$$\Theta_i(t) \triangleq [\Gamma_0(t) \ \Gamma_1(t) \ \dots \ \Gamma_{i-1}(t)], \quad 1 \leq i \leq n+1,$$

$$\Gamma_0(t) \triangleq B(t),$$

$$\Gamma_k(t) \triangleq -A(t)\Gamma_{k-1}(t) + \dot{\Gamma}_{k-1}(t), \quad 1 \leq k \leq n.$$

Then, the *controllability matrix* of $(A(t), B(t))$ is $\Theta_n(t)$ and the pair $(A(t), B(t))$ is *uniformly controllable* on the time interval $[0, 1]$ if, for all $t \in [0, 1]$, $\text{rank } \Theta_n(t) = n$ [31]. Recall that the controllability property in [17] requires that the controllability Gramian

$$N(t_1, t_0) \triangleq \int_{t_0}^{t_1} \Phi_A(t_0, \tau) B(\tau) B^\top(\tau) \Phi_A(t_0, \tau)^\top d\tau \quad (7)$$

is nonsingular for all $0 \leq t_0 < t_1 \leq 1$, where $\Phi_A(t, \tau)$ is the state transition matrix of $A(t)$ (see also (12) below). This property is, in fact, equivalent to the statement that the pair $(A(t), B(t))$ is *totally controllable* on the time interval $[0, 1]$ [32], which holds if and only if, for all $0 \leq t_0 < t_1 \leq 1$, there exists $t \in (t_0, t_1)$ such that $\text{rank } \Theta_n(t) = n$ [33]. Clearly, uniform controllability is a stronger property than total

controllability. The pair $(A(t), B(t))$ is *index invariant* on the interval $[0, 1]$ if, for each $i \in \{1, 2, \dots, n+1\}$, $\text{rank } \Theta_i(t)$ is constant for all $t \in [0, 1]$, and $\text{rank } \Theta_n(t) = \text{rank } \Theta_{n+1}(t)$ [34]. In the sequel, we refer to these properties as follows.

Condition C1: The pair $(A(t), B(t))$ is uniformly controllable on the interval $[0, 1]$.

Condition C2: The pair $(A(t), B(t))$ is index invariant on the interval $[0, 1]$.

Condition C3: The pair $(A(t), B(t))$ is totally controllable on the interval $[0, 1]$.

Theorem 1. *Let Conditions C1 and C2 hold. Then, the state covariance of the linear stochastic system (1) is controllable on $[0, 1]$ and also on any subinterval of $[0, 1]$.*

Proof. The key idea to prove Theorem 1 is to reduce a time-varying matrix pair to a time-invariant one via a time-varying coordinate transformation and a state feedback control. Without loss of generality, assume $B(t)$ has full column rank for all $t \in [0, 1]$. Since Conditions C1 and C2 hold, there exists a time-varying coordinate transformation that brings the pair $(A(t), B(t))$ into its canonical form [35], [36]. That is, there exist nonsingular matrices $P(t) \in \mathbb{R}^{n \times n}$ and $Q(t) \in \mathbb{R}^{p \times p}$ such that the new state $\tilde{x}(t) = P(t)x(t)$ satisfies

$$d\tilde{x}(t) = \tilde{A}(t)\tilde{x}(t)dt + \tilde{B}\tilde{u}(t)dt + \tilde{C}(t)dm(t),$$

where $\tilde{A}(t) = (P(t)A(t) + \dot{P}(t))P^{-1}(t)$, $\tilde{B} = P(t)B(t)Q(t)$, $\tilde{C}(t) = P(t)C(t)$, $\tilde{u}(t) = Q^{-1}(t)u(t)$, and the pair $(\tilde{A}(t), \tilde{B})$ is in the canonical form

$$\tilde{A}(t) = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \dots & \tilde{A}_{1p} \\ \tilde{A}_{21} & \tilde{A}_{22} & \dots & \tilde{A}_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{A}_{p1} & \tilde{A}_{p2} & \dots & \tilde{A}_{pp} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} b_1 & 0 & \dots & 0 \\ 0 & b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_p \end{bmatrix}$$

$$\tilde{A}_{ii} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 \\ \times & \dots & \times & \times \end{bmatrix}, \quad \tilde{A}_{ij} = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \dots & 0 \\ \times & \dots & \times \end{bmatrix}, \quad b_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$\tilde{A}_{ij}(t) \in \mathbb{R}^{n_i \times n_j}, \quad i, j \in \{1, 2, \dots, p\}, \quad n_1 + \dots + n_p = n.$$

With the state feedback control $\tilde{u}(t) = \tilde{K}(t)\tilde{x}(t)$, where $\tilde{K}(t) = Q^{-1}(t)K(t)P^{-1}(t)$, the new state covariance

$$\begin{aligned} \tilde{\Sigma}(t) &\triangleq \mathbb{E} \left[(\tilde{x}(t) - \mathbb{E}[\tilde{x}(t)]) (\tilde{x}(t) - \mathbb{E}[\tilde{x}(t)])^\top \right] \\ &= P(t)\Sigma(t)P^\top(t) \end{aligned}$$

satisfies

$$\begin{aligned} \dot{\tilde{\Sigma}} &= (\tilde{A}(t) + \tilde{B}\tilde{K}(t))\tilde{\Sigma} + \tilde{\Sigma}(\tilde{A}(t) + \tilde{B}\tilde{K}(t))^\top \\ &\quad + \tilde{C}(t)D(t)\tilde{C}^\top(t). \end{aligned}$$

From the previous expressions of $\tilde{A}(t)$ and $\tilde{B}$, it follows that there exists

$$\tilde{F}(t) = -[\tilde{a}_1^\top(t) \quad \tilde{a}_2^\top(t) \quad \dots \quad \tilde{a}_p^\top(t)]^\top \in \mathbb{R}^{p \times n},$$

where $\tilde{a}_i(t) \in \mathbb{R}^{1 \times n}$ is the $(\sum_{j=1}^i n_j)$ th row of $\tilde{A}(t)$ for $i \in \{1, \dots, p\}$, such that $(\tilde{A}(t) + \tilde{B}\tilde{F}(t), \tilde{B})$ is a time-invariant matrix pair. Notice that the initial and terminal constraints (2) and (3) in the new coordinates are

$$\begin{aligned} \mathbb{E}[\tilde{x}(0)] &= 0, \quad \mathbb{E}[\tilde{x}(0)\tilde{x}^\top(0)] = \tilde{\Sigma}_0 = P(0)\Sigma_0P^\top(0) \succ 0, \\ \mathbb{E}[\tilde{x}(1)] &= 0, \quad \mathbb{E}[\tilde{x}(1)\tilde{x}^\top(1)] = \tilde{\Sigma}_1 = P(1)\Sigma_1P^\top(1) \succ 0. \end{aligned}$$

Clearly, if $\tilde{\Sigma}(t) \succ 0$ for all $t \in [0, 1]$, then $\Sigma(t) \succ 0$ for all $t \in [0, 1]$. Therefore, the controllability of the state covariance for the time-varying system (1) follows from that for the time-invariant system $(\tilde{A}(t) + \tilde{B}\tilde{F}(t), \tilde{B})$ [18]. Since Conditions C1 and C2 hold, it follows that $(A(t), B(t))$ is also uniformly controllable and index invariant on any subinterval. ■

Remark 1. *Although Conditions C1 and C2 facilitate the proof of Theorem 1, these conditions can be relaxed. Indeed, the more relaxed controllability condition, namely, Condition C3, is the only controllability condition that will be needed to solve Problems P1 and P2 and establish all results in this paper (see Theorem 2 and Corollary 1 in Section V below). The intuition behind the claim above is that the state covariance of the system (1) can be controlled via controlling all possible (state) trajectories of the system. Since Condition C3 ensures that the state of the system is controllable on any subinterval of $[0, 1]$, it should be sufficient for the state covariance of the system to be controllable on any subinterval.*

V. OPTIMAL CONTROL OF THE STATE COVARIANCE

In this section, the existence and uniqueness of the optimal control for Problem P2 are established. As a byproduct, the controllability condition for Problem P1 is relaxed.

Using the “completion of squares” argument [17]–[19], a candidate optimal control law can be derived³ as follows.

Lemma 1. *Assume $\Pi(t)$ and $\Sigma(t)$ satisfy, for $t \in [0, 1]$,*

$$\dot{\Pi} = -A^\top(t)\Pi - \Pi A(t) + \Pi B(t)R^{-1}(t)B^\top(t)\Pi, \quad (8)$$

$$\begin{aligned} \dot{\Sigma} &= (A(t) - B(t)R^{-1}(t)B^\top(t)\Pi(t))\Sigma + C(t)D(t)C^\top(t) \\ &\quad + \Sigma(A(t) - B(t)R^{-1}(t)B^\top(t)\Pi(t))^\top, \quad (9) \end{aligned}$$

$$\Sigma(0) = \Sigma_0 \succ 0, \quad \Sigma(1) = \Sigma_1 \succ 0. \quad (10)$$

Then, the state feedback control

$$u^*(t) = -R^{-1}(t)B^\top(t)\Pi(t)x(t) \quad (11)$$

is optimal with respect to the cost functional (4), subject to the boundary constraints (2), (3).

We point out that as long as a covariance steering problem satisfies the following properties: 1) the system model is linear, 2) the cost function is the expected value of a quadratic form, 3) the disturbance is purely additive with a positive semi-definite covariance matrix, then, the optimal solution is always governed by coupled Riccati-Lyapunov equations with split boundary conditions on the latter.

³For the details of the derivation, see Sections II and V.A in [29].

Next, we show the following main result by analyzing the solution to the coupled matrix ODEs (8), (9), (10).

Theorem 2. *Let Condition C3 hold. For any given $\Sigma_0, \Sigma_1 \succ 0$, the unique optimal control that solves Problem P2 is given by (11), where $\Pi(t)$ is the unique solution to (8), (9), (10).*

Proof. Theorem 2 follows immediately from Corollary 3 and Lemma 5, which are provided in Section V-C. ■

Corollary 1. *Let Condition C3 hold. Then, the state covariance of the linear stochastic system (1) is controllable on $[0, 1]$ and also on any subinterval of $[0, 1]$.*

Proof. Theorem 2 shows that there exists a unique optimal control to Problem P2 for any $\Sigma_0, \Sigma_1 \succ 0$. It follows that the state covariance of (1) is controllable on $[0, 1]$. Since Condition C3 implies that $(A(t), B(t))$ is totally controllable on any subinterval of $[0, 1]$, the state covariance of (1) is also controllable on any subinterval. ■

First, it is straightforward to check that, for $R(t) \succ 0$, if Condition C3 holds, then the pair $(A(t), B(t)R^{-\frac{1}{2}}(t))$ is also totally controllable on $[0, 1]$. As $R(t)$ can always be recovered by a time-varying coordinate transformation, for simplicity, we assume $R(t) \equiv I_p$ for the rest of this section. The proof structure of Theorem 2 is organized as follows. To begin with, a complete solution $\Pi(t)$ to the Riccati equation (8) is presented in Section V-A. Then, the solution $\Sigma(t)$ to the linear equation (9) with two initial conditions $\Pi(0)$ and $\Sigma(0)$ is provided in Section V-B, which will be used to define a map from $\Pi(0)$ to $\Sigma(1)$ for a given $\Sigma(0)$. Lastly, the map is shown to be a bijection in Section V-C, which establishes the existence and uniqueness of the solution to the coupled ODEs (8) and (9) with the split boundary conditions (10).

A. Solution to the Riccati Differential Equation

Let $\Phi_A(t, s)$ denote the state transition matrix of $A(t)$, which satisfies

$$\frac{\partial}{\partial t} \Phi_A(t, s) = A(t)\Phi_A(t, s), \quad \Phi_A(s, s) = I_n. \quad (12)$$

Let the matrix $N(t, s)$ as in (7). If Condition C3 holds, then $N(t, s) \succ 0$ for all $0 \leq s < t \leq 1$.

Lemma 2. *Let Condition C3 hold. Given $\Pi(s)$ for some $s \in [0, 1]$, (8) admits a unique solution $\Pi(t)$ on $[0, 1]$ if and only if*

$$N(0, s)^{-1} \prec \Pi(s) \prec N(1, s)^{-1}, \quad (13)$$

where $N(0, 0^+)^{-1} = -\infty$ and $N(1, 1^-)^{-1} = +\infty$ ⁴. More-

⁴Positive infinity of the $n \times n$ positive semidefinite cone, written $+\infty$, is the limit of a sequence of $n \times n$ positive definite matrices whose eigenvalues all grow to $+\infty$. Likewise, for $-\infty$. Notice that as $s \rightarrow 0^+$, all eigenvalues of $N(0, s)$ go to 0^- , and thus all eigenvalues of $N(0, s)^{-1}$ go to $-\infty$. Likewise, as $s \rightarrow 1^-$, all eigenvalues of $N(1, s)$ go to 0^+ , and thus all eigenvalues of $N(1, s)^{-1}$ go to $+\infty$.

over,

$$\begin{aligned} \Pi(t) &= \Phi_A(s, t)^T \Pi(s) (I_n - N(t, s) \Pi(s))^{-1} \Phi_A(s, t) \\ &= \Phi_A(s, t)^T (I_n - \Pi(s) N(t, s))^{-1} \Pi(s) \Phi_A(s, t), \end{aligned} \quad (14)$$

and

$$N(0, t)^{-1} \prec \Pi(t) \prec N(1, t)^{-1}, \quad t \in [0, 1]. \quad (15)$$

Proof. First, recall that (8) admits a unique solution $\Pi(t)$ on $[0, 1]$ if and only if $\Phi_A(t, s) - \Phi_A(t, s)N(t, s)\Pi(s)$ is invertible for all $t \in [0, 1]$, and the solution $\Pi(t)$ is given by (14) [37].

Next, we show (13). Since $\Phi_A(t, s)$ is invertible, it suffices to show that $I_n - N(t, s)\Pi(s)$ is invertible for all $t \in [0, 1]$ if and only if (13) holds.

(Necessity) Let $I_n - N(t, s)\Pi(s)$ be invertible for all $t \in [0, 1]$. Then, $M(t) \triangleq I_n - N(t, s)^{\frac{1}{2}}\Pi(s)N(t, s)^{\frac{1}{2}}$ is invertible for all $t \in [s, 1]$. Since $M(s) = I_n \succ 0$ and $M(t)$ is continuous on the interval $[s, 1]$, it follows that $M(t) \succ 0$ for all $t \in [s, 1]$. In particular, $M(1) \succ 0$, that is, $I_n \succ N(1, s)^{\frac{1}{2}}\Pi(s)N(1, s)^{\frac{1}{2}}$. Hence, $\Pi(s) \prec N(1, s)^{-1}$. The inequality $\Pi(s) \succ N(0, s)^{-1}$ can be shown in an analogous way, and thus is omitted.

(Sufficiency) Let (13) hold. We will show that $I_n - N(t, s)\Pi(s)$ is invertible for all $t \in [0, 1]$. When $t = s$, we have $I_n - N(s, s)\Pi(s) = I_n$, which is invertible. When $t \in (s, 1]$, and since $0 \prec N(t, s) \preceq N(1, s)$, it follows from (13) that $N(t, s)^{-1} \succeq N(1, s)^{-1} \succ \Pi(s)$. Hence, by multiplying $N(t, s)^{\frac{1}{2}}$ on both sides of the above inequality, we obtain $I_n \succ N(t, s)^{\frac{1}{2}}\Pi(s)N(t, s)^{\frac{1}{2}}$. Since $N(t, s)^{\frac{1}{2}}\Pi(s)N(t, s)^{\frac{1}{2}}$ is similar to $N(t, s)\Pi(s)$, all the eigenvalues of $N(t, s)\Pi(s)$ are strictly less than 1. Therefore, $I_n - N(t, s)\Pi(s)$ is invertible for all $t \in (s, 1]$. The case when $t \in [0, s)$ can be shown in an analogous way, and thus is omitted.

Next, we show (15). Notice that, for any fixed r , $N(r, t)^{-1}$ satisfies the same Riccati differential equation as $\Pi(t)$, that is,

$$\begin{aligned} \frac{\partial}{\partial t} N(r, t)^{-1} &= -A^T(t)N(r, t)^{-1} - N(r, t)^{-1}A(t) \\ &\quad + N(r, t)^{-1}B(t)B^T(t)N(r, t)^{-1}. \end{aligned} \quad (16)$$

The monotonicity of the matrix Riccati differential equation [38] implies that, for any $s \in [0, 1]$ such that $N(0, s)^{-1} \prec \Pi(s) \prec N(1, s)^{-1}$, it follows that, for all $t \in [0, 1]$, $N(0, t)^{-1} \prec \Pi(t) \prec N(1, t)^{-1}$. ■

Lemma 2 states that the solution of the matrix Riccati differential equation (8) exists on a time interval if and only if it has upper and lower bounds in terms of its coefficient matrices on the interval. This result, in turn, characterizes the two end-points of the maximal interval of existence of the solution, at which the finite escape phenomenon takes place.

Corollary 2. *Assume that $(A(t), B(t))$ is totally controllable for $t \in \mathbb{R}$. Let $\mathcal{I}_s \subset \mathbb{R}$ be the maximal interval of existence of the solution to (8), starting from $\Pi(s) = \Pi_s$. Then, $\mathcal{I}_s =$*

(t_0, t_1) , where

$$t_0 \triangleq \inf \{t \mid t < s, N(t, s)^{-1} \prec \Pi_s\},$$

$$t_1 \triangleq \sup \{t \mid t > s, N(t, s)^{-1} \succ \Pi_s\}.$$

B. Solution to the State Covariance Equation

Armed with the complete solution of $\Pi(t)$, we obtain an explicit expression for the state transition matrix $\Phi_{A-BB^\top\Pi}(t, s)$, which satisfies $\Phi_{A-BB^\top\Pi}(s, s) = I_n$, and

$$\frac{\partial}{\partial t} \Phi_{A-BB^\top\Pi}(t, s) = (A(t) - B(t)B^\top(t)\Pi(t))\Phi_{A-BB^\top\Pi}(t, s).$$

Then, we may provide an explicit solution of $\Sigma(t)$ over $[0, 1]$.

Lemma 3. *Let $\Pi(t)$ exist on $[0, 1]$. The state transition matrix of $A(t) - B(t)B^\top(t)\Pi(t)$ is given, for $s, t \in [0, 1]$, by*

$$\Phi_{A-BB^\top\Pi}(t, s) = \Phi_A(t, s) - \Phi_A(t, s)N(t, s)\Pi(s). \quad (17)$$

Given $\Sigma(0) = \Sigma_0 \succeq 0$, then, for $t \in [0, 1]$,

$$\begin{aligned} \Sigma(t) &= \Phi_{A-BB^\top\Pi}(t, 0)\Sigma_0\Phi_{A-BB^\top\Pi}(t, 0)^\top \\ &+ \int_0^t \Phi_{A-BB^\top\Pi}(t, s)C(s)D(s)C^\top(s)\Phi_{A-BB^\top\Pi}(t, s)^\top ds, \end{aligned} \quad (18)$$

where $\Phi_{A-BB^\top\Pi}(t, s)$ is given by (17) and $\Pi(t)$ is given by (14). In particular, for any $s \in [0, 1]$, as $\Pi(s) \rightarrow N(1, s)^{-1}$, then $\Sigma(1) \rightarrow 0_{n \times n}$. When $\Sigma_0 \succ 0$, as $\Pi(s) \rightarrow N(0, s)^{-1}$, then $\Sigma(1) \rightarrow +\infty$.

Proof. One can readily check that

$$\Phi_A(s, s) - \Phi_A(s, s)N(s, s)\Pi(s) = I_n.$$

Since,

$$\frac{\partial}{\partial t} N(t, s) = \Phi_A(s, t)B(t)B^\top(t)\Phi_A(s, t)^\top,$$

in view of (14), we obtain that

$$\begin{aligned} \frac{\partial}{\partial t} (\Phi_A(t, s) - \Phi_A(t, s)N(t, s)\Pi(s)) \\ = (A(t) - B(t)B^\top(t)\Pi(t)) \\ \times (\Phi_A(t, s) - \Phi_A(t, s)N(t, s)\Pi(s)). \end{aligned}$$

This completes the proof of (17).

It is not difficult to check using direct differentiation that (18) is the solution of (9). By (14) and the continuity of Π , it follows that, as $\Pi(s) \rightarrow N(1, s)^{-1}$, then the corresponding solutions of the Riccati equations for $\Pi(t)$ and $N(1, t)^{-1}$ from (8) and (16), respectively, satisfy $\Pi(t) \rightarrow N(1, t)^{-1}$ for each $t \in [0, 1]$. By the dominated convergence theorem, it follows that, as $\Pi(t) \rightarrow N(1, t)^{-1}$ for all $t \in [0, 1]$, then $\Sigma(1) \rightarrow 0_{n \times n}$. Similarly, as $\Pi(s) \rightarrow N(0, s)^{-1}$, then $\Pi(t) \rightarrow N(0, t)^{-1}$ for each $t \in [0, 1]$. When $\Sigma_0 \succ 0$, as $\Pi(0) \rightarrow N(0, 0^+)^{-1} = -\infty$, then $\Sigma(1) \rightarrow +\infty$. ■

For the special case when $\Sigma_0 \succ 0$ and $C(t)D(t)C^\top(t) = B(t)B^\top(t)$ for all $t \in [0, 1]$, we have that

$$\begin{aligned} &\int_0^1 \Phi_{A-BB^\top\Pi}(1, \tau)B(\tau)B^\top(\tau)\Phi_{A-BB^\top\Pi}(1, \tau)^\top d\tau \\ &= \int_0^1 (\Phi_A(1, \tau) - \Phi_A(1, \tau)N(1, \tau)\Pi(\tau))B(\tau)B^\top(\tau) \\ &\quad \times (\Phi_A(1, \tau)^\top - \Pi(\tau)N(1, \tau)\Phi_A(1, \tau)^\top) d\tau \\ &= -N(0, 1) + \Phi_A(1, 1)N(1, 1)\Pi(1)N(1, 1)\Phi_A(1, 1)^\top \Big|_0^1 \\ &= -N(0, 1) - \Phi_A(1, 0)N(1, 0)\Pi(0)N(1, 0)\Phi_A(1, 0)^\top. \end{aligned}$$

For notational simplicity, in the sequel let $N(1, 0)$ be denoted by N_{10} , and let $\Pi(0)$ be denoted by Π_0 . Then,

$$\begin{aligned} \Phi_A(0, 1)\Sigma(1)\Phi_A(0, 1)^\top &= N_{10}(N_{10}^{-1} - \Pi_0)N_{10} \\ &\quad + N_{10}(N_{10}^{-1} - \Pi_0)\Sigma_0(N_{10}^{-1} - \Pi_0)N_{10} \\ &= N_{10}\Sigma_0^{-\frac{1}{2}} \left(\Sigma_0^{\frac{1}{2}}(N_{10}^{-1} - \Pi_0)\Sigma_0(N_{10}^{-1} - \Pi_0)\Sigma_0^{\frac{1}{2}} \right. \\ &\quad \left. + \Sigma_0^{\frac{1}{2}}(N_{10}^{-1} - \Pi_0)\Sigma_0^{\frac{1}{2}} \right) \Sigma_0^{-\frac{1}{2}} N_{10} \\ &= N_{10}\Sigma_0^{-\frac{1}{2}} \left(\left[\Sigma_0^{\frac{1}{2}}(N_{10}^{-1} - \Pi_0)\Sigma_0^{\frac{1}{2}} + \frac{I}{2} \right]^2 - \frac{I}{4} \right) \Sigma_0^{-\frac{1}{2}} N_{10}. \end{aligned}$$

Therefore, given $\Sigma(1) = \Sigma_1 \succ 0$, $\Pi(0)$ is unique and is given by

$$\begin{aligned} \Pi(0) &= \frac{\Sigma_0^{-1}}{2} + N_{10}^{-1} \\ &\quad - \Sigma_0^{-\frac{1}{2}} \left(\frac{I}{4} + \Sigma_0^{\frac{1}{2}} N_{10}^{-1} \Phi_A(0, 1)\Sigma_1\Phi_A(0, 1)^\top N_{10}^{-1} \Sigma_0^{\frac{1}{2}} \right) \Sigma_0^{-\frac{1}{2}}. \end{aligned}$$

This is the same solution reported in [17].

C. Existence and Uniqueness of the Optimal Control

We are ready to show the existence and uniqueness of the optimal control. Since $\Phi_{A-BB^\top\Pi}(1, s) = \Phi_{A-BB^\top\Pi}(1, 0)\Phi_{A-BB^\top\Pi}(0, s)$, Lemma 3 provides an explicit map from $\Pi(0)$ to $\Sigma(1)$. Specifically, define $f : \{\Pi_0 \in \mathbb{R}^{n \times n} \mid \Pi_0 = \Pi_0^\top \prec N_{10}^{-1}\} \rightarrow \{\Sigma_1 \in \mathbb{R}^{n \times n} \mid \Sigma_1 = \Sigma_1^\top \succ 0\}$ such that

$$\begin{aligned} f(\Pi_0) &= \Phi_{A_{10}}(I - N_{10}\Pi_0) \left[\Sigma_0 + \int_0^1 (I - N_{s0}\Pi_0)^{-1} \Phi_{A_{0s}} C_s D_s C_s^\top \Phi_{A_{0s}}^\top (I - \Pi_0 N_{s0})^{-1} ds \right] \\ &\quad \times (I - \Pi_0 N_{10}) \Phi_{A_{10}}^\top, \end{aligned} \quad (19)$$

where $\Phi_{A_{ts}} \triangleq \Phi_A(t, s)$, $N_{ts} \triangleq N(t, s)$, $C_s \triangleq C(s)$, and $D_s \triangleq D(s)$. Lemma 3 leads naturally to an alternative sufficient condition for optimality stated below.

Corollary 3. *Let Condition C3 hold. Assume there exists an $n \times n$ symmetric matrix $\Pi_0 \prec N_{10}^{-1}$ such that the terminal state covariance Σ_1 can be written as $\Sigma_1 = f(\Pi_0)$. Then, the optimal control is $u^*(t) = -B^\top(t)\Pi(t)x(t)$, where $\Pi(t)$ is the unique solution to (8) with $\Pi(0) = \Pi_0$.*

Next, we show that the map f defined by (19) establishes a one-to-one correspondence between $\Pi(0)$ and $\Sigma(1)$. The key idea behind the proof is to show that this map f satisfies all the criteria in Hadamard's global inverse function theorem [39]. For the sake of notational simplicity, let

$$\Phi_{\Pi_{10}} \triangleq \Phi_{A-BB^T\Pi}(1, 0) = \Phi_{A_{10}}(I - N_{10}\Pi_0),$$

and

$$T_{s0} \triangleq \begin{cases} (N_{s0}^{-1} - \Pi_0)^{-1} \succ 0, & s \in (0, 1], \\ 0_{n \times n}, & s = 0. \end{cases}$$

The vectorization $\text{vec}(H)$ of an $n \times n$ matrix $H = [h_{ij}]$ is $\text{vec}(H) \triangleq [h_{11} \dots h_{n1} \ h_{12} \dots h_{n2} \dots h_{1n} \dots h_{nn}]^T$.

Define the map $\bar{f} : \{ \text{vec}(\Pi_0) \in \mathbb{R}^{n^2} \mid \Pi_0 = \Pi_0^T \prec N_{10}^{-1} \} \rightarrow \{ \text{vec}(\Sigma_1) \in \mathbb{R}^{n^2} \mid \Sigma_1 = \Sigma_1^T \succ 0 \}$ such that

$$\bar{f}(\text{vec}(\Pi_0)) = \text{vec}(f(\Pi_0)), \quad (20)$$

where f is defined by (19).

Lemma 4. *The Jacobian of the map $\bar{f}$, defined by (20), at $\text{vec}(\Pi_0)$ is given by*

$$\begin{aligned} \partial \bar{f}(\text{vec}(\Pi_0)) &= -\Phi_{\Pi_{10}} \otimes \Phi_{\Pi_{10}} \left[\Sigma_0 \otimes T_{10} + T_{10} \otimes \Sigma_0 \right. \\ &\quad \left. + \int_0^1 P_s \otimes (T_{10} - T_{s0}) + (T_{10} - T_{s0}) \otimes P_s \, ds \right], \end{aligned} \quad (21)$$

where $\otimes$ denotes the Kronecker product and

$$P_s \triangleq (I - N_{s0}\Pi_0)^{-1} \Phi_{A_{0s}} C_s D_s C_s^T \Phi_{A_{0s}}^T (I - \Pi_0 N_{s0})^{-1} \succeq 0.$$

Proof. See the Appendix. ■

Lemma 5. *Let Condition C3 hold. For any given $\Sigma_0 \succ 0$, the map f in (19) is a homeomorphism. Thus, for any $\Sigma_1 \succ 0$, there exists a unique $\Pi_0 \prec N_{10}^{-1}$ such that $\Sigma_1 = f(\Pi_0)$.*

Proof. See the Appendix. ■

In general, it is not straightforward to get an analytic expression of the inverse map f^{-1} from $\Sigma(1)$ to $\Pi(0)$. Instead, we have to resort to numerical methods. Specifically, since the map f from $\Pi(0)$ to $\Sigma(1)$ and its Jacobian are known, we can use root-finding algorithms to find $\Pi(0)$ for given $\Sigma(1)$. When $C(t)D(t)C^T(t)$ is approximately equal to a scalar multiple of $B(t)R^{-1}(t)B^T(t)$, one can adopt perturbation methods to find an approximate solution to the coupled ODEs. Lastly, as explained in [18], the general optimal covariance steering problem can always be solved numerically by recasting it as a semidefinite program.

When $n > 1$, the map f from $\Pi(0)$ to $\Sigma(1)$ is not monotone in the Loewner order⁵. However, when $n = 1$, f is monotonically decreasing. Thus, we can find $\Pi(0)$ using, for

⁵Loewner order is a partial order on the set of $n \times n$ symmetric matrices in that, for two such matrices M_1 and M_2 , $M_1 \geq M_2$ if $M_1 \succeq M_2$, and $M_1 > M_2$ if $M_1 \succ M_2$.

example, bisection. In addition, for $n = 1$ we can also handle the pathological case of a singular initial covariance.

D. One-Dimensional Covariance Control with Singular Initial Covariance

In this subsection, we explain how to optimally steer the state covariance when the initial covariance is zero, that is, the initial state is deterministic, by exploiting the monotonicity of f when $n = 1$. Without loss of generality, assume $p = q = 1$. Let the one-dimensional variables be denoted by the lower-case letters of the corresponding matrix variables. With $n = 1$, Condition C3 reduces to the condition that, for all $0 \leq t_0 < t_1 \leq 1$, there exists $t \in (t_0, t_1)$ such that $b(t) \neq 0$.

When $n = 1$, let $\sigma(t; \pi_0)$ denote the state covariance (18) starting from $\pi(0) = \pi_0 < N(1, 0)^{-1}$ and $\sigma(0) = \sigma_0 \geq 0$. It follows from the monotonicity of the Riccati differential equation [38] that $\sigma(1; \pi_0)$ is continuous and monotonically decreasing in π_0 . The following result shows that with a singular initial state covariance $\sigma(0) = 0$, the desired terminal covariance may not be too large, depending on the initial noise.

Theorem 3. *Let Condition C3 hold and let $\sigma_0 = 0$. For any given $\sigma_1 \in (0, \eta)$, where*

$$\eta \triangleq \int_0^1 \left(\int_0^1 e^{\int_\tau^1 2a(s) \, ds} r^{-1}(\tau) b^2(\tau) \, d\tau \right)^2 e^{-\int_t^1 2a(\tau) \, d\tau} d(t) c^2(t) \, dt,$$

the unique optimal control for solving Problem P2 is given by (11). As $\pi(0) \rightarrow -\infty$, $\sigma(1) \rightarrow \eta^-$. Moreover⁶, if $c(0)d(0) \neq 0$, $\eta = +\infty$.

Proof. See the Appendix. ■

VI. NUMERICAL EXAMPLE

We illustrate the theoretical results using an example. Let us consider a small flying vehicle in heavy rain, which is modeled by a kinematic point subject to Poisson and Gaussian noise [40]. We only consider the motion of the vehicle along the vertical direction. Let $x(t)$ and $v(t)$ be the vertical position and velocity, respectively. The effect of rain on the vehicle is modeled by a nonhomogeneous compound Poisson process $h(t)$. Then, the vertical dynamics of the vehicle is given by

$$\begin{aligned} dx(t) &= v(t) \, dt, \\ dv(t) &= u(t) \, dt + 0.2 \, dw(t) + dh(t), \end{aligned}$$

where $h(t)$ has arrival rate $\lambda(t) = 5 - t$ and i.i.d. jump size $\chi \sim \mathcal{N}(-0.5, 0.1^2)$. Our goal is to control the covariance of the vehicle while hovering at a certain height from an initial covariance of $\Sigma_0 = \text{diag}[0.6, 0.6]$ to a target covariance of $\Sigma_1 = \text{diag}[0.2, 0.1]$ with the least effort $\mathbb{E} \left[\int_0^1 u^2(t) \, dt \right]$.

⁶If $c(0)d(0) = 0$, the value of η depends on the asymptotic behavior of $d(t)c^2(t)$ as $t \rightarrow 0^+$.

Assume $[x(0) \ v(0)]^T \sim \mathcal{N}([0 \ 0]^T, \Sigma_0)$. Figure 1 illustrates ten controlled sample paths in the phase space as a function of time. The three-standard deviation tolerance interval for $t \in [0, 1]$ is depicted as the shaded tube.

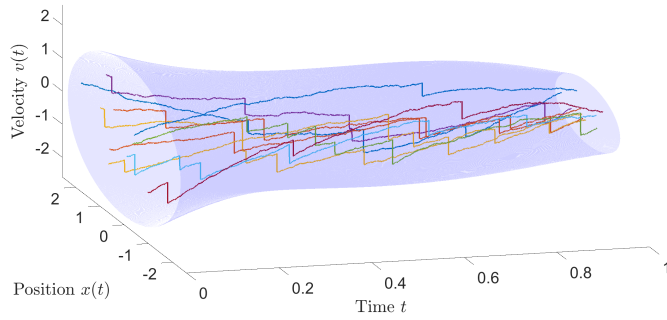


Fig. 1: Controlled position and velocity: 10 sample paths.

VII. CONCLUDING REMARKS

In this paper, the optimal control law is established for steering the state covariance of a general linear time-varying stochastic system subject to general martingale additive noise. The discrete-time counterpart of this result can be found in [41]. Interesting potential extensions include the development of efficient algorithms for solving coupled matrix ODEs (8), (9), (10) and the optimal control of the state covariance along the trajectory, or the covariance steering problem with partially fixed terminal covariance and/or free final time.

REFERENCES

- [1] A. F. Hotz and R. E. Skelton, "A covariance control theory," in *Proc. IEEE Conf. Decision Control*, Lauderdale, FL, 1985, pp. 552–557.
- [2] E. Collins and R. Skelton, "Covariance control of discrete systems," in *Proc. IEEE Conf. Decision Control*, Lauderdale, FL, 1985, pp. 542–547.
- [3] J. Ridderhof and P. Tsiotras, "Uncertainty quantification and control during Mars powered descent and landing using covariance steering," in *AIAA Guidance, Navigation, Control Conf.*, Kissimmee, FL, 2018.
- [4] J. Ridderhof, J. Pilipovsky, and P. Tsiotras, "Chance-constrained covariance control for low-thrust minimum-fuel trajectory optimization," in *AAS/AIAA Astrodynamics Specialist Conf.*, South Lake Tahoe, CA, 2020.
- [5] K. Okamoto and P. Tsiotras, "Optimal stochastic vehicle path planning using covariance steering," *IEEE Robot. Autom. Lett.*, vol. 4, no. 3, pp. 2276–2281, 2019.
- [6] D. Zheng, J. Ridderhof, P. Tsiotras, and A.-a. Agha-mohammadi, "Belief space planning: a covariance steering approach," in *Proc. IEEE Int. Conf. Robot. Autom.*, Philadelphia, PA, 2022, pp. 11 051–11 057.
- [7] J. Yin, Z. Zhang, E. Theodorou, and P. Tsiotras, "Trajectory distribution control for model predictive path integral control using covariance steering," in *Proc. IEEE Int. Conf. Robot. Autom.*, Philadelphia, PA, 2022, pp. 1478–1484.
- [8] A. D. Saravanos, A. G. Tzolovikos, E. Bakolas, and E. A. Theodorou, "Distributed covariance steering with consensus ADMM for stochastic multi-agent systems," in *Robot.: Sci. Syst.*, 2021.
- [9] Y. Chen, T. T. Georgiou, and M. Pavon, "Optimal transport in systems and control," *Annu. Rev. Control Robot. Auton. Syst.*, vol. 4, pp. 89–113, 2021.
- [10] Y. Chen and T. T. Georgiou, "Stochastic bridges of linear systems," *IEEE Trans. Autom. Control*, vol. 61, no. 2, pp. 526–531, 2016.
- [11] Y. Chen, T. T. Georgiou, and M. Pavon, "On the relation between optimal transport and Schrödinger bridges: a stochastic control viewpoint," *J. Optim. Theory Appl.*, vol. 169, pp. 671–691, 2016.
- [12] A. Hotz and R. E. Skelton, "Covariance control theory," *Int. J. Control*, vol. 46, no. 1, pp. 13–32, 1987.
- [13] E. Collins and R. Skelton, "A theory of state covariance assignment for discrete systems," *IEEE Trans. Autom. Control*, vol. 32, no. 1, pp. 35–41, 1987.
- [14] K. Yasuda, R. E. Skelton, and K. M. Grigoriadis, "Covariance controllers: a new parametrization of the class of all stabilizing controllers," *Automatica*, vol. 29, no. 3, pp. 785–788, 1993.
- [15] T. T. Georgiou, "The structure of state covariances and its relation to the power spectrum of the input," *IEEE Trans. Autom. Control*, vol. 47, no. 7, pp. 1056–1066, 2002.
- [16] G. Zhu, M. Rotea, and R. Skelton, "A convergent algorithm for the output covariance constraint control problem," *SIAM J. Control Optim.*, vol. 35, no. 1, pp. 341–361, 1997.
- [17] Y. Chen, T. T. Georgiou, and M. Pavon, "Optimal steering of a linear stochastic system to a final probability distribution, part I," *IEEE Trans. Autom. Control*, vol. 61, no. 5, pp. 1158–1169, 2016.
- [18] —, "Optimal steering of a linear stochastic system to a final probability distribution, part II," *IEEE Trans. Autom. Control*, vol. 61, no. 5, pp. 1170–1180, 2016.
- [19] —, "Optimal steering of a linear stochastic system to a final probability distribution, part III," *IEEE Trans. Autom. Control*, vol. 63, no. 9, pp. 3112–3118, 2018.
- [20] V. Ciccone, Y. Chen, T. T. Georgiou, and M. Pavon, "Regularized transport between singular covariance matrices," *IEEE Trans. Autom. Control*, vol. 66, no. 7, pp. 3339–3346, 2020.
- [21] E. Bakolas, "Finite-horizon covariance control for discrete-time stochastic linear systems subject to input constraints," *Automatica*, vol. 91, pp. 61–68, 2018.
- [22] I. M. Balci and E. Bakolas, "Covariance control of discrete-time Gaussian linear systems using affine disturbance feedback control policies," *arXiv:2103.14428*, 2021.
- [23] —, "Covariance steering of discrete-time stochastic linear systems based on Wasserstein distance terminal cost," *IEEE Control Syst. Lett.*, vol. 5, no. 6, pp. 2000–2005, 2021.
- [24] K. Okamoto, M. Goldshtein, and P. Tsiotras, "Optimal covariance control for stochastic systems under chance constraints," *IEEE Control Syst. Lett.*, vol. 2, no. 2, pp. 266–271, 2018.
- [25] V. Sivaramakrishnan, J. Pilipovsky, M. M. Oishi, and P. Tsiotras, "Distribution steering for discrete-time linear systems with general disturbances using characteristic functions," in *Proc. Amer. Control Conf.*, Atlanta, GA, 2022.
- [26] J. Pilipovsky and P. Tsiotras, "Covariance steering with optimal risk allocation," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 57, no. 6, pp. 3719–3733, 2021.
- [27] R. Brockett, "Notes on the control of the Liouville equation," *Control of Partial Differential Equations*, pp. 101–129, 2012.
- [28] E. Çinlar, *Probability and Stochastics*. Springer, 2011, vol. 261.
- [29] F. Liu and P. Tsiotras, "Optimal covariance steering for continuous-time linear stochastic systems with additive generic noise," *arXiv:2206.11201v1*, 2022.
- [30] P. E. Protter, *Stochastic Integration and Differential Equations*. Springer, 2003, vol. 21.
- [31] L. M. Silverman and H. Meadows, "Controllability and observability in time-variable linear systems," *SIAM J. Control*, vol. 5, no. 1, pp. 64–73, 1967.
- [32] E. Kreindler and P. Sarachik, "On the concepts of controllability and observability of linear systems," *IEEE Trans. Autom. Control*, vol. 9, no. 2, pp. 129–136, 1964.
- [33] A. Stubberud, "A controllability criterion for a class of linear systems," *IEEE Trans. Ind. Appl.*, vol. 83, no. 75, pp. 411–413, 1964.
- [34] A. Morse and L. Silverman, "Structure of index-invariant systems," *SIAM J. Control*, vol. 11, no. 2, pp. 215–225, 1973.
- [35] W. Wolovich, "On the stabilization of controllable systems," *IEEE Trans. Autom. Control*, vol. 13, no. 5, pp. 569–572, 1968.
- [36] C. Seal and A. Stubberud, "Canonical forms for multiple-input time-variable systems," *IEEE Trans. Autom. Control*, vol. 14, no. 6, pp. 704–707, 1969.
- [37] S. Kilicaslan and S. P. Banks, "Existence of solutions of Riccati differential equations for linear time varying systems," in *Proc. Amer. Control Conf.*, Baltimore, MD, 2010, pp. 1586–1590.
- [38] G. Freiling, G. Jank, and H. Abou-Kandil, "Generalized Riccati difference and differential equations," *Linear Algebra Its Appl.*, vol. 241, pp. 291–303, 1996.
- [39] S. G. Krantz and H. R. Parks, *The Implicit Function Theorem: History, Theory, and Applications*. Springer, 2002.

- [40] P. Cowpertwait, V. Isham, and C. Onof, "Point process models of rainfall: developments for fine-scale structure," *Proc. R. Soc. A: Math. Phys. Eng. Sci.*, vol. 463, no. 2086, pp. 2569–2587, 2007.
- [41] F. Liu, G. Rapakoulis, and P. Tsiotras, "Optimal covariance steering for discrete-time linear stochastic systems," *arXiv:2211.00618*, 2023.
- [42] H. V. Henderson and S. R. Searle, "On deriving the inverse of a sum of matrices," *SIAM Rev.*, vol. 23, no. 1, pp. 53–60, 1981.

APPENDIX

Proof of Lemma 4. We can compute the Jacobian of the map $\bar{f}$ defined by (20) as follows. Let $\Delta\Pi_0$ denote a small increment of Π_0 . Then, it follows from [42] that

$$(I - N_{s0}\Pi_0 - N_{s0}\Delta\Pi_0)^{-1} = (I - N_{s0}\Pi_0)^{-1} + (I - N_{s0}\Pi_0)^{-1}N_{s0}\Delta\Pi_0(I - N_{s0}\Pi_0)^{-1} + O(\|\Delta\Pi_0\|^2).$$

Hence, by collecting all the first order terms of $\Delta\Pi_0$, we have

$$\begin{aligned} f(\Pi_0 + \Delta\Pi_0) - f(\Pi_0) &= O(\|\Delta\Pi_0\|^2) \\ &\quad - \Phi_{A_{10}}N_{10}\Delta\Pi_0\left[\Sigma_0 + \int_0^1 P_s ds\right](I - \Pi_0N_{10})\Phi_{A_{10}}^\top \\ &\quad - \Phi_{A_{10}}(I - N_{10}\Pi_0)\left[\Sigma_0 + \int_0^1 P_s ds\right]\Delta\Pi_0N_{10}\Phi_{A_{10}}^\top \\ &\quad + \Phi_{A_{10}}(I - N_{10}\Pi_0)\left[\int_0^1 (I - N_{s0}\Pi_0)^{-1}N_{s0}\Delta\Pi_0P_s \right. \\ &\quad \left. + P_s\Delta\Pi_0N_{s0}(I - \Pi_0N_{s0})^{-1}ds\right](I - \Pi_0N_{10})\Phi_{A_{10}}^\top. \end{aligned}$$

Thus, we can write

$$\begin{aligned} f(\Pi_0 + \Delta\Pi_0) - f(\Pi_0) &= O(\|\Delta\Pi_0\|^2) \\ &\quad - \Phi_{\Pi_{10}}\left[T_{10}\Delta\Pi_0\Sigma_0 + \Sigma_0\Delta\Pi_0T_{10} \right. \\ &\quad \left. + \int_0^1 (T_{10} - T_{s0})\Delta\Pi_0P_s + P_s\Delta\Pi_0(T_{10} - T_{s0})ds\right]\Phi_{\Pi_{10}}^\top. \end{aligned}$$

Vectorizing both sides of the previous equation yields

$$\begin{aligned} \text{vec}\left(f(\Pi_0 + \Delta\Pi_0) - f(\Pi_0)\right) &= \partial\bar{f}(\text{vec}(\Pi_0))\text{vec}(\Delta\Pi_0) \\ &\quad + O(\|\Delta\Pi_0\|^2). \end{aligned}$$

It follows that

$$\begin{aligned} \partial\bar{f}(\text{vec}(\Pi_0)) &= -\Phi_{\Pi_{10}} \otimes \Phi_{\Pi_{10}} \left[\Sigma_0 \otimes T_{10} + T_{10} \otimes \Sigma_0 \right. \\ &\quad \left. + \int_0^1 P_s \otimes (T_{10} - T_{s0}) + (T_{10} - T_{s0}) \otimes P_s ds \right], \end{aligned}$$

thus completing the proof. $\blacksquare$

Proof of Lemma 5. Since T_{s0} is continuous in Π_0 and $s \in [0, 1]$, $\partial\bar{f}(\text{vec}(\Pi_0))$ is continuous in $\text{vec}(\Pi_0)$. First, we show that $\partial\bar{f}(\text{vec}(\Pi_0))$ is nonsingular at each $\Pi_0 \prec N_{10}^{-1}$. Since $\Phi_{\Pi_{10}}$ is nonsingular, $\Phi_{\Pi_{10}} \otimes \Phi_{\Pi_{10}}$ is nonsingular as well. It suffices to show that the term in the square brackets of (21), that is,

$$\begin{aligned} S &\triangleq \Sigma_0 \otimes T_{10} + T_{10} \otimes \Sigma_0 \\ &\quad + \int_0^1 P_s \otimes (T_{10} - T_{s0}) + (T_{10} - T_{s0}) \otimes P_s ds, \end{aligned}$$

is nonsingular. Notice that S is symmetric, since $\Sigma_0, T_{10} \succ 0$ and $P_s, T_{10} - T_{s0} \succeq 0$ are all symmetric. Let $X \neq 0$ be an $n \times n$ matrix, not necessarily symmetric. Then,

$$\begin{aligned} \text{vec}(X)^\top S \text{vec}(X) &= \text{trace}\left(X^\top T_{10} X \Sigma_0 + X^\top \Sigma_0 X T_{10} \right. \\ &\quad \left. + \int_0^1 X^\top (T_{10} - T_{s0}) X P_s + X^\top P_s X (T_{10} - T_{s0}) ds\right) \\ &\geq \text{trace}\left(T_{10}^{\frac{1}{2}} X \Sigma_0 X^\top T_{10}^{\frac{1}{2}} + T_{10}^{\frac{1}{2}} X^\top \Sigma_0 X T_{10}^{\frac{1}{2}}\right) > 0. \end{aligned}$$

Thus, $S \succ 0$, which implies that $\partial\bar{f}(\text{vec}(\Pi_0))$ is nonsingular at each $\Pi_0 \prec N_{10}^{-1}$.

Next, we show that the map f is proper, that is, for any compact subset $\mathcal{K} \subset \{\Sigma_1 \in \mathbb{R}^{n \times n} \mid \Sigma_1 = \Sigma_1^\top \succ 0\}$, the inverse image $f^{-1}(\mathcal{K}) \subset \{\Pi_0 \in \mathbb{R}^{n \times n} \mid \Pi_0 = \Pi_0^\top \prec N_{10}^{-1}\}$ is compact. Since f is continuous, the inverse image of a closed set is closed. Since $\mathcal{K}$ is bounded, in view of (19), the set

$$\left\{ \Phi_{A_{10}}(I - N_{10}\Pi_0)\Sigma_0(I - \Pi_0N_{10})\Phi_{A_{10}}^\top \mid \Pi_0 \in f^{-1}(\mathcal{K}) \right\}$$

is bounded. It follows that $f^{-1}(\mathcal{K})$ is bounded. Hence, $f^{-1}(\mathcal{K})$ is compact, and f is proper.

Since the set of positive definite matrices is convex, it is simply connected [39]. By Hadamard's global inverse function theorem [39], f is a homeomorphism, thus a bijection. $\blacksquare$

Proof of Theorem 3. By the continuity and monotonicity of $\pi(t; \pi_0)$ in π_0 for all $t \in [0, 1]$, it follows that, as $\pi_0 \rightarrow -\infty$, $\pi(t; \pi_0) \rightarrow N(0, t)^{-1}$ monotonically. By (18) and the monotone convergence theorem, it follows that for $\sigma_0 = 0$, $\lim_{\pi_0 \rightarrow -\infty} \sigma(1; \pi_0) = \eta^-$.

Let now $c(0)d(0) \neq 0$. By continuity, $d(t)c^2(t)$ is bounded below from zero on some interval $[0, \varepsilon]$ for $\varepsilon \in (0, 1]$. Let $Z > 0$ satisfy

$$e^{-\int_t^1 2a(\tau) d\tau} d(t)c^2(t) \geq Z > 0, \quad \forall t \in [0, \varepsilon],$$

and let

$$\rho \triangleq \int_0^1 e^{\int_\tau^1 2a(s) ds} r^{-1}(\tau) b^2(\tau) d\tau > 0.$$

Let $G \in \mathbb{R}$ and $\Omega > 0$ satisfy

$$2a(t) \leq G, \quad r^{-1}(t)b^2(t) \leq \Omega, \quad \forall t \in [0, \varepsilon].$$

Without loss of generality, assume ε is sufficiently small, such that, for all $t \in [0, \varepsilon]$, $e^{Gt} - 1 = Gt + O(t^2)$. Then, for all $t \in [0, \varepsilon]$,

$$\begin{aligned} \int_0^t e^{\int_\tau^t 2a(s) ds} r^{-1}(\tau) b^2(\tau) d\tau &\leq \int_0^t e^{G(t-\tau)} \Omega d\tau \\ &= \frac{\Omega(e^{Gt} - 1)}{G} \approx \Omega t. \end{aligned}$$

Therefore,

$$\eta \geq \int_0^\varepsilon \left(\frac{\rho}{\Omega t}\right)^2 Z dt = +\infty.$$

This completes the proof. $\blacksquare$