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From Attitude Estimation To Pose Estimation Using Dual Quaternions

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2.1 Introduction

Unit dual quaternions are an extension of the classical unit quaternions. Whereas unit quaternions represent the relative attitude between two frames, unit dual quaternions represent the relative pose (i.e., attitude and position) between two frames. Compared to other representations of pose, such as dual orthogonal 3-by-3 matrices, dual special unitary 2-by-2 matrices, and dual Pauli spin matrices, dual quaternions have been found to be the most efficient representation to perform basic pose transformations in terms of storage

requirements and number of operations [7]. Under the same metrics, dual quaternions have also been found to be more efficient than 4-by-4 homogeneous matrix transformations and Rodriguez parameters/translation vector pairs for solving the direct kinematic problem in robotics [1].

Several properties make dual quaternions an appealing representation for pose control. First, they allow attitude and position controllers to be written as a single compact control law. Moreover, they automatically take into account the natural coupling between the rotational and the translational motions. However, arguably their most useful property is that the dual quaternion representation of the 6-DOF (pose) kinematic and dynamic equations of motion has the same form as the quaternion representation of the 3-DOF (attitude-only) kinematic and dynamic equations of motion. Thanks to this property, it has also been shown that it is possible to extend attitude controllers with desirable properties into pose controller with analogous properties by almost simply replacing quaternions by dual quaternions [4, 6].

Although the usefulness of dual quaternions for pose control is undeniable, their usefulness for pose estimation, in particular in the framework of the Extended Kalman Filter (EKF), was questionable until recently. Existing EKF formulations with a unit dual quaternion in their state required 8 states to represent pose, instead of 6, and failed to properly address the two algebraic constraints associated with unit dual quaternions [2, 14]. However, the highly successful Quaternion Multiplicative EKF (Q-MEKF) based on unit quaternions for spacecraft attitude estimation, described in detail in Section XI of [10], requires only 3 states to represent attitude (instead of 4) and seamlessly addresses the algebraic constraint associated with unit quaternions. Hence, using the Q-MEKF as a starting point and the ability of dual quaternions to extend 3-DOF results into 6-DOF results, a Dual Quaternion Multiplicative EKF (DQ-MEKF) has been recently proposed [5] that does not suffer from the mentioned drawbacks.

In this chapter, strongly based on [5], the authors provide step-by-step derivations of the Q-MEKF and of the DQ-MEKF that highlight all the similarities and analogies between the Q-MEKF and the DQ-MEKF. It is their hope that this chapter will serve as a guide to other authors on how to extend other useful 3-DOF estimation results into novel 6-DOF estimation results.

2.2 Mathematical Preliminaries

This section establishes the nomenclature and conventions adopted in this chapter for using quaternions and dual quaternions. It also reviews the EKF algorithm to establish notation. This review is based on a similar review provided in [10].

2.2.1 Quaternions

A quaternion is defined as $q = q_0 + q_1i + q_2j + q_3k$, where $q_0, q_1, q_2, q_3 \in \mathbb{R}$ and $i^2 = j^2 = k^2 = -1$, $i = jk = -kj$, $j = ki = -ik$, and $k = ij = -ji$. It can be represented as the ordered pair $q = (q_0, \bar{q})$, where $q_0 \in \mathbb{R}$ is the *scalar part* of the quaternion and $\bar{q} = [q_1 \ q_2 \ q_3]^\top \in \mathbb{R}^3$ is the *vector part* of the quaternion. *Vector quaternions* and *scalar quaternions* are quaternions with zero scalar part and zero vector part, respectively. The set of quaternions, vector quaternions, and scalar quaternions are denoted as $\mathbb{H} = \{q : q = q_0 + q_1i + q_2j + q_3k, q_0, q_1, q_2, q_3 \in \mathbb{R}\}$, $\mathbb{H}^v = \{q \in \mathbb{H} : q_0 = 0\}$, and $\mathbb{H}^s = \{q \in \mathbb{H} : q_1 = q_2 = q_3 = 0\}$, respectively.

The following operations are defined for quaternions:

$$\begin{aligned} \text{Addition: } a + b &= (a_0 + b_0, \bar{a} + \bar{b}) \in \mathbb{H}, \\ \text{Multiplication by a scalar: } \lambda a &= (\lambda a_0, \lambda \bar{a}) \in \mathbb{H}, \\ \text{Multiplication: } ab &= (a_0b_0 - \bar{a} \cdot \bar{b}, a_0\bar{b} + b_0\bar{a} + \bar{a} \times \bar{b}) \in \mathbb{H}, \\ \text{Conjugation: } a^* &= (a_0, -\bar{a}) \in \mathbb{H}, \\ \text{Dot product: } a \cdot b &= (a_0b_0 + \bar{a} \cdot \bar{b}, 0_{3 \times 1}) \in \mathbb{H}^s, \\ \text{Cross product: } a \times b &= (0, b_0\bar{a} + a_0\bar{b} + \bar{a} \times \bar{b}) \in \mathbb{H}^v, \end{aligned} \quad (2.1)$$

where $a, b \in \mathbb{H}$, $\lambda \in \mathbb{R}$, and $0_{m \times n}$ is a m -by- n matrix of zeros. Note that the quaternion multiplication is not commutative. In fact, Lefferts, Markley, and Shuster [10] define (2.1) as ba , and not as ab as originally defined by Hamilton [8]. This chapter follows the original convention by Hamilton. Finally, the quaternions $(1, 0_{3 \times 1})$ and $(0, 0_{3 \times 1})$ are denoted by $\mathbf{1}$ and $\mathbf{0}$, respectively.

The bijective mapping between the set of quaternions and $\mathbb{R}^4$ is denoted by $[\cdot] : \mathbb{H} \rightarrow \mathbb{R}^4$, where $[q] = [q_0 \ q_1 \ q_2 \ q_3]^\top$. Using this mapping, the cross product of $a \in \mathbb{H}^v$ with $b \in \mathbb{H}^v$ can be computed as $[a \times b] = [a]^\times [b]$, where $[\cdot]^\times : \mathbb{H}^v \rightarrow \mathbb{R}^{4 \times 4}$ is defined as

$$[a]^\times = \begin{bmatrix} 0 & 0_{1 \times 3} \\ 0_{3 \times 1} & \bar{a}^\times \end{bmatrix}, \quad \text{where} \quad \bar{a}^\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}. \quad (2.2)$$

Likewise, the left quaternion multiplication of $a \in \mathbb{H}$ with $b \in \mathbb{H}$ can be computed as $[ab] = \llbracket a \rrbracket [b]$, where $\llbracket \cdot \rrbracket : \mathbb{H} \rightarrow \mathbb{R}^{4 \times 4}$ is defined as

$$\llbracket a \rrbracket = \begin{bmatrix} a_0 & -\bar{a}^\top \\ \bar{a} & \tilde{a} \end{bmatrix}, \quad \text{where} \quad \tilde{a} = a_0 I_{3 \times 3} + \bar{a}^\times. \quad (2.3)$$

The relative orientation of a frame fixed to a body, the B-frame, with respect to another frame, the I-frame, can be represented by the *unit* quaternion $q_{B/I} = \left(\cos(\frac{\phi}{2}), \sin(\frac{\phi}{2})\bar{n} \right)$, where the B-frame is said to be rotated with respect to the I-frame about the unit vector $\bar{n}$ by an angle ϕ . The set of unit quaternions is defined as $\mathbb{H}^u = \{q \in \mathbb{H} : q \cdot q = qq^* = q^*q = \mathbf{1}\}$. From the constraint

associated with $\mathbb{H}^u$, assuming that $-180 < \phi < 180$ deg, the scalar part of a unit quaternion can be computed from

$$q_0 = \sqrt{1 - \|\bar{q}\|^2}, \quad (2.4)$$

where $\|\cdot\|$ denotes the usual Euclidean norm in $\mathbb{R}^3$.

The coordinates of a vector in the B-frame, $\bar{v}^B$, can be calculated from the coordinates of the same vector in the I-frame, $\bar{v}^I$, and vice-versa, via $v^B = q_{B/I}^* v^I q_{B/I}$ and $v^I = q_{B/I} v^B q_{B/I}^*$, where $v^B = (0, \bar{v}^B)$ and $v^I = (0, \bar{v}^I)$. This is equivalent to $\bar{v}^B = R(q_{B/I}) \bar{v}^I$ and $\bar{v}^I = R(q_{B/I}^*) \bar{v}^B$, where $R(q_{B/I})$ and $R(q_{B/I}^*)$ are the rotation matrices constructed from $q_{B/I}$ and $q_{B/I}^*$, respectively.

The rotational kinematic equations of the body frame with respect to another frame can be written as

$$\dot{q}_{B/I} = \frac{1}{2} \omega_{B/I}^I q_{B/I} = \frac{1}{2} q_{B/I} \omega_{B/I}^B, \quad (2.5)$$

where $\omega_{Y/Z}^X = (0, \bar{\omega}_{Y/Z}^X)$ and $\bar{\omega}_{Y/Z}^X$ is the angular velocity of the Y-frame with respect to the Z-frame expressed in the X-frame.

2.2.2 Dual Quaternions

A dual quaternion is defined as $\mathbf{q} = q_r + \epsilon q_d$, where ϵ is the *dual unit*. The dual unit satisfies $\epsilon^2 = 0$ and $\epsilon \neq 0$. The quaternions $q_r, q_d \in \mathbb{H}$ are the *real part* and the *dual part* of the dual quaternion, respectively. *Dual vector quaternions* and *dual scalar quaternions* are dual quaternions formed from vector quaternions (i.e., $q_r, q_d \in \mathbb{H}^v$) and scalar quaternions (i.e., $q_r, q_d \in \mathbb{H}^s$), respectively. The set of dual quaternions, dual scalar quaternions, and dual vector quaternions are denoted as $\mathbb{H}_d = \{\mathbf{q} : \mathbf{q} = q_r + \epsilon q_d, q_r, q_d \in \mathbb{H}\}$, $\mathbb{H}_d^s = \{\mathbf{q} : \mathbf{q} = q_r + \epsilon q_d, q_r, q_d \in \mathbb{H}^s\}$, and $\mathbb{H}_d^v = \{\mathbf{q} : \mathbf{q} = q_r + \epsilon q_d, q_r, q_d \in \mathbb{H}^v\}$, respectively.

The following operations are defined for dual quaternions:

$$\text{Addition: } \mathbf{a} + \mathbf{b} = (a_r + b_r) + \epsilon(a_d + b_d) \in \mathbb{H}_d,$$

$$\text{Multiplication by a scalar: } \lambda \mathbf{a} = (\lambda a_r) + \epsilon(\lambda a_d) \in \mathbb{H}_d,$$

$$\text{Multiplication: } \mathbf{ab} = (a_r b_r) + \epsilon(a_r b_d + a_d b_r) \in \mathbb{H}_d,$$

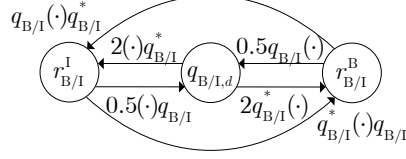
$$\text{Conjugation: } \mathbf{a}^* = a_r^* + \epsilon a_d^* \in \mathbb{H}_d,$$

$$\text{Dot product: } \mathbf{a} \cdot \mathbf{b} = a_r \cdot b_r + \epsilon(a_d \cdot b_r + a_r \cdot b_d) \in \mathbb{H}_d^s,$$

$$\text{Cross product: } \mathbf{a} \times \mathbf{b} = a_r \times b_r + \epsilon(a_d \times b_r + a_r \times b_d) \in \mathbb{H}_d^v,$$

where $\mathbf{a}, \mathbf{b} \in \mathbb{H}_d$ and $\lambda \in \mathbb{R}$. Note that the dual quaternion multiplication is not commutative. In this chapter, the dual quaternions $1 + \epsilon 0$ and $0 + \epsilon 0$ are denoted by $\mathbf{1}$ and $\mathbf{0}$, respectively.

The bijective mapping between the set of dual quaternions and $\mathbb{R}^8$ is denoted by $[\cdot] : \mathbb{H}_d \rightarrow \mathbb{R}^8$, where $[\mathbf{q}] = [[q_r]^\top [q_d]^\top]^\top$. Using this mapping, the left


FIGURE 2.1

 Relation between $r_{B/I}^B$, $q_{B/I,d}$, and $r_{B/I}^I$.

dual quaternion multiplication of $\mathbf{a} \in \mathbb{H}_d$ with $\mathbf{b} \in \mathbb{H}_d$ can be computed as $[\mathbf{ab}] = [\mathbf{a}][\mathbf{b}]$, where $[\cdot] : \mathbb{H}_d \rightarrow \mathbb{R}^{8 \times 8}$ is defined as

$$[\mathbf{a}] = \begin{bmatrix} [a_r] & 0_{4 \times 4} \\ [a_d] & [a_r] \end{bmatrix}. \quad (2.6)$$

It is convenient to define the vector part of a dual quaternion $\bar{\cdot} : \mathbb{H}_d \rightarrow \mathbb{R}^6$ as $\bar{\mathbf{a}} = [\bar{a}_r^\top \ \bar{a}_d^\top]^\top \in \mathbb{R}^6$ and $\bar{\cdot}^\times : \mathbb{H}_d \rightarrow \mathbb{R}^{6 \times 6}$ as

$$\bar{\mathbf{a}}^\times = \begin{bmatrix} \bar{a}_r^\times & 0_{3 \times 3} \\ \bar{a}_d^\times & \bar{a}_r^\times \end{bmatrix}. \quad (2.7)$$

Finally, it is also convenient to define $[\tilde{\cdot}] : \mathbb{H}_d \rightarrow \mathbb{R}^{6 \times 6}$ as

$$[\tilde{\mathbf{a}}] = \begin{bmatrix} [\tilde{a}_r] & 0_{3 \times 3} \\ [\tilde{a}_d] & [\tilde{a}_r] \end{bmatrix}, \quad (2.8)$$

$(\cdot)_r : \mathbb{H}_d \rightarrow \mathbb{H}$ as $(\mathbf{a})_r = a_r$, and $(\cdot)_d : \mathbb{H}_d \rightarrow \mathbb{H}$ as $(\mathbf{a})_d = a_d$.

The attitude and position (i.e., pose) of frame fixed to a body with respect to another frame can be represented by a *unit* quaternion $q_{B/I} \in \mathbb{H}^u$ and by a translation vector $\bar{r}_{B/I}^I \in \mathbb{R}^3$ or $\bar{r}_{B/I}^B \in \mathbb{R}^3$, where $\bar{r}_{Y/Z}^X$ is the translation vector from the origin of the Z-frame to the origin of the Y-frame expressed in the X-frame. Alternatively, the pose of the B-frame with respect to the I-frame can be represented more compactly by the *unit* dual quaternion

$$\mathbf{q}_{B/I} = q_{B/I,r} + \epsilon q_{B/I,d} = q_{B/I} + \epsilon \frac{1}{2} r_{B/I}^I q_{B/I} = q_{B/I} + \epsilon \frac{1}{2} q_{B/I} r_{B/I}^B, \quad (2.9)$$

where $r_{Y/Z}^X = (0, \bar{r}_{Y/Z}^X)$. Note that the dual part of $\mathbf{q}_{B/I}$, i.e., $q_{B/I,d}$, is a representation of the position of the B-frame with respect to the I-frame that is neither expressed in the B-frame nor in the I-frame. Given $\mathbf{q}_{B/I}$, the position of the B-frame with respect to the I-frame can be obtained in I-frame coordinates from $r_{B/I}^I = 2q_{B/I,d}q_{B/I}^*$ and in B-frame coordinates from $r_{B/I}^B = 2q_{B/I}^*q_{B/I,d}$. Figure 2.1 illustrates the relation between $r_{B/I}^B$, $q_{B/I,d}$, and $r_{B/I}^I$. Note that whereas the relation between $r_{B/I}^B$ and $r_{B/I}^I$ is quadratic in $q_{B/I}$, $q_{B/I,d}$ is related linearly in $q_{B/I}$ with $r_{B/I}^B$ and $r_{B/I}^I$.

The set of *unit* dual quaternions is defined as

$$\mathbb{H}_d^u = \{\mathbf{q} \in \mathbb{H}_d : \mathbf{q} \cdot \mathbf{q} = \mathbf{q} \mathbf{q}^* = \mathbf{q}^* \mathbf{q} = \mathbf{1}\} = \{\mathbf{q} \in \mathbb{H}_d : q_r \cdot q_r = 1 \text{ and } q_r \cdot q_d = 0\}. \quad (2.10)$$

From the two constraints associated with $\mathbb{H}_d^u$, assuming that $-180 < \phi < 180$ deg, the scalar parts of the real and dual parts of a unit dual quaternion can be computed from their respective vector parts from

$$q_{r,0} = \sqrt{1 - \|\bar{q}_r\|^2} \quad \text{and} \quad q_{d,0} = \frac{-\bar{q}_r^\top \bar{q}_d}{q_{r,0}}. \quad (2.11)$$

The rotational and translational kinematic equations of the body frame with respect to another frame can be written compactly in terms of dual quaternions as

$$\dot{\mathbf{q}}_{B/I} = \frac{1}{2} \boldsymbol{\omega}_{B/I}^I \mathbf{q}_{B/I} = \frac{1}{2} \mathbf{q}_{B/I} \boldsymbol{\omega}_{B/I}^B, \quad (2.12)$$

where $\boldsymbol{\omega}_{Y/Z}^X$ is the *dual velocity* of the Y-frame with respect to the Z-frame expressed in the X-frame, $\boldsymbol{\omega}_{B/I}^B \triangleq \boldsymbol{\omega}_{B/I}^B + \epsilon v_{B/I}^B$, $\boldsymbol{\omega}_{B/I}^I \triangleq \boldsymbol{\omega}_{B/I}^I + \epsilon(v_{B/I}^I - \boldsymbol{\omega}_{B/I}^I \times \mathbf{r}_{B/I}^I)$, $v_{Y/Z}^X = (0, \bar{v}_{Y/Z}^X)$, and $\bar{v}_{Y/Z}^X$ is the linear velocity of the origin of the Y-frame with respect to the Z-frame expressed in the X-frame. Note that the rotational and translational kinematic equations written in terms of dual quaternions, 2.12, have the same form as the rotational-only kinematic equations written in terms of quaternions, 2.5.

2.2.3 The Extended Kalman Filter

The state equation of the EKF is given by

$$\dot{x}_n(t) = f_n(x_n(t), t) + g_{n \times p}(x_n(t), t) w_p(t), \quad (2.13)$$

where $x_n(t) \in \mathbb{R}^n$ is the state and $w_p(t) \in \mathbb{R}^p$ is the process noise. The process noise is assumed to be a Gaussian white-noise process, whose mean and covariance are given by $E\{w_p(t)\} = 0_{p \times 1}$ and $E\{w_p(t)w_p^\top(\tau)\} = Q_{p \times p}(t)\delta(t - \tau)$, respectively, where $Q_{p \times p}(t) \in \mathbb{R}^{p \times p}$ is a symmetric positive semidefinite matrix. The initial mean and covariance of the state are given by $E\{x_n(t_0)\} \triangleq \hat{x}_n(t_0) = x_{n,0} \in \mathbb{R}^n$ and $E\{(x_n(t_0) - x_{n,0})(x_n(t_0) - x_{n,0})^\top\} \triangleq P_{n \times n}(t_0) = P_{n \times n,0} \in \mathbb{R}^{n \times n}$ and are assumed to be known.

2.2.3.1 Time Update

Given the initial mean of the state, the minimum covariance estimate of the state at a future time t in the absence of measurements is given by the conditional expectation $\hat{x}_n(t) = E\{x_n(t) | \hat{x}_n(t_0) = x_{n,0}\}$. This estimate satisfies the differential equation $\dot{\hat{x}}_n(t) = E\{f_n(x_n(t), t)\}$, which is typically approximated by

$$\dot{\hat{x}}_n(t) \approx f_n(\hat{x}_n(t), t). \quad (2.14)$$

In addition to the state estimate, also the covariance matrix of the state needs to be propagated. The covariance matrix of the state is given by $P_{n \times n}(t) = E\{\Delta x_n(t) \Delta x_n^\top(t)\} \in \mathbb{R}^{n \times n}$, where $\Delta x_n(t) = x_n(t) - \hat{x}_n(t) \in \mathbb{R}^n$

is the state error. As a first-order approximation, the derivative of the state error is given by

$$\frac{d}{dt}\Delta x_n(t) = F_{n \times n}(t)\Delta x_n(t) + G_{n \times p}(t)w_p(t), \quad (2.15)$$

and the covariance matrix of the state satisfies the Riccati equation

$$\begin{aligned} \dot{P}_{n \times n}(t) = & F_{n \times n}(t)P_{n \times n}(t) + P_{n \times n}(t)F_{n \times n}^\top(t) \\ & + G_{n \times p}(t)Q_{p \times p}(t)G_{n \times p}^\top(t), \end{aligned} \quad (2.16)$$

where

$$F_{n \times n}(t) \triangleq \left. \frac{\partial f_n(x_n, t)}{\partial x_n} \right|_{\hat{x}_n(t)} \quad \text{and} \quad G_{n \times p}(t) \triangleq g_{n \times p}(\hat{x}_n(t), t). \quad (2.17)$$

2.2.3.2 Measurement Update

Assume that a measurement is taken at time t_k that is related with the state of the EKF through the nonlinear output equation

$$z_m(t_k) = h_m(x_n(t_k)) + v_m(t_k) \in \mathbb{R}^m, \quad (2.18)$$

where $v_m(t_k) \in \mathbb{R}^m$ is the measurement noise assumed to be a discrete Gaussian white-noise process, whose mean and covariance are given by $E\{v_m(t_k)\} = 0_{m \times 1}$ and $E\{v_m(t_k)v_m^\top(t_\ell)\} = R_{m \times m}(t_k)\delta_{t_k t_\ell}$, where $R_{m \times m}(t_k) \in \mathbb{R}^{m \times m}$ is a symmetric positive definite matrix.

Immediately following the measurement at time t_k , the minimum variance estimate of $x_n(t_k)$ is given by

$$\hat{x}_n^+(t_k) = \hat{x}_n^-(t_k) + \Delta^* \hat{x}_n(t_k), \quad (2.19)$$

where

$$\Delta^* \hat{x}_n(t_k) = K_{n \times m}(t_k)[z_m(t_k) - \hat{z}_m(t_k)] \quad (2.20)$$

is the optimal Kalman state update, $\hat{z}_m(t_k) = E\{z_m(t_k)\} \approx h_m(\hat{x}_n^-(t_k))$, $\hat{x}_n^-(t_k)$ and $\hat{x}_n^+(t_k)$ are the predicted values of the state immediately before and after the measurement, and $K_{n \times m}(t_k)$ is the Kalman gain. The Kalman gain is given by

$$\begin{aligned} K_{n \times m}(t_k) = & P_{n \times n}^-(t_k)H_{m \times n}^\top(t_k)[H_{m \times n}(t_k)P_{n \times n}^-(t_k)H_{m \times n}^\top(t_k) \\ & + R_{m \times m}(t_k)]^{-1}, \end{aligned} \quad (2.21)$$

where $P_{n \times n}^-(t_k)$ is the predicted state covariance matrix immediately before the measurement and

$$H_{m \times n}(t_k) = \left. \frac{\partial h_m(x_n)}{\partial x_n} \right|_{\hat{x}_n^-(t_k)} \quad (2.22)$$

is the measurement sensitivity matrix.

Immediately after the measurement, the state covariance matrix is given by

$$P_{n \times n}^+(t_k) = (I_{n \times n} - K_{n \times m}(t_k)H_{m \times n}(t_k))P_{n \times n}^-(t_k). \quad (2.23)$$

2.3 Quaternion Multiplicative Extended Kalman Filter (Q-MEKF) for Spacecraft Attitude Estimation

In this section, the Q-MEKF is re-derived step-by-step using quaternion arithmetic, instead of matrix arithmetic as in [10]. This will hopefully make the similarities and analogies between the Q-MEKF and the DQ-MEKF much more evident when deriving the DQ-MEKF in Section 2.4.

2.3.1 Angular Velocity Measurement Model

The angular velocity measurement model [3, 10] is defined as

$$\omega_{B/I,m}^B = \omega_{B/I}^B + b_\omega + \eta_\omega, \quad (2.24)$$

where $\omega_{B/I,m}^B = (0, \bar{\omega}_{B/I,m}^B)$, $\bar{\omega}_{B/I,m}^B$ is a measurement of the angular velocity of the B-frame with respect to the I-frame expressed in the B-frame, $b_\omega = (0, \bar{b}_\omega)$, $\bar{b}_\omega \in \mathbb{R}^3$ is the bias of the angular velocity measurement, $\eta_\omega = (0, \bar{\eta}_\omega)$, $\bar{\eta}_\omega \in \mathbb{R}^3$ is the noise of the angular velocity measurement assumed to be a zero-mean Gaussian white noise process with $E\{\bar{\eta}_\omega\} = 0_{3 \times 1}$ and covariance $E\{\bar{\eta}_\omega(t)\bar{\eta}_\omega^T(\tau)\} = \bar{Q}_\omega(t)\delta(t - \tau)$, and $\bar{Q}_\omega(t) \in \mathbb{R}^{3 \times 3}$ is a symmetric positive semidefinite matrix. The bias is not constant, but assumed to be driven by another zero-mean Gaussian white noise process through

$$\dot{b}_\omega = \eta_{b_\omega}, \quad (2.25)$$

where $\eta_{b_\omega} = (0, \bar{\eta}_{b_\omega})$, $E\{\bar{\eta}_{b_\omega}\} = 0_{3 \times 1}$, $E\{\bar{\eta}_{b_\omega}(t)\bar{\eta}_{b_\omega}^T(\tau)\} = \bar{Q}_{b_\omega}(t)\delta(t - \tau)$, and $\bar{Q}_{b_\omega}(t) \in \mathbb{R}^{3 \times 3}$ is a symmetric positive semidefinite matrix.

2.3.2 Time Update

The state and process noise of the Q-MEKF are selected as

$$x_6 = \begin{bmatrix} \overline{\delta q_{B/I}} \\ \bar{b}_\omega \end{bmatrix} \quad \text{and} \quad w_6 = \begin{bmatrix} \bar{\eta}_\omega \\ \bar{\eta}_{b_\omega} \end{bmatrix}, \quad (2.26)$$

where $\overline{\delta q_{B/I}}$ is the vector part of the error quaternion $\delta q_{B/I} \in \mathbb{H}^u$ defined as [10]

$$\delta q_{B/I} = \hat{q}_{B/I}^* q_{B/I} \in \mathbb{H}^u, \quad (2.27)$$

i.e., $\delta q_{B/I} \in \mathbb{H}^u$ is the quaternion between the true quaternion $q_{B/I} \in \mathbb{H}^u$ and its current best guess $\hat{q}_{B/I} \in \mathbb{H}^u$. The latter is propagated using [11]

$$\frac{d}{dt}(\hat{q}_{B/I}) \approx \frac{1}{2}\hat{q}_{B/I}\hat{\omega}_{B/I}^B, \quad (2.28)$$

where, from (2.24),

$$\hat{\omega}_{B/I}^B \triangleq E\{\omega_{B/I}^B\} = E\{\omega_{B/I,m}^B - b_\omega - \eta_\omega\} = \omega_{B/I,m}^B - \hat{b}_\omega, \quad (2.29)$$

with $\hat{b}_\omega \triangleq E\{b_\omega\}$ and

$$\frac{d}{dt}(\hat{b}_\omega) = E\{\eta_{b_\omega}\} = 0. \quad (2.30)$$

For $-180 < \phi < 180$ deg, $\delta q_{B/I}$ is parameterized by $\overline{\delta q_{B/I}}$ [11] and the expected value of $\overline{\delta q_{B/I}}$ is required to be zero, i.e., $E\{\overline{\delta q_{B/I}}\} = 0_{3 \times 1}$. Hence, $E\{\delta q_{B/I}(\overline{\delta q_{B/I}})\} = 1$. Note that the current best guess of $q_{B/I}$, given by $\hat{q}_{B/I}$, is not defined as the standard expected value of the random variable $q_{B/I}$ as this would require the expectation to be defined with respect to a non-trivial probability density function in $\mathbb{H}^u$. As shown in [12], the definition of probability density function on $\mathbb{H}^u$ is not trivial.

To determine the state equation of the Q-MEKF, the time derivative of $\delta q_{B/I}$ needs to be calculated. Taking the time derivative of (2.27) yields

$$\frac{d}{dt}(\delta q_{B/I}) = \frac{d}{dt}(\hat{q}_{B/I}^*)q_{B/I} + \hat{q}_{B/I}^* \frac{d}{dt}(q_{B/I}). \quad (2.31)$$

Substituting (2.5) and (2.28) in (2.31) yields

$$\frac{d}{dt}(\delta q_{B/I}) \approx \frac{1}{2}(\hat{\omega}_{B/I}^B)^* \hat{q}_{B/I}^* q_{B/I} + \frac{1}{2}\hat{q}_{B/I}^* q_{B/I} \omega_{B/I}^B = -\frac{1}{2}\hat{\omega}_{B/I}^B \delta q_{B/I} + \frac{1}{2}\delta q_{B/I} \omega_{B/I}^B. \quad (2.32)$$

Combining (2.29) and (2.24) yields

$$\omega_{B/I}^B = \hat{\omega}_{B/I}^B + \hat{b}_\omega - b_\omega - \eta_\omega. \quad (2.33)$$

Finally, inserting (2.33) in (2.32) results in

$$\frac{d}{dt}(\delta q_{B/I}) \approx -\frac{1}{2}\hat{\omega}_{B/I}^B \delta q_{B/I} + \frac{1}{2}\delta q_{B/I} \hat{\omega}_{B/I}^B + \frac{1}{2}\delta q_{B/I} \hat{b}_\omega - \frac{1}{2}\delta q_{B/I} b_\omega - \frac{1}{2}\delta q_{B/I} \eta_\omega. \quad (2.34)$$

Thus, the state equations of the Q-MEKF are given by the vector parts of (2.34) and (2.25), i.e.,

$$f_6(x_6(t), t) = \begin{bmatrix} -\frac{1}{2}\hat{\omega}_{B/I}^B \delta q_{B/I} + \frac{1}{2}\delta q_{B/I} \hat{\omega}_{B/I}^B + \frac{1}{2}\delta q_{B/I} \hat{b}_\omega - \frac{1}{2}\delta q_{B/I} b_\omega \\ 0_{3 \times 1} \end{bmatrix}, \quad (2.35)$$

$$g_{6 \times 6}(x_6(t), t) = \begin{bmatrix} -\frac{1}{2}[\widehat{\delta q_{B/I}}] & 0_{3 \times 3} \\ 0_{3 \times 3} & I_{3 \times 3} \end{bmatrix}. \quad (2.36)$$

By replacing the scalar part of $\delta q_{B/I}$ through (2.4) in (2.35) and (2.36) and using (2.17), $F_{6 \times 6}(t)$ and $G_{6 \times 6}(t)$ can be determined to be

$$F_{6 \times 6}(t) = \begin{bmatrix} -\overline{\hat{\omega}_{B/I}^B}^\times & -\frac{1}{2}I_{3 \times 3} \\ 0_{3 \times 3} & 0_{3 \times 3} \end{bmatrix} \text{ and } G_{3 \times 3}(t) = \begin{bmatrix} -\frac{1}{2}I_{3 \times 3} & 0_{3 \times 3} \\ 0_{3 \times 3} & I_{3 \times 3} \end{bmatrix}. \quad (2.37)$$

Hence, for the time update of the Q-MEKF, $\hat{q}_{B/I}$, $\hat{\omega}_{B/I}^B$, and $\hat{b}_\omega$ are propagated using (2.28), (2.29), and (2.30), respectively, given $\hat{q}_{B/I}(t_0)$ and $\hat{b}_\omega(t_0)$. As for the covariance matrix of x_6 , it is propagated according to (2.16) given $P_{6 \times 6}(t_0)$ and where

$$Q_{6 \times 6}(t) = \begin{bmatrix} \overline{Q}_\omega(t) & 0_{3 \times 3} \\ 0_{3 \times 3} & \overline{Q}_{b_\omega}(t) \end{bmatrix}. \quad (2.38)$$

Numerical errors in the propagation of $\hat{q}_{B/I}$ through (2.28) can result in the violation of the algebraic constraint associated with $\mathbb{H}^u$. Hence, after each integration step, this algebraic constraint is enforced by calculating

$$[q_{B/I}] = \frac{[q_{B/I}]}{\|[q_{B/I}]\|}. \quad (2.39)$$

2.3.3 Measurement Update

It is assumed that a direct measurement of $q_{B/I}$ is available, e.g., as an output of a star tracker. Similarly to [9, 11], the output equation is defined as

$$(\hat{q}_{B/I}^-(t_k))^* q_{B/I,m}(t_k) = \overline{\delta q_{B/I}(t_k)} + v_3(t_k). \quad (2.40)$$

Using (2.22) to calculate the measurement sensitivity matrix yields

$$H_{3 \times 6}(t_k) = [I_{3 \times 3} \quad 0_{3 \times 3}]. \quad (2.41)$$

Hence, for the measurement update of the Q-MEKF, the Kalman gain is calculated from (2.21), whereas the optimal Kalman state update is calculated from (2.20) as

$$\Delta^* \hat{x}_6(t_k) \triangleq \begin{bmatrix} \Delta^* \overline{\delta \hat{q}_{B/I}(t_k)} \\ \Delta^* \hat{b}_\omega(t_k) \end{bmatrix} = K_{6 \times 3} \overline{(\hat{q}_{B/I}^-(t_k))^* q_{B/I,m}(t_k)}. \quad (2.42)$$

The estimate of the state at time t_k after the measurement is then calculated from

$$\hat{q}_{B/I}^+(t_k) = \hat{q}_{B/I}^-(t_k) \Delta^* \delta \hat{q}_{B/I}(t_k), \quad (2.43)$$

$$\bar{b}_\omega^+(t_k) = \bar{b}_\omega^-(t_k) + \Delta^* \bar{b}_\omega(t_k), \quad (2.44)$$

where $\Delta^* \delta \hat{q}_{B/I}$ is defined as the unit dual quaternion

$$\Delta^* \delta \hat{q}_{B/I} = \left(\sqrt{1 - \|\Delta^* \overline{\delta \hat{q}_{B/I}}\|^2}, \Delta^* \overline{\delta \hat{q}_{B/I}} \right). \quad (2.45)$$

Finally, the covariance matrix of the state immediately after the measurement at t_k is computed from (2.23).

2.4 Dual Quaternion Multiplicative Extended Kalman Filter (DQ-MEKF) for Spacecraft Pose Estimation

The DQ-MEKF is derived in this section following the same steps used in Section 2.3 to derive the Q-MEKF. Similarities and analogies between the derivation of the Q-MEKF and of the DQ-MEKF are highlighted.

2.4.1 Dual Velocity Measurement Model

The dual velocity measurement model is defined analogously to the angular velocity measurement (2.24) as

$$\boldsymbol{\omega}_{B/I,m}^B = \boldsymbol{\omega}_{B/I}^B + \mathbf{b}_\omega + \boldsymbol{\eta}_\omega, \quad (2.46)$$

where $\boldsymbol{\omega}_{B/I,m}^B = \boldsymbol{\omega}_{B/I}^B + \epsilon v_{B/I,m}^B \in \mathbb{H}_d^v$, $v_{B/I,m}^B = (0, \bar{v}_{B/I,m}^B)$, $\bar{v}_{B/I,m}^B$ is a measurement of the linear velocity of the origin of the B-frame with respect to the I-frame expressed in the B-frame, $\mathbf{b}_\omega = b_\omega + \epsilon b_v$ is the dual bias, $b_v = (0, \bar{b}_v)$, $\bar{b}_v \in \mathbb{R}^3$ is the bias of the linear velocity measurement, $\boldsymbol{\eta}_\omega = \eta_\omega + \epsilon \eta_v$, $\eta_v = (0, \bar{\eta}_v)$, $\bar{\eta}_v \in \mathbb{R}^3$ is the noise of the linear velocity measurement assumed to be a Gaussian white-noise process with $E\{\bar{\eta}_\omega\} = 0_{6 \times 1}$ and covariance

$$E\{\bar{\eta}_\omega(t)\bar{\eta}_\omega^\top(\tau)\} = \bar{Q}_\omega(t)\delta(t-\tau) = \begin{bmatrix} \bar{Q}_\omega(t) & \bar{Q}_{\omega v}(t) \\ \bar{Q}_{\omega v}(t) & \bar{Q}_v(t) \end{bmatrix} \delta(t-\tau), \quad (2.47)$$

and $\bar{Q}_\omega(t) \in \mathbb{R}^{6 \times 6}$ is a symmetric positive semidefinite matrix. Analogously to (2.25), the dual bias is assumed to be driven by another zero-mean Gaussian white noise process through

$$\dot{\mathbf{b}}_\omega = \boldsymbol{\eta}_{b_\omega}, \quad (2.48)$$

where $\boldsymbol{\eta}_{b_\omega} = (0, \bar{\eta}_{b_\omega}) + \epsilon(0, \bar{\eta}_{b_v})$, $E\{\bar{\eta}_{b_\omega}\} = 0_{6 \times 1}$,

$$E\{\bar{\eta}_{b_\omega}(t)\bar{\eta}_{b_\omega}^\top(\tau)\} = \bar{Q}_{b_\omega}(t)\delta(t-\tau) = \begin{bmatrix} \bar{Q}_{b_\omega}(t) & \bar{Q}_{b_\omega b_v}(t) \\ \bar{Q}_{b_\omega b_v}(t) & \bar{Q}_{b_v}(t) \end{bmatrix} \delta(t-\tau), \quad (2.49)$$

and $\bar{Q}_{b_\omega}(t) \in \mathbb{R}^{6 \times 6}$ is a symmetric positive semidefinite matrix.

If the I-frame is inertial, $\boldsymbol{\omega}_{B/I}^B$ should be interpreted as the inertial angular and linear velocities of the satellite expressed in the B-frame. In that case, $\boldsymbol{\omega}_{B/I}^B$ can be measured using a combination of, say, rate-gyros, Doppler radar, and GPS. On the other hand, if the I-frame is not inertial, $\boldsymbol{\omega}_{B/I}^B$ should be interpreted as the relative angular and linear velocities of the satellite with respect to a frame attached, for example, to another satellite. In that case, $\boldsymbol{\omega}_{B/I}^B$ can be measured using a combination of, say, rate-gyros on both satellites [9], Doppler radar, differential GPS, and LIDAR.

2.4.2 Time Update

Analogously to the states of the Q-MEKF, the state and process noise of the DQ-MEKF are selected as

$$x_{12} = \begin{bmatrix} \delta \mathbf{q}_{B/I} \\ \hat{\mathbf{b}}_{\omega} \end{bmatrix} \quad \text{and} \quad w_{12} = \begin{bmatrix} \bar{\boldsymbol{\eta}}_{\omega} \\ \bar{\boldsymbol{\eta}}_{b_{\omega}} \end{bmatrix}, \quad (2.50)$$

where the dual error quaternion $\delta \mathbf{q}_{B/I} \in \mathbb{H}_d^u$ is defined analogously to the error quaternion (2.27) as

$$\delta \mathbf{q}_{B/I} = \hat{\mathbf{q}}_{B/I}^* \mathbf{q}_{B/I} \in \mathbb{H}_d^u, \quad (2.51)$$

i.e., $\delta \mathbf{q}_{B/I} \in \mathbb{H}_d^u$ is the dual quaternion between the actual dual quaternion $\mathbf{q}_{B/I} \in \mathbb{H}_d^u$ and its current best guess $\hat{\mathbf{q}}_{B/I} \in \mathbb{H}_d^u$. Analogously to the propagation of $\hat{\mathbf{q}}_{B/I} \in \mathbb{H}^u$ (2.28), $\hat{\mathbf{q}}_{B/I}$ is propagated using

$$\frac{d}{dt}(\hat{\mathbf{q}}_{B/I}) \approx \frac{1}{2} \hat{\mathbf{q}}_{B/I} \hat{\boldsymbol{\omega}}_{B/I}^B, \quad (2.52)$$

where, from Eq. (2.46),

$$\hat{\boldsymbol{\omega}}_{B/I}^B \triangleq E \{ \boldsymbol{\omega}_{B/I}^B \} = E \{ \boldsymbol{\omega}_{B/I,m}^B - \mathbf{b}_{\omega} - \boldsymbol{\eta}_{\omega} \} = \boldsymbol{\omega}_{B/I,m}^B - \hat{\mathbf{b}}_{\omega}, \quad (2.53)$$

with $\hat{\mathbf{b}}_{\omega} \triangleq E \{ \mathbf{b}_{\omega} \}$ and

$$\frac{d}{dt}(\hat{\mathbf{b}}_{\omega}) = E \{ \boldsymbol{\eta}_{b_{\omega}} \} = \mathbf{0}. \quad (2.54)$$

Analogously to the Q-MEKF, for $-180 < \phi < 180$ deg, $\delta \mathbf{q}_{B/I}$ is parameterized by $\overline{\delta \mathbf{q}_{B/I}}$ and the expected value of $\overline{\delta \mathbf{q}_{B/I}}$ is required to be zero, i.e., $E \{ \overline{\delta \mathbf{q}_{B/I}} \} = \mathbf{0}_{6 \times 1}$. Hence, $E \{ \delta \mathbf{q}_{B/I} (\overline{\delta \mathbf{q}_{B/I}}) \} = \mathbf{1}$. As $\hat{\mathbf{q}}_{B/I}$ is not defined as the standard expected value of the random variable $\mathbf{q}_{B/I}$, $\hat{\mathbf{q}}_{B/I}$ is also not defined as the standard expected value of the random variable $\mathbf{q}_{B/I}$, since this would require the expectation to be defined with respect to a non-trivial probability density function in $\mathbb{H}_d^u$.

The calculation of the time derivative of $\delta \mathbf{q}_{B/I}$ is analogous to the calculation of the time derivative of $\delta \mathbf{q}_{B/I}$ (2.34) and yields

$$\delta \dot{\mathbf{q}}_{B/I} \approx -\frac{1}{2} \hat{\boldsymbol{\omega}}_{B/I}^B \delta \mathbf{q}_{B/I} + \frac{1}{2} \delta \mathbf{q}_{B/I} \hat{\boldsymbol{\omega}}_{B/I}^B + \frac{1}{2} \delta \mathbf{q}_{B/I} \hat{\mathbf{b}}_{\omega} - \frac{1}{2} \delta \mathbf{q}_{B/I} \mathbf{b}_{\omega} - \frac{1}{2} \delta \mathbf{q}_{B/I} \boldsymbol{\eta}_{\omega}. \quad (2.55)$$

Thus, the state equations of the DQ-MEKF are given by the vector parts of (2.55) and (2.48), i.e.,

$$f_{12}(x_{12}(t), t) = \begin{bmatrix} -\frac{1}{2} \hat{\boldsymbol{\omega}}_{B/I}^B \delta \mathbf{q}_{B/I} + \frac{1}{2} \delta \mathbf{q}_{B/I} \hat{\boldsymbol{\omega}}_{B/I}^B + \frac{1}{2} \delta \mathbf{q}_{B/I} \hat{\mathbf{b}}_{\omega} - \frac{1}{2} \delta \mathbf{q}_{B/I} \mathbf{b}_{\omega} \\ \mathbf{0}_{6 \times 1} \end{bmatrix}, \quad (2.56)$$

$$g_{12 \times 12}(x_{12}(t), t) = \begin{bmatrix} -\frac{1}{2} [\widetilde{\delta \mathbf{q}_{B/I}}] & \mathbf{0}_{6 \times 6} \\ \mathbf{0}_{6 \times 6} & I_{6 \times 6} \end{bmatrix}. \quad (2.57)$$

By replacing the scalar parts of $\delta q_{B/I,r}$ and $\delta q_{B/I,d}$ through (2.11) in (2.56) and (2.57) and using (2.17), $F_{12 \times 12}(t)$ and $G_{12 \times 12}(t)$ can be determined to be

$$F_{12 \times 12}(t) = \begin{bmatrix} -\hat{\omega}_{B/I}^{\times} & -\frac{1}{2}I_{6 \times 6} \\ 0_{6 \times 6} & 0_{6 \times 6} \end{bmatrix} \text{ and } G_{12 \times 12}(t) = \begin{bmatrix} -\frac{1}{2}I_{6 \times 6} & 0_{6 \times 6} \\ 0_{6 \times 6} & I_{6 \times 6} \end{bmatrix}. \quad (2.58)$$

Hence, for the time update of the DQ-MEKF, $\hat{q}_{B/I}$, $\hat{\omega}_{B/I}^B$, and $\hat{b}_{\omega}$ are propagated using (2.52), (2.53), and (2.54), respectively, given $\hat{q}_{B/I}(t_0)$ and $\hat{b}_{\omega}(t_0)$. As for the covariance matrix of x_{12} , it is propagated according to (2.16) given $P_{12 \times 12}(t_0)$ and where

$$Q_{12 \times 12}(t) = \begin{bmatrix} \overline{Q}_{\omega}(t) & 0_{6 \times 6} \\ 0_{6 \times 6} & \overline{Q}_{b_{\omega}}(t) \end{bmatrix}. \quad (2.59)$$

Note that the matrices (2.58) and (2.59) have the same form as (2.37) and (2.38), respectively.

Analogously to the Q-MEKF, numerical errors in the propagation of $\hat{q}_{B/I}$ through (2.52) can result in the violation of the algebraic constraints associated with $\mathbb{H}_q^u$. Hence, after each integration step, these algebraic constraints are enforced by calculating

$$[q_{B/I,r}] = \frac{[q_{B/I,r}]}{\|[q_{B/I,r}]\|} \quad \text{and} \quad [q_{B/I,d}] = \left(I_{4 \times 4} - \frac{[q_{B/I,r}][q_{B/I,r}]^{\top}}{\|[q_{B/I,r}]\|^2} \right) [q_{B/I,d}], \quad (2.60)$$

where the latter equation corresponds to the projection of $[q_{B/I,d}]$ onto the subspace orthogonal to $[q_{B/I,r}]$.

2.4.3 Measurement Update

Analogously to the Q-MEKF, it is assumed that a direct measurement of $q_{B/I}$ is available. If the I-frame is a moving frame, this measurement can come, for example, from a vision-based system. If the I-frame is an inertial frame, this measurement can come, for example, from a combination of a star tracker and a GPS. If the pose measurement is available in terms of a quaternion and a translation vector, then the corresponding dual quaternion can be computed from (2.9). Then, the output equation is defined analogously to (2.40) as

$$\overline{(\hat{q}_{B/I}^-(t_k))^* q_{B/I,m}(t_k)} = \overline{\delta q_{B/I}(t_k)} + v_6(t_k). \quad (2.61)$$

Using (2.22) to calculate the measurement sensitivity matrix yields

$$H_{6 \times 12}(t_k) = \begin{bmatrix} I_{6 \times 6} & 0_{6 \times 6} \end{bmatrix}, \quad (2.62)$$

which is analogous to (2.41).

Hence, for the measurement update of the DQ-MEKF, the Kalman gain is

calculated from (2.21), whereas the optimal Kalman state update is calculated from (2.20) as

$$\Delta^* \hat{x}_{12}(t_k) \triangleq \begin{bmatrix} \Delta^* \overline{\delta \hat{q}_{B/I}}(t_k) \\ \Delta^* \overline{\hat{b}_\omega}(t_k) \end{bmatrix} = K_{12 \times 6} \overline{(\hat{q}_{B/I}^-(t_k))^* q_{B/I,m}(t_k)}. \quad (2.63)$$

The estimate of the state at time t_k after the measurement is then calculated from

$$\hat{q}_{B/I}^+(t_k) = \hat{q}_{B/I}^-(t_k) \Delta^* \delta \hat{q}_{B/I}(t_k), \quad (2.64)$$

$$\hat{b}_\omega^+(t_k) = \hat{b}_\omega^-(t_k) + \Delta^* \hat{b}_\omega(t_k), \quad (2.65)$$

where $\Delta^* \delta \hat{q}_{B/I}$ is defined as the unit dual quaternion

$$(\sqrt{1 - \|\Delta^* \overline{\delta \hat{q}_{B/I,r}}\|^2}, \Delta^* \overline{\delta \hat{q}_{B/I,r}}) + \epsilon \left(\frac{-\Delta^* \overline{\delta \hat{q}_{B/I,r}}^\top \Delta^* \overline{\delta \hat{q}_{B/I,d}}}{\sqrt{1 - \|\Delta^* \overline{\delta \hat{q}_{B/I,r}}\|^2}}, \Delta^* \overline{\delta \hat{q}_{B/I,d}} \right). \quad (2.66)$$

Finally, the covariance matrix of the state immediately after the measurement at t_k is computed from (2.23).

Any measurement that is a nonlinear function of the state of the DQ-MEKF, i.e., any measurement that satisfies (2.18), can be used in the measurement update. If another measurement is used, only the measurement sensitivity matrix (2.62) needs to be recalculated. For example, if the measurements are $q_{B/I,m}$ and $r_{B/I,m}^I$, then the output equation is defined as

$$\begin{bmatrix} \overline{(\hat{q}_{B/I}^-(t_k))^* q_{B/I,m}(t_k)} \\ r_{B/I,m}^I(t_k) \end{bmatrix} = \begin{bmatrix} \overline{\delta q_{B/I,r}(t_k)} \\ 2 \left(\hat{q}_{B/I} \delta q_{B/I} \right)_d \delta q_{B/I}^* \hat{q}_{B/I} \end{bmatrix} + v_6(t_k), \quad (2.67)$$

since $r_{B/I}^I = 2q_{B/I,d}q_{B/I}^* = 2 \left(\hat{q}_{B/I} \delta q_{B/I} \right)_d \delta q_{B/I}^* \hat{q}_{B/I}$. By replacing the scalar parts of $\delta q_{B/I,r}$ and $\delta q_{B/I,d}$ through (2.11) in (2.67), the new sensitivity matrix can be determined to be

$$H_{6 \times 12}(t_k) = \begin{bmatrix} I_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} \\ 0_{3 \times 3} & 2R((\hat{q}_{B/I}^-)^*) & 0_{3 \times 3} & 0_{3 \times 3} \end{bmatrix}. \quad (2.68)$$

2.4.4 Special Case: No Velocity Measurements

A special case of particular interest is when pose measurements are available, but velocity measurements are not. Although velocity measurements are not available, angular and linear velocity estimates might be required for pose stabilization or tracking [4]. It is shown that this case can be handled by modifying the inputs and parameters of the DQ-MEKF algorithm, but not its structure or equations. This version of the DQ-MEKF is specially suited for satellite proximity operations where the relative pose is measured using vision-based systems, which typically do not provide direct relative velocity

measurements. In this scenario, the I-frame is a moving frame attached to the target satellite.

If velocity measurements are not available, but estimates are required, $\omega_{B/I,m}^B$ and η_ω are set to zero in (2.46). This results in

$$\mathbf{b}_\omega = -\omega_{B/I}^B \quad (2.69)$$

and $\overline{Q}_\omega = 0_{6 \times 6}$. The dual velocity estimate is still given by (2.53), which now has the form $\hat{\omega}_{B/I}^B = -\hat{\mathbf{b}}_\omega$, and the time derivative of $\mathbf{b}_\omega$ is still calculated as in (2.48). However, since $\mathbf{b}_\omega$ is now expected to be time-varying and not constant, the effect of the noise η_{b_ω} might have to be increased by changing $\overline{Q}_{b_\omega}$.

In summary, this special case can be handled by just setting $\omega_{B/I,m}^B$ and $\overline{Q}_\omega$ to zero and, if necessary, by changing $\overline{Q}_{b_\omega}$.

2.5 Experimental Results

The DQ-MEKF without velocity measurements is validated experimentally on the Autonomous Spacecraft Testing of Robotic Operations in Space (ASTROS) facility at the School of Aerospace Engineering of Georgia Tech [13]. This experimental facility includes a 5-DOF platform supported on hemispherical and linear air-bearings moving over a flat epoxy floor in order to simulate as best as possible the frictionless environment of space. It also includes a VICON motion capture system mounted on an aluminum grid above the experimental area. The VICON system measures the pose of the platform with respect to a reference frame fixed to the room. The standard deviations of $q_{B/I,m}$ and $\hat{r}_{B/I,m}^1$ are smaller than 7×10^{-5} and 1 mm, respectively.

The ground truth for the pose was obtained from VICON measurements at 100 Hz. The ground truth for the linear velocity was obtained by passing the position measurements through a Linear Time-Invariant (LTI) system with transfer matrix $H(s) = \frac{3s}{s+3} I_{3 \times 3}$. The ground truth for the angular velocity was obtained by passing the quaternion measurements through a LTI system with transfer matrix $H(s) = \frac{3s}{s+3} I_{4 \times 4}$ and by using the relation $\omega_{B/I}^B = 2q_{B/I}^* \dot{q}_{B/I}$.

The DQ-MEKF was fed pose measurements from the VICON system at 0.5 Hz modeled through output equation (2.67). The DQ-MEKF was initialized with the covariance matrices: $P_{12 \times 12}(0) = 1 \times 10^{-9} I_{12 \times 12}$, $Q_{12 \times 12} = \text{diag}([0, 0, 0, 0, 0, 0, 1 \times 10^{-3}, 1 \times 10^{-3}, 1 \times 10^{-3}, 1 \times 10^{-1}, 1 \times 10^{-1}, 1 \times 10^{-1}])$, and $R_{6 \times 6} = \text{diag}([1.4 \times 10^{-6}, 1.4 \times 10^{-6}, 1.4 \times 10^{-6}, 2.25 \times 10^{-6}, 2.25 \times 10^{-6}, 2.25 \times 10^{-6}])$.

The pose estimated by the DQ-MEKF is compared with the ground truth in Fig. 2.2. Note that the motion only starts around 20 sec after the beginning of the experiment.

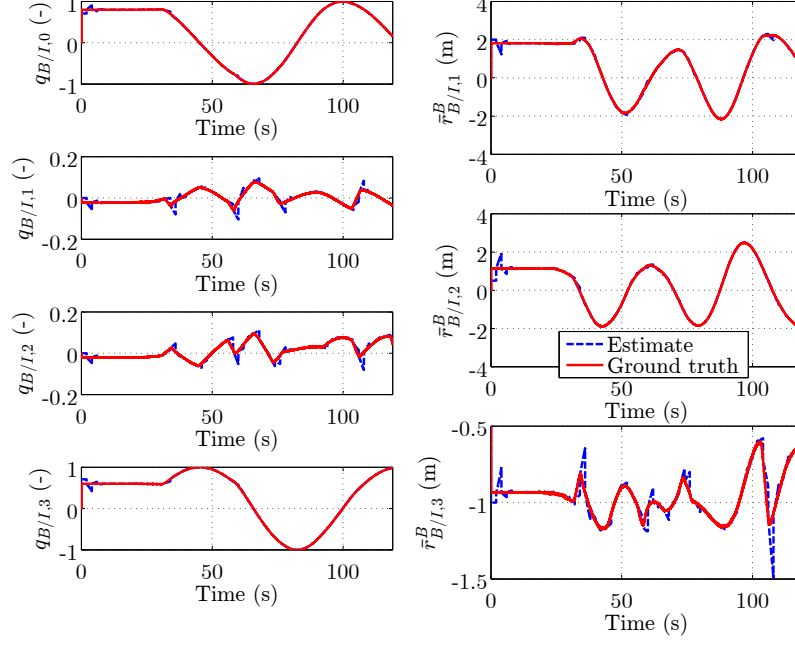


FIGURE 2.2
Estimated and true pose.

The Root-Mean-Square (RMS) attitude, position, angular velocity, and linear velocity estimation errors after 20 sec obtained with the DQ-MEKF and with two alternative EKF formulations are given in Table 2.1. The first alternative EKF formulation, hereby referred to as the QV-AEKF, is an additive EKF, where the state contains the vector part of the unit error quaternion and the position vector of the body with respect to the inertial frame expressed in the body frame. The second alternative EKF formulation, hereby referred to as the SQV-AEKF, is essentially the QV-AEKF split into two additive EKFs, one for the attitude and another one for the position. The QV-AEKF and SQV-AEKF are derived in detail in [5]. The three filters were fed the same measurements, were initialized with the same initial estimate of the state, and were tuned with the same covariance matrices.

Note that the RMS attitude and angular velocity estimation errors obtained with the three filters are the same. This is not surprising since the DQ-MEKF, the QV-AEKF, and the SQV-AEKF represent and update the attitude in the same way and the attitude is independent from the position. The SQV-AEKF exhibits the highest RMS position and linear velocity estimation errors. This is understandable since the DQ-MEKF and the QV-AEKF take into consideration the fact that the position vector of the body with respect to the inertial frame expressed in the body frame depends on the attitude

of the body, whereas the SQV-AEKF does not. Moreover, the RMS position and linear velocity estimation errors obtained with the DQ-MEKF are smaller than the ones obtained with the QV-AEKF. This can be justified in part by the fact that the relation between $r_{B/I}^B$ and $r_{B/I}^I$ is quadratic in $q_{B/I}$, whereas the relation between $q_{B/I,d}$ and $r_{B/I}^I$ is linear in $q_{B/I}$. Hence, the linearization error committed when linearizing the output equations of the QV-AEKF and of the DQ-MEKF with respect to $\delta q_{B/I}$ is smaller in the DQ-MEKF case.

TABLE 2.1

RMS Estimation Error	DQ-MEKF	QV-AEKF	SQV-AEKF
Attitude (deg)	2.22	2.22	2.22
Position (mm)	69.5	70.8	122.8
Angular Velocity (deg/s)	1.91	1.91	1.91
Linear Velocity (mm/s)	22.2	22.7	80.7

2.6 Conclusion

Hopefully after reading this chapter, the reader will agree that dual quaternions are not only a great tool for pose control, but can also be a great tool for pose estimation. The experimental results presented in this chapter show that, compared to the more traditional QV-AEKF, the DQ-MEKF has similar computational cost, but can provide better pose estimation. As mentioned in the introduction, the authors also hope that this chapter will serve as a guide to other authors on how to extend other useful attitude estimation results into higher dimensional pose estimation results.