

Recursively Feasible Stochastic Model Predictive Control for Time-Varying Linear Systems Subject to Unbounded Disturbances¹

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Abstract

Model predictive control solves a constrained optimization problem online in order to compute an implicit closed-loop control policy. Recursive feasibility—guaranteeing that the optimal control problem will have a solution at every time step—is an important property to guarantee the success of any model predictive control approach. However, recursive feasibility is difficult to establish in a stochastic setting and, in particular, in the presence of disturbances having unbounded support (e.g., Gaussian noise). The problem is further exacerbated for time-varying systems, in which case recursive feasibility must be established also in a robust sense, over all possible future time-varying system matrices, as well as in a stochastic sense, over all potential disturbance realizations. This work presents a method for ensuring the recursive feasibility of a convex, affine-feedback stochastic model predictive control problem formulation for systems with time-varying system matrices and unbounded (Gaussian) disturbances using ideas from covariance steering stochastic model predictive control. It is additionally shown that the proposed approach ensures that the closed-loop operation of the system will satisfy the desired chance constraints, and that the stochastic model predictive control problem may be formulated as a convex program so that it may be solved efficiently in real-time.

Key words: Stochastic optimal control; Model predictive control; Convex optimization; Linear time-varying systems; Uncertain systems.

1 Introduction

Model predictive control (MPC) is an attractive optimization-based method for controlling autonomous systems owing to its ability to handle input and state constraints while ensuring convergence under real-time computational limitations [4]. Stochastic MPC (SMPC), in particular, has gained interest as a method to provide probabilistic guarantees of constraint satisfaction and convergence to a neighborhood of the target in the presence of random disturbances. However, several challenges of stochastic MPC still exist. Whereas deterministic MPC optimizes over control actions at each time step, stochastic MPC, instead, optimizes over control *policies* as a means of controlling the dispersion of trajectories over time. Since optimizing over arbitrary

policies is not easily tractable, typical formulations consider an affine feedback policy [25]. In some cases, the feedback gain is calculated offline and is fixed; a more optimal approach is to design the feedback policy online in conjunction with a feed-forward control action. This, however, may lead to challenges in terms of preserving the convexity of the problem formulation. Another issue of SMPC is ensuring convergence. Whereas for deterministic MPC problems it is possible to show asymptotic convergence for the case of linear systems, for SMPC problems, only convergence to a neighborhood of the origin can be shown since the system is subject to a persistent disturbance [24]. Finally, providing recursive feasibility guarantees for SMPC approaches is challenging when the noise is sampled from a distribution with unbounded support (e.g., Gaussian distribution). Existing approaches addressing this issue typically consider either a truncated Gaussian distribution or relax the problem formulation [6, 11, 17, 22, 25].

In our recent work [18], we formulated a stochastic MPC

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approach for linear *time-varying* systems based on the recent theoretical developments in finite-horizon optimal covariance steering [2, 3, 8–10, 12, 26, 27]. The covariance steering problem addresses the issue of how to optimally drive the first two moments of a linear system from some initial condition to a given final condition over a finite horizon. This problem is closely related to the questions of recursive feasibility and stability of SMPC, since bounds on the first two moments of the state distribution may be used to ensure chance constraint satisfaction and convergence. Using results from covariance steering, in [18] we formulated an online affine-feedback solution to the SMPC problem that can be efficiently solved using convex programming, which we refer to as covariance steering stochastic MPC (CS-SMPC). Assuming recursive feasibility, the method provides asymptotic tracking convergence for the nominal system and convergence to a neighborhood of the reference trajectory for the stochastic system. However, the question of ensuring recursive feasibility, and consequently closed-loop constraint satisfaction, of CS-SMPC was not addressed in [18] and thus is the focus of this paper.

1.1 Literature Review

Most papers addressing the issue of recursive feasibility for stochastic systems assume that the disturbance is drawn from a distribution with bounded support (e.g., truncated Gaussian distribution). In [5, 6, 21, 22], the authors consider a linear time-invariant (LTI) system subject to bounded additive disturbances, and design an SMPC controller based on a fixed stabilizing feedback gain that is computed offline. In [20, 28], the authors consider an SMPC problem for an LTI system subject to additive Gaussian disturbances. The authors in these papers optimize over the space of affine feedback policies; however, they also approximate the Gaussian with a truncated distribution, essentially formulating a robust MPC control problem.

One of the few works that considered disturbances drawn from a distribution with unbounded support is [11], in which the authors optimize over the space of initial conditions and affine feedback policies in order to achieve recursive feasibility. The drawback of this strategy is that if the current state would lead to infeasibility, an open loop control must be used, discarding the most recent state information. However, a more recent work [19] notes that the relaxation of the initialization proposed in [11] is still the state-of-the-art and has been utilized in numerous theoretical extensions and applications. The authors of [19] propose to improve the initialization strategy by allowing for a continuous interpolation between using the most recent state measurement and relying on open-loop prediction of the state, referred to as initial state interpolation. The authors of [16, 17] also extended the work of [11]. Instead of optimizing over the state initialization, they propose a method referred to as indirect feedback, where the state measurement is included only in the cost function, while the constraints are

given purely in terms of the nominal system through a tube-based constraint-tightening approach. While both of these methods are able to show recursive feasibility, their analysis is limited to time-invariant systems and can be overly conservative due to the reliance on a pre-computed feedback policy.

The authors of [29] consider unbounded disturbances by taking a different approach. They address the issue of recursive feasibility and establish convergence for an unconstrained variation of the problem by softening the chance constraints using a penalty function (i.e., by imposing them through the cost function).

A similar problem formulation to the current paper is found in [1], which considers a stochastic LTI system subject to bounded parametric uncertainty and unbounded additive disturbances. However, as in [7], the authors of [1] assume that a stabilizing feedback gain is given beforehand (Assumption 1 in [1]) and then proceed to develop their recursive feasibility analysis for the closed-loop LTI system. Thus, the authors of [1] take a robust approach and do not allow for online optimization of the feedback gains, which can lead to significant conservatism, especially for LTV systems. Moreover, the approach in [1] does not provide any insights on how to compute the optimal robust feedback gains for a system subject to parametric uncertainty. In this paper, we formulate a semidefinite program (17), the solution of which provides a means of retrieving a robust feedback gain that will ensure a necessary invariance condition on the uncertainty of the stochastic system is satisfied (Lemma 4). This feedback gain and the corresponding invariance condition are reminiscent to the requirements of Assumption 1 in [1], in which the stabilizing feedback gain must satisfy a similar robust invariance equation.

Very few works deal with SMPC for linear time-varying (LTV) systems, and those that do, only consider bounded disturbances. Considerable challenges hamper the direct application of existing recursive feasibility results to LTV systems. For example, the results obtained in [11] require a pre-computed stabilizing feedback gain to design the terminal constraints. However, finding a single feedback gain that is stabilizing over all possible time-varying future system parameters is challenging.

One of the few works addressing LTV systems is [14] which considers a robust MPC problem formulated with an online affine-feedback policy. However, as mentioned above, the authors limit their analysis to bounded disturbances. The authors of [15, 23] consider a linear parameter-varying system that approximates a non-linear deterministic system. They show that recursive feasibility can be ensured using robust tube-based and invariant set-based approaches, respectively, provided the parameters lie in a compact set.

As summarized in Table 1, the existing literature on

the recursive feasibility of SMPC for systems subject to unbounded disturbances is extremely sparse, especially for the less-conservative case in which an affine-feedback policy is computed online. Furthermore, the overlap between results that consider both unbounded disturbances *and* time-varying systems is virtually non-existent. In this work, we fill this gap in the literature by establishing sufficient conditions for the recursive feasibility of SMPC for affine time-varying systems subject to unbounded (Gaussian) disturbances. Moreover, we provide a practical approach for computing the terminal constraint necessary in order to ensure that these conditions hold in practice. We show that our method results in satisfaction of the chance constraints during closed-loop operation of the system, as desired. Finally, we show that these results hold for the particular formulation of the SMPC problem used by CS-SMPC in [18], which optimizes online over the space of affine feedback policies by solving a convex program.

Table 1
Comparison of existing results on recursive feasibility of stochastic MPC.

Disturbances	Bounded (Robust) [5, 6, 20–22, 28] [14, 15, 23]	Unbounded (Stochastic) [1, 11, 16, 17, 19, 29]
Feedback	Offline (Tube) [1, 5, 6, 17, 22] [16, 19, 21]	Online (Affine-feedback) [11, 14, 20, 28, 29]
System	LTI [5, 6, 11, 16, 17] [1, 19–22, 28, 29]	LTV [14, 15, 23]

2 Problem Formulation

We consider an affine time-varying (ATV)³ system, as in [18], given by

$$x_{k+1} = A_k x_k + B_k u_k + D_k w_k + r_k, \quad (1)$$

where $x_k \in \mathbb{R}^{n_x}$ and $u_k \in \mathbb{R}^{n_u}$ are the fully observable state and control action at time-step k , respectively, and $w_k \sim \mathcal{N}(\mathbf{0}_{n_w}, I_{n_w})$ is an i.i.d. disturbance (that is, $\mathbb{E}[w_{k_1} w_{k_2}^\top] = \mathbb{E}[w_{k_1}] \mathbb{E}[w_{k_2}^\top] = 0$ for all $k_1 \neq k_2$). The system parameter matrices/vector are given by $A_k \in \mathcal{A} \subseteq \mathbb{R}^{n_x \times n_x}$, $B_k \in \mathcal{B} \subseteq \mathbb{R}^{n_x \times n_u}$, $D_k \in \mathcal{D} \subseteq \mathbb{R}^{n_x \times n_w}$, and $r_k \in \mathcal{R} \subseteq \mathbb{R}^{n_x}$, where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D}$, and $\mathcal{R}$ are such that

$$S_k = \begin{bmatrix} A_k & B_k & D_k & r_k \end{bmatrix} \in \mathcal{S} = \text{co}\{S^1, \dots, S^{N_p}\} \quad (2)$$

$$\mathcal{S} = \left\{ \sum_{\ell=1}^{N_p} \lambda_\ell S^\ell : \sum_{\ell=1}^{N_p} \lambda_\ell = 1, \lambda_\ell \geq 0 \right\}, \quad (3)$$

where $\text{co}\{\cdot\}$ refers to the convex hull and where $S^\ell = \begin{bmatrix} A^\ell & B^\ell & D^\ell & r^\ell \end{bmatrix}$, for all $\ell = 1, \dots, N_p$. The system (1) is

³ We consider an ATV system as the affine term is useful for modeling autonomous vehicles as seen in Sec. 5 and in [18].

subject to N_x (state) and N_u (input) polytopic chance constraints, given by

$$\Pr(\alpha_{x,i}^\top x_k \leq \beta_{x,i}) \geq 1 - p_{x,i}, \quad (4a)$$

$$\Pr(\alpha_{u,j}^\top u_k \leq \beta_{u,j}) \geq 1 - p_{u,j}, \quad (4b)$$

for $i = 1, 2, \dots, N_x$ and $j = 1, 2, \dots, N_u$, where the linear inequalities define polytopes given by $\mathcal{X} \triangleq \bigcap_{i=1}^{N_x} \{x : \alpha_{x,i}^\top x \leq \beta_{x,i}\}$, $\mathcal{U} \triangleq \bigcap_{j=1}^{N_u} \{u : \alpha_{u,j}^\top u \leq \beta_{u,j}\}$. The state and control actions of system (1) are associated with a cost function given by

$$\ell_k(x_k, u_k) = (x_k - x_{g_k})^\top Q_k (x_k - x_{g_k}) + u_k^\top R_k u_k, \quad (5)$$

where, for all $k = 0, 1, \dots$, $Q_k \succeq 0$ and $R_k \succ 0$ are known bounded matrices, and $x_{g_k} \in \mathcal{X}$ is a known target sequence to track. The model in (1) implies that the system matrices are known over the control horizon, a common assumption, which, for the sake of completeness, we formalize below. This is a weak assumption, which is satisfied in practice, e.g., when the ATV system (1) is derived from linearizing a nonlinear system.

Assumption 1 *At time k , the system matrices A_t , B_t , D_t , and r_t are known over the prediction horizon $t = k, \dots, k + N - 1$.*

To design an optimal control policy for system (1), define the following CS-SMPC problem given by

$$\min_{\rho_k^{N-1}(\cdot)} J_N(\mu_k, \Sigma_k; \rho_k^{N-1}(\cdot)) = \sum_{t=k}^{k+N-1} \mathbb{E}[\ell_t(x_{t|k}, u_{t|k})], \quad (6a)$$

subject to

$$x_{t+1|k} = A_t x_{t|k} + B_t u_{t|k} + D_t w_t + r_t, \quad (6b)$$

$$x_{k|k} \sim \mathcal{N}(\mu_k, \Sigma_k), \quad w_t \sim \mathcal{N}(\mathbf{0}_{n_w}, I_{n_w}) \quad (6c)$$

$$u_{t|k} = \rho_{t|k}(x_{t|k}), \quad (6d)$$

$$\Pr(\alpha_{x,i}^\top x_{t|k} \leq \beta_{x,i}) \geq 1 - p_{x,i}, \quad i = 1, \dots, N_x, \quad (6e)$$

$$\Pr(\alpha_{u,j}^\top u_{t|k} \leq \beta_{u,j}) \geq 1 - p_{u,j}, \quad j = 1, \dots, N_u, \quad (6f)$$

$$\mathbb{E}[x_{k+N|k}] \in \mathcal{X}_f^\mu, \quad (6g)$$

$$\mathbb{E}[(x_{k+N|k} - \mathbb{E}[x_{k+N|k}]) (\star)^\top] \preceq \Sigma_f, \quad (6h)$$

where, for all $t = k, \dots, k + N - 1$, $x_{t|k}$ and $u_{t|k}$ are the predicted state and control action, respectively, at time t computed at the present time k , $(\star)$ represents repeated terms, and where $\mathcal{X}_f^\mu \subseteq \mathcal{X}$ and $\Sigma_f \succ 0$ must be designed to ensure the recursive feasibility of (6). The initial state is given by $x_{k|k}$ and, as is often the case in SMPC, is sampled from a random distribution which may depend on the most recent state observation x_k . The selection of μ_k and Σ_k are described below. The optimization is performed over the sequence of planned state-feedback control policies $\rho_k^{N-1}(\cdot) = \{\rho_{k|k}(\cdot), \dots, \rho_{k+N-1|k}(\cdot)\}$. Since the future state sequence $\mathbf{x}_k^{N-1} = \{x_{k|k}, \dots, x_{k+N-1|k}\}$ consists

of random variables, the sequence of predicted future control actions, given by $\mathbf{u}_k^{N-1} = \{u_{k|k}, \dots, u_{k+N-1|k}\}$, where $u_{t|k} = \rho_{t|k}(x_{t|k})$, is also stochastic.

Problem (6) is a finite-horizon optimal control problem representing a tractable approximation of the corresponding infinite-horizon optimal control problem, and is solved in a receding horizon fashion to generate an optimal closed-loop control policy for the system. Let the optimal cost of Problem (6) at time k be denoted by $J_N^*(\mu_k, \Sigma_k)$ and let the corresponding optimal solution be denoted by $\rho_k^{*N-1}(\cdot) = \{\rho_{k|k}^*(\cdot), \dots, \rho_{k+N-1|k}^*(\cdot)\}$, generating the predicted sequences of stochastic control actions $\mathbf{u}_k^{*N-1} = \{u_{k|k}^*, \dots, u_{k+N-1|k}^*\}$ and states $\mathbf{x}_k^{*N} = \{x_{k|k}^*, \dots, x_{k+N|k}^*\}$, where $u_{t|k}^* = \rho_{t|k}^*(x_{t|k}^*)$, $x_{t+1|k}^* = A_t x_{t|k}^* + B_t u_{t|k}^* + D_t w_t + r_t$, and $x_{k|k}^* \sim \mathcal{N}(\mu_k, \Sigma_k)$. The sequence $\mathbf{x}_k^{*N}$ has the corresponding optimal predicted moment sequences denoted by $\mu_k^{*N} = \{\mu_{k|k}^*, \dots, \mu_{k+N|k}^*\}$ and $\Sigma_k^{*N} = \{\Sigma_{k|k}^*, \dots, \Sigma_{k+N|k}^*\}$, where $x_{t|k}^* \sim \mathcal{N}(\mu_{t|k}^*, \Sigma_{t|k}^*)$ and where $\mu_{k|k}^* = \mu_k$ and $\Sigma_{k|k}^* = \Sigma_k$. The control command applied to system (1) at time k is then given by

$$u_k = u_{k|k}^*. \quad (7)$$

In [18], we showed that Problem (6) can be cast as a convex program for a suitable parameterization of the control policy (see Theorem 2 of [18]). Additionally, we showed that if system (1) is controlled by (7), then the nominal system ($w_k \equiv 0$) converges asymptotically to the reference trajectory x_{g_k} (see Theorem 3 of [18]) and the stochastic system (1) converges to a neighborhood of the reference trajectory on average (that is, the average stage cost is bounded as $k \rightarrow \infty$, see Theorem 4 of [18]). However, both of these results rely on the condition that a solution to (6) exists at every time $k = 0, 1, \dots$, using the initialization (6c), where μ_k and Σ_k are given by

$$(\mu_k, \Sigma_k) = \begin{cases} (x_0, 0), & k = 0, \\ (\mu_{k|k-1}^*, \Sigma_{k|k-1}^*), & k > 0, \end{cases} \quad (8)$$

where

$$\mu_{k+1|k}^* = A_k \mu_k + B_k \bar{u}_k + r_k, \quad (9a)$$

$$\begin{aligned} \Sigma_{k+1|k}^* &= A_k \Sigma_k A_k^\top + A_k \Sigma_{xu_k} B_k^\top + B_k \Sigma_{xu_k}^\top A_k^\top \\ &\quad + B_k \Sigma_{u_k} B_k^\top + D_k D_k^\top, \end{aligned} \quad (9b)$$

and where $\bar{u}_k = \mathbb{E}[u_k | \mu_k, \Sigma_k]$, $\Sigma_{xu_k} = \mathbb{E}[(x_k - \mu_k)(u_k - \bar{u}_k)^\top | \mu_k, \Sigma_k]$ and $\Sigma_{u_k} = \mathbb{E}[(u_k - \bar{u}_k)(u_k - \bar{u}_k)^\top | \mu_k, \Sigma_k]$.

The aim of this work is to investigate conditions implying the feasibility of Problem (6) under the initialization (8), as we have already established convergence results for this problem, conditioned on feasibility, in Theorems 3

and 4 of [18]. In Section 3, we derive sufficient conditions for $\mathcal{X}_f^\mu$ and Σ_f such that Problem (6) is recursively feasible, and in Section 4 we show how to compute $\mathcal{X}_f^\mu$ and Σ_f to meet these conditions. We then show that our method allows for a convex problem formulation and ensures satisfaction of the chance constraints in practice.

Remark 2 In [18] we allowed Problem (6) to be initialized with $\mu_k = x_k$, $\Sigma_k = 0$ instead of (8) if the following two conditions are satisfied: Problem (6) is feasible under such an initialization, and also its solution provides a lower optimal cost than that under (8) (i.e., $J_N^*(x_k, 0) \leq J_N^*(\mu_{k|k-1}^*, \Sigma_{k|k-1}^*)$). Therefore, feasibility under (8) implies feasibility under the initialization proposed in [18] and thus it provides a sufficient condition for the stability results in Theorems 3 and 4 of [18].

3 Recursive Feasibility Conditions

We are now ready to introduce the main result of the paper, which establishes sufficient conditions for the recursive feasibility of Problem (6).

Theorem 3 Let $\mathcal{X}_f^\mu$ and Σ_f be such that, given $\mu \in \mathcal{X}_f^\mu$ and $\Sigma \preceq \Sigma_f$, there exists a $\rho(\cdot)$ such that, for all $A \in \mathcal{A}$, $B \in \mathcal{B}$, $D \in \mathcal{D}$, $r \in \mathcal{R}$,

$$\Pr(\alpha_{x,i}^\top x \leq \beta_{x,i}) \geq 1 - p_{x,i}, \quad i = 1, \dots, N_x, \quad (10a)$$

$$\Pr(\alpha_{u,j}^\top u \leq \beta_{u,j}) \geq 1 - p_{u,j}, \quad j = 1, \dots, N_u, \quad (10b)$$

$$A\mu + B\bar{u} + r \in \mathcal{X}_f^\mu, \quad (10c)$$

$$\begin{aligned} A\Sigma A^\top + A\Sigma_{xu} B^\top + B\Sigma_{xu}^\top A^\top \\ + B\Sigma_u B^\top + D D^\top \preceq \Sigma_f, \end{aligned} \quad (10d)$$

where $x \sim \mathcal{N}(\mu, \Sigma)$, $u = \rho(x)$, $\bar{u} = \mathbb{E}[u | \mu, \Sigma]$, $\Sigma_{xu} = \mathbb{E}[(x - \mu)(u - \bar{u})^\top | \mu, \Sigma]$ and $\Sigma_u = \mathbb{E}[(u - \bar{u})(u - \bar{u})^\top | \mu, \Sigma]$. Then, assuming Problem (6) is feasible at time $k = 0$ using the initialization (8), it will remain feasible under initialization (8) for all future time steps $k > 0$.

PROOF. It is sufficient to show that feasibility at an arbitrary time-step k implies feasibility at the next time-step $k+1$. To that end, let an optimal solution $\rho_k^{*N-1}(\cdot)$ to Problem (6) at time k exist with corresponding optimal control, state, and moment sequences $\mathbf{u}_k^{*N-1}$, $\mathbf{x}_k^{*N}$, μ_k^{*N} , and Σ_k^{*N} . We claim that a feasible solution at time $k+1$ is given by

$$\begin{aligned} \rho_{k+1}^{N-1}(\cdot) &= \{\rho_{k+1|k+1}(\cdot), \dots, \rho_{k+N|k+1}(\cdot)\} \\ &= \{\rho_{k+1|k}^*(\cdot), \dots, \rho_{k+N-1|k}^*(\cdot), \rho(\cdot)\}, \end{aligned} \quad (11)$$

generating the control sequence

$$\mathbf{u}_{k+1}^{N-1} = \{u_{k+1|k+1}, \dots, u_{k+N|k+1}\}, \quad (12)$$

and the predicted state sequence

$$\mathbf{x}_{k+1}^N = \{x_{k+1|k+1}, \dots, x_{k+N+1|k+1}\}, \quad (13)$$

with corresponding moment sequences

$$\begin{aligned}\mu_{k+1}^N &= \{\mu_{k+1|k+1}, \dots, \mu_{k+N+1|k+1}\} \\ &= \{\mu_{k+1|k}^*, \mu_{k+2|k}^*, \dots, \mu_{k+N|k}^*, \mu_{k+N+1|k+1}\},\end{aligned}\quad (14)$$

$$\begin{aligned}\Sigma_{k+1}^N &= \{\Sigma_{k+1|k+1}, \dots, \Sigma_{k+N+1|k+1}\} \\ &= \{\Sigma_{k+1|k}^*, \Sigma_{k+2|k}^*, \dots, \Sigma_{k+N|k}^*, \Sigma_{k+N+1|k+1}\},\end{aligned}\quad (15)$$

where $u_{t|k+1} = \rho_{t|k+1}(x_{t|k+1})$, $x_{t|k+1} \sim \mathcal{N}(\mu_{t|k+1}, \Sigma_{t|k+1})$, for $t = k+1, \dots, k+N+1$.

To show that (11) results in a feasible solution of Problem (6), notice that under initialization (8), $x_{k+1|k+1} \sim \mathcal{N}(\mu_{k+1|k}^*, \Sigma_{k+1|k}^*)$. Then, the sequences $\{u_{k+1|k+1}, \dots, u_{k+N-1|k+1}\}, \{x_{k+1|k+1}, \dots, x_{k+N-1|k+1}\}$ necessarily satisfy (6e) and (6f) as, due to (8), they will be distributed the same as those derived from a feasible solution of (6). Moreover, $x_{k+N|k+1}$ will be distributed according to $\mathcal{N}(\mu_{k+N|k}^*, \Sigma_{k+N|k}^*)$ and will satisfy (6e) because it necessarily satisfies the terminal constraints (6g)-(6h) as a solution of Problem (6), which, under (10a), implies satisfaction of (6e). Finally, due to (10), and since, as a solution of (6), $\mu_{k+N|k}^* \in \mathcal{X}_f^\mu$ and $\Sigma_{k+N|k}^* \preceq \Sigma_f$, it is known that there exists ρ such that $u_{k+N|k+1} = \rho_{k+N|k+1}(x_{k+N|k+1}) = \rho(x_{k+N|k+1})$ satisfying (6f), and such that $x_{k+N+1|k+1}$ satisfies (6g)-(6h). $\square$

We have thus established sufficient conditions for the recursive feasibility of Problem (6) under the initialization (8). Next, we show how (10) can be used to design the terminal set $\mathcal{X}_f^\mu$ and the terminal covariance Σ_f so as to ensure that the Problem (6) is recursively feasible. As a direct consequence of that result, we will show that the convex CS-SMPC formulation presented in [18] is also recursively feasible.

4 Terminal Constraint Design

To compute the terminal set $\mathcal{X}_f^\mu$ and the terminal covariance Σ_f satisfying (10), we introduce the candidate control parameterization

$$u_{t|k} = \rho_{t|k}(x_{t|k}) = v_{t|k} + L_{t|k}(x_{t|k} - \mu_{t|k}), \quad (16)$$

where $v_{t|k} \in \mathbb{R}^{n_u}$ and $L_{t|k} \in \mathbb{R}^{n_u \times n_x}$. Under (16), $\bar{u}_{t|k} = v_{t|k}$, $\Sigma_{xu_{t|k}} = \Sigma_{t|k} L_{t|k}^\top$, and $\Sigma_{u_{t|k}} = L_{t|k} \Sigma_{t|k} L_{t|k}^\top$.

Lemma 4 *A terminal covariance matrix Σ_f satisfying (10d) is given by the solution to the semidefinite program (SDP)*

$$\begin{aligned}\min_{\Sigma, Z} \quad & \text{tr}(\Sigma) \\ \text{subject to} \quad & \end{aligned}\quad (17a)$$

$$\begin{bmatrix} \Sigma - D^\ell D^{\ell\top} & A^\ell \Sigma + B^\ell Z \\ \Sigma A^{\ell\top} + Z^\top B^{\ell\top} & \Sigma \end{bmatrix} \succeq 0, \quad (17b)$$

for all $\ell = 1, \dots, N_p$, where $\Sigma \succeq \mathbf{0}_{n_x \times n_x}$ and $Z \in \mathbb{R}^{n_u \times n_x}$, and where $\text{tr}(\cdot)$ is the trace operation.

PROOF. Under the control policy parameterization (16), $\rho(x) = v + L(x - \mu)$, and (10d) reduces to the existence of a matrix $L \in \mathbb{R}^{n_u \times n_x}$, such that $(A + BL)\Sigma(A + BL)^\top + DD^\top \preceq \Sigma_f$, for all $A \in \mathcal{A}$, $B \in \mathcal{B}$, $D \in \mathcal{D}$, and $\Sigma \preceq \Sigma_f$. Thus, a terminal covariance matrix Σ_f satisfying (10d), is given by a matrix Σ satisfying

$$\Sigma \succeq (A + BL)\Sigma(A + BL)^\top + DD^\top, \quad (18)$$

for all $A \in \mathcal{A}$, $B \in \mathcal{B}$, $D \in \mathcal{D}$ and for a particular choice of $L \in \mathbb{R}^{n_u \times n_x}$. Similar to the approach in [13], using the Schur complement, (18) may be formulated as a linear matrix inequality (LMI) given by

$$M = \begin{bmatrix} \Sigma & (A + BL) & D \\ (A + BL)^\top & \Sigma^{-1} & 0 \\ D^\top & 0 & I \end{bmatrix} \succeq 0, \quad (19)$$

for all $A \in \mathcal{A}$, $B \in \mathcal{B}$, $D \in \mathcal{D}$.

Using the Schur complement and (2), it may be seen that satisfaction of (19) is ensured by

$$\begin{bmatrix} \Sigma - D^\ell D^{\ell\top} & (A^\ell + B^\ell L) \\ (A^\ell + B^\ell L)^\top & \Sigma^{-1} \end{bmatrix} \succeq 0, \quad (20)$$

for all $\ell = 1, \dots, N_p$. Finally, (20) may be converted to the equivalent LMI given by (17b), which is convex in Σ and Z , by multiplying from both sides by $\text{blkdiag}(I, \Sigma)$ and using a change of variables given by $Z = L\Sigma$. Thus, Σ_f satisfying (10d) may be found by solving the SDP (17). $\square$

Let the optimal solution of Problem (17) be given by Σ^* , Z^* . The optimal control gain L^* may be recovered from $L^* = Z^* \Sigma^{*-1}$, provided $\Sigma^* \succ 0$. We are now ready to introduce the second main result, which shows how to compute $\mathcal{X}_f^\mu$ and Σ_f so that the conditions of Theorem 3 are satisfied.

Theorem 5 *Let $\Sigma_f = \Sigma^*$ be given by (17), and let $\mathcal{X}_f^\mu$ be the largest set such that $\mathcal{X}_f^\mu \subseteq \mathcal{X}_{\text{safe}}$ and such that, for all $\ell = 1, \dots, N_p$,*

$$\mu \in \mathcal{X}_f^\mu \implies \exists v \in \mathcal{U}_{\text{safe}} : A^\ell \mu + B^\ell v + r^\ell \in \mathcal{X}_f^\mu, \quad (21)$$

where,

$$\mathcal{X}_{\text{safe}} = \bigcap_{i=1}^{N_x} \left\{ \mu : \alpha_{x,i}^\top \mu \leq \tilde{\beta}_{x,i} \right\}, \quad (22a)$$

$$\mathcal{U}_{\text{safe}} = \bigcap_{j=1}^{N_u} \left\{ v : \alpha_{u,j}^\top v \leq \tilde{\beta}_{u,j} \right\}, \quad (22b)$$

and where

$$\tilde{\beta}_{x,i} = \beta_{x,i} - \sqrt{\alpha_{x,i}^\top \Sigma_f \alpha_{x,i}} \Phi^{-1}(1 - p_{x,i}), \quad (23a)$$

$$\tilde{\beta}_{u,j} = \beta_{u,j} - \sqrt{\alpha_{u,j}^\top L^* \Sigma_f L^{*\top} \alpha_{u,j}} \Phi^{-1}(1 - p_{u,j}), \quad (23b)$$

where $\Phi^{-1}(\cdot)$ is the inverse cumulative distribution function of the standard normal distribution. Then, the conditions (10) of Theorem 3 for the recursive feasibility of Problem (6) are satisfied.

PROOF. Let $\rho(x) = v + L^*(x - \mu)$ where $x \sim \mathcal{N}(\mu, \Sigma)$. Then, as shown in Lemma 3 of [18], the chance constraints (10a) and (10b) may be reformulated as the deterministic constraints

$$\alpha_{x,i}^\top \mu + \sqrt{\alpha_{x,i}^\top \Sigma \alpha_{x,i}} \Phi^{-1}(1 - p_{x,i}) \leq \beta_{x,i}, \quad (24a)$$

$$\alpha_{u,j}^\top v + \sqrt{\alpha_{u,j}^\top L^* \Sigma L^{*\top} \alpha_{u,j}} \Phi^{-1}(1 - p_{u,j}) \leq \beta_{u,j}, \quad (24b)$$

for all $i = 1, \dots, N_x$ and $j = 1, \dots, N_u$. It may be seen from (22) that the deterministic constraints (24) are satisfied for all $\Sigma \preceq \Sigma_f$, $\mu \in \mathcal{X}_{\text{safe}}$, and $v \in \mathcal{U}_{\text{safe}}$. Moreover, since $\mathcal{X}_f^\mu \subseteq \mathcal{X}_{\text{safe}}$, the chance constraints (10a) and (10b) are satisfied for all $\Sigma \preceq \Sigma_f$, $\mu \in \mathcal{X}_f^\mu$ and $v \in \mathcal{U}_{\text{safe}}$. Next, due to (2) and (16), it may be seen that (10c) is ensured for all $A \in \mathcal{A}$, $B \in \mathcal{B}$, and $r \in \mathcal{R}$ by the existence of a $v \in \mathbb{R}^{n_u}$ such that $A^\ell \mu + B^\ell v + r^\ell \in \mathcal{X}_f^\mu$, for all $\ell = 1, \dots, N_p$ and for all $\mu \in \mathcal{X}_f^\mu$. Thus (21) is sufficient for satisfaction of (10c). Finally, by Lemma 4, setting Σ_f to the optimal solution of (17) ensures satisfaction of (10d). $\square$

Remark 6 The set $\mathcal{X}_f^\mu$, given by (21), is often referred to as a maximal robust controlled invariant set and may be efficiently computed using the iterative procedure presented in, e.g., [4, 14]. Although any robust controlled invariant set is sufficient, using the maximal such set minimizes conservatism as, ideally, the terminal set should be as large as possible while ensuring recursive feasibility.

4.1 Closed-loop Chance Constraint Satisfaction

Next, we show that the proposed approach ensures the chance constraints (4) will be satisfied by the system under closed-loop operation with (7), as desired.

Theorem 7 The control policy (7) generated by solving Problem (6) with the terminal constraints Σ_f and $\mathcal{X}_f^\mu$

given by (17) and (21), respectively, and using the initialization (8), ensures that the system (1) satisfies

$$\Pr(\alpha_{x,i}^\top x_k \leq \beta_{x,i} | x_0) \geq 1 - p_{x,i}, \quad (25a)$$

$$\Pr(\alpha_{u,j}^\top u_k \leq \beta_{u,j} | x_0) \geq 1 - p_{u,j}, \quad (25b)$$

for $i = 1, 2, \dots, N_x$ and $j = 1, 2, \dots, N_u$ and $k = 0, 1, 2, \dots$, provided Problem (6) is feasible at time $k = 0$.

PROOF. Examining the constraints (6c), (6e), and (6f), it is clear that recursive feasibility of Problem (6) under Theorems 3 and 5 and control policy (7) imply

$$\Pr(\alpha_{x,i}^\top x_k \leq \beta_{x,i} | \mu_k, \Sigma_k) \geq 1 - p_{x,i}, \quad (26a)$$

$$\Pr(\alpha_{u,j}^\top u_k \leq \beta_{u,j} | \mu_k, \Sigma_k) \geq 1 - p_{u,j}, \quad (26b)$$

for all $k = 0, 1, \dots$, where the problem is initialized with $x_k \sim \mathcal{N}(\mu_k, \Sigma_k)$. Moreover, under initialization (8) which depends on the moment dynamics (9) and the applied control policy (7), we have

$$\mu_{k+1} = A_k \mu_k + B_k \bar{u}_k^* + r_k, \quad (27a)$$

$$\begin{aligned} \Sigma_{k+1} = & A_k \Sigma_k A_k^\top + A_k \Sigma_{x u_k}^* B_k^\top + B_k \Sigma_{x u_k}^{*\top} A_k^\top \\ & + B_k \Sigma_{u_k}^* B_k^\top + D_k D_k^\top, \end{aligned} \quad (27b)$$

for all $k = 0, 1, \dots$, where $\bar{u}_k^* = \mathbb{E}[u_k | \mu_k, \Sigma_k]$, $\Sigma_{x u_k}^* = \mathbb{E}[(x_k - \mu_k)(u_k - \bar{u}_k^*)^\top | \mu_k, \Sigma_k]$, and $\Sigma_{u_k}^* = \mathbb{E}[(u_k - \bar{u}_k^*)(u_k - \bar{u}_k^*)^\top | \mu_k, \Sigma_k]$, and where $\mu_0 = x_0$ and $\Sigma_0 = 0$. Note that (27) are precisely the moment dynamics for the system (1) operating in closed-loop under the control policy (7). Therefore, since the problem is initialized with $x_k \sim \mathcal{N}(\mu_k, \Sigma_k)$, given x_0 , $\rho_k^{*N-1}(\cdot)$ and the moment dynamics evolve deterministically. Thus,

$$\mu_k = \mathbb{E}[x_k | \mu_0, \Sigma_0] = \mathbb{E}[x_k | x_0], \quad (28a)$$

$$\begin{aligned} \Sigma_k = & \mathbb{E}[(x_k - \mu_k)(x_k - \mu_k)^\top | \mu_0, \Sigma_0] \\ = & \mathbb{E}[(x_k - \mu_k)(x_k - \mu_k)^\top | x_0], \end{aligned} \quad (28b)$$

for all $k = 0, 1, \dots$. Therefore, (26) implies (25). $\square$

In some cases, (e.g. [11, 18]) it may be desirable to reduce conservatism by replacing the static initialization strategy (8) with a dynamic strategy

$$(\mu_k, \Sigma_k) = \begin{cases} (x_k, 0), & x_k \in \mathcal{X}_{\text{feas}}, \\ (\mu_{k|k-1}^*, \Sigma_{k|k-1}^*), & \text{otherwise,} \end{cases} \quad (29)$$

where $\mathcal{X}_{\text{feas}} \subseteq \mathcal{X}$ is the set of states for which Problem (6) is feasible with $\mu_k = x_k$ and $\Sigma_k = 0$. In this case, (25) is not inherently satisfied. Rather, this initialization strategy ensures a different notion of closed-loop chance constraint satisfaction.

Corollary 8 *The control policy (7) generated by solving Problem (6) with terminal constraints Σ_f and $\mathcal{X}_f^\mu$ given by (17) and (21), respectively, and using the initialization (29), ensures recursive feasibility and that the closed-loop operation of system (1) will satisfy*

$$\Pr(\alpha_{x,i}^\top x_k \leq \beta_{x,i} | x_\tau) \geq 1 - p_{x,i}, \quad (30a)$$

$$\Pr(\alpha_{u,j}^\top u_k \leq \beta_{u,j} | x_\tau) \geq 1 - p_{u,j}, \quad (30b)$$

for $i = 1, 2, \dots, N_x$ and $j = 1, 2, \dots, N_u$ and $k \geq \tau \geq 0$, provided $x_0 \in \mathcal{X}_{feas}$, and where τ is the most-recent time-step at which $x_\tau \in \mathcal{X}_{feas}$.

PROOF. Recursive feasibility is ensured since the first case of initialization (29) is ensured for $k = 0$ and, due to Theorems 3 and 5, at least the second case will be ensured for $k > 0$. Closed-loop satisfaction of the chance-constraints (30) follows from a generalization of Theorem 7 by replacing the fixed case x_0 with the dynamic case x_τ as the last state on which the problem was re-conditioned. $\square$

Remark 9 *Note that the initialization (8) is more conservative than (29) and ensures (25), which is conditioned on the initial state x_0 . On the other hand, initialization (29) is less conservative, but ensures (30), which is conditioned on x_τ , which, typically, is a more recent state.*

4.2 Convex Formulation

Theorem 5 shows how to compute the terminal constraints under the control parameterization (16), so as to ensure recursive feasibility. However, formulating Problem (6) with the affine state feedback parameterization (16) results in a nonconvex program [11, 26]. **Next, we show that by a change of variables, we may arrive at a convex reformulation of Problem (6) using the more general control parameterization (31).**

Theorem 10 *Any state-feedback policy of the form (16) may be represented by an affine feedback policy, given by*

$$u_{t|k} = v_{t|k} + \sum_{i=k}^t K_{t,i|k} y_{i|k}, \quad (31a)$$

$$y_{t+1|k} = A_t y_{t|k} + D_t w_t, \quad y_{k|k} = x_k - \mu_k, \quad (31b)$$

where $y_{t|k} \in \mathbb{R}^{n_x}$ is an auxiliary variable that tracks the error evolution of the uncontrolled system. Moreover, using the parameterization (31), Problem (6) may be cast as a convex program.

PROOF. The predicted dynamics of system (1) may be alternatively written as

$$X_{k|k}^N = \bar{A}_k^N x_{k|k} + \bar{B}_k^N U_{k|k}^{N-1} + \bar{D}_k^N W_k^{N-1} + \bar{r}_k^N, \quad (32)$$

where $X_{k|k}^N = [x_{k|k}^\top, x_{k+1|k}^\top, \dots, x_{k+N|k}^\top]^\top$, $U_{k|k}^{N-1} = [u_{k|k}^\top, u_{k+1|k}^\top, \dots, u_{k+N-1|k}^\top]^\top$, $W_k^{N-1} = [w_k^\top, w_{k+1}^\top, \dots, w_{k+N-1}^\top]^\top$, where $\bar{A}_k^N \in \mathbb{R}^{(N+1)n_x \times n_x}$, $\bar{B}_k^N \in \mathbb{R}^{(N+1)n_x \times Nn_u}$, $\bar{D}_k^N \in \mathbb{R}^{(N+1)n_x \times Nn_w}$, $\bar{r}_k^N \in \mathbb{R}^{(N+1)n_x}$, and $\bar{B}_k^N$ and $\bar{D}_k^N$ are lower block triangular. The exact expressions for the system matrices $\bar{A}_k^N$, $\bar{B}_k^N$, $\bar{D}_k^N$ may be found in [12], for example.

We introduce the parameterization (16) and split (32) into the nominal and error dynamics as follows

$$X_{k|k}^N = \bar{X}_{k|k}^N + \tilde{X}_{k|k}^N, \quad (33a)$$

$$\bar{X}_{k|k}^N = \bar{A}_k^N \mu_k + \bar{B}_k^N V_{k|k}^{N-1} + \bar{r}_k^N, \quad (33b)$$

$$\begin{aligned} \tilde{X}_{k|k}^N &= \bar{A}_k^N \tilde{x}_k + \bar{B}_k^N L_{k|k}^{N-1} \tilde{X}_{k|k}^N + \bar{D}_k^N W_k^{N-1} \\ &= (I - \bar{B}_k^N L_{k|k}^{N-1})^{-1} (\bar{A}_k^N \tilde{x}_k + \bar{D}_k^N W_k^{N-1}), \end{aligned} \quad (33c)$$

where $\tilde{x}_k = x_k - \mu_k$, $V_{k|k}^{N-1} = [v_{k|k}^\top, \dots, v_{k+N-1|k}^\top]^\top$, and

$$L_{k|k}^{N-1} = \begin{bmatrix} L_{k|k} & 0 & \dots & 0 & 0 \\ 0 & L_{k+1|k} & \dots & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & L_{k+N-1|k} & 0 \end{bmatrix}. \quad (33d)$$

Therefore, $U_{k|k}^{N-1} = V_{k|k}^{N-1} + L_{k|k}^{N-1} (I - \bar{B}_k^N L_{k|k}^{N-1})^{-1} (\bar{A}_k^N \tilde{x}_k + \bar{D}_k^N W_k^{N-1})$.

Introducing the change of variables

$$K_{k|k}^{N-1} = L_{k|k}^{N-1} (I - \bar{B}_k^N L_{k|k}^{N-1})^{-1}, \quad (34a)$$

$$Y_{k|k}^N = (\bar{A}_k^N \tilde{x}_k + \bar{D}_k^N W_k^{N-1}), \quad (34b)$$

yields the control parameterization $U_{k|k}^{N-1} = K_{k|k}^{N-1} Y_{k|k}^N + V_{k|k}^{N-1}$, where $K_{k|k}^{N-1} =$

$$\begin{bmatrix} K_{k,k|k} & 0 & \dots & 0 & 0 \\ K_{k+1,k|k} & K_{k+1,k+1|k} & \dots & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ K_{k+N-1,k|k} & K_{k+N-1,k+1|k} & \dots & K_{k+N-1,k+N-1|k} & 0 \end{bmatrix},$$

where the lower triangular structure of $K_{k|k}^{N-1}$ results from the block lower triangular structure of $\bar{B}_k^N$.

The parameterization given by (34) is the augmented form of the affine feedback policy (31). In Theorem 2 of [18], we have shown that Problem (6) may be cast as a convex program by introducing the affine feedback parameterization given by (31). $\square$

Remark 11 In [18], the feedback is applied only with respect to the current error (that is, $u_{t|k} = v_{t|k} + K_{t|k}y_{t|k}$). However, this is done merely to limit the number of decision variables and improve computation time. The proof of convexity in [18] may clearly be extended to the more general case of (31).

The final convex program is given as

$$\begin{aligned} \min_{V_{k|k}^{N-1}, K_{k|k}^{N-1}} \quad & J_N(\mu_k, \Sigma_k; V_{k|k}^{N-1}, K_{k|k}^{N-1}) \\ = \quad & (\bar{A}_k^N \mu_k + \bar{B}_k^N V_{k|k}^{N-1} + \bar{r}_k^N - X_g)^\top \bar{Q}(\star) + V_{k|k}^{N-1\top} \bar{R} V_{k|k}^{N-1} \\ & + \text{tr}[(I + \bar{B}_k^N K_{k|k}^{N-1})^\top \bar{Q}(\star) + K_{k|k}^{N-1\top} \bar{R} K_{k|k}^{N-1}] \Sigma_y, \end{aligned} \quad (35a)$$

subject to

$$\begin{aligned} \alpha_{x,i}^\top E_{t-k} \left(\bar{A}_k^N \mu_k + \bar{B}_k^N V_{k|k}^{N-1} + \bar{r}_k^N \right) \\ + \|\Sigma_y^{1/2} (I + \bar{B}_k^N K_{k|k}^{N-1})^\top E_{t-k}^\top \alpha_{x,i}\| \Phi^{-1}(1 - p_{x,i}) \\ - \beta_{x,i} \leq 0, \quad i = 1, \dots, N_x, \end{aligned} \quad (35b)$$

$$\begin{aligned} \alpha_{u,j}^\top F_{t-k} V_{k|k}^{N-1} + \|\Sigma_y^{1/2} K_{k|k}^{N-1\top} F_{t-k}^\top \alpha_{u,j}\| \Phi^{-1}(1 - p_{u,j}) \\ - \beta_{u,j} \leq 0, \quad j = 1, \dots, N_u, \end{aligned} \quad (35c)$$

$$E_N \left(\bar{A}_k^N \mu_k + \bar{B}_k^N V_{k|k}^{N-1} + \bar{r}_k^N \right) \in \mathcal{X}_f^\mu, \quad (35d)$$

$$\Sigma_f \succeq E_N(I + \bar{B}_k^N K_{k|k}^{N-1}) \Sigma_y (I + \bar{B}_k^N K_{k|k}^{N-1})^\top E_N^\top, \quad (35e)$$

where $\Sigma_y = \bar{A}_k^N \Sigma_k \bar{A}_k^{N\top} + \bar{D}_k^N \bar{D}_k^{N\top}$, $X_g = [x_{gk}^\top, \dots, x_{gk+N}^\top]^\top$, $\bar{Q} = \text{blkdiag}(Q_k, \dots, Q_{k+N-1}, 0_{n_x \times n_x})$, and $\bar{R} = \text{blkdiag}(R_k, \dots, R_{k+N-1})$, and E_{t-k} and F_{t-k} are such that $x_{t|k} = E_{t-k} X_{k|k}^N$ and $u_{t|k} = F_{t-k} U_{k|k}^{N-1}$.

In summary, using Theorem 3, we have established sufficient conditions to guarantee recursive feasibility of Problem (6). Theorem 5 shows how the terminal constraints may be designed so as to satisfy these conditions in practice. Theorem 7 establishes that the chance constraints will be satisfied under closed-loop operation, as desired. Finally, Theorem 10 shows that the results of the previous theorems extend to the convex problem formulation (35) utilized in [18].

5 Numerical Example

We evaluate the proposed approach in a realistic vehicle lateral control example. We consider a nonlinear kinematic bicycle model, which, as in, e.g., [30], is linearized around a reference trajectory to obtain an ATV system. The nonlinear kinematic bicycle model, shown in Fig. 1, is given by the following

$$\dot{e}_\psi = \dot{\psi} - \dot{\psi}_{\text{ref}} + \dot{\nu}_y / \nu_x, \quad (36a)$$

$$\dot{e}_y = \nu_y \cos e_\psi + \nu_x \sin e_\psi, \quad (36b)$$

$$\dot{s} = (\nu_x \cos e_\psi - \nu_y \sin e_\psi) / (1 - e_y \rho), \quad (36c)$$

where $\nu_y = \nu_x \delta \frac{\ell_r}{\ell_f + \ell_r}$, $\dot{\psi} = \tan \delta \frac{\nu_x}{\ell_f + \ell_r}$, and where e_ψ is the heading error with respect to the reference path, ψ is the vehicle's absolute heading, ψ_{ref} is the heading of the reference path, ν_y is the vehicle's lateral velocity, ν_x is the vehicle's longitudinal velocity, e_y is the vehicle's lateral

error with respect to the reference path, s is the vehicle's longitudinal position with respect to the reference path, ρ is the curvature of the reference path, δ is the front steering angle, and ℓ_f and ℓ_r are the locations of the vehicle's center of mass. Given a velocity profile and a

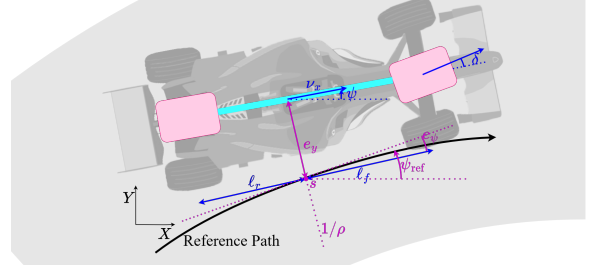


Fig. 1. Kinematic Bicycle Model in Curvilinear Coordinates

reference path, and assuming δ and e_ψ remain small, an affine approximation for the lateral motion is given by

$$\dot{e}_\psi = \frac{\nu_x}{\ell_f + \ell_r} \delta - \nu_x \rho + \dot{\delta} \frac{\ell_r}{\ell_f + \ell_r}, \quad (37a)$$

$$\dot{e}_y = \frac{\ell_r}{\ell_f + \ell_r} \nu_x \delta + \nu_x e_\psi. \quad (37b)$$

We define the state $x = [\delta, e_\psi, e_y]^\top$ and control $u = \dot{\delta}$ and use Euler integration with a time-step of $\Delta t = 0.1$ s to obtain the affine system $x_{k+1} = A(\nu_{x_k})x_k + B(\nu_{x_k})u_k + Dw_k + r(\nu_{x_k}, \rho_k)$, where, for the given reference trajectory $\mathbf{z}_{\text{ref}} = \{[\nu_{x_0}, \rho_0]^\top, \dots, [\nu_{x_T}, \rho_T]^\top\}$,

$$A(\nu_{x_k}) = A_k = \begin{bmatrix} 1 & 0 & 0 \\ \frac{\nu_{x_k}}{\ell_f + \ell_r} \Delta t & 1 & 0 \\ \frac{\ell_r}{\ell_f + \ell_r} \nu_{x_k} \Delta t & \nu_{x_k} \Delta t & 1 \end{bmatrix}, \quad (38a)$$

$$B(\nu_{x_k}) = B_k = [\Delta t, \frac{\ell_r}{\ell_f + \ell_r} \Delta t, 0]^\top, \quad (38b)$$

$$D = \text{diag}(\theta_\delta \Delta t, \theta_\psi \Delta t, \theta_y \Delta t), \quad (38c)$$

$$r(\nu_{x_k}, \rho_k) = r_k = [0, -\rho_k \nu_{x_k} \Delta t, 0]^\top, \quad (38d)$$

where we have added the additive noise term to account for errors owing to the linear approximation.

We define $\mathcal{X} = \{x \in \mathbb{R}^3 : [-\pi/4, -\pi/4, -2]^\top \leq x \leq [\pi/4, \pi/4, 2]^\top\}$, and $\mathcal{U} = \{u \in \mathbb{R} : -1 \leq u \leq 1\}$, and we let $\mathbf{z}_{\text{ref}}$ satisfy $[\nu_{\min}, \rho_{\min}]^\top \leq z \leq [\nu_{\max}, \rho_{\max}]^\top$, for all $z \in \mathbf{z}_{\text{ref}}$, where $\nu_{\min} = 1.0$, $\nu_{\max} = 20.0$, $\rho_{\min} = -0.025$, $\rho_{\max} = 0.025$. Thus, $\mathcal{S} = \text{co}\{S(\nu_{\min}, \rho_{\min}), S(\nu_{\max}, \rho_{\min}), S(\nu_{\min}, \rho_{\max}), S(\nu_{\max}, \rho_{\max})\}$, where $S(v, \rho) = [A(v), B(v), D, r(v, \rho)]$, and $[A_k, B_k, D, r_k] \in \mathcal{S}$ for all $k = 0, \dots, T$. We set the maximum probability of constraint violations to $p_{x,i} = 0.025$ for $i = 1, \dots, 6$ and $p_{u,j} = 0.05$ for $j = 1, 2$. The length of the control horizon is $N = 4$, the cost matrices are given by $Q = \text{diag}(1, 1, 1)$, $R = 100$, and

$x_{gk} = \mathbf{0}_{3 \times 1}$ for all $k = 0, 1, \dots, T$. The vehicle dimensions are $\ell_f = \ell_r = 2.4$. We set the noise parameters as $\theta_\delta = \theta_\psi = \theta_y = 0.1$ and the simulation length $T = 50$.

The proposed approach is compared with two naïve implementations of CS-SMPC over 20 Monte Carlo trials in which uncertainty enters the system purely through the additive disturbances. The first implementation uses nominal terminal sets given by $\mathcal{X}_{f,\text{nom}}^\mu$ and $\Sigma_{f,\text{nom}}$ computed using an LTI system given by taking the mean values of ν_{x_k} and ρ_k over $k = 0, \dots, T$, and the second uses no terminal constraints at all. The proposed approach is evaluated with three initialization schemes compared in Table 2: (8) which is the most conservative and ensures Theorem 7, a probabilistic reachable set (PRS) method similar to [17] which is less conservative and also ensures Theorem 7⁴, and (29) which ensures Corollary 8. Due to the weighting of R , (5) is dominated by the control cost, so the more conservative methods incur the highest cost. The three approaches are simulated over 20 Monte Carlo trials with the results, including the computed robust terminal set as well as the closed-loop CS-SMPC trajectories, shown in Figs. 2-3.

Table 2

Average total cost for different initialization schemes, computed by summing (5) over the simulation length T and averaging over the 20 Monte Carlo trials.

Initialization	(8)	PRS	(29)
Average Total Cost	343	261	228
Std. Dev. Total Cost	35.9	34.2	17.3

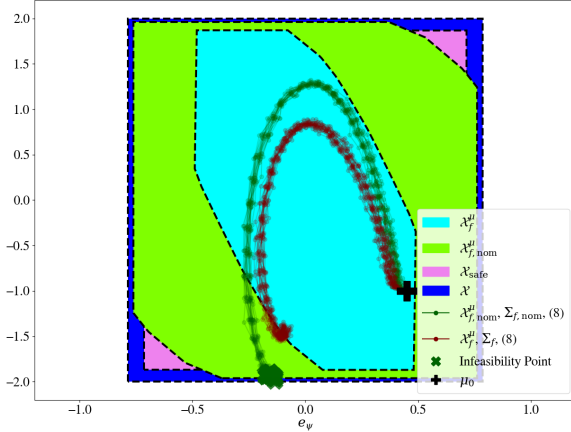


Fig. 2. Computed robust terminal set $\mathcal{X}_f^\mu$, compared with a nominal terminal set for a single set of system parameters $\mathcal{X}_{f,\text{nom}}^\mu$, and the tightened state constraints $\mathcal{X}_{\text{safe}}$. Monte Carlo closed-loop CS-SMPC trajectories with robust $\mathcal{X}_f^\mu$ and Σ_f compared with using nominal terminal constraints.

⁴ Additional methods to reduce conservatism while ensuring Theorem 7 include indirect feedback [16] and initial state interpolation [19] which also lie between the extremes of initialization schemes (8) and (29) in terms of conservatism vs. performance.

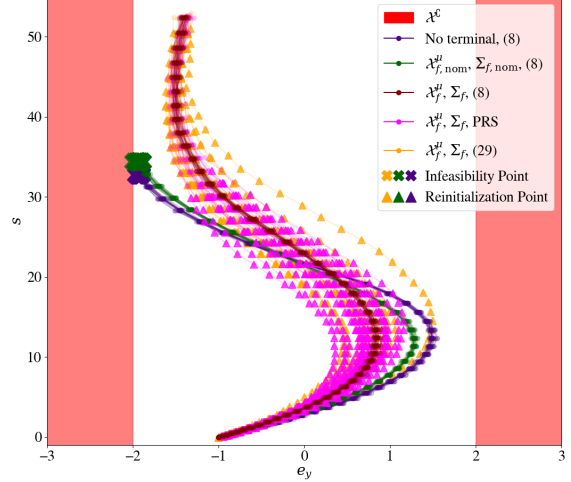


Fig. 3. Monte Carlo closed-loop CS-SMPC trajectories in curvilinear coordinates with robust $\mathcal{X}_f^\mu$ and Σ_f for different initialization schemes, compared with CS-SMPC using nominal and no terminal constraints, respectively.

It may be seen from Fig. 2 that the robust terminal set is a subset of the nominal terminal set, $\mathcal{X}_f^\mu \subseteq \mathcal{X}_{f,\text{nom}}^\mu$, as the robust terminal set is the intersection of all nominal control invariant sets for any system parameters. Likewise, the terminal set $\mathcal{X}_f^\mu$ is a subset of $\mathcal{X}_{\text{safe}}$ as it is the set for which any value of the mean state may be contained within $\mathcal{X}_{\text{safe}}$ at all future time steps, illustrating the additional conservatism necessitated by the robust terminal constraints.

Figure 3 shows the closed loop trajectories in curvilinear coordinates for the vehicle. The colored crosses indicate points at which no feasible solution to the optimal control problem exists and the corresponding trajectory ends. This occurs when the vehicle reaches a state from which no control policy exists within the search space that will prevent it from violating the state constraints in the near future. It may be seen that the closed-loop trajectories using CS-SMPC with the robust terminal constraints ($\mathcal{X}_f^\mu, \Sigma_f$) obey the state constraints and do not lead to infeasibility of Problem (6). When CS-SMPC uses the incorrect terminal constraints ($\mathcal{X}_{f,\text{nom}}^\mu, \Sigma_{f,\text{nom}}$), it may be seen from the green crosses, that Problem (6) does not remain recursively feasible for all trajectories, due to the lack of invariance over all system parameters.

6 Conclusion

We have established the first recursive feasibility result in the literature for SMPC using convex affine feedback for ATV systems subject to unbounded disturbances. We first presented a theorem giving sufficient conditions for the recursive feasibility of the ATV SMPC optimization problem. Then, we introduced a result showing how the terminal constraints may be calculated to satisfy these conditions. Finally, we showed that these results extend to the CS-SMPC convex problem formulation used in

our previous work, for which we have already established convergence guarantees.

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