



Brief paper

Partial attitude synchronization for networks of underactuated spacecraft[☆]

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ABSTRACT

In this paper, the problem of partial attitude synchronization between spacecraft with fewer than three control torques is formulated and solved for the first time. Considering a network of N underactuated spacecraft, we propose two control strategies to align the uncontrolled axes of the spacecraft such that they all point along the same inertial direction. The first control law achieves this objective using inertial angular velocity feedback and, as a result, the final, common, pointing direction of the uncontrolled axes of all spacecraft remains fixed in inertial space. The final direction may also be prescribed with the presence of a leader. The second control law achieves the same objective by relying upon only relative angular velocity feedback. In this case the final pointing direction of the uncontrolled axes is no longer fixed in the inertial space. The control laws utilize a unique attitude parameterization capable of decoupling the motion between the uncontrolled axis of each spacecraft and the spacecraft's angular orientation about this axis. Subject to the proposed control laws, it is shown that the spacecraft network's partial consensus state is almost globally asymptotically stable.

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1. Introduction

Spacecraft attitude synchronization has been previously explored in numerous works. The majority of the literature to date has assumed complete control authority for all spacecraft, and has focused on complete attitude convergence of a spacecraft cluster. Several results using a variety of different approaches have been proposed over the years to solve this problem (Abdessameud, Ayebe, & Polushin, 2012; Abdessameud & Tayebi, 2009; Bai, Arcak, & Wen, 2008; d. Queiroz, Kapila, & Yan, 2000; Dimarogonas, Tsiotras, & Kyriakopoulos, 2006, 2009; Erdong, Xiaolei, & Zhaowei, 2008; Lawton & Beard, 2002; Nair & Leonard, 2004, 2007; Ren, 2010; Sarlette, Sepulchre, & Leonard, 2009; VanDyke & Hall, 2006; Wang, Hadaegh, & Lau, 1999). For instance, in de Queiroz et al. (2000) and Wang et al. (1999) the utilization of a leader–follower approach was used to solve problems related to formation flying. Formation control laws for maintaining attitude alignment was also presented in Lawton and Beard (2002) by requiring that spacecraft communication be restricted to a simple ring structure. The

problem of spacecraft attitude stabilization in a network where each spacecraft may spin about its unstable axis was solved in Nair and Leonard (2004). Attitude tracking control laws which globally asymptotically stabilize the attitude of the spacecraft within a formation was developed and analyzed in VanDyke and Hall (2006). The stabilization of rigid bodies so that their orientations and angular velocity vectors align was shown using a graph theoretic approach in Dimarogonas et al. (2006). The synchronization of both the translation and rotation of rigid body networks was addressed in Nair and Leonard (2007). Extensions using energy shaping methods were provided in Sarlette et al. (2009). Multiple robust controllers which achieved attitude coordinated control of a spacecraft formation was proven in Erdong et al. (2008). Some works have also expanded on the cooperative attitude control to handle scenarios where only a subset of the spacecraft have knowledge of an attitude Ref. Dimarogonas et al. (2009) and Ren (2010). Other results propose control laws that omit the need for an inertial Ref. Bai et al. (2008), control laws that omit the need for angular velocity measurements (Abdessameud & Tayebi, 2009; Ren, 2010), and control laws that incorporate communication delays (Abdessameud et al., 2012; Erdong et al., 2008).

In addition to these papers, attitude control of a single spacecraft has also been explored for the scenario in which complete attitude control is unavailable. Publications such as Coron and Keraï (1996), Krishnan, McClamroch, and Reyhanoglu (1995), Tsiotras, Corless, and Longuski (1995), Tsiotras and Doumchenko (2000), Tsiotras and Longuski (1994), Tsiotras and Luo (2000) and Wang,

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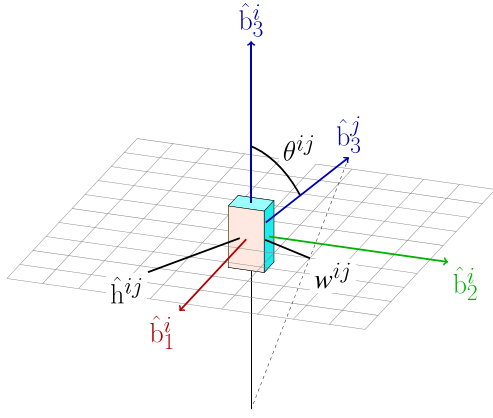


Fig. 1. Visualization of the stereographic projection of $\hat{\mathbf{b}}_3^j$ in the $\hat{\mathbf{b}}^i$ frame.

Jia, Jin, and Xu (2013) have focused on the underactuated spacecraft control problem, where torques about only two axes are available, addressing not only the theoretical limitations of this problem, but also the practical utility for spacecraft that have experienced irreversible failures. Only the results in Meng, Dimarogonas, and Johansson (2017) have addressed partial synchronization of underactuated spacecraft, as far as the authors know. They demonstrate that the partial alignment of the underactuated axes of multiple spacecraft is possible, although it requires direct manipulation of the angular velocities and therefore provides more of a kinematical control approach of the attitude as opposed to a dynamical one.

The main objective of this paper is to present a new class of control laws for the partial attitude synchronization between all spacecraft in an N -spacecraft network where for each spacecraft the control law influences only two of the three principle body axes, with the purpose of aligning the third *uncontrolled* axes of all spacecraft. The relative or absolute orientations of the spacecraft about these uncontrolled axes is assumed to be inconsequential to the control objective. Controlling a single axis (as opposed to all three axes) may be of interest in many applications. For example, this can be the situation where the synchronized pointing of the sensor boresight (e.g., telescope) or high-gain antenna for a multi-spacecraft system is desired, without the need for complete full attitude control of the spacecraft. As shown in this paper, such pointing requirements can be achieved using only two control torques per spacecraft. Note that even if full actuation is available for most spacecraft, the proposed control laws can be utilized in cases of actuator failures.

We propose two different control laws. The first control law ensures that all spacecraft will eventually point their uncontrolled axes along the same *fixed* inertial direction given nearly any set of initial orientations. The first control law also allows the final fixed pointing direction to be specified if one spacecraft is given knowledge of the desired consensus direction. As a result of the second control law, the final, common, pointing direction of the uncontrolled axes is not necessarily restricted to a fixed inertial direction and instead may move in inertial space.

2. Problem formulation and background theory

2.1. Attitude parameterization

The relative orientation between two spacecraft in this paper is described using the non-standard w - z attitude parameters introduced by Tsiotras and Longuski in Tsiotras and Longuski (1995, 1996). The z parameter represents an angle of rotation taken in

the positive direction about the third axis of the initial body frame (assumed here to be underactuated), while the w parameter offers a unique way to describe a vector's orientation with respect to a certain reference frame via a stereographic projection onto a complex plane in the frame resulting from the initial rotation defined by z . The precise definition of the z and the w parameters is given below. For further details regarding this kinematic parameterization of the attitude and its connection with other common attitude parameterizations, please refer to Tsiotras and Longuski (1995, 1996). The use of w - z attitude parameterization is essential to solve the problems in this paper, as it allows us to separate the motion of interest (pointing of the axis) from the spacecraft motion that is irrelevant to us (rotation about this axis). Nonetheless, given the known formulas relating the w - z parameters with more standard attitude parameters such as quaternions or Eulerian angles (Tsiotras & Longuski, 1995), the control laws proposed in this paper can be expressed in terms of these classical attitude parameters as well, if needed.

The w - z attitude parameterization was originally defined in a way that describes the attitude discrepancy between two reference frames (Tsiotras & Longuski, 1995, 1996). To this end, let two body-fixed reference frames consisting of the unit vectors $\{\hat{\mathbf{b}}_1^i, \hat{\mathbf{b}}_2^i, \hat{\mathbf{b}}_3^i\}$ and $\{\hat{\mathbf{b}}_1^j, \hat{\mathbf{b}}_2^j, \hat{\mathbf{b}}_3^j\}$, where $\hat{\mathbf{b}}^i$ represents the frame of body i and $\hat{\mathbf{b}}^j$ represents the frame of body j . The rotation matrix that transforms vectors from frame $\hat{\mathbf{b}}^j$ to frame $\hat{\mathbf{b}}^i$ will be denoted by R_j^i and can be decomposed into two successive rotations according to the two parameters w^{ij} and z^{ij} as follows (Tsiotras & Longuski, 1995, 1996)

$$R_j^i(w^{ij}, z^{ij}) = R_j^{j'}(w^{ij})R_j^{i'}(z^{ij}), \quad (1)$$

where j' represents the intermediate frame $\hat{\mathbf{b}}^{j'}$ after the first rotation by z^{ij} . The matrix $R_j^{j'}(z^{ij})$ is the initial rotation about the $\hat{\mathbf{b}}_3^j$ axis in the positive direction by an angle z^{ij} , resulting in the intermediate frame $\hat{\mathbf{b}}^{j'}$. From this intermediate frame, we desire that the third axis of the frame $\hat{\mathbf{b}}^{j'}$ (also the third axis of the intermediate frame) be described with respect to the final body frame $\hat{\mathbf{b}}^i$. This is accomplished by describing the $\hat{\mathbf{b}}_3^j$ vector in the $\hat{\mathbf{b}}^i$ frame as $\hat{\mathbf{b}}_3^j = a^{ij}\hat{\mathbf{b}}_1^i + b^{ij}\hat{\mathbf{b}}_2^i + c^{ij}\hat{\mathbf{b}}_3^i$. This vector is then stereographically projected onto the $\hat{\mathbf{b}}_1^i$ - $\hat{\mathbf{b}}_2^i$ plane to obtain a new vector with components w_1^{ij} and w_2^{ij} , defined below

$$w_1^{ij} = \frac{b^{ij}}{1 + c^{ij}}, \quad w_2^{ij} = \frac{-a^{ij}}{1 + c^{ij}}. \quad (2)$$

The parameters w_1^{ij} and w_2^{ij} can be used to describe how far to rotate or "tilt" the $\hat{\mathbf{b}}_3^j$ axis away from the $\hat{\mathbf{b}}_3^i$ axis about the vector $\hat{\mathbf{h}}^{ij} = \hat{\mathbf{b}}_3^j \times \hat{\mathbf{b}}_3^i$ as depicted in Fig. 1. The rotation matrix $R_j^{i'}(w^{ij})$ in (1), where $w^{ij} = [w_1^{ij}, w_2^{ij}]^T \in \mathbb{R}^2$, then describes the rotation about the unit vector $\hat{\mathbf{h}}^{ij}$, and its expression can be found in Tsiotras and Longuski (1995).

As shown in Fig. 1, the w parameters offer a measure of the $\hat{\mathbf{b}}_3^j$ axis discrepancy from the $\hat{\mathbf{b}}_3^i$ axis in the form of a stereographic projection. Specifically, when the $\hat{\mathbf{b}}_3^j$ and $\hat{\mathbf{b}}_3^i$ axes are aligned, we have that $w_1^{ij} = w_2^{ij} = 0$. The angle θ^{ij} between the unit vectors $\hat{\mathbf{b}}_3^j$ and $\hat{\mathbf{b}}_3^i$ can be easily computed from the w parameters as follows

$$\theta^{ij} = \arccos \left(\frac{1 - (w_1^{ij})^2 - (w_2^{ij})^2}{1 + (w_1^{ij})^2 + (w_2^{ij})^2} \right). \quad (3)$$

Clearly, $\theta^{ij} = 0$ if and only if $w_1^{ij} = w_2^{ij} = 0$.

Next, consider the kinematics of the w parameters. To this end, let the angular velocity of the $\hat{\mathbf{b}}^j$ frame with respect to the $\hat{\mathbf{b}}^i$ frame,

expressed in the $\hat{\mathbf{b}}^i$ frame, be denoted as $\vec{\omega}^{ij} = \omega_1^{ij}\hat{\mathbf{b}}_1^i + \omega_2^{ij}\hat{\mathbf{b}}_2^i + \omega_3^{ij}\hat{\mathbf{b}}_3^i$. Letting $\omega^i = [\omega_1^i, \omega_2^i, \omega_3^i]^\top \in \mathbb{R}^3$ represent the angular velocity of the $\hat{\mathbf{b}}^i$ frame with respect to the inertial frame, also expressed in the $\hat{\mathbf{b}}^i$ frame, and similarly for ω^j , the components of $\vec{\omega}^{ij}$ are found by

$$\omega^{ij} = \omega^i - R_j^i \omega^j, \quad (4)$$

where $\omega^{ij} = [\omega_1^{ij}, \omega_2^{ij}, \omega_3^{ij}]^\top \in \mathbb{R}^3$. Using Eq. (2) and the derivative of the $\hat{\mathbf{b}}_3^j$ unit vector expressed in the $\hat{\mathbf{b}}^i$ frame (Kane, Likins, & Levinson, 1983; Shuster, 1993; Wie, 2008), the differential equations for the parameters w_1^{ij} and w_2^{ij} are given as follows

$$\begin{bmatrix} \dot{w}_1^{ij} \\ \dot{w}_2^{ij} \end{bmatrix} = \frac{1 + (w_1^{ij})^2 + (w_2^{ij})^2}{2} [\check{R}_j^i(w^{ij})]^\top \omega^{ij}, \quad (5)$$

where $\check{R}_j^i(w^{ij})$ is defined as the matrix consisting of the first two columns of the rotation matrix $R_j^i(w^{ij})$.

2.2. Some tools from algebraic graph theory

Since this paper involves networks of spacecraft interacting with each other, concepts from algebraic graph theory will be utilized to reveal system behavior while under the influence of the proposed control laws. Primarily, we make use of the Laplacian and incidence matrices of a graph (Bapat, 2014; Godsil & Royle, 2001; Mesbahi & Egerstedt, 2010).

Communication between the spacecraft will be encoded using a communication graph $\mathcal{G}$ that is assumed to be connected. The graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ is given by the vertex set $\mathcal{V} = \{1, \dots, N\}$ having as vertices the spacecraft, and the edge set $\mathcal{E} = \{(j, i) \in \mathcal{V} \times \mathcal{V} \mid j \in \mathcal{N}_i\} = \{\varepsilon_1, \dots, \varepsilon_M\}$ consisting of M directed links (j, i) , where the set $\mathcal{N}_i \subset \mathcal{V}$ consists of all spacecraft $j \in \mathcal{V}$ from which spacecraft $i \in \mathcal{V}$ may receive information. A graph $\mathcal{G}$ with a directed set of edges is a *directed* graph. The graph $\mathcal{G}$ is *undirected* when $j \in \mathcal{N}_i$ if and only if $i \in \mathcal{N}_j$, for all $i, j \in \mathcal{V}$, $i \neq j$. Given a directed graph $\mathcal{G}$, the incidence matrix, denoted by $\mathcal{D}(\mathcal{G}) \in \mathbb{R}^{N \times M}$, is defined as

$$[\mathcal{D}(\mathcal{G})]_{ie} = \begin{cases} 1, & \text{if } (j, i) = \varepsilon_e \in \mathcal{E}, \\ -1, & \text{if } (i, j) = \varepsilon_e \in \mathcal{E}, \\ 0, & \text{otherwise.} \end{cases} \quad (6)$$

The incidence matrix $\mathcal{D}(\mathcal{G})$ for an undirected graph is the same as for a directed graph in which each edge in the undirected graph is arbitrarily chosen. Notice that each column of $\mathcal{D}(\mathcal{G})$ sums to 0 and contains a +1 (in-going) and a -1 (out-going) for each end of the corresponding edge. An undirected graph $\mathcal{G}$ is a tree if there is *exactly* one path between any two vertices in the graph, i.e., the graph is acyclic. A tree has $M = N - 1$ edges and the incidence matrix $\mathcal{D}(\mathcal{G})$ for a tree is full column rank. If a directed graph is such that for a vertex $i \in \mathcal{V}$, there is exactly one directed path from i to every $j \in \mathcal{V} \setminus i$, the graph is called an *arborescence* and takes the form of a directed rooted tree with i as the root; as a consequence, all vertices which are not the root have at most one in-going edge.

If we denote by $\mathcal{D}^+(\mathcal{G})$ the pseudo-inverse of $\mathcal{D}(\mathcal{G})$, then for a connected graph $\mathcal{G}$, it holds that $\mathbf{1}_N - \mathcal{D}(\mathcal{G})\mathcal{D}^+(\mathcal{G}) = (1/N)\mathbf{1}_{N \times N}$, where $\mathbf{1}_{N \times N}$ represents the $N \times N$ matrix consisting of all ones (Bapat, 2014). The Laplacian matrix for a graph $\mathcal{G}$, denoted by $\mathcal{L}(\mathcal{G}) \in \mathbb{R}^{N \times N}$, is defined as

$$[\mathcal{L}(\mathcal{G})]_{ij} = \begin{cases} \deg_{\text{in}}(i), & \text{if } i = j, \\ -1, & \text{if } (j, i) \in \mathcal{E}, \\ 0, & \text{otherwise,} \end{cases}$$

where $\deg_{\text{in}}(i)$ is the in-degree of vertex i . For both directed (in which case $\mathcal{L}(\mathcal{G})$ defined above is the in-degree Laplacian) and undirected graphs, the matrix $\mathcal{L}(\mathcal{G})$ is positive semidefinite. If $\mathcal{G}$

is connected then $\mathcal{L}(\mathcal{G})$ contains a single eigenvalue at 0 and the corresponding eigenvector is a multiple of the vector of ones $\mathbf{1}_N = [1, \dots, 1]^\top \in \mathbb{R}^N$.

3. N-spacecraft partial synchronization with inertial velocity feedback

We consider the problem of partial attitude synchronization of N underactuated spacecraft interacting through a communication graph $\mathcal{G}$. It will be assumed that $\mathcal{G}$ is a connected and undirected tree. The more general case when $\mathcal{G}$ has cycles is addressed in Section 3.2. Each spacecraft, denoted by i , belongs to the set $\mathcal{V} = \{1, 2, \dots, N\}$, and is in communication with a subset of $\mathcal{V}$ denoted by $\mathcal{N}_i \subset \mathcal{V} \setminus \{i\}$. Since $\mathcal{G}$ is a connected and undirected tree, we have $\mathcal{N}_i \cap \mathcal{N}_j \neq \emptyset$ for all $i \in \mathcal{V}$ and $j \in \mathcal{N}_i$ if and only if $i \in \mathcal{N}_j$. Associated with the spacecraft $i \in \mathcal{V}$ is the body frame $\hat{\mathbf{b}}^i = (\hat{\mathbf{b}}_1^i, \hat{\mathbf{b}}_2^i, \hat{\mathbf{b}}_3^i)$. In addition, spacecraft i is assumed to have the principle inertia tensor $I^i = \text{diag}(I_1^i, I_2^i, I_3^i)$, with inertias I_1^i , I_2^i , and I_3^i along the $\hat{\mathbf{b}}_1^i$, $\hat{\mathbf{b}}_2^i$, and $\hat{\mathbf{b}}_3^i$ axes, respectively. We adopt the notation

$$I^i \dot{\omega}^i = S(\omega^i) I^i \omega^i + u^i, \quad (7)$$

to refer to the rigid body dynamics specific to spacecraft i , where $u^i = [u_1^i, u_2^i, u_3^i]^\top \in \mathbb{R}^3$ represents the control torque influencing spacecraft i . Accordingly, $\omega^i = [\omega_1^i, \omega_2^i, \omega_3^i]^\top \in \mathbb{R}^3$ represents the inertial angular velocity of spacecraft i expressed in the $\hat{\mathbf{b}}^i$ frame as in (4), and $S(\cdot)$ represents the skew symmetric matrix representing the cross product between two vectors, i.e., $S(\vec{x})\vec{y} = -\vec{x} \times \vec{y}$.

Each spacecraft $i \in \mathcal{V}$ is underactuated, that is, control torques act only along the $\hat{\mathbf{b}}_1^i$ and $\hat{\mathbf{b}}_2^i$ axes, leaving the $\hat{\mathbf{b}}_3^i$ axis uncontrolled, i.e., $u_3^i \equiv 0$ for all $i \in \mathcal{V}$. The objective is to control the direction of the uncontrolled $\hat{\mathbf{b}}_3^i$ axes for each spacecraft so that the final orientations of $\hat{\mathbf{b}}_3^i$ of the spacecraft all point along some common, *fixed* inertial direction. The final direction may also be specified with the designation of a leader who provides the desired reference direction. This is to be achieved by applying the necessary torques on each spacecraft that depend only on the corresponding *inertial* angular velocities ω^i and the *partial attitude discrepancy* with all communicating spacecraft encoded by w^{ij} for $j \in \mathcal{N}_i$. In the presence of an assigned leader, the reference direction given is described in the frame of a satellite i as $\hat{\mathbf{b}}_3^d = a^{id}\hat{\mathbf{b}}_1^i + b^{id}\hat{\mathbf{b}}_2^i + c^{id}\hat{\mathbf{b}}_3^i$, and the corresponding w parameters describing this reference direction in the frame of spacecraft i are $w_1^{id} = b^{id}/(1 + c^{id})$ and $w_2^{id} = -a^{id}/(1 + c^{id})$. Henceforth, we will refer to this as Problem 1.

Before solving Problem 1, we present a few lemmas which will prove helpful in the sequel.

Lemma 1. Let $\hat{\mathbf{b}}^i$ and $\hat{\mathbf{b}}^j$ denote the two body reference frames for spacecraft i and j , respectively. In addition, let $\tilde{w}^{ij} = [w_1^{ij}, w_2^{ij}, 0]^\top \in \mathbb{R}^3$, let $\tilde{w}^{ji} = [w_1^{ji}, w_2^{ji}, 0]^\top \in \mathbb{R}^3$, and let $\check{R}_j^i \in \mathbb{R}^{2 \times 2}$ represent the upper left 2×2 submatrix of R_j^i . Then $\tilde{w}^{ij} = -\check{R}_j^i \tilde{w}^{ji}$, and thus $w^{ij} = -\check{R}_j^i w^{ji}$ and $(w^{ij})^\top w^{ij} = (w^{ji})^\top w^{ji}$.

Proof. The rotation matrix that transforms vectors from frame $\hat{\mathbf{b}}^j$ to frame $\hat{\mathbf{b}}^i$ is given by $R_j^i = R_j^i(w^{ij})R_j^j(z^{ij})$, where $R_j^i(w^{ij})$ and $R_j^j(z^{ij})$ as in (1) and similarly for $R_i^j = R_i^j(w^{ji})R_i^i(z^{ji})$. Using the fact that $R_j^i = [R_i^j]^\top$, it follows immediately that $-w_1^{ij} = w_1^{ji}\cos(z^{ij}) - w_2^{ji}\sin(z^{ij})$ and $-w_2^{ij} = w_1^{ji}\sin(z^{ij}) + w_2^{ji}\cos(z^{ij})$. The desired result follows by direct substitution.

The next lemma is a well known fact (Jeurissen, 1981). It is restated here without proof for the sake of convenience.

Lemma 2. Consider an arborescence $\mathcal{G}$ having N vertices and $M = N - 1$ edges and incidence matrix $\mathcal{D}(\mathcal{G}) \in \mathbb{R}^{N \times M}$. Let $\mathcal{D}_r(\mathcal{G}) \in \mathbb{R}^{M \times M}$ be the resulting matrix from $\mathcal{D}(\mathcal{G})$ after deleting its r th row. An ordering of vertices and edges always exists such that $\mathcal{D}_r(\mathcal{G})$ is upper triangular. Similarly, an ordering of vertices and edges always exists such that $\mathcal{D}_r(\mathcal{G})$ is lower triangular.

Lemma 3. Consider an undirected tree $\mathcal{G}$ having N vertices and $M = N - 1$ edges and incidence matrix $\mathcal{D}(\mathcal{G}) \in \mathbb{R}^{N \times M}$. Let the matrix $\tilde{\mathcal{D}}(\mathcal{G}) \in \mathbb{R}^{rN \times sM}$ be such that it has the same structure as $\mathcal{D}(\mathcal{G}) \otimes A$, where $A \in \mathbb{R}^{r \times s}$, $r \geq s$, constructed as follows. For every element in $\mathcal{D}(\mathcal{G})$, there corresponds an $r \times s$ submatrix in $\tilde{\mathcal{D}}(\mathcal{G})$ such that $\tilde{\mathcal{D}}(\mathcal{G})$ is given by

$$[\tilde{\mathcal{D}}(\mathcal{G})]_{\alpha, \beta} = \begin{cases} \Phi_{ie} & \text{if } (j, i) = \varepsilon_e \in \mathcal{E}, \\ \Psi_{ie} & \text{if } (i, j) = \varepsilon_e \in \mathcal{E}, \\ 0 & \text{otherwise,} \end{cases}$$

where α and β indicate a range of indices for the submatrix $[\tilde{\mathcal{D}}(\mathcal{G})]_{\alpha, \beta}$ where $r(i-1) + 1 \leq \alpha \leq ri$ and $s(e-1) + 1 \leq \beta \leq se$, and where $\Phi_{ie}, \Psi_{ie} \in \mathbb{R}^{r \times s}$. If either Φ_{ie} or Ψ_{ie} is full column rank for all $i = 1, \dots, N$ and $e = 1, \dots, M$, then an edge orientation exists such that $\tilde{\mathcal{D}}(\mathcal{G})$ is also full column rank.

Proof. Assume, without loss of generality, that Φ_{ie} is full column rank for all $i = 1, \dots, N$ and $e = 1, \dots, M$. The case when Ψ_{ie} is full column rank is treated similarly. Notice that since $\mathcal{G}$ is a tree, $\mathcal{D}(\mathcal{G})$ is full column rank. Furthermore, by Lemma 2, an ordering of vertices and edges always exists such that the upper $(N-1) \times M$ submatrix of $\mathcal{D}(\mathcal{G})$ is lower triangular. Furthermore, since $\mathcal{D}(\mathcal{G})$ may be constructed for an undirected graph through arbitrarily directing each link, we can ensure that each element on the diagonal of $\mathcal{D}(\mathcal{G})$ is $+1$. See also Eq. (6). It follows that the matrix $\tilde{\mathcal{D}}(\mathcal{G})$ can be constructed such that the upper $r(N-1) \times sM$ submatrix is lower block triangular where the blocks along the diagonal are the matrices Φ_{ie} . It follows immediately that $\tilde{\mathcal{D}}(\mathcal{G})$ is full column rank.

3.1. Main result for solving Problem 1

In this section, we provide an explicit feedback control law that solves Problem 1.

Proposition 4. Consider a cluster consisting of N spacecraft, and assume that the underlying communication graph $\mathcal{G}$ is a tree. Let $\eta^i = [\omega_1^i, \omega_2^i]^T \in \mathbb{R}^2$, and consider the feedback control law

$$\begin{bmatrix} u_1^i \\ u_2^i \end{bmatrix} = - \sum_{j \in \mathcal{N}_i} k_1^{ij} w^{ij} - k_2^i \eta^i - k_4^i w^{id}, \quad i \in \mathcal{V}, \quad (8)$$

where w^{id} is the reference direction described in the frame of satellite i , and where $k_1^{ij} = k_1^{ji} > 0$, $k_2^i > 0$, and $k_4^i = 0$ for all $i \in \mathcal{V}$ with $i \neq \ell$ and $j \in \mathcal{N}_i$, and $k_4^\ell > 0$ for a some $\ell \in \mathcal{V}$. Then the feedback control law (8) ensures that $\lim_{t \rightarrow \infty} w^{ij} = 0$, $\lim_{t \rightarrow \infty} \eta^i = 0$, and $\lim_{t \rightarrow \infty} w^{id} = 0$ for all $i \in \mathcal{V}$, $j \in \mathcal{N}_i$, and thus solves Problem 1.

Proof. Without loss of generality, we will take $\ell \equiv N$ and will assume that the edges $\varepsilon_e \in \mathcal{E}$, $e = 1, \dots, M$, have been directed such that the resulting graph forms an arborescence with vertex N as its root.

Consider the Lyapunov function candidate

$$V = \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{N}_i} \frac{1}{2} k_1^{ij} \ln(1 + \|w^{ij}\|^2) + \sum_{i \in \mathcal{V}} \frac{1}{2} (\omega^i)^T I^i \omega^i + k_4^N \ln(1 + \|w^{Nd}\|^2).$$

Using (4) and (5) along with Lemma 1, the derivative of $\ln(1 + \|w^{ij}\|^2)$ is found as

$$\frac{d}{dt} \ln(1 + \|w^{ij}\|^2) = (w^{ij})^T \dot{\eta}^i + (\dot{w}^{ij})^T w^{ij}. \quad (9)$$

Taking the time derivative of V along the trajectories of the system (7) with (8), and using (9), yields

$$\begin{aligned} \dot{V} &= \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{N}_i} k_1^{ij} ((w^{ij})^T \dot{\eta}^i - (\eta^i)^T \dot{w}^{ij}) \\ &\quad - \sum_{i \in \mathcal{V}} k_2^i (\eta^i)^T \dot{\eta}^i - k_4^N (\eta^N)^T \dot{w}^{Nd} + k_4^N (w^{Nd})^T \dot{\eta}^N \\ &= - \sum_{i \in \mathcal{V}} k_2^i (\eta^i)^T \dot{\eta}^i - k_4^N (\eta^N)^T \dot{w}^{Nd} + k_4^N (w^{Nd})^T \dot{\eta}^N \\ &= - \sum_{i \in \mathcal{V}} k_2^i (\eta^i)^T \dot{\eta}^i \leq 0. \end{aligned}$$

It follows that the system is Lyapunov stable and hence w^{Nd} , w^{ij} , and ω^i where $i \in \mathcal{V}$, $j \in \mathcal{N}_i$, are all bounded. From (7) with (8), it follows that $\mathcal{M} \triangleq \{(w^{Nd}, \eta^N), (w^{ij}, \eta^i) : \dot{V} \equiv 0\} = \{(w^{Nd}, 0), (w^{ij}, 0) : \sum_{j \in \mathcal{N}_i} k_1^{ij} w^{ij} + k_4^i w^{id} = 0, \forall i \in \mathcal{V}\}$. Since an undirected tree with N vertices and $M = N - 1$ edges has an incidence matrix $\mathcal{D}(\mathcal{G}) \in \mathbb{R}^{N \times M}$ that is full column rank (Mesbahi & Egerstedt, 2010), and since $k_4^i = 0$ for all $i \neq N$, the agreement subspace $\sum_{j \in \mathcal{N}_i} k_1^{ij} w^{ij} + k_4^i w^{id} = 0$ for all $i \in \mathcal{V}$ can be described as

$$\begin{bmatrix} \sum_{j \in \mathcal{N}_1} k_1^{1j} w^{1j} \\ \vdots \\ \sum_{j \in \mathcal{N}_M} k_1^{Mj} w^{Mj} \\ \sum_{j \in \mathcal{N}_N} k_1^{Nj} w^{Nj} + k_4^N w^{Nd} \end{bmatrix} = \mathbf{0} = \tilde{\mathcal{D}}(\mathcal{G}) \tilde{W}, \quad (10)$$

where $\tilde{\mathcal{D}}(\mathcal{G}) \in \mathbb{R}^{2N \times 2N}$ is given by

$$\tilde{\mathcal{D}}(\mathcal{G}) = \begin{bmatrix} \tilde{\mathcal{D}}(\mathcal{G}) & \begin{bmatrix} 0_{2M} \\ \mathbf{I}_2 \end{bmatrix} \end{bmatrix},$$

with $\tilde{\mathcal{D}}(\mathcal{G}) \in \mathbb{R}^{2N \times 2M}$ a block matrix having the same structure as $\mathcal{D}(\mathcal{G}) \otimes A$, where $A \in \mathbb{R}^{2 \times 2}$, given by

$$[\tilde{\mathcal{D}}(\mathcal{G})]_{\alpha, \beta} = \begin{cases} \mathbf{I}_2 & \text{if } (j, i) = \varepsilon_e \in \mathcal{E}, \\ -\tilde{r}_j^i & \text{if } (i, j) = \varepsilon_e \in \mathcal{E}, \\ 0 & \text{otherwise,} \end{cases} \quad (11)$$

where $2i - 1 \leq \alpha \leq 2i$ and $2e - 1 \leq \beta \leq 2e$, and where $\tilde{W} \in \mathbb{R}^{2N}$ is the vector

$$\tilde{W} = \begin{bmatrix} W \\ k_4^N w^{Nd} \end{bmatrix}, \quad (12)$$

with W such that $[W]_\beta = k_1^{ij} w^{ij}$ if $(j, i) = \varepsilon_e \in \mathcal{E}$. It follows from Lemma 1, Lemma 3, and the fact that $\tilde{\mathcal{D}}(\mathcal{G})$ is full rank that $\tilde{W} = 0$ and $w^{ij} = 0$ for all $i \in \mathcal{V}$, $j \in \mathcal{N}_i$. Thus, the agreement subspace $\mathcal{M}$ consists only of the point $w^{ij} = 0$ and $\eta^i = 0$ for all $i \in \mathcal{V}$, $j \in \mathcal{N}_i$. Since $k_4^N > 0$, the last entry in (12) yields $w^{Nd} = 0$ and $w^{ij} = 0$ imply that $w^{id} = 0$ for all $i \in \mathcal{V}$. By LaSalle's invariance principle, the proof is complete.

Corollary 5. Under the conditions of Proposition 4, if $k_4^i = 0$ for all $i \in \mathcal{V}$, then $\lim_{t \rightarrow \infty} w^{ij} = 0$ and $\lim_{t \rightarrow \infty} \eta^i = 0$ for all $i \in \mathcal{V}$, $j \in \mathcal{N}_i$.

3.2. Effect of cycles in the communication graph

In this section, the case in which the communication graph contains cycles is considered. It is shown that, contrary to the standard case where the underlying space is Euclidean, in our problem the existence of cycles is much more complicated. To illustrate the potential issues involved we further elaborate in this section on the case of a graph with a single cycle. It is shown that, in general, the existence of a cycle induces spurious non-consensus equilibria. In the sequel, we assume that the conditions of [Proposition 4](#) still hold, except from that fact that we now allow the communication graph to include a single cycle. Since $\tilde{\mathcal{D}}(\mathcal{G})$ in (11) is not full column rank, there may exist nontrivial solutions to (10), that is, there may exist solutions $W \neq \mathbf{0}$ corresponding to equilibria in the agreement subspace $\mathcal{M} = \{(w^{Nd}, 0), (w^{ij}, 0) : \sum_{j \in \mathcal{N}_i} k_1^{ij} w^{ij} + k_4^i w^{id} = 0, \forall i \in \mathcal{V}\}$ which are not consensus configurations. At such equilibria the underactuated spacecraft axes are not aligned with each other. We first show the following result for spacecraft communication graphs with a ring topology.

Lemma 6. Assume the conditions of [Proposition 4](#), assume that $k_4^i = 0$ for all $i \in \mathcal{V}$ and the underlying communication graph $\mathcal{G}$ is a ring. Then every solution to (10) satisfies $k_1^{ij} \|w^{ij}\| = k_1^{pq} \|w^{pq}\|$, for all $i, p \in \mathcal{V}, j \in \mathcal{N}_i, q \in \mathcal{N}_p$.

Proof. Since $\mathcal{G}$ is a ring, $|\mathcal{N}_i| = 2$ for all $i \in \mathcal{V}$. Letting $j, \ell \in \mathcal{N}_i$, the agreement subspace defined by (10) implies $k_1^{ij} w^{ij} = -k_1^{i\ell} w^{i\ell}$ and hence $k_1^{ij} \|w^{ij}\| = k_1^{i\ell} \|w^{i\ell}\|$. Utilizing [Lemma 1](#), it follows that $\|w^{i\ell}\| = \|w^{ji}\|$, which implies that $k_1^{ij} \|w^{ij}\| = k_1^{ji} \|w^{ji}\|$. Letting now $r \in \mathcal{N}_\ell$, it follows that $k_1^{i\ell} w^{i\ell} = -k_1^{r\ell} w^{r\ell}$ which implies $k_1^{ij} \|w^{ij}\| = k_1^{r\ell} \|w^{r\ell}\|$. Repeating the same procedure to w^{pq} , proves the desired result.

Proposition 7. Assume the conditions of [Proposition 4](#) hold and $k_4^i = 0$ for all $i \in \mathcal{V}$. Also assume that the underlying communication graph $\mathcal{G}$ contains a tree $\mathcal{G}_T = (\mathcal{V}_T, \mathcal{E}_T)$ with the property that every leaf of $\mathcal{G}_T$ except one is a leaf of $\mathcal{G}$ and each pair of these leaves in $\mathcal{G}_T$ is connected through a unique path in $\mathcal{G}$. Then, for all $(j, i) \in \mathcal{E}_T$, the solution to (10) satisfies $w^{ij} = 0$ for all $i, j \in \mathcal{V}_T$.

Proof. Assume that the subgraph $\mathcal{G}_T$ contains n_T nodes and e_T edges, and that the indexing of the nodes and edges are such that the first n_T of the total N nodes in $\mathcal{G}$ are in $\mathcal{G}_T$, and similarly for the first e_T of the total M edges of $\mathcal{G}$. Then the incidence matrix (6) can be partitioned as follows

$$\tilde{\mathcal{D}}(\mathcal{G}) = \begin{bmatrix} A & 0 \\ B & C \\ 0 & D \end{bmatrix}, \quad (13)$$

where $A \in \mathbb{R}^{2(n_T-1) \times 2e_T}$, $B \in \mathbb{R}^{2 \times 2e_T}$, $C \in \mathbb{R}^{2 \times 2(M-e_T)}$, $D \in \mathbb{R}^{2(N-n_T) \times 2(M-e_T)}$, and $[A^T, B^T]^T = \tilde{\mathcal{D}}(\mathcal{G}_T)$. Since $\mathcal{G}_T$ is a tree, by [Lemmas 2](#) and [3](#), A can be constructed such that it is full rank. Therefore, the solution W to (10) with $\tilde{\mathcal{D}}(\mathcal{G})$ as in (13) has the first $2e_T$ elements zero, indicating that the w parameters associated with the first e_T edges are zero, thus proving the desired result.

[Proposition 7](#) deals with graphs with one cycle such that the vertices of the cycle are roots of trees branching out to the leafs of $\mathcal{G}$. Next, we use this result to show that for such graphs there exist equilibrium configurations which may involve non-consensus states. Before doing that, we need to introduce the following notation. For a vector $W \in \mathbb{R}^{2M}$ with elements $[W]_r = k_1^{ij} w^{ij}$ where $r = 1, \dots, 2M$ we now define the corresponding vector $\mathbf{W} \in \mathbb{R}^M$ such that $[W]_e = \alpha^{ij}$ with $|\alpha^{ij}| = k_1^{ij} \|w^{ij}\|$ for $(j, i) = \varepsilon_e \in \mathcal{E}$ and $e = 1, \dots, M$.

Proposition 8. Let the conditions of [Proposition 4](#) hold and let $k_4^i = 0$ for all $i \in \mathcal{V}$. If the underlying communication graph $\mathcal{G}$ contains a single cycle, then the vector W corresponding to the solution W to (10) satisfies $\mathbf{0} = \mathcal{D}(\mathcal{G})\mathbf{W}$.

Proof. Let $\mathcal{G}_c = (\mathcal{V}_c, \mathcal{E}_c)$ be the single cycle in $\mathcal{G}$. If we remove the edges corresponding to the cycle, the remaining graph is a collection of trees each one rooted at one of the elements of $\mathcal{V}_c$. From [Proposition 7](#), $w^{ij} = 0$ for all $(j, i) \in \mathcal{E} \setminus \mathcal{E}_c$. Hence, the only nonzero w parameters are associated with edges in $\mathcal{E}_c$, and hence the solution to (10) concerning w^{ij} for all $(j, i) \in \mathcal{E}_c$ is identical to the case when $\mathcal{G} = \mathcal{G}_c$. If $j, \ell \in \mathcal{N}_i$ such that $(j, i), (\ell, i) \in \mathcal{E}_c$, then $k_1^{ij} w^{ij} = -k_1^{i\ell} w^{i\ell}$ (see the proof of [Lemma 6](#)). Any vector $k_1^{ij} w^{ij}$ can be written as $k_1^{ij} w^{ij} = R_z(z^{i*})[\alpha^{ij}, 0]^T$ for $\alpha \in \mathbb{R}, z^{i*} \in \mathbb{R}, |\alpha^{ij}| = k_1^{ij} \|w^{ij}\|$, and where R_z is the rotation matrix in the $\hat{\mathbf{b}}_1^i$ - $\hat{\mathbf{b}}_2^i$ plane. Therefore, there exists a matrix $R_* = \text{blkdiag}(R_z(z^{1*}), R_z(z^{2*}), \dots, R_z(z^{N*})) \in \mathbb{R}^{2N \times 2N}$, such that $\mathbf{0} = \tilde{\mathcal{D}}W = R_*^T \tilde{\mathcal{D}}W = (\mathcal{D} \otimes \mathbf{I}_2)(W \otimes [1, 0]^T)$, which implies that $\mathbf{0} = \mathcal{D}(\mathcal{G})\mathbf{W}$. The proof is now complete.

The set of all non-trivial (non-consensus) equilibria are contained in the null space of $\mathcal{D}(\mathcal{G})$ which is non-empty if the graph $\mathcal{G}$ contains a cycle. In such a case, the solution W of $\mathbf{0} = \mathcal{D}(\mathcal{G})\mathbf{W}$ corresponding to a non-consensus equilibrium configuration is given by $W = \kappa v_0$, where v_0 is the column vector of $\mathcal{V}$ associated with the zero singular value of $\mathcal{D}(\mathcal{G})$, obtained from the singular value decomposition $\mathcal{D}(\mathcal{G}) = \mathcal{U}S\mathcal{V}^*$. Note that, clearly, $\mathcal{D}(\mathcal{G})v_0 = 0$ and κ is a nonzero scaling factor computed from

$$\mathbf{I}_3 = \left(\prod_{i=2}^{n_c} R_i^{-1}(w^{i-1,i}, z^{i-1,i}) \right) R_1^{n_c}(w^{n_c,1}, z^{n_c,1}), \quad (14)$$

where $n_c = |\mathcal{V}_c|$, $w^{ij} = (1/k_1^{ij})R_z(z^{i*})[\alpha^{ij}, 0]^T$ and z^{i*} is arbitrary.

Remark 9. Note that if all edges in the graph that correspond to nonzero w^{ij} are part of a single cycle, then $k_1^{ij} w^{ij} = -k_1^{i\ell} w^{i\ell}$ for all $(j, i), (\ell, i) \in \mathcal{E}_c$ and [Lemma 1](#) implies that the underactuated axes of the spacecraft which are part of the cycle lie within a plane. It follows that the summation of all angles defined by (3) between the underactuated axes of the neighboring spacecraft in the cycle must be 2π . This fact is actually a special case of the condition given by (14).

3.2.1. The stability of non-consensus equilibria

In this section, the stability of the non-consensus equilibrium configurations is investigated. Because of [Proposition 7](#), all w parameters associated with edges that are not part of the cycle are zero, and therefore the result of [Lemma 6](#) holds. Therefore, and without loss of generality, the remainder of this section focuses on the case when $\mathcal{G}$ is a ring.

Below we provide a heuristic argument to show the instability of the non-consensus equilibria for a ring with three spacecraft. The case for cycles with more spacecraft follows using similar arguments. First, note that for two spacecraft i and j which do not have their underactuated axes aligned, a constant angle $0 < \theta^{ij} < \pi$ may be preserved if $\hat{\mathbf{b}}_3^i$ traces a circle on the unit sphere as shown in [Fig. 2\(a\)](#), where the arrows along the circle indicate the direction of motion from the circle that results in a decrease of θ^{ij} . For the case of three spacecraft i, j and k , the angles θ^{ij} , θ^{jk} , and θ^{ik} are greater than $\pi/2$, otherwise (10) is not satisfied. As can be seen from [Fig. 2\(b\)](#) and (c), due to the region between the two circles, the vector field has two components that point away from $\hat{\mathbf{b}}_3^i$ and are normal to the plane created by the vectors $\hat{\mathbf{b}}_3^j$ and $\hat{\mathbf{b}}_3^k$ where θ^{ij} and θ^{ik} both decrease, indicating a saddle point.

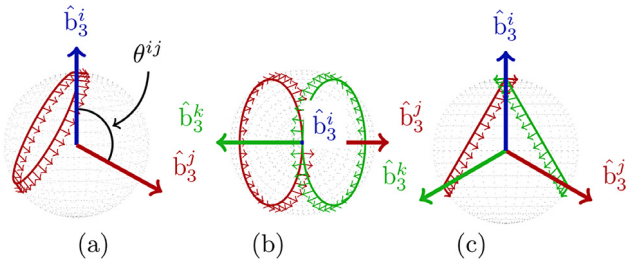


Fig. 2. Visualization of tracing angles on the unit sphere: (a) small circle traced by $\hat{\mathbf{b}}_3^i$ for a fixed $\hat{\mathbf{b}}_3^j$ and fixed θ^{ij} ; (b) plot of two small circles, one as in (a) and the other traced by $\hat{\mathbf{b}}_3^i$ for a fixed $\hat{\mathbf{b}}_3^k$ and fixed θ^{ik} , as seen from the positive $\hat{\mathbf{b}}_3^i$ axis; (c) is similar to (b) but seen normal to the plane formed by $\hat{\mathbf{b}}_3^i$, $\hat{\mathbf{b}}_3^j$, and $\hat{\mathbf{b}}_3^k$.

As an alternative to the heuristic argument shown, it is possible to show the instability of these spurious equilibria through linearization. The use of linearization to analytically show instability can become difficult, however, since the associated matrix becomes quite large. In the examples of Section 5 instability was thus confirmed numerically.

4. N -spacecraft partial synchronization with relative velocity feedback

In this section we consider the problem (henceforth referred to as Problem 2) of the partial synchronization of an N -spacecraft cluster in which the final pointing direction of the uncontrolled axes is the same for all spacecraft, but this direction may not be fixed in inertial space. We assume that the communication graph $\mathcal{G}$ has the same properties as for Problem 1, that is, the graph $\mathcal{G}$ is a connected and undirected tree.

The desired objective of Problem 2 is to synchronize the uncontrolled $\hat{\mathbf{b}}_3^i$ axes for all spacecraft so that the final orientation of $\hat{\mathbf{b}}_3^i$ for each spacecraft $i \in \mathcal{V}$ is along some common inertial direction, which, however, is not necessarily fixed. This can be achieved by applying the necessary torques to each spacecraft $i \in \mathcal{V}$, which depend however only upon the *partial relative angular velocities* defined as

$$\eta^{ij} = \eta^i - \tilde{R}_j^i \eta^j, \quad j \in \mathcal{N}_i, \quad (15)$$

where $\tilde{R}_j^i \in \mathbb{R}^{2 \times 2}$ represents the upper left 2×2 submatrix of R_j^i , and the *partial attitude discrepancy* described by w^{ij} for $j \in \mathcal{N}_i$. Next, we present a lemma that will help us solve Problem 2.

Lemma 10. Let Q_i represent the rotation matrix that transforms elements from the body frame $\hat{\mathbf{b}}^i$ to the inertial frame, let $\tilde{Q}_i \in \mathbb{R}^{3 \times 2}$ be the matrix consisting of the first two columns of Q_i , and let $\tilde{Q}_j$ and $\tilde{Q}_j$ be defined similarly for the body frame $\hat{\mathbf{b}}^j$. Then, $\tilde{Q}_i^T \tilde{Q}_i = \mathbf{I}_2$ and $\tilde{Q}_i^T \tilde{Q}_j = \tilde{R}_j^i$.

Proof. The proof follows immediately by performing the corresponding matrix multiplications, and thus it is omitted.

4.1. Main result for solving Problem 2

The explicit feedback control law that solves Problem 2 is given below.

Proposition 11. Consider a network consisting of N spacecraft, and assume that the corresponding communication graph $\mathcal{G}$ is a tree. Let

the feedback control law for each spacecraft $i \in \mathcal{V}$ be given by

$$\begin{bmatrix} u_1^i \\ u_2^i \end{bmatrix} = -k_3^i \begin{bmatrix} (I_2^i - I_3^i) \omega_2^i \omega_3^i \\ (I_3^i - I_1^i) \omega_1^i \omega_3^i \end{bmatrix} - \sum_{j \in \mathcal{N}_i} k_1 w^{ij} - \sum_{j \in \mathcal{N}_i} k_2 \eta^{ij},$$

where $k_1 > 0$, $k_2 > 0$, and $k_3^i = 1$ for $I_1^i \neq I_2^i$ and $k_3^i = \mu_i \in \mathbb{R}$ for $I_1^i = I_2^i$. Then the previous control ensures that $\lim_{t \rightarrow \infty} w^{ij} = 0$ and $\lim_{t \rightarrow \infty} \eta^{ij} = 0$, and thus solves Problem 2.

Proof. Consider the Lyapunov function candidate

$$V = \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{N}_i} \frac{1}{2} k_1 \ln(1 + \|w^{ij}\|^2) + \sum_{i \in \mathcal{V}} \frac{1}{2} (\eta^i)^T J^i \eta^i,$$

where $J^i = \text{diag}(I_1^i, I_2^i)$. Taking the derivative of V along the trajectories of the closed-loop system and using (9), (15), and Lemma 10 yields

$$\dot{V} = -k_2 \mathcal{H}^T \tilde{\mathcal{L}}(\mathcal{G}) \mathcal{H} \leq 0,$$

where $\mathcal{H} = [(\tilde{Q}_1 \eta^1)^T, \dots, (\tilde{Q}_N \eta^N)^T]^T \in \mathbb{R}^{3N}$, $\tilde{\mathcal{L}}(\mathcal{G}) = \mathcal{L}(\mathcal{G}) \otimes \mathbf{I}_3$, and $\mathcal{L}(\mathcal{G})$ is the graph Laplacian. Since $\dot{V} \leq 0$, it follows that the signals w^{ij} and η^i for all $i \in \mathcal{V}$, $j \in \mathcal{N}_i$, are bounded. Letting $\dot{V} \equiv 0$, from the properties of $\mathcal{L}(\mathcal{G})$, it follows that the null space of $\tilde{\mathcal{L}}(\mathcal{G})$ is $\mathbf{1}_N \otimes y$ where $y \in \mathbb{R}^3$, and thus $\tilde{Q}_i \eta^i = y$ for all $i \in \mathcal{V}$. This implies that $\dot{V} = 0$ if and only if $\tilde{Q}_i \eta^i = \tilde{Q}_j \eta^j$ for all $i, j \in \mathcal{V}$, which, after a left multiplication by $\tilde{Q}_i^T$ and using again Lemma 10, yields that $\eta^i = \tilde{R}_j^i \eta^j$ for all $i, j \in \mathcal{V}$. It follows that $\eta^{ij} = \eta^i - \tilde{R}_j^i \eta^j = 0$ for all $i, j \in \mathcal{V}$, indicating that $\mathcal{M} \triangleq \{(w^{ij}, \eta^{ij}) : \dot{V} \equiv 0\} = \{(w^{ij}, 0), \forall i \in \mathcal{V}, j \in \mathcal{N}_i\}$. Furthermore, note that $(\eta^i)^T \eta^i = (\eta^j)^T \eta^j$ for all $i, j \in \mathcal{V}$, and thus the following holds for all $i, j \in \mathcal{V}$: $0 = (\eta^i)^T \eta^i - (\eta^j)^T \eta^j = (\eta^i)^T \eta^j - (\eta^j)^T \eta^i = (\eta^i)^T \eta^j - (\eta^j)^T \eta^i = (\eta^i)^T \eta^j - (\eta^j)^T \eta^i = \omega_1^i w_2^j - \omega_2^i w_1^j$. Assume now that $w^{ij} \neq 0$ for some $i \in \mathcal{V}$, $j \in \mathcal{N}_i$, otherwise there is nothing to prove. Also assume that $\eta^i \neq 0$ for all $i \in \mathcal{V}$, otherwise $\sum_{j \in \mathcal{N}_i} w^{ij} = 0$ for all $i \in \mathcal{V}$, in which case the result follows using an argument similar to the proof of Problem 1. From $\omega_1^i w_2^j - \omega_2^i w_1^j = 0$ it then follows that $w^{ij} = \lambda_{ij} \eta^j$ for all $i, j \in \mathcal{V}$, where $\lambda_{ij} \neq 0$ for some $i \in \mathcal{V}$, $j \in \mathcal{N}_i$. Next, let $w^{ji} = \lambda_{ji} \eta^i$ and notice that left multiplying $w^{ij} = \lambda_{ij} \eta^j$ with $\tilde{R}_j^i$ leads to $-w^{ji} = \lambda_{ij} \eta^j$, where Lemma 1 and $\eta^i = \tilde{R}_j^i \eta^j$ were used. This implies that for each communication path between i and j , $i \in \mathcal{V}$, $j \in \mathcal{N}_i$, the proportionality constant in $w^{ij} = \lambda_{ij} \eta^j$ and $w^{ji} = \lambda_{ji} \eta^i$ are equal and opposite ($\lambda_{ij} = -\lambda_{ji}$) and thus a single constant $\lambda_{ij} = -\lambda_{ji} = \lambda_e$ may be associated with each undirected edge $(i, j) \in \mathcal{E}$. Letting $\sum_{j \in \mathcal{N}_i} \lambda_{ij} = \Lambda_i$, it follows that $\sum_{j \in \mathcal{N}_i} w^{ij} = \sum_{j \in \mathcal{N}_i} \lambda_{ij} \eta^j = \Lambda_i \eta^i$. The relationship between the vectors constructed by $\Gamma = [\Lambda_1, \dots, \Lambda_N]^T \in \mathbb{R}^N$ and $\gamma = [\lambda_1, \dots, \lambda_M]^T \in \mathbb{R}^M$, can then be expressed as $\Gamma = \mathcal{D}(\mathcal{G}) \gamma$, where $\mathcal{D}(\mathcal{G})$ is the incidence matrix for the communication graph $\mathcal{G}$. Since $\mathcal{G}$ is a tree (i.e., $\mathcal{D}(\mathcal{G})$ is full column rank) and at least one $\lambda_e \neq 0$, $\epsilon_e \in \mathcal{E}$, there are at least two $\Lambda_i \neq 0$, $i \in \mathcal{V}$. By taking the derivative of $(\eta^i)^T \eta^i = (\eta^j)^T \eta^j$ and using $\eta^{ij} = 0$ and $\omega_1^i w_2^j - \omega_2^i w_1^j = 0$, one obtains $(\eta^i)^T \dot{\eta}^i = (\eta^p)^T \dot{\eta}^p$ or that

$$\begin{aligned} -(\eta^i)^T (J^i)^{-1} k_1 \sum_{j \in \mathcal{N}_i} w^{ij} &= -(\eta^p)^T (J^p)^{-1} k_1 \sum_{q \in \mathcal{N}_p} w^{pq}, \\ -\Lambda_i k_1 (\eta^i)^T (J^i)^{-1} \eta^i &= -\Lambda_p k_1 (\eta^p)^T (J^p)^{-1} \eta^p, \end{aligned}$$

where $i, p \in \mathcal{V}$. Since $-k_1 (\eta^i)^T (J^i)^{-1} \eta^i < 0$ for all $i \in \mathcal{V}$ and since at least one $\Lambda_i \neq 0$, $i \in \mathcal{V}$, it follows necessarily that $\Lambda_i \neq 0$ and $\text{sign}(\Lambda_i) = \text{sign}(\Lambda_p)$ for all $i, p \in \mathcal{V}$. Using properties from graph theory, it can be easily shown that $N(\mathbf{I}_N - \mathcal{D}(\mathcal{G}) \mathcal{D}^+(\mathcal{G})) \Gamma = \mathbf{1}_{N \times N} \Gamma$ or that $N(\Gamma - \mathcal{D}(\mathcal{G}) \gamma) \gamma = \mathbf{1}_{N \times N} \Gamma$, and thus $\mathbf{0} = \mathbf{1}_N \Gamma$. However, since $\Lambda_i \neq 0$ and $\text{sign}(\Lambda_i) = \text{sign}(\Lambda_p)$, it follows that $\mathbf{1}_{N \times N} \Gamma \neq \mathbf{0}$, leading to a contradiction. Thus, necessarily $\Lambda_i = 0$ for all $i \in \mathcal{V}$. Since $\mathcal{D}(\mathcal{G})$ is full column

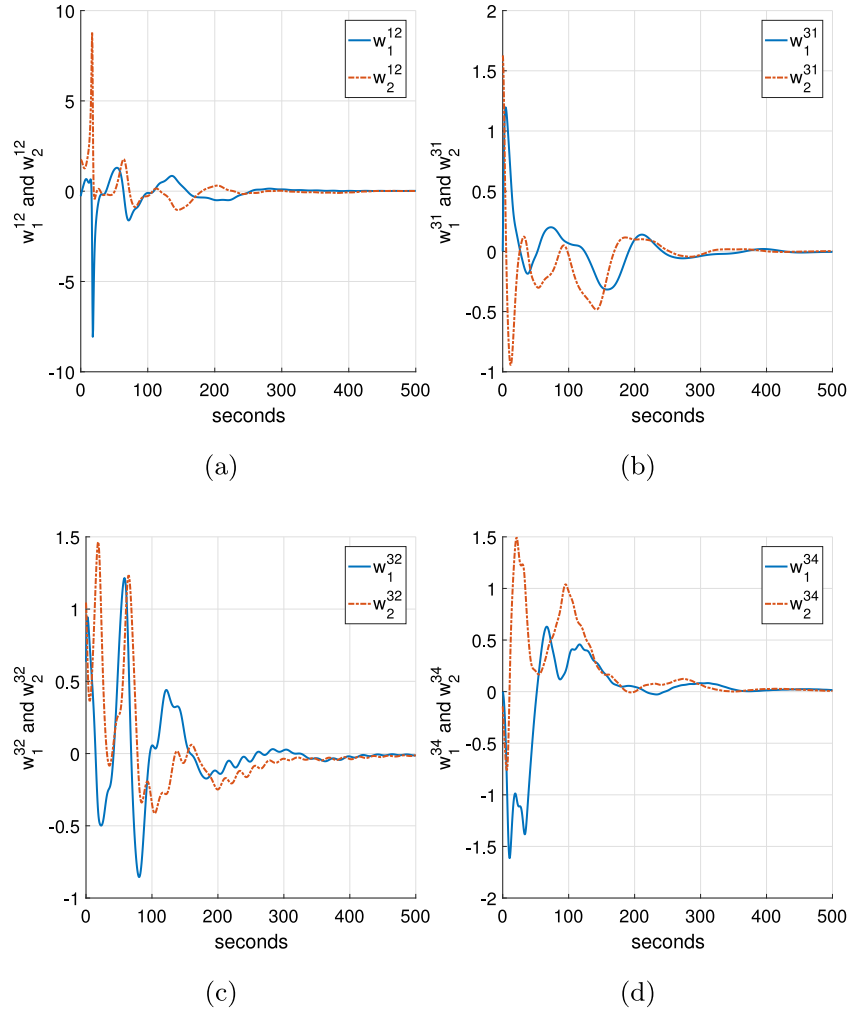


Fig. 3. The w parameter state histories for Problem 1.

rank and $\Gamma = \mathbf{0}$, it follows from $\Gamma = \mathcal{D}(\mathcal{G})\gamma$ that $\lambda_e = 0$ for all $(i, j) = \varepsilon_e \in \mathcal{E}$ which violates our assumption that $\lambda_{ij} \neq 0$. Since $\lambda_e = 0$ for all $(i, j) = \varepsilon_e \in \mathcal{E}$, it follows that $w^{ij} = 0$ for all $i \in \mathcal{V}, j \in \mathcal{N}_i$. Thus, the agreement subspace $\mathcal{M}$ consists only of the point where $w^{ij} = 0$ and $\eta^{ij} = 0$ for all $i \in \mathcal{V}, j \in \mathcal{N}_i$. By LaSalle's invariance principle, the proof is complete.

5. Simulation results

To demonstrate the previous results, three numerical simulations were performed to validate Proposition 4 and Proposition 11. For Proposition 4, two simulations were made: one in which the network is absent of a leader and another which contains a leader.

5.1. Simulation 1: 4-spacecraft network with inertial velocity feedback and without a leader

For Problem 1, we assume that the spacecraft communication topology is encoded in the sets $\mathcal{N}_i$, for $i \in \mathcal{V} = \{1, 2, 3, 4\}$, where $\mathcal{N}_1 \triangleq \{3\}$, $\mathcal{N}_2 \triangleq \{3\}$, $\mathcal{N}_3 \triangleq \{1, 2, 4\}$, $\mathcal{N}_4 \triangleq \{3\}$. The spacecraft have moments of inertia $I^1 = \text{diag}(45, 25, 15) \text{ kg m}^2$, $I^2 = \text{diag}(15, 35, 45) \text{ kg m}^2$, $I^3 = \text{diag}(84, 24, 15) \text{ kg m}^2$, and $I^4 = \text{diag}(60, 40, 15) \text{ kg m}^2$, and their initial angular velocities are $\omega^1(0) = (0.2, -0.1, 0.1)^T \text{ rad/s}$, $\omega^2(0) = (-0.1, -0.1, -0.2)^T \text{ rad/s}$, $\omega^3(0) = (0.1, 0.1, 0.1)^T \text{ rad/s}$, and $\omega^4(0) = (0.1, 0.3, 0.1)^T \text{ rad/s}$, respectively. The initial orientations given in unit

quaternions for Spacecraft 1, 2, 3, and 4 were chosen as $[1, 0, 0, 0]$, $[0.5547, 0.5547, -0.2774, 0.5547]$, $[0, -0.3162, 0, 0.9487]$, and $[0, -0.4472, 0, 0.8944]$, respectively. The control gains were set to $k_1 = 0.2$ and $k_2^i = 1$ for $i = 1, 2, 3, 4$.

The w parameters describing the $\hat{\mathbf{b}}_3$ axis of Spacecraft 2 in the frame of Spacecraft 1 and Spacecraft 1, 2, and 4 in the frame of Spacecraft 3 are plotted in Fig. 3. Similar behavior was observed for the other partial attitude parameters as well. Snapshots of the simulation's animation are shown in Fig. 4. The animation of Simulation 1 can be found at <http://dcs1.gatech.edu/movies/undersatsim1.avi>.

5.2. Simulation 2: 4-spacecraft network with inertial velocity feedback and with a leader

For the leader–follower version of Problem 1, the simulation for Problem 1 was modified so that the spacecraft represented by $i = 1$ serves as a reference direction in which its attitude remains constant. In this way, the spacecraft represented by $i = 3$ takes the role of the leader of the network because it has knowledge of the reference direction from Spacecraft 1. We achieve this by assuming that the spacecraft communication topology is encoded in the sets $\mathcal{N}_i$, for $i \in \mathcal{V} = \{2, 3, 4\}$, where $\mathcal{N}_2 \triangleq \{3\}$, $\mathcal{N}_3 \triangleq \{2, 4\}$, $\mathcal{N}_4 \triangleq \{3\}$. The spacecraft have moments of inertia and initial angular velocities identical to Simulation 1 with the exception that $\omega^1(0) = (0, 0, 0)^T \text{ rad/s}$. The initial orientations given in

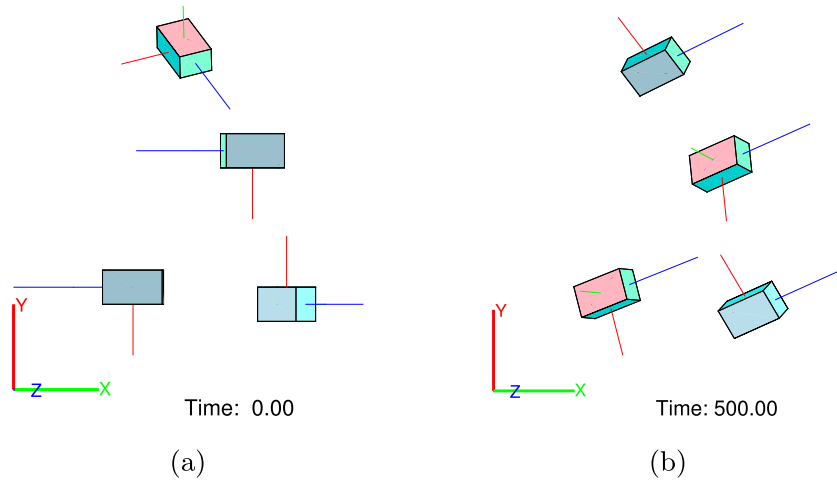


Fig. 4. Snapshots of initial and final orientations of the simulation for Problem 1.

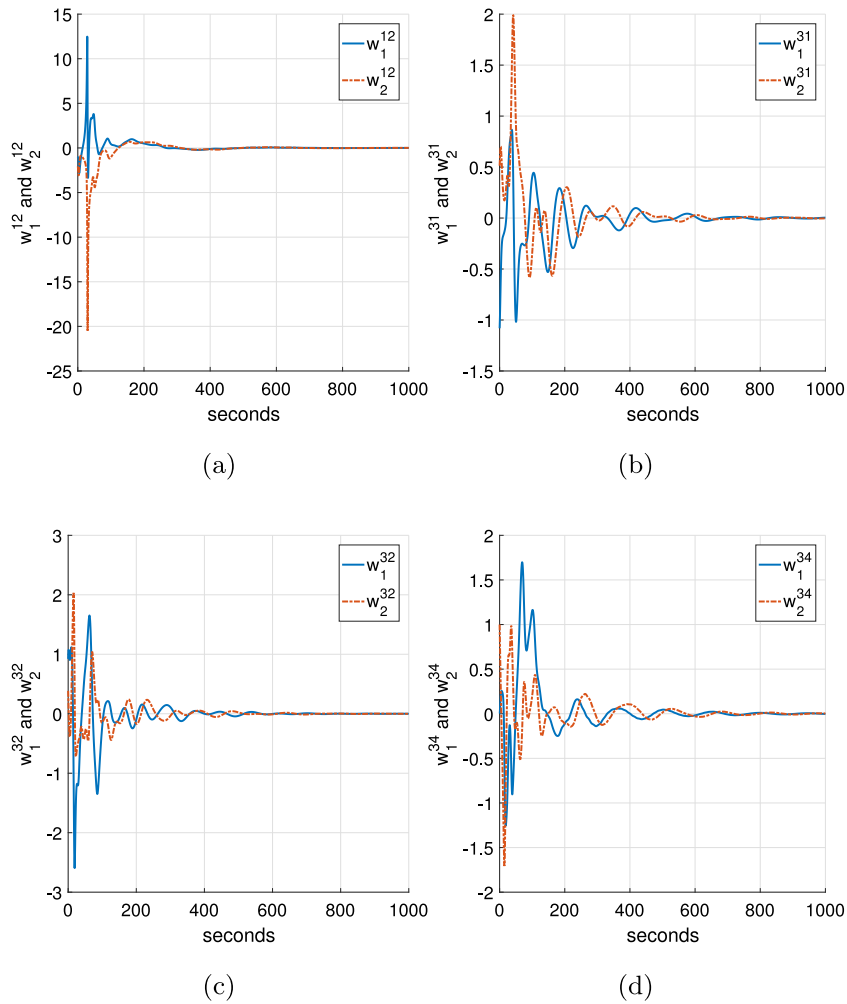


Fig. 5. The w parameter state histories for the extension to Problem 1.

unit quaternions for the reference Spacecraft 1 was chosen as $[0.6667, -0.6667, 0, -0.3333]$ and for the networked Spacecraft 2, 3, and 4 were chosen to be the same as in Simulation 1. The control gains were set to $k_1 = 0.2$, $k_2^i = 1$ for $i = 2, 3, 4$, and $k_4^3 = 0.2$.

The w parameters describing the $\hat{\mathbf{b}}_3$ axis of Spacecraft 2 in the frame of the reference Spacecraft 1 and Spacecraft 1, 2, and 4 in the frame of Spacecraft 3 are plotted in Fig. 5. Snapshots of the simulation's animation are shown in Fig. 6. The animation can be found at <http://dcsl.gatech.edu/movies/undersatsim3.avi>.

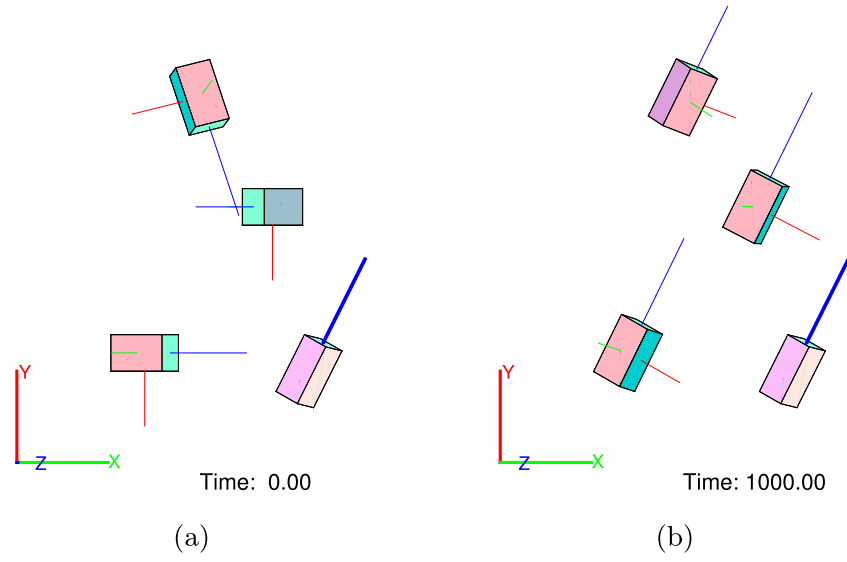


Fig. 6. Snapshots of initial and final orientations of the simulation for the extension to Problem 1.

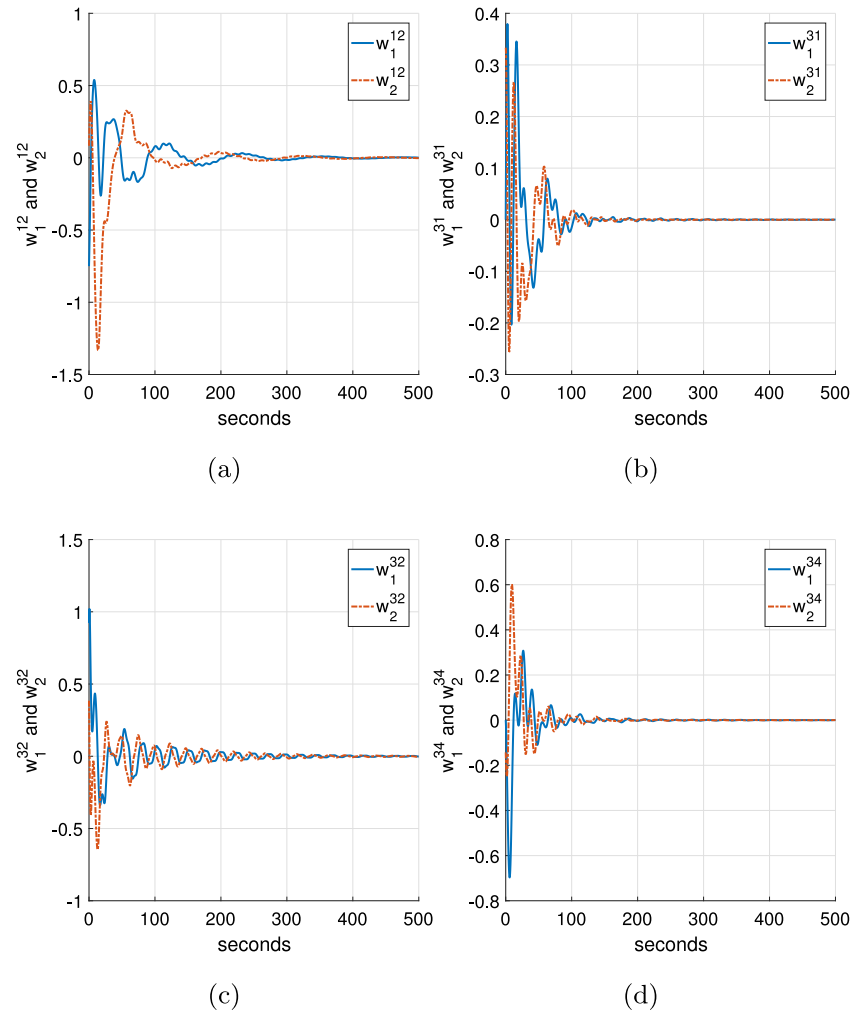


Fig. 7. The w parameter state histories for Problem 2.

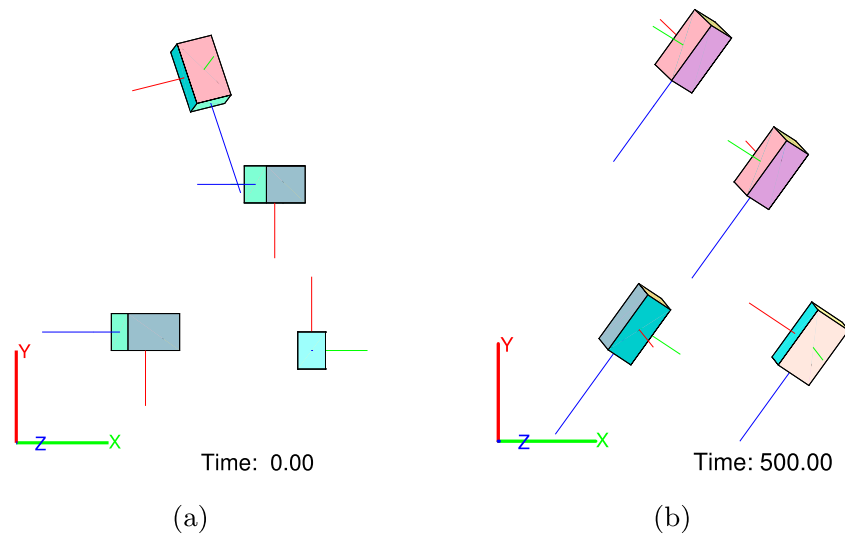


Fig. 8. Snapshots of initial and final orientations of the simulation for Problem 2.

5.3. Simulation 3: 4-spacecraft network with relative velocity feedback

For Problem 2, we assume four spacecraft communicating based on the same communication topology as in Simulation 1. The spacecraft have moments of inertia $I^1 = \text{diag}(25, 25, 15) \text{ kg m}^2$, $I^2 = \text{diag}(15, 35, 45) \text{ kg m}^2$, $I^3 = \text{diag}(34, 24, 15) \text{ kg m}^2$, and $I^4 = \text{diag}(30, 40, 15) \text{ kg m}^2$, and their initial angular velocities are $\omega^1(0) = (0.2, -0.1, 0.1)^T \text{ rad/s}$, $\omega^2(0) = (-0.1, -0.1, -0.2)^T \text{ rad/s}$, $\omega^3(0) = (0.1, 0.1, 0.3)^T \text{ rad/s}$, and $\omega^4(0) = (0.1, 0.3, 0.1)^T \text{ rad/s}$, respectively. The initial orientations given in unit quaternions for Spacecraft 1, 2, 3, and 4 were chosen to be the same as in Simulation 1. The control gains were set to $k_1 = 1.5$, $k_2 = 2$, $k_3^1 = 0$, and $k_3^i = 1$ for $i = 2, 3, 4$.

The w parameters describing the $\hat{b}_3$ axis of Spacecraft 2 in the frame of Spacecraft 1 and Spacecraft 1, 2, and 4 in the frame of Spacecraft 3 are plotted in Fig. 7. Similar behavior was observed for the other partial attitude parameters as well. Snapshots of the simulation's animation are shown in Fig. 8. The animation of Simulation 2 can be found in <http://dcs1.gatech.edu/movies/undersatsim2.avi>.

6. Conclusions and future work

In this paper we have proposed two problems related to the partial attitude stabilization for a cluster of multiple underactuated spacecraft. Although prior work has dealt with the coordinated attitude control of two or more spacecraft, this is the first work that addresses the problem of attitude synchronization among multiple underactuated spacecraft. Each of the spacecraft considered utilizes only two control torques acting on two of their corresponding principal axes, while the third principal axis is left uncontrolled. Only the relative partial attitude between the underactuated axes is used in the feedback control. Two cases are considered. In the first case, the spacecraft underactuated axes end up pointing along the same fixed inertial direction in space with the option of specifying this direction utilizing a leader, and the controller uses *inertial* partial velocities for feedback. It is also shown that if the communication graph contains a cycle there may exist multiple spurious non-consensus equilibria, but these are unstable. In the second case, the axes end up pointing along the same inertial direction, and the controller in this case uses only *relative* partial velocities for feedback. Although, for brevity, the second case is assumed leaderless, the results from Problem 2 can be easily extended to a

leader–follower formation, in case specific inertial directions need to be tracked by all spacecraft.

The problem considered in this paper encompasses many practical applications, such as, for instance, the pointing of communication antennas for multi-spacecraft systems without the need for complete control of each spacecraft. Future possible work includes the development of solutions that will bring these theoretical results closer to actual implementation, such as the incorporation of control saturation effects, dynamically changing communication graphs, and/or communication lag, among others.

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