

Reachability and Controllability Analysis of the State Covariance for Linear Stochastic Systems [★]

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Abstract

This paper studies the set of terminal state covariances that are reachable over a finite time horizon from a given initial state covariance for a linear stochastic system subject to additive noise. For discrete-time systems, a complete characterization of the set of reachable state covariances is given. For continuous-time systems, we present an upper bound on the set of reachable state covariances. Moreover, for both linear discrete-time and continuous-time systems, necessary and sufficient conditions are provided for the controllability of the state covariance over a finite horizon.

Key words: Covariance Steering, Reachability, Controllability, Finite Horizon, Riccati Differential Equation.

1 Introduction

For the past forty years, covariance control theory has emerged as a tool for addressing the growing need to quantify and control the uncertainty in dynamical systems. The study of covariance control theory first addressed the *infinite-horizon* state covariance assignment problem [1, 2], which aims to find the set of all admissible state-state covariances of a linear stochastic system subject to additive noise. In the same works, the parameterization of all state-feedback controllers for attaining an assignable state covariance is also presented. The optimal control law to achieve an admissible steady-state covariance with minimum control energy is provided in a closed form in [3].

In recent years, covariance steering theory, which aims to steer the state covariance from a given initial covariance to a desired terminal covariance over a *finite horizon*, has gained popularity in a variety of control and motion planning problems [4–8]. With these applications in mind, several optimal covariance steering problems have been studied for both continuous-time and discrete-time stochastic systems. For a continuous-time linear stochas-

tic system with a quadratic cost, when the noise coefficient matrix coincides with the control coefficient matrix, the optimal control law can be derived in closed form [9–11]. When the noise coefficient matrix differs from the control coefficient matrix, the optimal control law can be computed by recasting the optimal covariance steering problem as a semi-definite program [12–14]. In addition to the first two moments, the question of how to optimally steer the whole state density of a stochastic system between two prescribed endpoint densities with minimum control energy has also been studied [15, 16]. In the basic discrete-time version, the optimal control policy has been derived to steer the state covariance of a discrete-time linear stochastic system with different noise models and objective functions [17–19] or different types of constraints (e.g., chance constraints) [20–23]. Furthermore, the system matrices may be subject to unknown parameters [24] or can be completely unknown [25]. General distribution steering problems in discrete time have been tackled using either characteristic functions or power moments [26–28]. Multi-agent covariance and distribution steering problems have also been explored in [29, 30].

Despite the extensive research on optimal covariance steering theory, the more fundamental question regarding which terminal state covariances can be reached from a given initial state covariance via a proper control law remains largely unanswered, except for some special cases.

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One special case, where a complete analysis is available, is the controllability of the continuous-time state covariance. This case addresses the problem under which (sufficient) conditions a positive-definite terminal state covariance can be reached from a given initial positive-definite state covariance. Specifically, as it was shown in [31], for a continuous-time deterministic linear time-invariant (LTI) system $\dot{x}(t) = Ax(t) + Bu(t)$, its state covariance is controllable on a finite time interval if the matrix pair (A, B) is controllable. This result still holds even when the LTI system is subject to additive noise [32]. Similarly, for a linear time-varying stochastic system $dx(t) = (A(t)x(t) + B(t)u(t))dt + D(t)dw(t)$, its state covariance is controllable on the time interval $[0, T]$ if, for all $s \in [0, T]$, the matrix pair $(A(t), B(t))$ is controllable on $[s, T]$ [33]. Interestingly, either input delay or output feedback has been shown to incur a lower bound on the terminal state covariance [34, 35]. On the other hand, when there is no noise in a linear time-varying system $\dot{x}(t) = A(t)x(t) + B(t)u(t)$, where the pair $(A(t), B(t))$ is controllable on all subintervals of $[0, T]$, it is established in [15] via an optimal transport argument that any initial continuous distribution (i.e., a distribution without point mass) can be steered to any target continuous distribution. As pointed out in [31], the controllability/reachability of the state covariance for a linear system (A, B) can be studied through those of the state transition matrix Φ_{A+BF} for $A+BF(t)$, where the state feedback gain matrix $F(t)$ serves as the control input. A recent work [36] fixes a flaw in the proof of the controllability result in [31] and shows that at any terminal time $T > 0$, $\Phi_{A+BF}(t, 0)$ can be driven from the identity matrix at time $t = 0$ to any matrix with a positive determinant at time $t = T$ via an integrable $F(t)$, provided that (A, B) is controllable.

Another special case where controllability has been thoroughly analyzed is for the discrete-time stochastic systems with no noise. For a one-dimensional linear system $x_{k+1} = a_k x_k + u_k$, where a_k is a random variable, it has been shown in [27] that the state covariance (and higher power moments) is controllable in a finite time, provided that the control input is independent of the state. For a multi-dimensional, possibly nonlinear, system $x_{k+1} = f_k(x_k, u_k)$, it is possible to steer an initial distribution to a target distribution with the same cost as the optimal transport cost, as long as the system is controllable with a compact control set [37]. Not surprisingly, for LTI systems, the computation of the reachable sets is often computationally much more tractable than the general case. When there is additive noise added, the maximal backward reachable ambiguity set parameterized by the first and second moments under chance constraints on the state and the control input of the stochastic LTI system $x_{k+1} = Ax_k + Bu_k + Dw_k$ can be efficiently computed via semi-definite programming (SDP) [38].

In this paper, we study the more general problem of the

reachability of the terminal state covariance for linear *stochastic* systems; we also provide necessary and sufficient conditions for the controllability of the state covariance over a finite horizon.

Contributions: First, we provide a complete characterization of the set of reachable state covariances for a *discrete-time* linear stochastic system with and without additive noise (Theorem 1, Theorem 2, and Theorem 3). Second, we present an upper bound on the reachable set of terminal state covariances for a *continuous-time* linear stochastic system corrupted by additive noise (Theorem 5). Third, we establish necessary and sufficient conditions for the finite-horizon controllability of the state covariance for *both discrete-time* and *continuous-time* linear stochastic systems in the presence of additive noise (Theorem 4 and Theorem 7). In the process of our analysis, we also derive necessary and sufficient conditions for a Riccati differential equation to admit a unique solution on a given time interval (Theorem 6), which may be of independent interest. To the best of our knowledge, this is the first work addressing the reachability of the state covariance over a finite horizon, which complements the assignability results (over an infinite horizon) by Skelton et al. [1, 2].

The rest of the paper is organized as follows. The problems of interest are formulated in Section 2. The discrete-time and continuous-time results are presented in Section 3 and Section 4, respectively. A numerical example demonstrating the theory is given in Section 5. To help the exposition, some auxiliary lemmas used in the proofs of the main results are postponed to the Appendix.

2 Problem Formulation

Consider the discrete-time linear stochastic system

$$x_{k+1} = A_k x_k + B_k u_k + D_k w_k, \quad k = 0, 1, 2, \dots \quad (1)$$

where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^p$ is the control input, $\{w_k\}_{k \geq 0}$ is a sequence of independent, square-integrable q -dimensional random noise vectors with $\mathbb{E}[w_k] = 0$ and $\mathbb{E}[w_k w_k^T] = I_q$, the initial state x_0 is independent of the noise sequence $\{w_k\}_{k \geq 0}$, and $A_k \in \mathbb{R}^{n \times n}$, $B_k \in \mathbb{R}^{n \times p}$, and $D_k \in \mathbb{R}^{n \times q}$ are known coefficient matrices.

Assume that the initial state x_0 has a finite positive semi-definite covariance matrix $\Sigma_0 \succeq 0$. At any given time $k = 1, 2, \dots$, a state covariance $\bar{\Sigma}_k$ is said to be *reachable from* Σ_0 if there exists a finite-expected-energy control sequence $\{u_0, u_1, \dots, u_{k-1}\}$ such that, at each time step $i = 0, 1, \dots, k-1$, u_i depends only on i and on the past history of the states $x_0, x_1, \dots, x_i$, and drives the state covariance of (1) from Σ_0 at time 0 to $\bar{\Sigma}_k$ at exactly time k . In particular, the state covariance of (1) is *controllable from time 0 to k* if, for any pair of matrices $\bar{\Sigma}_0, \bar{\Sigma}_k \succeq 0$, there exists such a control sequence $\{u_0, u_1, \dots, u_{k-1}\}$ that steers the state covariance of the system from $\Sigma_0 =$

$\bar{\Sigma}_0$ to $\Sigma_k = \bar{\Sigma}_k$.

Problem 1 Find the set of reachable state covariances of the system (1) at time $k = 1, 2, \dots$ starting from a given initial state covariance $\Sigma_0 \succeq 0$ at time 0.

Problem 2 Find necessary and sufficient conditions for the state covariance of the system (1) to be controllable from time 0 to $k = 1, 2, \dots$.

We will also consider the continuous-time linear stochastic system

$$dx(t) = A(t)x(t)dt + B(t)u(t)dt + D(t)dw(t), \quad t \in [0, T] \quad (2)$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^p$ is the control input, $w(t) \in \mathbb{R}^q$ is an independent, square-integrable, martingale noise signal with $d\mathbb{E}[w(t)w(t)^\top] = I_q dt$, the initial state $x(0)$ is independent of the noise $\{w(t)\}_{t \geq 0}$, and $A(t) \in \mathbb{R}^{n \times n}$, $B(t) \in \mathbb{R}^{n \times p}$, and $D(t) \in \mathbb{R}^{n \times q}$ are known coefficient matrices.

Similarly to the discrete-time case, we assume that the initial state $x(0)$ has a finite positive semi-definite covariance matrix $\Sigma_0 \succeq 0$. At any given time $T > 0$, a state covariance $\bar{\Sigma}_T$ is said to be *reachable* from Σ_0 if there exists a finite-expected-energy control u that, for each time $t \in [0, T]$, depends only on t and on the past history of the states $\{x(s) : 0 \leq s \leq t\}$, and drives the state covariance of (2) from Σ_0 at time 0 to $\bar{\Sigma}_T$ at time T . In particular, the state covariance of (2) is *controllable* from time 0 to T if, for any pair of matrices $\bar{\Sigma}_0, \bar{\Sigma}_T \succ 0$, there exists such a control u that steers the state covariance of the system from $\Sigma(0) = \bar{\Sigma}_0$ to $\Sigma(T) = \bar{\Sigma}_T$.

Problem 3 Find the set of reachable state covariances of the system (2) at time $T > 0$, starting from a given initial state covariance $\Sigma_0 \succeq 0$ at time 0.

Problem 4 Find necessary and sufficient conditions for the state covariance of the system (2) to be controllable from time 0 to $T > 0$.

In this paper, we will provide complete answers to Problem 1, Problem 2, and Problem 4, as well as a partial answer to Problem 3.

For clarity, some notation used in this paper is briefly summarized below. Let M be an $n \times m$ real-valued matrix. The column span of M is denoted by $\text{range } M$, which is a (linear) subspace in $\mathbb{R}^n$. Let $\mathcal{Z}$ be a subspace in $\mathbb{R}^n$. The *orthogonal complement* of $\mathcal{Z}$, denoted by $\mathcal{Z}^\perp$, is the set of all vectors in $\mathbb{R}^n$ that are orthogonal to every vector in $\mathcal{Z}$. It can be verified that $\mathcal{Z}^\perp$ is also a subspace in $\mathbb{R}^n$. For any real matrix M , it is straightforward to verify that $\ker M = (\text{range } M^\top)^\perp$. If M is symmetric, we further have $\ker M = (\text{range } M)^\perp$. The *orthogonal projection* onto the subspace $\mathcal{Z}$ is the projection along $\mathcal{Z}^\perp$ onto $\mathcal{Z}$. Let Z be any matrix whose columns form a basis of $\mathcal{Z}$. Then, the orthogonal projection matrix onto $\mathcal{Z}$ is

given by $Z(Z^\top Z)^{-1}Z^\top$. For an $n \times n$ symmetric matrix M , its *square root* $M^{\frac{1}{2}}$ is given by $M^{\frac{1}{2}} = P\Lambda^{\frac{1}{2}}P^\top$, where $\Lambda^{\frac{1}{2}} = \text{diag}[\sqrt{\lambda_1}, \dots, \sqrt{\lambda_j}, i\sqrt{\lambda_{j+1}}, \dots, i\sqrt{\lambda_k}, 0, \dots, 0]$ and i is the imaginary unit, where P is an orthogonal matrix and $\Lambda = \text{diag}[\lambda_1, \dots, \lambda_j, -\lambda_{j+1}, \dots, -\lambda_k, 0, \dots, 0]$ is a real diagonal matrix with $\lambda_1, \dots, \lambda_k > 0$ such that $M = P\Lambda P^\top$.

3 Discrete-Time Case

In this section, we first show that it suffices to consider the class of randomized state-feedback controls for the system (1). Then, we characterize the set of all reachable state covariances for the special case when (1) has no noise, that is, $D_k \equiv 0$. Next, we extend the result to the general case when $D_k \neq 0$. Lastly, from the reachability of the state covariance, we derive a necessary and sufficient condition for the controllability of the state covariance of (1) over a finite horizon.

3.1 Randomized State-Feedback Controls

First, we recall a result from [39] that characterizes a positive semi-definite matrix, which will be used multiple times in this paper.

Proposition 1 Let M be a symmetric matrix partitioned as

$$M = \begin{bmatrix} M_1 & M_2 \\ M_2^\top & M_3 \end{bmatrix},$$

where M_1 and M_3 are square. Then, $M \succeq 0$ if and only if $M_1 \succeq 0$, $\text{range } M_2 \subseteq \text{range } M_1$, and $M_3 - M_2^\top M_1^- M_2 \succeq 0$, where M_1^- is a generalized inverse (not necessarily unique) of M_1 satisfying $M_1 M_1^- M_1 = M_1$.

Let the *generalized Schur complement* of the block M_1 in the matrix M be denoted by $M/M_1 \triangleq M_3 - M_2^\top M_1^- M_2$. Clearly, M_1^- can be chosen as the (unique) Moore–Penrose inverse M_1^+ of M_1 . We will use Proposition 1 to show that we only need to consider randomized state feedback controls to steer the state covariance of (1).

Let Σ_k denote the covariance of the state x_k at time k in (1). The state covariance then satisfies

$$\Sigma_{k+1} = A_k \Sigma_k A_k^\top + B_k \Sigma_k^{ux} A_k^\top + A_k \Sigma_k^{ux^\top} B_k^\top + B_k \Sigma_k^{uu} B_k^\top + D_k D_k^\top, \quad (3)$$

where Σ_k^{ux} is the cross covariance of u_k and x_k , and Σ_k^{uu} is the covariance of u_k . They are both treated as “control” terms in equation (3). Since the covariance matrix of the

stacked vector $[x_k^\top \ u_k^\top]^\top$ is $\begin{bmatrix} \Sigma_k & \Sigma_k^{ux^\top} \\ \Sigma_k^{ux} & \Sigma_k^{uu} \end{bmatrix}$, which is always positive semi-definite, it follows from Proposition 1 that there exists a deterministic matrix $F_k \in \mathbb{R}^{p \times n}$, which

may depend on the distributions of u_k and x_k , such that $\Sigma_k^{ux} = F_k \Sigma_k$ and $\Sigma_k^{uu} \succeq \Sigma_k^{ux} \Sigma_k^+ \Sigma_k^{ux\top} = F_k \Sigma_k F_k^\top$. Thus, (3) can be written equivalently as

$$\Sigma_{k+1} = (A_k + B_k F_k) \Sigma_k (A_k + B_k F_k)^\top + B_k V_k B_k^\top + D_k D_k^\top, \quad (4)$$

where $V_k = \Sigma_k^{uu} - F_k \Sigma_k F_k^\top \succeq 0$. Hence, we have reached the following conclusion.

Proposition 2 *At time $K = 1, 2, \dots$, if a terminal state covariance $\bar{\Sigma}_K$ of the discrete-time system (1) is reachable from a given initial state covariance Σ_0 , then, $\bar{\Sigma}_K$ is also reachable via a randomized state feedback control $u_k = F_k x_k + \nu_k$, for some $F_k \in \mathbb{R}^{p \times n}$, where $\nu_k \in \mathbb{R}^p$, ($k = 0, 1, \dots, K-1$) is a sequence of independent, square-integrable random vectors with $\mathbb{E}[\nu_k] = 0$ and $\mathbb{E}[\nu_k \nu_k^\top] \succeq 0$.*

Proof: Apply the control $u_k = F_k x_k + \nu_k$ to system (1), where $\mathbb{E}[\nu_k \nu_k^\top] = V_k \succeq 0$. We then obtain that $x_{k+1} = (A_k + B_k F_k) x_k + B_k \nu_k + D_k w_k$. Since $\mathbb{E}[\nu_k] = 0$, $\mathbb{E}[w_k] = 0$, $\mathbb{E}[\nu_k \nu_k^\top] = V_k$, and $\mathbb{E}[w_k w_k^\top] = I_q$, it follows that the state covariance of the closed-loop system satisfies (4). In light of the equivalence of the state covariance equations (3) and (4), Proposition 2 holds. $\blacksquare$

3.2 Reachability of the State Covariance without Noise

Without additive noise in (1), we have $D_k \equiv 0$. In light of (4), in this case the state covariance of (1) propagates according to

$$\Sigma_{k+1} = (A_k + B_k F_k) \Sigma_k (A_k + B_k F_k)^\top + B_k V_k B_k^\top, \quad (5)$$

where $F_k \in \mathbb{R}^{p \times n}$ and $V_k \in \mathbb{R}^{p \times p}$, $V_k \succeq 0$.

Let $\Phi_A(k, i)$ be the state transition matrix of A_s from time i to k . That is, $\Phi_A(k, k) \triangleq I_n$ and, for $i < k$, $\Phi_A(k, i) \triangleq A_{k-1} A_{k-2} \cdots A_i$. Let $G(k, i)$ be the reachability Gramian of the matrix pair (A_s, B_s) from time i to k . That is,

$$G(k, i) \triangleq \begin{cases} 0_{n \times n}, & i = k, \\ \sum_{j=i}^{k-1} \Phi_A(k, j+1) B_j B_j^\top \Phi_A(k, j+1)^\top, & i < k. \end{cases}$$

Theorem 1 *When $D_i \equiv 0$ and the system (1) starts from an initial state covariance $\Sigma_0 \succeq 0$, the set of all reachable state covariances at time $k = 1, 2, \dots$ is*

$$\mathcal{R}_{k,0} \triangleq \{ \Sigma_k \succeq 0 : P_{k,0} \Sigma_k P_{k,0} = P_{k,0} \Phi_A(k, 0) \Sigma_0 \Phi_A(k, 0)^\top P_{k,0} \},$$

where $P_{k,0}$ is the orthogonal projection onto $\ker G(k, 0)$.

Proof: We show Theorem 1 by induction on k . When $k = 1$, since $\text{range } G(1, 0) = \text{range } B_0$, it follows from (5)

and Lemma 1 in the Appendix that Σ_1 is a positive semi-definite matrix with $P_{1,0} \Sigma_1 P_{1,0} = P_{1,0} A_0 \Sigma_0 A_0^\top P_{1,0}$. Hence, Theorem 1 holds for $k = 1$.

Now, assume that Theorem 1 holds for $k = s$. Then, it follows from Lemma 1 in the Appendix that $\bar{\Sigma}_s = \Phi_A(s, 0) \Sigma_0 \Phi_A(s, 0)^\top + G(s, 0) \bar{X} + \bar{X}^\top G(s, 0)$ is reachable at time s , where $\bar{X} \in \mathbb{R}^{n \times n}$. It follows from (5) and Lemma 1 that $\bar{\Sigma}_{s+1} = A_s \bar{\Sigma}_s A_s^\top + B_s \bar{Y} + \bar{Y}^\top B_s^\top$ is reachable at time $s+1$, where $\bar{Y} \in \mathbb{R}^{p \times n}$. Hence, $\bar{\Sigma}_{s+1} = \Phi_A(s+1, 0) \Sigma_0 \Phi_A(s+1, 0)^\top + A_s G(s, 0) \bar{X} A_s^\top + A_s \bar{X}^\top G(s, 0) A_s^\top + B_s \bar{Y} + \bar{Y}^\top B_s^\top$. Since $G(s+1, 0) = A_s G(s, 0) A_s^\top + B_s B_s^\top$ and $G(s, 0) \succeq 0$, we have that $\text{range } G(s+1, 0) = \text{range } A_s G(s, 0) A_s^\top + \text{range } B_s B_s^\top = \text{range } A_s G(s, 0) + \text{range } B_s$. It follows that $P_{s+1,0} \bar{\Sigma}_{s+1} P_{s+1,0} = P_{s+1,0} \Phi_A(s+1, 0) \Sigma_0 \Phi_A(s+1, 0)^\top P_{s+1,0}$. That is, $\bar{\Sigma}_{s+1} \in \mathcal{R}_{s+1,0}$. We have thus shown that all reachable state covariances at time $s+1$ belong to the set $\mathcal{R}_{s+1,0}$.

Next, we show the other direction, i.e., each element of $\mathcal{R}_{s+1,0}$ is a reachable state covariance at time $s+1$. Let $\bar{\Sigma}_{s+1} \in \mathcal{R}_{s+1,0}$. Then, it follows from Lemma 1 in the Appendix that

$$\bar{\Sigma}_{s+1} = \begin{bmatrix} \Phi_A(s+1, 0) & A_s G(s, 0) & B_s \end{bmatrix} \times \begin{bmatrix} \Sigma_0 & \star_{12} & \star_{13} \\ \star_{21} & \star_{22} & \star_{23} \\ \star_{31} & \star_{32} & \star_{33} \end{bmatrix} \begin{bmatrix} \Phi_A(s+1, 0)^\top \\ G(s, 0) A_s^\top \\ B_s^\top \end{bmatrix},$$

where the matrix in the middle of the right-hand side is a positive semi-definite matrix with Σ_0 on its top left block, and the $\star$ entries are irrelevant. In light of Lemma 1 in the Appendix we have that

$$\bar{\Sigma}_s = \begin{bmatrix} \Phi_A(s, 0) & G(s, 0) \end{bmatrix} \begin{bmatrix} \Sigma_0 & \star_{12} \\ \star_{21} & \star_{22} \end{bmatrix} \begin{bmatrix} \Phi_A(s, 0)^\top \\ G(s, 0) \end{bmatrix} \in \mathcal{R}_{s,0},$$

thus $\bar{\Sigma}_s$ is reachable by the induction assumption. Since $\bar{\Sigma}_{s+1} = A_s \bar{\Sigma}_s A_s^\top + B_s \bar{Z} + \bar{Z}^\top B_s^\top \succeq 0$ for some $\bar{Z} \in \mathbb{R}^{p \times n}$, it follows from (5) and Lemma 1 that $\bar{\Sigma}_{s+1}$ is reachable. Thus, Theorem 1 holds for $k = s+1$. $\blacksquare$

Remark 1 Let $\mu_k \triangleq \mathbb{E}[x_k] \in \mathbb{R}^n$ denote the mean value of the state x_k of the system (1). Note that, at time $k = 1, 2, \dots$, the mean state μ_k is reachable from some initial mean state μ_0 if and only if $\mu_k \in \Phi_A(k, 0) \mu_0 + \text{range } G(k, 0)$. Clearly, at time k , the set of all reachable mean values of the state of the system (1) is $\{ \mu_k \in \mathbb{R}^n : P_{k,0} \mu_k = P_{k,0} \Phi_A(k, 0) \mu_0 \}$, which bears a close resemblance to the set $\mathcal{R}_{k,0}$ of reachable state covariances in Theorem 1. In addition, if $\mu_0 = 0$, the set of all reachable mean states of (1) at time k is given by $\text{range } G(k, 0)$, which is a linear subspace in $\mathbb{R}^n$. Likewise, if the initial state covariance $\Sigma_0 = 0$ and the system (1) has no additive noise (i.e., $w_k \equiv 0$ or $D_k \equiv 0$), the set of

all reachable state covariances of (1) at time k is given by cone $G(k, 0) \triangleq \{\Sigma \succeq 0 : \text{range } \Sigma = \text{range } G(k, 0)\}$, which is a convex subcone in the cone of $n \times n$ positive semi-definite matrices. Notice, from (3), that randomness in Σ_k , in addition to the randomness due to the initial condition Σ_0 and additive noise w_k , comes as a result of the randomized control sequence $u_0, u_1, \dots, u_{k-1}$.

3.3 Reachability of the State Covariance with Noise

Under the influence of additive noise, the state covariance of (1) propagates as in (4), that is,

$$\Sigma_{k+1} = (A_k + B_k F_k) \Sigma_k (A_k + B_k F_k)^\top + B_k V_k B_k^\top + D_k D_k^\top,$$

where $F_k \in \mathbb{R}^{p \times n}$ and $V_k \in \mathbb{R}^{p \times p}$, $V_k \succeq 0$.

Theorem 2 When system (1) starts from a given initial state covariance $\Sigma_0 \succeq 0$ at time 0, the set of all reachable state covariances at time $k = 1, 2, \dots$ is $\mathcal{R}_k \triangleq \sum_{i=0}^k \mathcal{R}_{k,i}$, where $\mathcal{R}_{k,0}$ is defined in Theorem 1 and, for $i = 1, 2, \dots, k$,

$$\begin{aligned} \mathcal{R}_{k,i} &\triangleq \{\Sigma_{k,i} \succeq 0 : \\ &P_{k,i} \Sigma_{k,i} P_{k,i} = P_{k,i} \Phi_A(k, i) D_{i-1} D_{i-1}^\top \Phi_A(k, i)^\top P_{k,i}\}, \end{aligned}$$

with $P_{k,i}$ being the orthogonal projection onto $\ker G(k, i)$.

Proof: We show Theorem 2 by induction on k . When $k = 1$, it follows from (4) and Theorem 1 that the set of reachable state covariances is $\mathcal{R}_{1,0} + D_0 D_0^\top = \mathcal{R}_{1,0} + \mathcal{R}_{1,1} = \mathcal{R}_1$.

Assume that Theorem 2 holds for $k = s$. That is, at time s , the set of all reachable state covariances is $\mathcal{R}_s$. It follows from (4) that, at time $s + 1$, the set of all reachable state covariances is $\{\Sigma_{s+1} \succeq 0 : \Sigma_{s+1} = (A_s + B_s F_s) \bar{\Sigma}_s (A_s + B_s F_s)^\top + B_s V_s B_s^\top +$

$D_s D_s^\top$, where $F_s \in \mathbb{R}^{p \times n}$, $V_s \in \mathbb{R}^{p \times p}$, $V_s \succeq 0$, $\bar{\Sigma}_s \in \mathcal{R}_s\}$. Since $\mathcal{R}_s = \sum_{i=0}^s \mathcal{R}_{s,i}$, it follows from Lemma 2 in the Appendix that the set of reachable state covariances at time $s + 1$ is equal to $D_s D_s^\top + \sum_{i=0}^s \mathcal{S}_{s+1,i}$, where $\mathcal{S}_{s+1,i} \triangleq \{\Sigma_{s+1,i} \succeq 0 : \Sigma_{s+1,i} = (A_s + B_s F_{s,i}) \bar{\Sigma}_{s,i} (A_s + B_s F_{s,i})^\top + B_s V_{s,i} B_s^\top, F_{s,i} \in \mathbb{R}^{p \times n}, V_{s,i} \in \mathbb{R}^{p \times p}, V_{s,i} \succeq 0, \bar{\Sigma}_{s,i} \in \mathcal{R}_{s,i}\}$. In light of Theorem 1 and equation (5), it is clear that, for all $i = 0, 1, \dots, s$, $\mathcal{S}_{s+1,i} = \mathcal{R}_{s+1,i}$. Since $\mathcal{R}_{s+1,s+1} = \{D_s D_s^\top\}$, it follows that, at time $s + 1$, the set of reachable state covariances is $\sum_{i=0}^{s+1} \mathcal{R}_{s+1,i} = \mathcal{R}_{s+1}$. Thus, Theorem 2 holds for $k = s + 1$, which completes the proof. $\blacksquare$

Remark 2 Notice that the set $\mathcal{R}_{k,k}$ contains a single element, namely, $D_{k-1} D_{k-1}^\top$. Thus, Theorem 2 is consistent with the fact that $\Sigma_k \succeq D_{k-1} D_{k-1}^\top$, which is reported in [40]. Intuitively, at each time $i = 1, 2, \dots, k$, we can treat $D_{i-1} D_{i-1}^\top \succeq 0$ as the recurrent “initial” state covariance starting from time i to time k . In this sense, Σ_0 can be regarded as $D_{-1} D_{-1}^\top$.

The next result illustrates an alternative expression of the set $\mathcal{R}_k$ of the reachable state covariances at time k .

Theorem 3 The set $\mathcal{R}_k$ defined in Theorem 2 can be written, equivalently, as $\mathcal{R}_k = \bigcap_{i=0}^k \mathcal{T}_{k,i}$, where

$$\begin{aligned} \mathcal{T}_{k,0} &\triangleq \left\{ \Sigma_k \succeq 0 : P_{k,0} \Sigma_k P_{k,0} = P_{k,0} \left[\Phi_A(k, 0) \Sigma_0 \Phi_A(k, 0)^\top \right. \right. \\ &\quad \left. \left. + \sum_{j=1}^k \Phi_A(k, j) D_{j-1} D_{j-1}^\top \Phi_A(k, j)^\top \right] P_{k,0} \right\}, \end{aligned}$$

and, for all $i = 1, 2, \dots, k$,

$$\mathcal{T}_{k,i} \triangleq \left\{ \Sigma_k \succeq 0 : P_{k,i} \Sigma_k P_{k,i} \succeq \right.$$

$$\begin{aligned} \bar{\Sigma}_k &= \begin{bmatrix} \Phi_A(k, 0) & \Phi_A(k, 1) & \cdots & A_{k-1} & I_n & \Phi_A(k, 1) B_0 & \Phi_A(k, 2) B_1 & \cdots & B_{k-1} \end{bmatrix} \\ &\times \begin{bmatrix} \Sigma_0 & & & & & & & & \\ & D_0 D_0^\top & & & & & & & \\ & & \ddots & & & & & & \\ & & & D_{k-2} D_{k-2}^\top & & & & & \\ & & & & D_{k-1} D_{k-1}^\top & & & & \end{bmatrix} \begin{bmatrix} \star_{1,k+2} & \star_{1,k+3} & \cdots & \star_{1,2k+1} \\ \star_{2,k+2} & \star_{2,k+3} & \cdots & \star_{2,2k+1} \\ \vdots & \ddots & \ddots & \vdots \\ \star_{k,k+2} & \star_{k,k+3} & \ddots & \star_{k,2k+1} \\ \star_{k+1,k+2} & \star_{k+1,k+3} & \cdots & \star_{k+1,2k+1} \end{bmatrix} \begin{bmatrix} \Phi_A(k, 0)^\top \\ \Phi_A(k, 1)^\top \\ \vdots \\ A_{k-1}^\top \\ I_n \\ B_0^\top \Phi_A(k, 1)^\top \\ B_1^\top \Phi_A(k, 2)^\top \\ \vdots \\ B_{k-1}^\top \end{bmatrix}, \quad (6) \\ &\begin{bmatrix} \star_{1,k+2}^\top & \star_{2,k+2}^\top & \cdots & \star_{k,k+2}^\top & \star_{k+1,k+2}^\top & \star_{k+2,k+2}^\top & \star_{k+2,k+3}^\top & \cdots & \star_{k+2,2k+1}^\top \\ \star_{1,k+3}^\top & \star_{2,k+3}^\top & \ddots & \star_{k,k+3}^\top & \star_{k+1,k+3}^\top & \star_{k+2,k+3}^\top & \star_{k+3,k+3}^\top & \cdots & \star_{k+3,2k+1}^\top \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \star_{1,2k+1}^\top & \star_{2,2k+1}^\top & \cdots & \star_{k,2k+1}^\top & \star_{k+1,2k+1}^\top & \star_{k+2,2k+1}^\top & \star_{k+3,2k+1}^\top & \cdots & \star_{2k+1,2k+1}^\top \end{bmatrix} \end{aligned}$$

$$P_{k,i} \left[\sum_{j=i}^k \Phi_A(k,j) D_{j-1} D_{j-1}^\top \Phi_A(k,j)^\top \right] P_{k,i} \Big\},$$

with $P_{k,i}$ being the orthogonal projection onto $\ker G(k,i)$.

Proof: Let $\bar{\Sigma}_k \in \mathcal{R}_k$. Then, $\bar{\Sigma}_k = \sum_{i=0}^k \bar{\Sigma}_{k,i}$, where $\bar{\Sigma}_{k,i} \in \mathcal{R}_{k,i}$. Since, for all $0 \leq i \leq j \leq k$, $\text{range } G(k,j) \subseteq \text{range } G(k,i)$, it follows that

$$P_{k,i} \bar{\Sigma}_{k,j} P_{k,i} = P_{k,i} \Phi_A(k,j) D_{j-1} D_{j-1}^\top \Phi_A(k,j)^\top P_{k,i}.$$

Hence, for all $i = 0, 1, \dots, k$, $\bar{\Sigma}_k \in \mathcal{T}_{k,i}$. Since $\bar{\Sigma}_k \in \bigcap_{i=0}^k \mathcal{T}_{k,i}$, we have $\mathcal{R}_k \subseteq \bigcap_{i=0}^k \mathcal{T}_{k,i}$.

Next, we show the other direction. Let $\bar{\Sigma}_k \in \bigcap_{i=0}^k \mathcal{T}_{k,i}$. Since $\text{range } G(k,0) = \sum_{i=1}^k \text{range } \Phi_A(k,i) B_{i-1}$, it follows that $P_{k,0}$ is also the orthogonal projection onto $(\sum_{i=1}^k \text{range } \Phi_A(k,i) B_{i-1})^\perp$. Then, it follows from Lemma 1 in the Appendix and the fact $\bar{\Sigma}_k \in \mathcal{T}_{k,0}$, that equation (6) holds, where the matrix in the middle of the right-hand side of (6) is a positive semi-definite matrix, denoted by Ω , with its upper left block equal to $\text{blkdiag}[\Sigma_0, D_0 D_0^\top, \dots, D_{k-1} D_{k-1}^\top]$. In light of Proposition 1, it follows from equation (6) and the fact $\bar{\Sigma}_k \in \mathcal{T}_{k,1}$, that $\star_{2,k+2} = \star_{3,k+2} = \dots = \star_{k+1,k+2} = 0$. Similarly, for $i = 2, 3, \dots, k$, the fact $\bar{\Sigma}_k \in \mathcal{T}_{k,i}$ implies that $\star_{i+1,k+i+1} = \star_{i+2,k+i+1} = \dots = \star_{k+1,k+i+1} = 0$ in addition to the previously known zero entries on the left of the same row(s) in the upper right block of Ω . According to this partition, we denote the upper left (respectively, upper right, bottom left, bottom right) block of Ω by Ω_{UL} (respectively, Ω_{UR} , Ω_{BL} , Ω_{BR}). Since $\Omega \succeq 0$, it follows from Proposition 1 that $\text{range } \Omega_{UR} \subseteq \text{range } \Omega_{UL}$ and $\Omega_{BR} \succeq \Omega_{BL} \Omega_{UL}^+ \Omega_{UR}$. Since $\text{range } \Omega_{UR} \subseteq \text{range } \Omega_{UL}$, we have $\text{range}[\star_{1,k+2} \ \star_{1,k+3} \ \dots \ \star_{1,2k+1}] \subseteq \text{range } \Sigma_0$ and $\text{range}[\star_{i+2,k+i+3} \ \star_{i+2,k+i+4} \ \dots \ \star_{i+2,2k+1}] \subseteq \text{range } D_i D_i^\top$ for $i = 0, 1, \dots, k-2$. Since $\Omega_{BR} \succeq \Omega_{BL} \Omega_{UL}^+ \Omega_{UR}$, we have that

$$\begin{aligned} \Omega_{BR} &\succeq \begin{bmatrix} \star_{1,k+2}^\top \\ \star_{1,k+3}^\top \\ \vdots \\ \star_{1,2k+1}^\top \end{bmatrix} \Sigma_0^+ \begin{bmatrix} \star_{1,k+2} & \star_{1,k+3} & \dots & \star_{1,2k+1} \end{bmatrix} \\ &+ \begin{bmatrix} \star_{2,k+3}^\top \\ \vdots \\ \star_{2,2k+1}^\top \end{bmatrix} (D_0 D_0^\top)^+ \begin{bmatrix} \star_{2,k+3} & \dots & \star_{2,2k+1} \end{bmatrix} \\ &+ \dots + \star_{k,2k+1}^\top (D_{k-2} D_{k-2}^\top)^+ \star_{k,2k+1}. \end{aligned}$$

Therefore, we can take $\bar{\Sigma}_{k,k} = D_{k-1} D_{k-1}^\top$,

$$\begin{aligned} \bar{\Sigma}_{k,k-1} &= \begin{bmatrix} A_{k-1} & B_{k-1} \end{bmatrix} \\ &\times \begin{bmatrix} D_{k-2} D_{k-2}^\top & \star_{k,2k+1} \\ \star_{k,2k+1}^\top & \star_{k,2k+1}^\top (D_{k-2} D_{k-2}^\top)^+ \star_{k,2k+1} \end{bmatrix} \begin{bmatrix} A_{k-1}^\top \\ B_{k-1}^\top \end{bmatrix}, \\ \bar{\Sigma}_{k,k-2} &= \begin{bmatrix} A_{k-1} A_{k-2} & A_{k-1} B_{k-2} & B_{k-1} \end{bmatrix} \Omega_{k-2} \begin{bmatrix} A_{k-2}^\top A_{k-1}^\top \\ B_{k-2}^\top A_{k-1}^\top \\ B_{k-1}^\top \end{bmatrix}, \end{aligned}$$

where,

$$\begin{aligned} \Omega_{k-2} &= \begin{bmatrix} D_{k-3} D_{k-3}^\top & \star_{k-1,2k} & \star_{k-1,2k+1} \\ \star_{k-1,2k}^\top & \star & \star \\ \star_{k-1,2k+1}^\top & \star & \star \end{bmatrix}, \\ \Omega_{k-2} / D_{k-3} D_{k-3}^\top &= 0, \end{aligned}$$

and

$$\begin{aligned} \bar{\Sigma}_{k,1} &= \begin{bmatrix} \Phi_A(k,1) & \Phi_A(k,2) B_1 & \dots & B_{k-1} \end{bmatrix} \Omega_1 \begin{bmatrix} \Phi_A(k,1)^\top \\ B_1^\top \Phi_A(k,2)^\top \\ \vdots \\ B_{k-1}^\top \end{bmatrix}, \end{aligned}$$

where,

$$\Omega_1 = \begin{bmatrix} D_0 D_0^\top & \star_{2,k+3} & \dots & \star_{2,2k+1} \\ \star_{2,k+3}^\top & \star & \dots & \star \\ \vdots & \vdots & \ddots & \vdots \\ \star_{2,2k+1}^\top & \star & \dots & \star \end{bmatrix}, \quad \Omega_1 / D_0 D_0^\top = 0.$$

Finally, let

$$\begin{aligned} \bar{\Sigma}_{k,0} &= \bar{\Sigma}_k - \sum_{i=1}^k \bar{\Sigma}_{k,i} = \\ &\begin{bmatrix} \Phi_A(k,0) & \Phi_A(k,1) B_0 & \dots & B_{k-1} \end{bmatrix} \Omega_0 \begin{bmatrix} \Phi_A(k,0)^\top \\ B_0^\top \Phi_A(k,1)^\top \\ \vdots \\ B_{k-1}^\top \end{bmatrix}, \end{aligned}$$

where,

$$\Omega_0 = \begin{bmatrix} \Sigma_0 & \star_{1,k+2} & \cdots & \star_{1,2k+1} \\ \star_{1,k+2}^\top & \star & \cdots & \star \\ \vdots & \vdots & \ddots & \vdots \\ \star_{1,2k+1}^\top & \star & \cdots & \star \end{bmatrix}, \quad \Omega_0/\Sigma_0 \succeq 0.$$

In light of Proposition 1, we have, for all $i = 0, 1, \dots, k-2$, $\Omega_i \succeq 0$. It follows from Lemma 1 in the Appendix that, for all $i = 0, 1, \dots, k$, $\bar{\Sigma}_{k,i} \in \mathcal{R}_{k,i}$. Thus, $\bar{\Sigma}_k \in \mathcal{R}_k$. Therefore, $\bigcap_{i=0}^k \mathcal{T}_{k,i} \subseteq \mathcal{R}_k$. $\blacksquare$

3.4 Controllability of the State Covariance

In this subsection, necessary and sufficient conditions for the controllability of the state covariance of (1) are given. These resemble the condition for the controllability of (A_k, B_k) , equivalently, of the mean value of the state of (1).

The following assumption is needed to derive one of the conditions for the controllability of the state covariance, which ensures that the deterministic system $x_{k+1} = A_k x_k + B_k u_k$, where $k = 0, 1, 2, \dots$, is time reversible.

Assumption 1 For all $k = 0, 1, 2, \dots$, the matrix A_k is invertible.

Let $\hat{G}(k, i)$ denote the controllability Gramian of the matrix pair (A_s, B_s) from time i to k . That is, $\hat{G}(k, i) \triangleq \Phi_A(k, i)G(k, i)\Phi_A(k, i)^\top$, where $G(k, i)$ is the reachability Gramian defined above Theorem 1 and $\Phi_A(k, i) \triangleq (\Phi_A(k, i))^{-1} = A_i^{-1}A_{i+1}^{-1} \cdots A_{k-1}^{-1}$. Note that, under Assumption 1, $\hat{G}(k, i) \succ 0$ if and only if $G(k, i) \succ 0$.

Theorem 4 The following statements are equivalent.

- i) The state covariance of the system (1) is controllable from time 0 to k .
- ii) $G(k, 0) \succ 0$ and $\text{range } \Phi_A(k, i)D_{i-1} \subseteq \text{range } G(k, i)$ for all $i = 1, 2, \dots, k$.
- iii) Under Assumption 1, $\hat{G}(k, 0) \succ 0$ and, for all $i = 1, 2, \dots, k$, $\text{range } D_{i-1} \subseteq \text{range } \hat{G}(k, i)$.

Proof: It follows from the definition of the controllability Gramian that ii) $\iff$ iii). Next, we show that i) $\iff$ ii). Let $\mathcal{S}^+$ denote the set of all $n \times n$ positive semi-definite matrices. In light of Theorem 2, the state covariance of (1) is controllable from time 0 to k if and only if $\mathcal{R}_k = \mathcal{S}^+$.

If $G(k, 0) \succ 0$, then, $P_{k,0} = 0$. It follows that $\mathcal{R}_{k,0} = \mathcal{S}^+$. If, for each $i = 1, 2, \dots, k$, $\text{range } \Phi_A(k, i)D_{i-1} \subseteq \text{range } G(k, i)$, then, we have $0_{n \times n} \in \mathcal{R}_{k,i} \subseteq \mathcal{S}^+$. It follows that $\mathcal{R}_k = \sum_{i=0}^k \mathcal{R}_{k,i} = \mathcal{S}^+$. Thus, ii) $\implies$ i).

Next, we show the other direction. It follows from (4) that

$$\Sigma_k = \Phi_{A+BF}(k, 0)\Sigma_0\Phi_{A+BF}(k, 0)^\top + \sum_{i=1}^k \Phi_{A+BF}(k, i) \times (D_{i-1}D_{i-1}^\top + B_{i-1}V_{i-1}B_{i-1}^\top)\Phi_{A+BF}(k, i)^\top, \quad (7)$$

where $\Phi_{A+BF}(k, i)$ is the state transition matrix of $A_s + B_s F_s$ from time $i = 0, 1, \dots, k$ to time k . If $\mathcal{R}_k = \mathcal{S}^+$, then, clearly, for any $\Sigma_0 \succ 0$, there exists $F_i \in \mathbb{R}^{p \times n}$ and $V_i \in \mathbb{R}^{p \times p}$, $V_i \succeq 0$, $i = 0, 1, \dots, k$, such that $\Sigma_k = 0_{n \times n}$. Let $V_i \equiv 0$. In view of (7), it follows that there exists F_s such that $\Phi_{A+BF}(k, 0) = 0$ and, for all $i = 1, 2, \dots, k$, $\Phi_{A+BF}(k, i)D_{i-1} = 0$. Since the deterministic system $x_{s+1} = (A_s + B_s F_s)x_s$, where $s = 0, 1, 2, \dots$, satisfies $x_k = \Phi_{A+BF}(k, 0)x_0$, it follows from $\Phi_{A+BF}(k, 0) = 0$ that the pair (A_s, B_s) is controllable from time 0 to k . Thus, $G(k, 0) \succ 0$. Similarly, for each $i = 1, 2, \dots, k$, the fact that there exists F_s such that $\Phi_{A+BF}(k, i)D_{i-1} = 0$ implies that $\text{range } \Phi_A(k, i)D_{i-1} \subseteq \text{range } G(k, i)$. Therefore, i) $\implies$ ii). $\blacksquare$

Remark 3 Notice that in the second statement of Theorem 4, for $i = k$, we obtain that $\text{range } \Phi_A(k, k)D_{k-1} = \text{range } D_{k-1} \subseteq \text{range } G(k, k) = 0$, i.e., $D_{k-1} = 0$. In other words, if the state covariance of the system (1) is controllable from time 0 to k , we must have $D_{k-1} = 0$. Recall that the matrix pair (A_s, B_s) is controllable from time 0 to k if and only if $G(k, 0) \succ 0$, which is equivalent to $\hat{G}(k, 0) \succ 0$ under Assumption 1. Moreover, in case $G(k, 0) \succeq 0$, for a given initial state $\bar{x}_0 \in \mathbb{R}^n$, there exists a control that steers the state of the deterministic system $x_{s+1} = A_s x_s + B_s u_s$ from $x_0 = \bar{x}_0$ to $x_k = 0$ if and only if $\Phi_A(k, 0)\bar{x}_0 \in \text{range } G(k, 0)$, which is equivalent to $\bar{x}_0 \in \text{range } \hat{G}(k, 0)$ under Assumption 1. Clearly, for each $i = 1, 2, \dots, k$, if we treat $D_{i-1}D_{i-1}^\top \succeq 0$ as the recurrent “initial” state covariance starting from time i to time k , then, the conditions in Theorem 4 resemble the conditions for the controllability of the matrix pair (A_s, B_s) .

Corollary 1 Assume that there is no additive noise in the system (1), that is, $D_i \equiv 0$. Then, the state covariance of (1) is controllable from time 0 to k if and only if the matrix pair (A_s, B_s) is controllable from time 0 to k .

Corollary 2 Assume that in the system (1), the additive noise is non-degenerate at time $i = 0, 1, \dots, k-2$ and is zero at time $k-1$. That is, for all $i = 0, 1, \dots, k-2$, $\text{rank } D_i = n$, and $D_{k-1} = 0$. Then, under Assumption 1, the state covariance of (1) is controllable from time 0 to k if and only if $\text{rank } B_{k-1} = n$.

Proof: Suppose that, for all $i = 0, 1, \dots, k-2$, $\text{rank } D_i = n$, and $D_{k-1} = 0$. In light of Theorem 4, we need to show that under Assumption 1, $\text{rank } B_{k-1} = n$ if and only if $G(k, 0) \succ 0$ and $\text{range } \Phi_A(k, i)D_{i-1} \subseteq \text{range } G(k, i)$ for all $i = 1, 2, \dots, k$.

Let $\text{rank } B_{k-1} = n$. Then, $G(k, k-1) = B_{k-1}B_{k-1}^\top \succ 0$. For $0 \leq i \leq j \leq k$, we have $G(k, i) \succeq G(k, j)$. Hence, for all $0 \leq i \leq k-1$, we have $G(k, i) \succeq G(k, k-1) \succ 0$. It follows that $\text{range } \Phi_A(k, i)D_{i-1} \subseteq \text{range } G(k, i)$ for all $i = 1, 2, \dots, k-1$. Since $D_{k-1} = 0$, it holds trivially that $\text{range } \Phi_A(k, k)D_{k-1} \subseteq \text{range } G(k, k)$.

Let $G(k, 0) \succ 0$ and $\text{range } \Phi_A(k, i)D_{i-1} \subseteq \text{range } G(k, i)$ for all $i = 1, 2, \dots, k$. In particular, $\text{range } \Phi_A(k, k-1)D_{k-2} \subseteq \text{range } G(k, k-1)$. That is, $\text{range } A_{k-1}D_{k-2} \subseteq \text{range } B_{k-1}B_{k-1}^\top$. As A_{k-1} is invertible and $\text{rank } D_{k-2} = n$, $\text{range } A_{k-1}D_{k-2} = \mathbb{R}^n$. Therefore, $\text{rank } B_{k-1} = n$. ■

4 Continuous-Time Case

In this section, we derive results regarding the controllability of the continuous-time covariance equation. We first show that it suffices to consider the class of state-feedback controls for the system (2). Then, we give an upper bound on the set of all reachable state covariances. Finally, we derive a necessary and sufficient condition for the controllability of the state covariance of (2) on a finite horizon.

4.1 State Feedback Controls

Let $\Sigma(t)$ denote the covariance of the state $x(t)$ at time t . It is clear that the state covariance of (2) satisfies

$$\begin{aligned} \dot{\Sigma}(t) = & A(t)\Sigma(t) + \Sigma(t)A(t)^\top \\ & + B(t)\Sigma^{ux}(t) + \Sigma^{ux}(t)^\top B(t)^\top + D(t)D(t)^\top, \end{aligned} \quad (8)$$

where $\Sigma^{ux}(t)$ is the cross-covariance of $u(t)$ and $x(t)$. Note that $\Sigma^{ux}(t)$ can be treated as the “control” in equation (8). Note also that, contrary to the discrete-time case, there is no Σ^{uu} term in (8). It follows from Proposition 1 that, for each $t \in [0, T]$, there exists a deterministic matrix $F(t) \in \mathbb{R}^{p \times n}$, which may depend on the distributions of $u(t)$ and $x(t)$, such that $\Sigma^{ux}(t) = F(t)\Sigma(t)$. Hence, (8) can be written as

$$\begin{aligned} \dot{\Sigma}(t) = & [A(t) + B(t)F(t)]\Sigma(t) + \Sigma(t)[A(t) + B(t)F(t)]^\top \\ & + D(t)D(t)^\top. \end{aligned} \quad (9)$$

Thus, we have the following result.

Proposition 3 *At time $T > 0$, if a terminal state covariance $\bar{\Sigma}_T$ of the continuous-time system (2) is reachable from a given initial state covariance $\Sigma(0) = \Sigma_0$, then, $\bar{\Sigma}_T$ is also reachable via a state feedback control $u(t) = F(t)x(t)$, for some $F(t) \in \mathbb{R}^{p \times n}$, $t \in [0, T]$.*

4.2 Upper Bound on Reachable State Covariances

Let $\Phi_A(T, t)$ denote the state transition matrix from time t to T of the matrix $A(\tau)$. Let

$$G(T, t) \triangleq \int_t^T \Phi_A(T, \tau)B(\tau)B(\tau)^\top \Phi_A(T, \tau)^\top d\tau, \quad (10)$$

be the *reachability Gramian* of the matrix pair $(A(\tau), B(\tau))$ from time t to T . It follows immediately from (9) that

$$\begin{aligned} \Sigma(T) = & \Phi_{A+BF}(T, 0)\Sigma(0)\Phi_{A+BF}(T, 0)^\top \\ & + \int_0^T \Phi_{A+BF}(T, \tau)D(\tau)D(\tau)^\top \Phi_{A+BF}(T, \tau)^\top d\tau. \end{aligned} \quad (11)$$

Notice from (11) that the crux of characterizing the set of reachable state covariances of (2) is the following: For a given matrix pair $(A(t), B(t))$ on $[0, T]$, which state transition matrix $\Phi_{A+BF}(T, \tau)$ from any time $\tau \in [0, T]$ to T is reachable via a proper choice of the feedback gain $F(t)$? It was shown recently in [36] that for the special case when (A, B) is a constant controllable pair, any square matrix with a positive determinant can be reached as $\Phi_{A+BF}(t, 0)$ on any time interval $[0, t]$ for $t > 0$. For the general case when $(A(t), B(t))$ is time-varying and may not be controllable, we do not have a complete answer to this question at this moment. Instead, we provide an upper bound on the set of reachable state transition matrices $\Phi_{A+BF}(T, \tau)$ in Lemma 3 in the Appendix, and utilize it to bound the set of reachable state covariances as follows.

Theorem 5 *Let $\Sigma_0 \succeq 0$ at time 0. The set of reachable state covariances at time $T > 0$, denoted by $\mathcal{R}_T$, satisfies $\mathcal{R}_T \subseteq \mathcal{R}_T^+ \subseteq \bigcap_{t \in [0, T]} \mathcal{T}_{T, t}$, where,*

$$\begin{aligned} \mathcal{R}_T^+ \triangleq & \{ \Sigma_T \succeq 0 : \Sigma_T = \\ & [\Phi_A(T, 0) + G(T, 0)K(0)]\Sigma_0[\Phi_A(T, 0) + G(T, 0)K(0)]^\top \\ & + \int_0^T [\Phi_A(T, \tau) + G(T, \tau)K(\tau)]D(\tau)D(\tau)^\top \\ & \times [\Phi_A(T, \tau) + G(T, \tau)K(\tau)]^\top d\tau \}, \end{aligned}$$

where $K(t) \in \mathbb{R}^{n \times n}$ is such that $\Phi_A(T, t) + G(T, t)K(t)$ has a positive determinant for all $t \in [0, T]$,

$$\begin{aligned} \mathcal{T}_{T, 0} \triangleq & \left\{ \Sigma_T \succeq 0 : P_{T, 0}\Sigma_T P_{T, 0} = P_{T, 0} \left[\Phi_A^{T, 0}\Sigma_0(\Phi_A^{T, 0})^\top \right. \right. \\ & \left. \left. + \int_0^T \Phi_A^{T, \tau}D(\tau)D(\tau)^\top(\Phi_A^{T, \tau})^\top d\tau \right] P_{T, 0} \right\}, \end{aligned}$$

and, for all $t \in (0, T]$,

$$\mathcal{T}_{T,t} \triangleq \left\{ \Sigma_T \succeq 0 : P_{T,t} \Sigma_T P_{T,t} \succeq P_{T,t} \left[\int_t^T \Phi_A^{T,\tau} D(\tau) D(\tau)^\top (\Phi_A^{T,\tau})^\top d\tau \right] P_{T,t} \right\},$$

where, for all $t \in [0, T]$, $P_{T,t}$ is the orthogonal projection onto $\ker G(T, t)$ and $\Phi_A^{T,t} \triangleq \Phi_A(T, t)$.

Proof: It follows immediately from Lemma 3 in the Appendix that $\mathcal{R}_T \subseteq \mathcal{R}_T^+$. For all $0 \leq t \leq \tau \leq T$, we have $\text{range } G(T, \tau) \subseteq \text{range } G(T, t) = \ker P_{T,t}$, it follows that $P_{T,t} G(T, \tau) = 0$. Thus, $\mathcal{R}_T^+ \subseteq \bigcap_{t \in [0, T]} \mathcal{T}_{T,t}$. ■

4.3 Controllability of the State Covariance

In this section, we provide necessary and sufficient conditions for the controllability of the state covariance of (2). To this end, let $\hat{G}(T, t)$ denote the *controllability Gramian* of the matrix pair $(A(\tau), B(\tau))$ from time t to T . That is, $\hat{G}(T, t) \triangleq \Phi_A(t, T) G(T, t) \Phi_A(t, T)^\top$, where $G(T, t)$ is the reachability Gramian given by (10). Clearly, $\hat{G}(T, t) \succ 0$ if and only if $G(T, t) \succ 0$.

It is well known that the Riccati differential equation

$$\dot{\Pi} = -A(t)^\top \Pi - \Pi A(t) + \Pi B(t) B(t)^\top \Pi \quad (12)$$

may have “finite escape” times and therefore the solution may not exist on the whole interval $[0, T]$. Below, we provide a necessary and sufficient condition for (12) to admit a solution on $[0, T]$, that will be used to prove the main result of this section, and which may be of independent interest. It is worth pointing out that, unlike the condition in [12, Lemma 2], the condition below does not require $(A(t), B(t))$ to be totally controllable or completely controllable, thus it is more general than the result of [12].

Theorem 6 *The following statements are equivalent.*

- i) *The Riccati differential equation (12) with initial condition $\Pi(0)$ admits a unique solution $\Pi(t)$ on $[0, T]$.*
- ii) *For all $t \in [0, T]$, the eigenvalues of $I_n - \hat{G}(t, 0) \Pi(0)$ are all positive.*
- iii) *The eigenvalues of $I_n - \hat{G}(T, 0) \Pi(0)$ are all positive.*
- iv) *For all $t \in [0, T]$, $\hat{G}(t, 0)^{\frac{1}{2}} \Pi(0) \hat{G}(t, 0)^{\frac{1}{2}} \prec I_n$.*
- v) *$\hat{G}(T, 0)^{\frac{1}{2}} \Pi(0) \hat{G}(T, 0)^{\frac{1}{2}} \prec I_n$.*

Furthermore, the solution of the Riccati equation is given by

$$\Pi(t) = \Phi_A(0, t)^\top \Pi(0) (I_n - \hat{G}(t, 0) \Pi(0))^{-1} \Phi_A(0, t). \quad (13)$$

The proof of Theorem 6 is given in the Appendix. Next, we present the main result of this section on the controllability of the state covariance of (2).

Theorem 7 *The following statements are equivalent.*

- i) *The state covariance of the system (2) is controllable on the time interval $[0, T]$.*
- ii) *$G(T, 0) \succ 0$ and $\text{range } \Phi_A(T, t) D(t) \subseteq \text{range } G(T, t)$ for all $t \in [0, T]$.*
- iii) *$\hat{G}(T, 0) \succ 0$ and $\text{range } D(t) \subseteq \text{range } \hat{G}(T, t)$ for all $t \in [0, T]$.*

Proof: If the state covariance of (2) is controllable on $[0, T]$, then, for any $\bar{\Sigma}_0 \succ 0$ and any $\bar{\Sigma}_T \succ 0$ arbitrarily close to zero, there exists $u(t) = F(t)x(t)$ that steers the system from $\Sigma(0) = \bar{\Sigma}_0$ to $\Sigma(T) = \bar{\Sigma}_T$. In view of (11), it follows that $\Phi_{A+BF}(T, 0) \bar{\Sigma}_0$ is arbitrarily close to zero and, for all $t \in [0, T]$, $\Phi_{A+BF}(T, t) D(t)$ is arbitrarily close to zero. If $\text{range } G(T, 0) \neq \mathbb{R}^n$, one can select $x_0 \in \mathbb{R}^n$ such that $\Phi_A(T, 0)x_0 \notin \text{range } G(T, 0)$. Then, starting from the initial state $x(0) = x_0$, the set of all reachable states of the corresponding deterministic system $\dot{x}(t) = A(t)x(t) + B(t)u(t)$ at time T is the affine subspace $\Phi_A(T, 0)x_0 + \text{range } G(T, 0)$. Since this affine subspace does not contain zero, it follows that the final state $x(T)$ cannot be arbitrarily close to zero, thus contradicting the fact that, for some x_T arbitrarily close to zero, there exists $u(t) = F(t)x(t)$ that steers the deterministic system from $x(0) = x_0 \in \text{range } \bar{\Sigma}_0$ to $x(T) = x_T = \Phi_{A+BF}(T, 0)x_0$. Hence, $G(T, 0) \succ 0$. Similarly, for a given $t \in [0, T]$, the fact that there exists $F(t)$ such that $\Phi_{A+BF}(T, t) D(t)$ is arbitrarily close to zero implies that $\text{range } \Phi_A(T, t) D(t) \subseteq \text{range } G(T, t)$. Thus, $i) \implies ii)$.

Clearly, from the definitions of the reachability and controllability Gramians it follows that $ii) \iff iii)$.

To show $iii) \implies i)$, we first show that if $iii)$ holds, then, for any given $\bar{\Sigma}_0, \bar{\Sigma}_T \succ 0$, there exists an optimal control that steers the state covariance of the system (2) from $\Sigma(0) = \bar{\Sigma}_0$ to $\Sigma(T) = \bar{\Sigma}_T$ with the minimum expected control energy. It is first shown in [14] that such an optimal control is $u(t) = -B(t)^\top \Pi(t)x(t)$, provided that $\Pi(t)$ solves the following coupled matrix differential equations

$$\dot{\Pi} = -A(t)^\top \Pi - \Pi A(t) + \Pi B(t) B(t)^\top \Pi, \quad (14a)$$

$$\begin{aligned} \dot{\Sigma} = & (A(t) - B(t) B(t)^\top \Pi(t)) \Sigma \\ & + \Sigma (A(t)^\top - \Pi(t) B(t) B(t)^\top) + D(t) D(t)^\top, \end{aligned} \quad (14b)$$

$$\Sigma(0) = \bar{\Sigma}_0, \quad \Sigma(T) = \bar{\Sigma}_T. \quad (14c)$$

Thus, it suffices to show that if $iii)$ holds, then, for given $\bar{\Sigma}_0, \bar{\Sigma}_T \succ 0$, the coupled equations (14) have a solution. In light of Theorem 6, the Riccati differential equation (14a) with initial condition $\Pi(0)$ admits the unique solu-

tion $\Pi(t) = \Phi_A(0, t)^\top \Pi(0) (I_n - \hat{G}(t, 0) \Pi(0))^{-1} \Phi_A(0, t)$ on $[0, T]$ if and only if the eigenvalues of $I_n - \hat{G}(T, 0) \Pi(0)$ are all positive. Since $\hat{G}(T, 0) \succ 0$, this condition reduces to $\Pi_0 \prec \hat{G}(T, 0)^{-1}$.

Next, we borrow some results from [12] to show that, if *iii*) holds, then, for given $\bar{\Sigma}_0, \bar{\Sigma}_T \succ 0$, the coupled matrix differential equations (14) admit a unique solution. In view of [12, Lemma 3], let the state transition matrix of $A(\tau) - B(\tau)B(\tau)^\top \Pi(\tau)$ from time s to t be denoted by

$$\Phi_\Pi(t, s) \triangleq \Phi_{A-BB^\top \Pi}(t, s) = \Phi_A(t, s) (I_n - \hat{G}(t, s) \Pi(s)). \quad (15)$$

Define the sets $\mathcal{P} \triangleq \{\Pi_0 \in \mathbb{R}^{n \times n} : \Pi_0 \prec \hat{G}(T, 0)^{-1}\}$ and $\mathcal{M} \triangleq \{\Sigma_T \in \mathbb{R}^{n \times n} : \Sigma_T \succ 0\}$. As in [12, Section V.C], given an initial state covariance $\bar{\Sigma}_0 \succ 0$ of the system (2), we can define the map $f : \mathcal{P} \rightarrow \mathcal{M}$, according to

$$\begin{aligned} f(\Pi_0) = & \Phi_A(T, 0) (I_n - \hat{G}(T, 0) \Pi_0) \left[\bar{\Sigma}_0 \right. \\ & + \int_0^T (I_n - \hat{G}(\tau, 0) \Pi_0)^{-1} \Phi_A(0, \tau) D(\tau) D(\tau)^\top \\ & \quad \times \Phi_A(0, \tau)^\top (I_n - \Pi_0 \hat{G}(\tau, 0))^{-1} d\tau \Big] \\ & \quad \times (I_n - \Pi_0 \hat{G}(T, 0)) \Phi_A(T, 0)^\top, \end{aligned} \quad (16)$$

where, for notational simplicity, we write Π_0 for $\Pi(0)$.

Let $\text{vec}(M)$ denote the vectorization of the matrix M . Define the vectorized map $\bar{f}$ such that $\bar{f}(\text{vec}(\Pi_0)) = \text{vec}(f(\Pi_0))$, where the domain and codomain of $\bar{f}$ are the vectorized $\mathcal{P}$ and $\mathcal{M}$, respectively. Following the same procedure as in the proof of [12, Lemma 4], it can be shown that the Jacobian $J_{\bar{f}}$ of $\bar{f}$ at $\text{vec}(\Pi_0)$ is

$$\begin{aligned} J_{\bar{f}}(\text{vec}(\Pi_0)) = & -\Phi_\Pi(T, 0) \otimes \Phi_\Pi(T, 0) \left[\bar{\Sigma}_0 \otimes Q(T) \right. \\ & + Q(T) \otimes \bar{\Sigma}_0 + \int_0^T U(\tau) \otimes (Q(T) - Q(\tau)) \\ & \quad \left. + (Q(T) - Q(\tau)) \otimes U(\tau) d\tau \right], \end{aligned}$$

where $\otimes$ denotes the Kronecker product, for $t \in [0, T]$, $Q(t) \triangleq (I_n - \hat{G}(t, 0) \Pi_0)^{-1} \hat{G}(t, 0)$, and

$$\begin{aligned} U(t) \triangleq & (I_n - \hat{G}(t, 0) \Pi_0)^{-1} \Phi_A(0, t) D(t) D(t)^\top \\ & \times \Phi_A(0, t)^\top (I_n - \Pi_0 \hat{G}(t, 0))^{-1} \succeq 0. \end{aligned}$$

Since $\hat{G}(T, 0) \succ 0$ and $\Pi_0 \prec \hat{G}(T, 0)^{-1}$, we have $Q(T) \succ 0$. From [18, Lemma 13] it follows that, for all $\Pi_0 \in \mathcal{P}$

and $t \in [0, T]$, we have $Q(T) \succeq Q(t) \succeq 0$. Using the same technique as in [12, Lemma 5], it is not difficult to show that the Jacobian $J_{\bar{f}}(\text{vec}(\Pi_0))$ is nonsingular at each $\text{vec}(\Pi_0)$ with $\Pi_0 \in \mathcal{P}$.

Finally, we show that for any given $\bar{\Sigma}_0 \succ 0$, the map f given by (16) is a homeomorphism. Therefore, for given $\bar{\Sigma}_0, \bar{\Sigma}_T \succ 0$, the coupled matrix differential equations (14) admit a unique solution. In order to show that f is a homeomorphism, we need to check the conditions in Hadamard's global inverse function theorem [41]. First, $\mathcal{P}$ and $\mathcal{M}$ are smooth, connected manifolds of the same dimension. Second, $J_{\bar{f}}(\text{vec}(\Pi_0))$ is continuous and nonsingular at each $\text{vec}(\Pi_0)$ for which $\Pi_0 \in \mathcal{P}$. Third, since $\mathcal{M}$ is convex, it is simply connected. Next, we show that f is proper. Clearly, the boundedness of $f(\Pi_0)$ implies the boundedness of $\Phi_A(T, 0) (I_n - \hat{G}(T, 0) \Pi_0) \bar{\Sigma}_0 (I_n - \Pi_0 \hat{G}(T, 0)) \Phi_A(T, 0)^\top$. Since $\Phi_A(T, 0)$, $\hat{G}(T, 0)$, and $\bar{\Sigma}_0$ are all invertible, it follows that Π_0 is bounded as well. Finally, we need to show that if $\mathcal{S} \subset \mathcal{M}$ is closed and bounded in $\mathbb{R}^{n \times n}$, then, the inverse image $f^{-1}(\mathcal{S}) \subset \mathcal{P}$ is closed in $\mathbb{R}^{n \times n}$. We claim that $f^{-1}(\mathcal{S})$ is bounded away from the boundary $\partial \mathcal{P}$ of $\mathcal{P}$. It then follows from the continuity of f that $f^{-1}(\mathcal{S})$ is closed in $\mathbb{R}^{n \times n}$. In light of Hadamard's global inverse function theorem, f is a homeomorphism.

Next, we show that $f^{-1}(\mathcal{S})$ is bounded away from $\partial \mathcal{P}$. Note that it suffices to show that f maps $\partial \mathcal{P}$ to $\partial \mathcal{M}$ in the limit. In other words, we will show that, as Π_0 approaches $\partial \mathcal{P}$, then, $f(\Pi_0)$ approaches $\partial \mathcal{M}$. If this statement holds, then it follows that $f^{-1}(\mathcal{S})$ is bounded away from $\partial \mathcal{P}$, since $\mathcal{S}$ is bounded away from $\partial \mathcal{M}$. The first case of Π_0 approaching $\partial \mathcal{P}$ is that the smallest eigenvalue of Π_0 goes to $-\infty$. Since $\hat{G}(T, 0) \succ 0$, it follows from Lemma 4 that one eigenvalue of $I_n - \hat{G}(T, 0) \Pi_0$ goes to $+\infty$. Since $\Phi_A(T, 0)$ is invertible and $\bar{\Sigma}_0 \succ 0$, one eigenvalue of $\Phi_A(T, 0) (I_n - \hat{G}(T, 0) \Pi_0) \bar{\Sigma}_0 (I_n - \Pi_0 \hat{G}(T, 0)) \Phi_A(T, 0)^\top$ goes to $+\infty$. It follows that the largest eigenvalue of $f(\Pi_0)$ goes to $+\infty$. The second case of Π_0 approaching $\partial \mathcal{P}$ is that the smallest eigenvalue of $\hat{G}(T, 0)^{-1} - \Pi_0$ goes to zero. Let $\bar{P} \in \partial \mathcal{P}$ such that $\bar{P} \preceq \hat{G}(T, 0)^{-1}$ and the smallest eigenvalue of $\hat{G}(T, 0)^{-1} - \bar{P}$ is zero. Next, we show that, as $\Pi_0 \rightarrow \bar{P}$, then the smallest eigenvalue of $f(\Pi_0)$ goes to zero.

It follows from (14b) and (15) that

$$\begin{aligned} \Phi_A(0, T) f(\Pi_0) \Phi_A(0, T)^\top = & (I_n - \hat{G}(T, 0) \Pi_0) \bar{\Sigma}_0 (I_n - \Pi_0 \hat{G}(T, 0)) \\ & + \int_0^T \Phi_A(0, \tau) (I_n - \hat{G}(T, \tau) \Pi_\tau) D(\tau) D(\tau)^\top \\ & \times (I_n - \Pi_\tau \hat{G}(T, \tau)) \Phi_A(0, \tau)^\top d\tau, \end{aligned} \quad (17)$$

where, for $t \in [0, T]$, $\Pi_t = \Pi(t)$ is the solution to the Riccati differential equation (14a) starting from $\Pi(0) = \Pi_0$.

Since $\Phi_A(0, T)$ is invertible and $\bar{\Sigma}_0 \succ 0$, it follows from the above equation that if the smallest eigenvalue of $f(\Pi_0)$ goes to zero, then, one eigenvalue of $I_n - \hat{G}(T, 0)\Pi_0$ goes to zero. In light of Lemma 4 in the Appendix, only the positive eigenvalues of Π_0 can possibly render $I_n - \hat{G}(T, 0)\Pi_0$ singular. Hence, it suffices to consider the case when $\Pi_0, \bar{P} \succ 0$. Since $0 \prec \bar{P} \preceq \hat{G}(T, 0)^{-1}$, we can write $\bar{P} = (\hat{G}(T, 0) + \Gamma\Gamma^\top)^{-1}$, where $\Gamma \in \mathbb{R}^{n \times \ell}$, $\Gamma\Gamma^\top \succeq 0$. In view of the Woodbury matrix identity [42], we have $(\hat{G}(T, 0) + \Gamma\Gamma^\top)^{-1} = \hat{G}(T, 0)^{-1} - \hat{G}(T, 0)^{-1}\Gamma(I_\ell + \Gamma^\top\hat{G}(T, 0)^{-1}\Gamma)^{-1}\Gamma^\top\hat{G}(T, 0)^{-1}$. Thus, as $\Pi_0 \rightarrow \bar{P}$, then, $I_n - \hat{G}(T, 0)\Pi_0 \rightarrow I_n - \hat{G}(T, 0)(\hat{G}(T, 0) + \Gamma\Gamma^\top)^{-1} = \Gamma(I_\ell + \Gamma^\top\hat{G}(T, 0)^{-1}\Gamma)^{-1}\Gamma^\top\hat{G}(T, 0)^{-1}$. Since $\Gamma\Gamma^\top \succeq 0$ is singular, its left null space is nonempty. Let $z \in \ker(\Gamma\Gamma^\top)$ be a nonzero vector, then, $z^\top\Gamma = 0$. It follows that, as $\Pi_0 \rightarrow \bar{P}$, then, $I_n - \hat{G}(T, 0)\Pi_0$ approaches some matrix for which z is in its left null space.

Next, we show that, as $\Pi_0 \rightarrow \bar{P}$, then, for each $t \in [0, T]$, $\Phi_A(0, t)(I_n - \hat{G}(T, t)\Pi_t)D(t)$ approaches some matrix for which z is in its left null space as well. To this end, let $t \in [0, T]$ be fixed. Since $\text{range } D(t) \subseteq \text{range } \hat{G}(T, t)$ and $\Phi_A(0, t)^\top$ is invertible, it suffices to consider the left null space of $\Phi_A(0, t)(I_n - \hat{G}(T, t)\Pi_t)\hat{G}(T, t)\Phi_A(0, t)^\top$. Let $W \triangleq \Phi_A(0, t)\hat{G}(T, t)\Phi_A(0, t)^\top = \hat{G}(T, 0) - \hat{G}(t, 0) \succeq 0$. From the equation $\Pi_t = \Phi_A(0, t)^\top\Pi_0(I_n - \hat{G}(t, 0)\Pi_0)^{-1}\Phi_A(0, t)$, it follows that as $\Pi_0 \rightarrow \bar{P}$, then, $\Pi_t \rightarrow \Phi_A(0, t)^\top(\bar{P}^{-1} - \hat{G}(t, 0))^{-1}\Phi_A(0, t) = \Phi_A(0, t)^\top(W + \Gamma\Gamma^\top)^{-1}\Phi_A(0, t)$.¹ Consequently,

$$\Phi_A(0, t)(I_n - \hat{G}(T, t)\Pi_t)\hat{G}(T, t)\Phi_A(0, t)^\top \rightarrow W - W(W + \Gamma\Gamma^\top)^{-1}W.$$

From the fact that $\ker(W + \Gamma\Gamma^\top) \subseteq \ker W = (\text{range } W)^\perp$, it follows that $W(W + \Gamma\Gamma^\top)^{-1}W = W(W + \Gamma\Gamma^\top)^+W$ is well defined, where M^+ denotes the pseudo-inverse of the matrix M . Thus, it suffices to show that z is in the left null space of $W - W(W + \Gamma\Gamma^\top)^+W$, that is, $z^\top(W - W(W + \Gamma\Gamma^\top)^+W) = 0$.

Let $\text{rank } W = r$ and $\text{rank } [W \ \Gamma] = m$, where $r \leq m \leq n$. With an appropriate coordinate transformation, we can

¹ Since $W \succeq 0$ and $\Gamma\Gamma^\top \succeq 0$, $W + \Gamma\Gamma^\top \succeq 0$ may be singular. We can define the inverse of a positive semi-definite matrix on $\mathbb{R}^{n \times n}$, where $\mathbb{R} \triangleq \mathbb{R} \cup \{-\infty, +\infty\}$ is the extended real number line as follows: for $M \succeq 0$, we can write $M = V\Lambda V^\top$, where V is orthogonal and $\Lambda = \text{diag}[\lambda_1, \dots, \lambda_i, 0, \dots, 0]$ with $\lambda_1, \dots, \lambda_i > 0$, then, its inverse is $M^{-1} \triangleq V\Lambda^{-1}V^\top$, where $\Lambda^{-1} \triangleq \text{diag}[1/\lambda_1, \dots, 1/\lambda_i, +\infty, \dots, +\infty]$.

partition W and Γ as follows:

$$W_{n \times n} = \begin{bmatrix} W_1 & 0_{r \times (m-r)} & 0_{r \times (n-m)} \\ 0_{(m-r) \times r} & 0_{(m-r) \times (m-r)} & 0_{(m-r) \times (n-m)} \\ 0_{(n-m) \times r} & 0_{(n-m) \times (m-r)} & 0_{(n-m) \times (n-m)} \end{bmatrix},$$

$$\Gamma_{n \times \ell} = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ 0_{(n-m) \times \ell} \end{bmatrix},$$

where $W_1 \in \mathbb{R}^{r \times r}$, $W_1 \succ 0$, $\Gamma_1 \in \mathbb{R}^{r \times \ell}$, and $\Gamma_2 \in \mathbb{R}^{(m-r) \times \ell}$. In view of Lemma 5 in the Appendix, we have that

$$W - W(W + \Gamma\Gamma^\top)^+W = \begin{bmatrix} \Gamma_1 P_2(I_\ell + P_2 \Gamma_1^\top W_1^{-1} \Gamma_1 P_2)^{-1} P_2 \Gamma_1^\top & 0_{r \times (n-r)} \\ 0_{(n-r) \times r} & 0_{(n-r) \times (n-r)} \end{bmatrix},$$

where $P_2 \triangleq I_\ell - \Gamma_2^\top(\Gamma_2 \Gamma_2^\top)^{-1} \Gamma_2$ is the orthogonal projection onto $(\text{range } \Gamma_2)^\perp$. Let $z \in \mathbb{R}^n$ be partitioned in the same way as Γ . That is, let $z^\top = [z_1^\top \ z_2^\top \ z_3^\top]$, where $z_1 \in \mathbb{R}^r$, $z_2 \in \mathbb{R}^{m-r}$, and $z_3 \in \mathbb{R}^{n-m}$. Since $z^\top\Gamma = 0$, we have $z_1^\top\Gamma_1 + z_2^\top\Gamma_2 = 0$. Since $\Gamma_2 P_2 = 0$, it follows that $z_1^\top\Gamma_1 P_2 = z_2^\top\Gamma_2 P_2 = 0$. We can verify that $z^\top(W - W(W + \Gamma\Gamma^\top)^+W) = [z_1^\top\Gamma_1 P_2(I_\ell + P_2 \Gamma_1^\top W_1^{-1} \Gamma_1 P_2)^{-1} P_2 \Gamma_1^\top \ 0_{1 \times (n-r)}] = 0$. Since $t \in [0, T]$ is arbitrary, in light of (17), we have shown that, as $\Pi_0 \rightarrow \bar{P}$, then, $\Phi_A(0, T)f(\Pi_0)\Phi_A(0, T)^\top$ approaches a matrix for which z lies in its left null space. Since $\Phi_A(0, T)$ is invertible, it follows that, as the smallest eigenvalue of $\hat{G}(T, 0)^{-1} - \Pi_0$ goes to zero, then, $f(\Pi_0)$ approaches a singular matrix, that is, the smallest eigenvalue of $f(\Pi_0)$ goes to zero. Therefore, for any given $\Sigma_0 \succ 0$, the map f given by (16) is a homomorphism. It follows immediately that, for given $\bar{\Sigma}_0, \bar{\Sigma}_T \succ 0$, the coupled matrix differential equations (14) admit a unique solution $\Pi(t)$. Consequently, if *iii*) holds, then, the optimal control that steers the state covariance of the system (2) from $\Sigma(0) = \bar{\Sigma}_0$ to $\Sigma(T) = \bar{\Sigma}_T$ with the minimum expected control energy exists, and is $u(t) = -B(t)^\top\Pi(t)x(t)$. Thus, the state covariance of (2) is controllable on $[0, T]$. Therefore, *iii*) $\implies$ *i*), which completes the proof. ■

It is well known that the continuous-time matrix pair $(A(s), B(s))$ is completely controllable [43] on the time interval $[t, T]$ if and only if $G(T, t) \succ 0$ [44]. Similarly to the discrete-time case, we have two corollaries.

Corollary 3 *Assume that there is no additive noise in the system (2), that is, $D(t) \equiv 0$. Then, the state covariance of (2) is controllable on the interval $[0, T]$ if and only if the matrix pair $(A(t), B(t))$ is completely controllable on $[0, T]$.*

Corollary 4 *Assume that the additive noise in the sys-*

tem (2) is non-degenerate, that is, for all $t \in [0, T]$, $\text{rank } D(t) = n$. Then, the state covariance of (2) is controllable on the interval $[0, T]$ if and only if, for all $t \in [0, T]$, the matrix pair $(A(s), B(s))$ is completely controllable on $[t, T]$.

5 Numerical Example

In this section, we first summarize the semi-definite programming (SDP) approaches given in [13, 18] for finding a control that steers the state covariance of either system (2) or system (1) from a given initial state covariance to a desired reachable state covariance. Then, a numerical example is given to illustrate our theoretical results. The SDP method for the continuous-time system (2) is described below.

Let $\bar{\Sigma}_0$ be the initial state covariance and let $\bar{\Sigma}_T$ be the desired terminal state covariance. When $\bar{\Sigma}_0 \succ 0$, it follows from replacing T with t in (11) that $\Sigma(t) \succ 0$ for all $t \in [0, T]$. When $\bar{\Sigma}_0 \succeq 0$ and $D(t)D(t)^\top \succeq 0$, we may have $\Sigma(t) \succeq 0$. For $t \in [0, T]$, let $Y(t) = \Sigma^{uu}(t)$, let $U(t) = \Sigma^{ux}(t)$, and choose $R(t) \in \mathbb{R}^{p \times p}$, $R(t) \succ 0$. The SDP problem is [13]

$$\begin{aligned} & \min_{\Sigma, U, Y} \int_0^T \text{trace}(R(t)Y(t)) dt \\ & \text{such that} \\ & \dot{\Sigma} = A(t)\Sigma + \Sigma A(t)^\top + B(t)U(t) + U(t)^\top B(t)^\top \\ & \quad + D(t)D(t)^\top, \\ & \begin{bmatrix} \Sigma(t) & U(t)^\top \\ U(t) & Y(t) \end{bmatrix} \succeq 0, \quad \Sigma(0) = \bar{\Sigma}_0, \quad \Sigma(T) = \bar{\Sigma}_T. \end{aligned}$$

Since $R \succ 0$, it follows that, if $Y_1 \succeq Y_2 \succeq 0$, then,

$$\begin{aligned} \text{trace}(RY_1) &= \text{trace}(R^{\frac{1}{2}}Y_1R^{\frac{1}{2}}) \\ &\geq \text{trace}(R^{\frac{1}{2}}Y_2R^{\frac{1}{2}}) = \text{trace}(RY_2). \end{aligned}$$

In view of the fact that Y does not appear in the equality constraints, we know that the optimal solution of the SDP problem, which minimizes the previous cost function, satisfies $Y(t) = U(t)\Sigma^{-1}(t)U(t)^\top$, provided that $\Sigma(t) \succ 0$. After solving the above SDP problem, we find $F(t) = U(t)\Sigma^{-1}(t)$. In case of $\Sigma(t) \succeq 0$, it follows from Proposition 1 that we should replace $\Sigma^{-1}(t)$ with the Moore–Penrose inverse $\Sigma^+(t)$ in the above equations for $Y(t)$ and $F(t)$. In light of Proposition 3, the state feedback control $u(t) = F(t)x(t)$ will steer the state covariance of the system (2) from $\bar{\Sigma}_0$ to $\bar{\Sigma}_T$.

The SDP approach for the discrete-time system (1) is summarized as follows. More details can be found in [18]. Let $\bar{\Sigma}_0$ be the given initial state covariance and let $\bar{\Sigma}_k$ be the desired terminal state covariance. For each $i = 0, 1, \dots, k-1$, let $Y_i = \Sigma_i^{uu}$, let $U_i = \Sigma_i^{ux}$, and choose

$R_i \in \mathbb{R}^{p \times p}$, $R_i \succ 0$. The SDP problem is [18]

$$\begin{aligned} & \min_{\Sigma_i, U_i, Y_i} \sum_{i=0}^{k-1} \text{trace}(R_i Y_i), \\ & \text{such that, for all } k = 0, 1, \dots, N-1, \\ & \Sigma_{i+1} = A_i \Sigma_i A_i^\top + B_i U_i A_i^\top + A_i U_i^\top B_i^\top \\ & \quad + B_i Y_i B_i^\top + D_i D_i^\top, \\ & \begin{bmatrix} \Sigma_i & U_i^\top \\ U_i & Y_i \end{bmatrix} \succeq 0, \quad \Sigma_0 = \bar{\Sigma}_0, \quad \Sigma_k = \bar{\Sigma}_k. \end{aligned}$$

After the above SDP problem is solved, for $i = 0, 1, \dots, k-1$, we find $F_i \in \mathbb{R}^{p \times n}$ and $V_i \in \mathbb{R}^{p \times p}$ such that $U_i = F_i \Sigma_i$ and $V_i = Y_i - F_i \Sigma_i F_i^\top \succeq 0$. In light of Proposition 2, the randomized state feedback control $u_i = F_i x_i + \nu_i$ will steer the state covariance of the system (1) from $\bar{\Sigma}_0$ to $\bar{\Sigma}_k$, where $\nu_i \in \mathbb{R}^p$, $i = 0, 1, \dots, k-1$, is a sequence of independent, square-integrable random vectors with $\mathbb{E}[\nu_i] = 0$ and $\mathbb{E}[\nu_i \nu_i^\top] = V_i$.

Next, we use an example to illustrate our theoretical results. Consider the two-dimensional linear stochastic system

$$x_{k+1} = Ax_k + Bu_k + Dw_k, \quad k = 0, 1, 2, \dots, \quad (18)$$

where,

$$A = \begin{bmatrix} 1 & 0.2 \\ 0 & 0.96 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0.4 & 0.2 \\ 0 & 0.3 \end{bmatrix}.$$

Note that (A, B) is not controllable and hence the state covariance of the system (18) is not controllable from Theorem 4. Let $k = 30$. According to Theorem 2 or Theorem 3, all reachable state covariance matrices at time $k = 30$ are semi-definite matrices having the (2,2) entry fixed at 1.3079. For this example, we therefore choose $\mathbb{E}[x_0] = [0 \ 0]^\top$, $\mathbb{E}[x_{30}] = [30 \ 0]^\top$, while the initial and desired terminal state covariances are given by

$$\bar{\Sigma}_0 = \begin{bmatrix} 5 & -1 \\ -1 & 3 \end{bmatrix}, \quad \bar{\Sigma}_{30} = \begin{bmatrix} 0.5 & 0.2 \\ 0.2 & 1.3079 \end{bmatrix}.$$

We choose $R_i \equiv 1$ in the cost function of the SDP formulation. Figure 1 shows the trajectories of ten sample paths of (18) and their covariance ellipses, where the initial and target three-standard-deviation tolerance regions are marked with red ellipses, and the three-standard-deviation tolerance region at each time step is marked with a blue ellipse. As shown in the figure, the final covariance matrix is achieved even if the set of reachable covariances is not the whole space of positive semi-definite matrices.

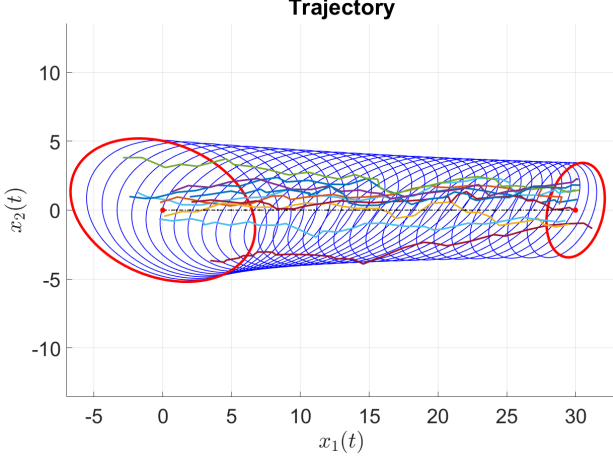


Fig. 1. Ten sample paths of (18) and their three-standard-deviation ellipses.

6 Concluding Remarks

This paper characterizes the set of reachable terminal state covariances for discrete-time and continuous-time linear stochastic systems with additive noise, starting from a given initial state covariance. Specifically, a complete characterization of the reachable set of terminal state covariances is given for discrete-time systems, while an upper bound on the reachable set of terminal state covariances is provided for continuous-time systems. We establish necessary and sufficient conditions for the controllability of the state covariance for both discrete-time and continuous-time systems.

A natural extension of this work is to determine the exact set of reachable terminal state covariances for continuous-time linear stochastic systems, which hinges on the reachability of the state transition matrix Φ_{A+BF} via the control input $F(t)$. We also observe a close resemblance between the reachability of the mean state in a stochastic setting (or the reachability of the state in a deterministic setting) and the reachability of the state covariance. It would be interesting to find out whether the reachability results for higher moments of the state, if they exist, also bear such a resemblance.

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A Appendix

Lemma 1 *Let $H \in \mathbb{R}^{n \times n}$, $J \in \mathbb{R}^{n \times m}$, and $\bar{\Sigma} \in \mathbb{R}^{n \times n}$ with $\bar{\Sigma} \succeq 0$. The following sets are equal.*

- i) $\mathcal{S}_1 \triangleq \{\Sigma \succeq 0 : \Sigma = H\bar{\Sigma}H^\top + JX + X^\top J^\top, X \in \mathbb{R}^{m \times n}\}.$
- ii) $\mathcal{S}_2 \triangleq \{\Sigma \succeq 0 : P\Sigma P = PH\bar{\Sigma}H^\top P\}, \text{ where } P \in \mathbb{R}^{n \times n} \text{ is the orthogonal projection onto } (\text{range } J)^\perp.$
- iii) $\mathcal{S}_3 \triangleq \left\{ \Sigma \succeq 0 : \Sigma = \begin{bmatrix} H & J \end{bmatrix} \begin{bmatrix} \bar{\Sigma} & \bar{\Sigma}F^\top \\ F\bar{\Sigma} & F\bar{\Sigma}F^\top + V \end{bmatrix} \begin{bmatrix} H^\top \\ J^\top \end{bmatrix}, \right.$
 $\left. F \in \mathbb{R}^{m \times n}, V \in \mathbb{R}^{m \times m}, V \succeq 0 \right\}.$
- iv) $\mathcal{S}_4 \triangleq \{\Sigma \succeq 0 : \Sigma = (H + JF)\bar{\Sigma}(H + JF)^\top + J\bar{V}J^\top, F \in \mathbb{R}^{m \times n}, \bar{V} \in \mathbb{R}^{m \times m}, \bar{V} \succeq 0\}.$

Proof: Clearly, $\mathcal{S}_1 = \mathcal{S}_2$, $\mathcal{S}_3 = \mathcal{S}_4$, and $\mathcal{S}_4 \subseteq \mathcal{S}_1$. We only need to show that $\mathcal{S}_1 \subseteq \mathcal{S}_4$. Let $Q \in \mathcal{S}_1$. Then, for some $\bar{X} \in \mathbb{R}^{m \times n}$, $Q = H\bar{\Sigma}H^\top + J\bar{X} + \bar{X}^\top J^\top \succeq 0$. If $\text{range } H \not\subseteq (\text{range } J)^\perp$, we can write $H = H_1 + H_2$, where $\text{range } H_1 \subseteq (\text{range } J)^\perp$ and $\text{range } H_2 \subseteq \text{range } J$. Since $\text{range } H_2 \subseteq \text{range } J$, there exists $\bar{Y} \in \mathbb{R}^{m \times n}$ such that $H_2 = J\bar{Y}$. It follows that

$$\begin{aligned} Q &= (H_1 + H_2)\bar{\Sigma}(H_1^\top + H_2^\top) + J\bar{X} + \bar{X}^\top J^\top \\ &= H_1\bar{\Sigma}H_1^\top + J\bar{Y}\bar{\Sigma}H_1^\top + H_1\bar{\Sigma}\bar{Y}^\top J^\top + J\bar{Y}\bar{\Sigma}\bar{Y}^\top J^\top \\ &\quad + J\bar{X} + \bar{X}^\top J^\top \\ &= H_1\bar{\Sigma}H_1^\top + J\bar{Z} + \bar{Z}^\top J^\top \succeq 0, \end{aligned}$$

where $\bar{Z} \triangleq \bar{Y}\bar{\Sigma}H_1^\top + \bar{X} + \frac{1}{2}\bar{Y}\bar{\Sigma}\bar{Y}^\top J^\top \in \mathbb{R}^{m \times n}$. Hence, without loss of generality, we may assume that $\text{range } H \subseteq (\text{range } J)^\perp$. Since $Q \succeq 0$, it follows from Proposition 1 that $\text{range } \bar{X}^\top \subseteq \text{range } H\bar{\Sigma}H^\top + \text{range } J$. Hence, there exist matrices $L \in \mathbb{R}^{m \times n}$ and $\bar{W} = \bar{W}^\top \in \mathbb{R}^{m \times m}$ such that $Q = H\bar{\Sigma}H^\top + J\bar{L}H\bar{\Sigma}H^\top + H\bar{\Sigma}H^\top \bar{L}^\top J^\top + J\bar{W}J^\top$. Since $\text{range } H \subseteq (\text{range } J)^\perp$ and $Q \succeq 0$, it follows from Proposition 1 that

$$\begin{aligned} J\bar{W}J^\top &\succeq J\bar{L}H\bar{\Sigma}H^\top(H\bar{\Sigma}H^\top)^\perp H\bar{\Sigma}H^\top \bar{L}^\top J^\top \\ &= J\bar{L}H\bar{\Sigma}H^\top \bar{L}^\top J^\top. \end{aligned}$$

Thus, there exists a matrix $\bar{V} \succeq 0$ such that $J\bar{W}J^\top = J\bar{L}H\bar{\Sigma}H^\top\bar{L}^\top J^\top + J\bar{V}J^\top$. Let $\bar{F} = \bar{L}H$. We then have

$$Q = H\bar{\Sigma}H^\top + J\bar{F}\bar{\Sigma}H^\top + H\bar{\Sigma}\bar{F}^\top J^\top + J\bar{F}\bar{\Sigma}\bar{F}^\top J^\top + J\bar{V}J^\top \\ = (H + J\bar{F})\bar{\Sigma}(H + J\bar{F})^\top + J\bar{V}J^\top \in \mathcal{S}_4.$$

Therefore, $\mathcal{S}_1 \subseteq \mathcal{S}_4$, which completes the proof. $\blacksquare$

Lemma 2 Let $H \in \mathbb{R}^{n \times n}$, $J \in \mathbb{R}^{n \times m}$, and $\bar{\Sigma}_1, \bar{\Sigma}_2 \in \mathbb{R}^{n \times n}$, $\bar{\Sigma}_1, \bar{\Sigma}_2 \succeq 0$. The following two sets are equal.

- i) $\mathcal{S}_5 \triangleq \{\Sigma \succeq 0 : \Sigma = (H + JF)(\bar{\Sigma}_1 + \bar{\Sigma}_2)(H + JF)^\top + J\bar{V}J^\top, F \in \mathbb{R}^{m \times n}, V \in \mathbb{R}^{m \times m}, V \succeq 0\}$.
- ii) $\mathcal{S}_6 \triangleq \{\Sigma \succeq 0 : \Sigma = (H + JF_1)\bar{\Sigma}_1(H + JF_1)^\top + J\bar{V}_1J^\top + (H + JF_2)\bar{\Sigma}_2(H + JF_2)^\top + J\bar{V}_2J^\top, F_1, F_2 \in \mathbb{R}^{m \times n}, V_1, V_2 \in \mathbb{R}^{m \times m}, V_1, V_2 \succeq 0\}$.

Proof: Let $Q \in \mathcal{S}_5$. Then, $Q = (H + J\bar{F})(\bar{\Sigma}_1 + \bar{\Sigma}_2)(H + J\bar{F})^\top + J\bar{V}J^\top$ for some $\bar{F} \in \mathbb{R}^{m \times n}$, $\bar{V} \in \mathbb{R}^{m \times m}$, $\bar{V} \succeq 0$. Let $\bar{F}_1 = \bar{F}_2 = \bar{F}$. Let $\bar{V}_1, \bar{V}_2 \in \mathbb{R}^{m \times m}$ be positive semi-definite matrices such that $\bar{V}_1 + \bar{V}_2 = \bar{V}$. Hence, $Q = (H + J\bar{F}_1)\bar{\Sigma}_1(H + J\bar{F}_1)^\top + (H + J\bar{F}_2)\bar{\Sigma}_2(H + J\bar{F}_2)^\top + J\bar{V}_1J^\top + J\bar{V}_2J^\top \in \mathcal{S}_6$. Thus, $\mathcal{S}_5 \subseteq \mathcal{S}_6$.

Next, we show that $\mathcal{S}_6 \subseteq \mathcal{S}_5$. Let $Q \in \mathcal{S}_6$. Then, for some $\bar{F}_1, \bar{F}_2 \in \mathbb{R}^{m \times n}$, $\bar{V}_1, \bar{V}_2 \in \mathbb{R}^{m \times m}$, $\bar{V}_1, \bar{V}_2 \succeq 0$,

$$Q = \begin{bmatrix} H & J \end{bmatrix} \left(\begin{bmatrix} \bar{\Sigma}_1 & \bar{\Sigma}_1\bar{F}_1^\top \\ \bar{F}_1\bar{\Sigma}_1 & \bar{F}_1\bar{\Sigma}_1\bar{F}_1^\top + \bar{V}_1 \end{bmatrix} + \begin{bmatrix} \bar{\Sigma}_2 & \bar{\Sigma}_2\bar{F}_2^\top \\ \bar{F}_2\bar{\Sigma}_2 & \bar{F}_2\bar{\Sigma}_2\bar{F}_2^\top + \bar{V}_2 \end{bmatrix} \right) \begin{bmatrix} H^\top \\ J^\top \end{bmatrix}. \quad (\text{A.1})$$

Since $\bar{\Sigma}_1, \bar{\Sigma}_2 \succeq 0$, it follows that $\text{range}(\bar{\Sigma}_1\bar{F}_1^\top + \bar{\Sigma}_2\bar{F}_2^\top) \subseteq \text{range}\bar{\Sigma}_1 + \text{range}\bar{\Sigma}_2 = \text{range}(\bar{\Sigma}_1 + \bar{\Sigma}_2)$. Thus, there exists $\bar{F} \in \mathbb{R}^{m \times n}$ such that $\bar{\Sigma}_1\bar{F}_1^\top + \bar{\Sigma}_2\bar{F}_2^\top = (\bar{\Sigma}_1 + \bar{\Sigma}_2)\bar{F}^\top$. Since the block matrix in the parenthesis of the right-hand side of (A.1) is positive semi-definite, it follows from Proposition 1 that $\bar{F}_1\bar{\Sigma}_1\bar{F}_1^\top + \bar{V}_1 + \bar{F}_2\bar{\Sigma}_2\bar{F}_2^\top + \bar{V}_2 \succeq \bar{F}(\bar{\Sigma}_1 + \bar{\Sigma}_2)\bar{F}^\top$. Hence, for some $\bar{V} \in \mathbb{R}^{m \times m}$, $\bar{V} \succeq 0$, we can write

$$Q = \begin{bmatrix} H & J \end{bmatrix} \begin{bmatrix} \bar{\Sigma}_1 + \bar{\Sigma}_2 & (\bar{\Sigma}_1 + \bar{\Sigma}_2)\bar{F}^\top \\ \bar{F}(\bar{\Sigma}_1 + \bar{\Sigma}_2) & \bar{F}(\bar{\Sigma}_1 + \bar{\Sigma}_2)\bar{F}^\top + \bar{V} \end{bmatrix} \begin{bmatrix} H^\top \\ J^\top \end{bmatrix} \in \mathcal{S}_5.$$

Thus, $\mathcal{S}_6 \subseteq \mathcal{S}_5$. $\blacksquare$

Lemma 3 For $t \in [0, T]$ and $F(t) \in \mathbb{R}^{p \times n}$, let $\Phi_{A+BF}(T, t)$ denote the state transition matrix of $A(\tau) + B(\tau)F(\tau)$ from time t to T . Then, $\Phi_{A+BF}(T, t) \in \{\Phi_A(T, t) + G(T, t)K(t) : K(t) \in \mathbb{R}^{n \times n}, \det(\Phi_A(T, t) + G(T, t)K(t)) > 0\}$, where $G(T, t)$ is given by (10).

Proof: Given $T > 0$, $\Phi_{A+BF}(T, t)$ satisfies

$$\dot{\Phi}_{A+BF}(T, t) = -\Phi_{A+BF}(T, t)[A(t) + B(t)F(t)], \\ \Phi_{A+BF}(T, T) = I_n.$$

We can check from direct differentiation that

$$\Phi_{A+BF}(T, t) = \Phi_A(T, t) + \int_t^T \Phi_{A+BF}(T, \tau)B(\tau)F(\tau)\Phi_A(\tau, t) d\tau.$$

We claim that

$$\text{range} \int_t^T \Phi_{A+BF}(T, \tau)B(\tau)F(\tau)\Phi_A(\tau, t) d\tau \subseteq \text{range} G(T, t). \quad (\text{A.2})$$

Thus, there exists $K(t) \in \mathbb{R}^{n \times n}$, such that

$$\Phi_{A+BF}(T, t) = \Phi_A(T, t) + G(T, t)K(t),$$

which is invertible. As the determinant of $\Phi_{A+BF}(T, t)$ is continuous in t and $\Phi_{A+BF}(T, T) = I_n$, it follows that $\det(\Phi_{A+BF}(T, t)) > 0$.

To show the claim (A.2) notice that, since $G(T, t)$ is symmetric, it follows that $(\text{range} G(T, t))^\perp = \ker G(T, t)$. Hence, to show (A.2), it suffices to show that

$$\ker G(T, t) \subseteq \left(\text{range} \int_t^T \Phi_{A+BF}(T, \tau)B(\tau)F(\tau)\Phi_A(\tau, t) d\tau \right)^\perp. \quad (\text{A.3})$$

Let $x \in \ker G(T, t)$. As the reachability Gramian of the pair $(A(\tau) + B(\tau)F(\tau), B(\tau))$ has the same range and kernel as that of $(A(\tau), B(\tau))$ on $[t, T]$, x also lies in the kernel of the reachability Gramian of $(A(\tau) + B(\tau)F(\tau), B(\tau))$ on $[t, T]$. It follows immediately that

$$x^\top \int_t^T \Phi_{A+BF}(T, \tau)B(\tau)B(\tau)^\top \Phi_{A+BF}(T, \tau)^\top d\tau x = 0.$$

It follows that, for $\tau \in [t, T]$, $x^\top \Phi_{A+BF}(T, \tau)B(\tau) \equiv 0$. Thus,

$$x^\top \int_t^T \Phi_{A+BF}(T, \tau)B(\tau)F(\tau)\Phi_A(\tau, t) d\tau = 0.$$

Therefore,

$$x \in \left(\text{range} \int_t^T \Phi_{A+BF}(T, \tau)B(\tau)F(\tau)\Phi_A(\tau, t) d\tau \right)^\perp,$$

and hence (A.3) holds. $\blacksquare$

Lemma 4 Let M_1 and M_2 be $n \times n$ real symmetric matrices. Then,

$$\begin{aligned} \text{spec}(M_1 M_2) &= \text{spec}(M_2 M_1) = \text{spec}(M_1^{\frac{1}{2}} M_2 M_1^{\frac{1}{2}}) \\ &= \text{spec}(M_2^{\frac{1}{2}} M_1 M_2^{\frac{1}{2}}), \end{aligned}$$

where $\text{spec}(M)$ denotes the spectrum of the matrix M , $M^{\frac{1}{2}}$ denotes the square root of M . Additionally, if $M_1 \succeq 0$ or $M_2 \succeq 0$, then $\text{spec}(M_1 M_2) \subset \mathbb{R}$.

Proof: Since M_1 is real symmetric, we can write $M_1 = P \bar{\Lambda} P^\top$, where P is an orthogonal matrix and $\bar{\Lambda}$ is a real diagonal matrix [42]. Without loss of generality, assume that $\bar{\Lambda} \triangleq \text{diag}[\bar{\lambda}_1, \dots, \bar{\lambda}_k, 0, \dots, 0]$, where $\bar{\lambda}_1, \dots, \bar{\lambda}_k \neq 0$. Let $\varepsilon > 0$, and define $\Gamma_\varepsilon \triangleq \text{diag}[0, \dots, 0, \varepsilon, \dots, \varepsilon]$, where the zero and nonzero diagonal entries of Γ_ε complement those of $\bar{\Lambda}$. Clearly, $\bar{\Lambda}^{\frac{1}{2}} + \Gamma_\varepsilon$ is invertible. Let $R_\varepsilon \triangleq P(\bar{\Lambda}^{\frac{1}{2}} + \Gamma_\varepsilon)P^\top$, which is also invertible. Then, for any $\varepsilon > 0$, $M_1 M_2$ is similar to $R_\varepsilon^{-1}(M_1 M_2)R_\varepsilon = M_1^{\frac{1}{2}} M_2 (M_1^{\frac{1}{2}} + P \Gamma_\varepsilon P^\top)$. Thus, we have $\text{spec}(M_1 M_2) = \text{spec}(M_1^{\frac{1}{2}} M_2 (M_1^{\frac{1}{2}} + P \Gamma_\varepsilon P^\top))$. It is clear that as $\varepsilon \rightarrow 0$, then, $P \Gamma_\varepsilon P^\top \rightarrow 0_{n \times n}$. Since the eigenvalues of a matrix are continuous in terms of the values of its entries [45], it follows that, for any $\varepsilon > 0$, $\text{spec}(M_1^{\frac{1}{2}} M_2 (M_1^{\frac{1}{2}} + P \Gamma_\varepsilon P^\top)) = \text{spec}(M_1^{\frac{1}{2}} M_2 M_1^{\frac{1}{2}})$. Therefore, as $\varepsilon \rightarrow 0$, we have $\text{spec}(M_1 M_2) = \text{spec}(M_1^{\frac{1}{2}} M_2 M_1^{\frac{1}{2}})$. Since M_1 and M_2 are symmetric, we have that $\text{spec}(M_1 M_2) = \text{spec}(M_2 M_1)$. If $M_1 \succeq 0$, then $M_1^{\frac{1}{2}} M_2 M_1^{\frac{1}{2}}$ is real symmetric, thus has all real eigenvalues. ■

Proof of Theorem 6: We know from [46] that (12) with initial condition $\Pi(0)$ admits the unique solution (13) on $[0, T]$ if and only if, for all $t \in [0, T]$, $I_n - \hat{G}(t, 0)\Pi(0)$ is invertible. It follows immediately from Lemma 4, that the spectrum of $I_n - \hat{G}(t, 0)\Pi(0)$ equals that of $I_n - \hat{G}(t, 0)^{\frac{1}{2}}\Pi(0)\hat{G}(t, 0)^{\frac{1}{2}}$. Since, at $t = 0$, $I_n - \hat{G}(0, 0)^{\frac{1}{2}}\Pi(0)\hat{G}(0, 0)^{\frac{1}{2}} = I_n \succ 0$, it suffices to show that, for all $t \in [0, T]$, $I_n - \hat{G}(t, 0)^{\frac{1}{2}}\Pi(0)\hat{G}(t, 0)^{\frac{1}{2}} \succ 0$ if and only if $I_n - \hat{G}(T, 0)^{\frac{1}{2}}\Pi(0)\hat{G}(T, 0)^{\frac{1}{2}} \succ 0$. The necessity is obvious. We now show the sufficiency.

Let $\text{rank } \hat{G}(T, 0) = r \leq n$. Let $L \triangleq [L_1 \ L_2] \in \mathbb{R}^{n \times n}$ be an orthogonal matrix² with $L_1 \in \mathbb{R}^{n \times r}$, $L_2 \in \mathbb{R}^{n \times (n-r)}$, $\text{range } L_1 = \text{range } \hat{G}(T, 0)$, and $\text{range } L_2 = \ker \hat{G}(T, 0)$. Since $\text{range } \hat{G}(T, 0)^{\frac{1}{2}} = \text{range } \hat{G}(T, 0)$ and $LL^\top = L_1 L_1^\top + L_2 L_2^\top = I_n$, multiplying $I_n - \hat{G}(T, 0)^{\frac{1}{2}}\Pi(0)\hat{G}(T, 0)^{\frac{1}{2}} \succ 0$ by $L^\top$ on the left and by L on the right yields

$$I_r - L_1^\top \hat{G}(T, 0)^{\frac{1}{2}} L_1 L_1^\top \Pi(0) L_1 L_1^\top \hat{G}(T, 0)^{\frac{1}{2}} L_1 \succ 0.$$

² Notice that such a matrix L is not unique. Any choice of such L will do.

Since $L_1^\top \hat{G}(T, 0)^{\frac{1}{2}} L_1 \succ 0$, it follows immediately from the above inequality that

$$L_1^\top \Pi(0) L_1 \prec (L_1^\top \hat{G}(T, 0)^{\frac{1}{2}} L_1)^{-2}. \quad (\text{A.4})$$

Since, for all $t \in [0, T]$, $\hat{G}(t, 0) \preceq \hat{G}(T, 0)$, it is clear that $L_1^\top \hat{G}(t, 0) L_1 \preceq L_1^\top \hat{G}(T, 0) L_1$. Since $\text{range } \hat{G}(t, 0)^{\frac{1}{2}} = \text{range } \hat{G}(t, 0) \subseteq \text{range } \hat{G}(T, 0)$, we have $\hat{G}(t, 0)^{\frac{1}{2}} L_2 = 0$ and

$$\begin{aligned} \hat{G}(t, 0) &= \hat{G}(t, 0)^{\frac{1}{2}} \hat{G}(t, 0)^{\frac{1}{2}} = \hat{G}(t, 0)^{\frac{1}{2}} L L^\top \hat{G}(t, 0)^{\frac{1}{2}} \\ &= \hat{G}(t, 0)^{\frac{1}{2}} L_1 L_1^\top \hat{G}(t, 0)^{\frac{1}{2}}. \end{aligned}$$

Then, it follows from $L_1^\top \hat{G}(t, 0) L_1 \preceq L_1^\top \hat{G}(T, 0) L_1$ and the above equation that $(L_1^\top \hat{G}(t, 0)^{\frac{1}{2}} L_1)^2 \preceq (L_1^\top \hat{G}(T, 0)^{\frac{1}{2}} L_1)^2$. Since $L_1^\top \hat{G}(T, 0)^{\frac{1}{2}} L_1 \succ 0$, we know that the largest singular value of

$$(L_1^\top \hat{G}(T, 0)^{\frac{1}{2}} L_1)^{-1} (L_1^\top \hat{G}(t, 0)^{\frac{1}{2}} L_1)$$

is no greater than 1 [42]. Thus, we have

$$(L_1^\top \hat{G}(t, 0)^{\frac{1}{2}} L_1) (L_1^\top \hat{G}(T, 0)^{\frac{1}{2}} L_1)^{-2} (L_1^\top \hat{G}(t, 0)^{\frac{1}{2}} L_1) \preceq I_r. \quad (\text{A.5})$$

To show that, $I_n - \hat{G}(t, 0)^{\frac{1}{2}} \Pi(0) \hat{G}(t, 0)^{\frac{1}{2}} \succ 0$, for all $t \in [0, T]$, it suffices to show that $L^\top (I_n - \hat{G}(t, 0)^{\frac{1}{2}} \Pi(0) \hat{G}(t, 0)^{\frac{1}{2}}) L \succ 0$. Since $\hat{G}(t, 0)^{\frac{1}{2}} L_2 = 0$ and $\hat{G}(t, 0)^{\frac{1}{2}} = \hat{G}(t, 0)^{\frac{1}{2}} L L^\top = \hat{G}(t, 0)^{\frac{1}{2}} L_1 L_1^\top$, it suffices to show that, for all $t \in [0, T]$, $I_r - L_1^\top \hat{G}(t, 0)^{\frac{1}{2}} L_1 L_1^\top \Pi(0) L_1 L_1^\top \hat{G}(t, 0)^{\frac{1}{2}} L_1 \succ 0$.

Let $z \in \mathbb{R}^r$ be a nonzero vector. Then, we can write $z = z_1 + z_2$, where $z_1 \in \text{range}(L_1^\top \hat{G}(t, 0)^{\frac{1}{2}} L_1)$, $z_2 \in \ker(L_1^\top \hat{G}(t, 0)^{\frac{1}{2}} L_1)$, $z_1^\top z_2 = 0$, $z_1 \neq 0$ or $z_2 \neq 0$. If $z_1 = 0$, $z^\top (I_r - L_1^\top \hat{G}(t, 0)^{\frac{1}{2}} L_1 L_1^\top \Pi(0) L_1 L_1^\top \hat{G}(t, 0)^{\frac{1}{2}} L_1) z = z_2^\top z_2 > 0$. If $z_1 \neq 0$, it follows from (A.4), (A.5), and $L_1^\top \hat{G}(t, 0)^{\frac{1}{2}} L_1 z_1 \neq 0$ that

$$\begin{aligned} &z^\top (I_r - L_1^\top \hat{G}(t, 0)^{\frac{1}{2}} L_1 L_1^\top \Pi(0) L_1 L_1^\top \hat{G}(t, 0)^{\frac{1}{2}} L_1) z \\ &= z_1^\top z_1 + z_2^\top z_2 \\ &\quad - z_1^\top L_1^\top \hat{G}(t, 0)^{\frac{1}{2}} L_1 L_1^\top \Pi(0) L_1 L_1^\top \hat{G}(t, 0)^{\frac{1}{2}} L_1 z_1 \\ &> z_1^\top z_1 + z_2^\top z_2 \\ &\quad - z_1^\top L_1^\top \hat{G}(t, 0)^{\frac{1}{2}} L_1 (L_1^\top \hat{G}(T, 0)^{\frac{1}{2}} L_1)^{-2} L_1^\top \hat{G}(t, 0)^{\frac{1}{2}} L_1 z_1 \\ &\geq z_1^\top z_1 + z_2^\top z_2 - z_1^\top z_1 = z_2^\top z_2 \geq 0. \end{aligned}$$

Therefore, $I_r - L_1^\top \hat{G}(t, 0)^{\frac{1}{2}} L_1 L_1^\top \Pi(0) L_1 L_1^\top \hat{G}(t, 0)^{\frac{1}{2}} L_1 \succ 0$, for all $t \in [0, T]$, thus completing the proof. ■

Lemma 5 Let $W \in \mathbb{R}^{m \times m}$, $W \succeq 0$ having rank r , and let $\Gamma \in \mathbb{R}^{m \times \ell}$ be such that $\text{rank}[W \ \Gamma] = m$, where

$r \leq m \leq r + \ell$. With a suitable coordinate transformation, we can write $W = \begin{bmatrix} W_1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\Gamma = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix}$, where $W_1 \in \mathbb{R}^{r \times r}$, $W_1 \succ 0$, $\Gamma_1 \in \mathbb{R}^{r \times \ell}$, $\Gamma_2 \in \mathbb{R}^{(m-r) \times \ell}$, and $\text{rank } \Gamma_2 = m - r$. Let $P_2 \triangleq I_\ell - \Gamma_2^\top (\Gamma_2 \Gamma_2^\top)^{-1} \Gamma_2$ be the orthogonal projection along $\text{range } \Gamma_2^\top$ onto its orthogonal complement $\ker \Gamma_2 \subseteq \mathbb{R}^\ell$. Then, we have

$$W(W + \Gamma \Gamma^\top)^{-1} W = \begin{bmatrix} W_1 - \Gamma_1 P_2 (I_\ell + P_2 \Gamma_1^\top W_1^{-1} \Gamma_1 P_2)^{-1} P_2 \Gamma_1^\top & 0 \\ 0 & 0 \end{bmatrix}. \quad (\text{A.6})$$

Proof: Since $\text{rank } [W \ \Gamma] = m$, we have $W + \Gamma \Gamma^\top \succ 0$ and Γ_2 has full row rank. In particular, $\Gamma_2 \Gamma_2^\top \succ 0$. Since

$$W + \Gamma \Gamma^\top = \begin{bmatrix} W_1 + \Gamma_1 \Gamma_1^\top & \Gamma_1 \Gamma_2^\top \\ \Gamma_2 \Gamma_1^\top & \Gamma_2 \Gamma_2^\top \end{bmatrix}, \text{ it follows that [42]}$$

$$(W + \Gamma \Gamma^\top)^{-1} = \begin{bmatrix} (W_1 + \Gamma_1 \Gamma_1^\top - \Gamma_1 \Gamma_2^\top (\Gamma_2 \Gamma_2^\top)^{-1} \Gamma_2 \Gamma_1^\top)^{-1} & \star \\ \star & \star \end{bmatrix},$$

where the $\star$ entries are irrelevant. Since $\Gamma_1 \Gamma_1^\top - \Gamma_1 \Gamma_2^\top (\Gamma_2 \Gamma_2^\top)^{-1} \Gamma_2 \Gamma_1^\top = \Gamma_1 (I_\ell - \Gamma_2^\top (\Gamma_2 \Gamma_2^\top)^{-1} \Gamma_2) \Gamma_1^\top = \Gamma_1 P_2 \Gamma_1^\top = \Gamma_1 P_2^2 \Gamma_1^\top$, it follows from the Woodbury matrix identity [42] that $(W_1 + \Gamma_1 P_2^2 \Gamma_1^\top)^{-1} = W_1^{-1} - W_1^{-1} \Gamma_1 P_2 (I_\ell + P_2 \Gamma_1^\top W_1^{-1} \Gamma_1 P_2)^{-1} P_2 \Gamma_1^\top W_1^{-1}$. Thus, equation (A.6) holds. $\blacksquare$



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