

Planetary Entry in a Randomly Perturbed Atmosphere

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As an entry vehicle descends into the atmosphere of a planet, variations in the atmospheric density from predicted values will cause the vehicle to deviate from the expected trajectory. The probability distribution of the vehicle state following atmospheric flight depends on the probabilistic structure of the density variations and on the control actions taken by the vehicle. In this paper, we model the atmospheric density as a random process with the altitude as the independent variable. By assuming that the entry vehicle is descending, we recast atmospheric disturbances in altitudes below the vehicle as stochastic perturbations arriving in time with an intensity related to the vehicle sink rate. This transformation allows us to derive a stochastic differential equation (SDE) describing the joint time-evolution of both the vehicle state and the atmospheric density at the vehicle position. Next, we linearly approximate the dynamics of the state deviation from a provided nominal trajectory. An approximate probability distribution of the vehicle state is then provided by explicitly integrating the covariance with respect to the linearized dynamics. We apply Linear Quadratic (LQ) optimal control, with optimized weights, to solve for the bank angle corrections that minimize the final range variance while satisfying state and control maximum variance constraints. Finally, we demonstrate the proposed method with a numerical example of a Martian entry mission. Based on an extensive Monte Carlo trial with density and initial state dispersions, it is observed that the proposed method results in lower final state variances compared to the Apollo direct entry guidance when adapted for the Mars Science Laboratory (MSL).

I. Introduction

PLANETARY entry is the process during which a spacecraft enters and flies through the atmosphere of a planet with the objective of landing. Entry guidance is responsible for steering the vehicle to the target landing site while controlling the dissipation of the vehicle's kinetic energy. The standard approach to entry guidance, which was developed for the Apollo program, is for the vehicle to fly at a trimmed angle of attack with positive lift, and then for the vehicle to steer by banking to direct the lift vector [1]; see Figure 1a.

The vertical component of the lift vector, and thus the magnitude of the bank angle, is set by the longitudinal control logic, whereas the sign of the bank angle (i.e., bank left or bank right) is set by the lateral control logic. It follows that the longitudinal control logic can command smooth corrections by adjusting the bank angle, but the lateral logic can only command discrete bank-reversals during which the sign of the bank angle is flipped. Furthermore, the authority of the lateral control depends on the longitudinal control, since the horizontal component of the lift decreases as the vertical component of lift increases. This is accounted for by limiting the allowable bank angles that can be commanded by the longitudinal guidance logic [2].

The Apollo entry guidance scheme consisted of a skip phase followed by a direct entry phase. The longitudinal guidance logic for the direct entry phase, which was later adopted for the Mars Science Laboratory (MSL) entry guidance, commands bank angles that null out the error between the desired range-to-go and the estimated range-to-go, which is determined by the navigated states and by the adjoint equations evaluated along a reference trajectory [1, 3–5]. As a separate process, the lateral guidance logic commands bank-reversals when the lateral track error passes a predefined threshold [2]. Proposed modern approaches to entry guidance, including the Orion entry guidance, follow the same general idea, except that the targeting-error prediction is done numerically in place of the linear approximation via the adjoint equations [6, 7]. These methods, which are referred to as numerical-predictor correctors (NPCs), do not require a fixed reference trajectory, and can be modified to include path and control constraints [8]. The Shuttle entry guidance, in contrast, tracked a drag profile as a function of the planet-relative velocity, and the projected range was computed as a function of the drag profile [9, 10]. Subsequent research has expanded on drag-profile tracking entry guidance for Earth and Mars entry [11–13].

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Common among these entry guidance methods is that uncertainty is controlled implicitly: the guidance is closed-loop, but corrective steering actions are designed to remove the predicted targeting error based on a deterministic model of the entry system. If, on the other hand, uncertainty is included in the problem formulation, then it is possible to explicitly control the uncertainty along the trajectory of the entry vehicle. In the case of linear feedback control, for example, the feedback gains become the inputs that steer the evolution of the state probability distribution [14, 15].

Considering the entry guidance problem from a stochastic point of view has several benefits. First, the objective function and the constraints take a clear and concise form: minimize the variance of the final range error while ensuring that the variance of the other trajectory errors satisfy prescribed bounds. Furthermore, the limits on the allowable bank angle can be similarly treated by constraining the variance of the closed-loop control signal. The explicit bounds on the control variance allow for more aggressive feedback control actions to be taken while still ensuring feasibility. Finally, the predicted trajectory dispersions can also be used for onboard decision making. For instance, numerical-predictor corrector guidance algorithms often include margins on the planned control inputs as to allow for future closed-loop corrections [16]; with an uncertainty propagation model, these margins could be set dynamically as a function of the probability distribution of future closed-loop controls [17].

In this paper, we take a step towards applying uncertainty control theory for planetary entry by constructing a stochastic process model for the density variations in the planet's atmosphere and then studying the resulting closed-loop longitudinal entry trajectories. The density is assumed to be radially symmetric about the planet so that only the altitude is required to specify the density at any point in the atmosphere. As a result, the density, as a function of altitude, is assumed to be the solution to a stochastic differential equation (SDE). Using this model, a vehicle descending through the atmosphere does not know the precise value the density will take for altitudes below the current vehicle altitude, but it has knowledge of the probability *distribution* over the possible values of the density. As time progresses and the vehicle descends, the actual density process is experienced by the vehicle at a rate which is proportional to how fast the vehicle is descending. For example, in the limiting case when the vehicle is in level flight, the experienced density process is constant from the perspective of the vehicle. Based on this observation, we derive an SDE that models the coupled time-evolution of the vehicle states and of the atmospheric density as a function of the position of the vehicle.

We proceed to linearize the coupled vehicle-density dynamics with respect to a provided reference trajectory. Based on this linearized model, the probability distribution of the vehicle states remains Gaussian, and therefore the approximate vehicle state distribution can be obtained for the entire trajectory by solving a single set of ordinary differential equations (ODEs) for the covariance matrix. We observe that in numerical simulations the linear-Gaussian model nearly exactly matches Monte Carlo results and, furthermore, it can be computed on the order of milliseconds, compared to tens or hundreds of seconds for a similar Monte Carlo analysis. Finally, we apply Linear Quadratic (LQ) optimal control to solve for the optimal closed-loop bank commands to minimize the final range-error variance. In order to handle state and control constraints, we perform a constrained nonlinear optimization over the LQ weights. The proposed method is demonstrated with a numerical example of a Martian entry of an MSL-like vehicle, and its performance is compared to the Apollo-derived MSL entry guidance described in Ref. [3].

II. Longitudinal Motion of Atmospheric Entry

Consider a spacecraft in non-climbing atmospheric flight over a spherical body at a radial position r , planet-relative velocity V , and planet-relative flight path angle γ . The longitudinal motion of the spacecraft is described by the ordinary differential equations

$$\dot{r} = V \sin \gamma, \quad (1a)$$

$$\dot{V} = -\frac{\rho V^2 S C_D}{2m} - \frac{\mu \sin \gamma}{r^2}, \quad (1b)$$

$$\dot{\gamma} = \frac{\rho V S C_L}{2m} \cos \sigma - \left(\frac{\mu}{r^2} - \frac{V^2}{r} \right) \frac{\cos \gamma}{V}, \quad (1c)$$

$$\dot{R} = V \cos \gamma, \quad (1d)$$

where R is the range traveled, μ is the gravitational parameter of the planet, ρ is the atmospheric density, C_L and C_D are the lift and drag coefficients, m is the spacecraft mass, and S is the spacecraft reference area. The atmospheric density is a function of the radius, and the aerodynamic coefficients are a function of both the velocity and the local atmospheric conditions (e.g., temperature, molar mass). We remark that our assumption that the vehicle is not climbing is a technical requirement for the following analysis on the atmospheric density perturbations, and hence we will not explicitly enforce

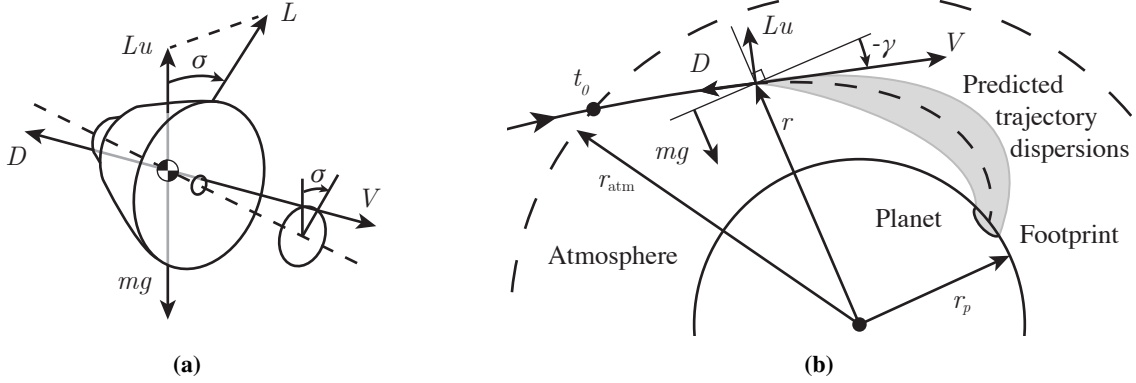


Fig. 1 (a) Bank angle control with lift L , drag D , gravitational acceleration g , bank angle σ , and control $u = \cos \sigma$. (b) Description of atmospheric entry with predicted trajectory dispersions.

this as a constraint.

The in-plane component of the lift is controlled by rolling the vehicle about its velocity vector by a bank angle σ , where a zero bank angle corresponds to lift-up. We assume that the response time to roll commands is sufficiently small so that we may consider the control input to be the vertical fraction of the lift magnitude, given by

$$u = \cos \sigma. \quad (2)$$

Let the vehicle state in (1) be given as $z = (r, V, \gamma, R)$. At a time t_0 , suppose that the vehicle state is Gaussian distributed as

$$z_{t_0} = \bar{z}_0 \sim \mathcal{N}(\bar{z}_0, \Sigma_0), \quad (3)$$

where both the vector $\bar{z}_0$ and the positive semi-definite matrix Σ_0 are known. We are interested in solving the following uncertainty quantification problem: For a stochastic model of atmospheric density perturbations, determine the probability distribution of the vehicle state z_t at all future times $t > t_0$. Furthermore, based on this model, we want to design a feedback controller for bank angle corrections so that the final range error is minimized. This problem is depicted graphically in Figure 1b. We begin by formalizing the stochastic model of the atmospheric density.

III. Stochastic Atmospheric Density Model

A. Density as a Random Process

Let $s > 0$ be the *sink distance* defined as a function of the altitude given by

$$s(h) = h_{\max} - h, \quad (4)$$

where $h_{\max}$ is a fixed reference altitude. It will be convenient to use the sink distance instead of the altitude as the independent variable of the atmospheric density process. This follows from our assumption that the vehicle is descending into the atmosphere, which implies that the sink distance is monotonic with time along descending vehicle trajectories. Since the sink distance and the altitude both equivalently represent radial distance, we will often use the two terms interchangeably. For a fixed minimum altitude $h_{\min}$, let the maximum sink distance be $s_f = s(h_{\min}) = h_{\max} - h_{\min}$. We define the density as a stochastic process $(\rho_s)_{s \in [0, s_f]}$ with values at each sink distance given by

$$\rho_s = \bar{\rho}_s(1 + \delta\rho_s), \quad (5)$$

where the *mean density* $\bar{\rho} : [0, s_f] \rightarrow (0, \infty)$ is a deterministic function and the *density variation* $(\delta\rho_s)_{s \in [0, s_f]}$ is a zero-mean stochastic process. This decomposition is unique: setting $\bar{\rho}_s = \mathbb{E}(\rho_s)$ and $\delta\rho_s = \rho_s / \mathbb{E}(\rho_s) - 1$, and noting that by this definition $\mathbb{E}(\delta\rho_s) = 0$, we obtain the expression (5) for any random process ρ with non-zero mean. In the present paper, we will assume that the mean density obeys a hydrostatic equilibrium, and hence the mean density is given as the solution to the hydrostatic equilibrium equation

$$\frac{d}{ds} \bar{\rho}_s = -\frac{\bar{\rho}_s}{H_s}, \quad \bar{\rho}_{s_f} = \bar{\rho}_{\text{ref}}, \quad (6)$$

where H_s is the scale height, which is a function of the sink distance, and where the mean density at the minimum altitude, $\bar{\rho}_{\text{ref}}$, is known. We model the density variation process as the solution to the SDE

$$d(\delta\rho_s) = -\frac{2}{H_s}\delta\rho_s ds + \sqrt{\psi_s}d\tilde{w}_s, \quad \delta\rho_0 \sim \mathcal{N}(0, q_0), \quad (7)$$

where $\psi_s > 0$ is a given function, $\tilde{w}$ is a Brownian motion, and $q_0 \geq 0$ is the known variance of the density variation at the zero sink (i.e., $h = h_{\text{max}}$) position. This implies that the density variation is an Ornstein-Uhlenbeck process [18]. We will write $q(s)$ to denote the variance of $\delta\rho_s$.

Remark 1. We set the drift coefficient of the density variation process (7) to be equal to $-2/H_s$ so that the resulting density process will have an analytical expression, and also because the resulting behavior appears to be very similar to the GRAM model [19]. However, the following analysis will hold if any arbitrary, negative-definite drift term is used.

Remark 2. Increasing ψ_s in (7) has the effect of increasing the volatility of sample paths of the density variation. Furthermore, when the scale height is a fixed function of the sink distance, or, more generally, when the drift coefficient is a fixed function, the function ψ_s completely determines the distribution of the density variation process. To this end, we can set a profile for ψ_s over the interval $[0, s_f]$ to achieve the desired statistics of the density variation trajectories.

Remark 3. For constant $H_s \equiv H_0$ and $\psi_s \equiv \psi_0$, it can be shown that

$$\lim_{s \rightarrow \infty} \mathbb{E}(\delta\rho_s^2) = \frac{\psi_0 H_0}{4}. \quad (8)$$

Next, we derive an SDE representation for the density process based on (5), (6), and (7). We apply Itô's formula to the function $F(s, x) = \bar{\rho}_s(1 + x)$, and substitute $\rho_s = F(s, \delta\rho_s)$, to obtain

$$d\rho_s = \frac{\partial F}{\partial s}(s, \delta\rho_s) ds + \frac{\partial F}{\partial x}(s, \delta\rho_s) d(\delta\rho_s) \quad (9)$$

$$= \frac{\bar{\rho}_s}{H_s}(1 + \delta\rho_s) ds + \bar{\rho}_s \left(-\frac{2}{H_s}\delta\rho_s ds + \sqrt{\psi_s}d\tilde{w}_s \right) \quad (10)$$

$$= \frac{\bar{\rho}_s}{H_s} \left(\frac{\rho_s}{\bar{\rho}_s} \right) ds - \frac{2\bar{\rho}_s}{H_s} \left(\frac{\rho_s}{\bar{\rho}_s} - 1 \right) ds + \bar{\rho}_s \sqrt{\psi_s} d\tilde{w}_s. \quad (11)$$

After simplifying, we arrive at an SDE representation of the density process as a function of the sink distance given by

$$d\rho_s = -\frac{1}{H_s}(\rho_s - 2\bar{\rho}_s) ds + \bar{\rho}_s \sqrt{\psi_s} d\tilde{w}_s. \quad (12)$$

Thus, the density is also an Ornstein-Uhlenbeck process with a drift term determined by the mean density. By this construction, there is a non-zero probability that the density takes negative values. This may be handled in practice by taking $\rho \leftarrow \max(\rho, 0)$. Note that if the stochastic terms are removed from (12), then we recover the mean density equation. That is, informally, if we set $\rho_s = \bar{\rho}_s$ and $\psi_s = 0$, then $d\rho_s = (\bar{\rho}_s/H_s) ds = d\bar{\rho}_s$.

B. Density Process Evaluated Along Vehicle Trajectories

We have defined the density as a random process in the spatial dimension of the atmosphere. However, from the perspective of a vehicle descending through the atmosphere, the atmospheric density is a function of time, and thus the density variation process would manifest as a random process in time. In other words, the vehicle dynamics may be used to change the coordinates of the density from the spatial dimension of the atmosphere to the time dimension. The following lemma provides a formal justification for this coordinate change.

Lemma III.1. (Time-Changed Brownian Motion) Let $(\tilde{w}_s)$ be a Brownian motion and let $s : [t_0, t_f] \rightarrow [0, s_f]$ be a smooth, non-decreasing function. Then, there exists a Brownian motion (w_t) such that, almost surely, for all $t \in [t_0, t_f]$,

$$\tilde{w}_{s(t)} = \int_{t_0}^t \sqrt{\dot{s}(z)} dw_z. \quad (13)$$

Proof. The Dambis-Dubins-Schwarz theorem [18, 5.13] states that all continuous local martingales may be given as a time-changed Brownian motion: for a martingale (y_t) with $t \in [t_0, t_f]$, there exists a Brownian motion ζ such that, a.s., for all $t \in [t_0, t_f]$, $y_t = \zeta_{\xi(t)}$, where $\xi(t) = \langle y, y \rangle_t$ is the quadratic variation of y . * To show (13), we let the martingale (y_t) be given by

$$y_t = \int_{t_0}^t \sqrt{\dot{s}(a)} dw_a. \quad (14)$$

This martingale has quadratic variation $\langle y, y \rangle_t = s(t) - s(t_0)$. Since Brownian motion is a stationary process, we can drop the constant term $s(t_0)$, and then we take the time changed Brownian motion $\tilde{w}_{s(t)}$ to be $\zeta_{s(t)}$, which gives the desired result. $\square$

Remark 4. Lemma III.1 provides a formula to scale the time for a Brownian motion by any smooth, non-decreasing function. For the special case $s(t) = c^2 t$, where c is a positive constant, we recover the linear scaling invariance property of Brownian motion, namely, $\tilde{w}_{c^2 t} = c w_t$.

Theorem III.2. The density $\rho_t = \rho_{s(t)}$ evaluated along a descending vehicle trajectory $s(t) = h_{\max} - h(t)$ is the solution to the SDE

$$d\rho_t = \frac{V_t \sin \gamma_t}{H_s} (\rho_t - 2\bar{\rho}_s) dt + \bar{\rho}_s \sqrt{-\psi_s V_t \sin \gamma_t} dw_t, \quad \rho_{t_0} \sim \mathcal{N}(\bar{\rho}_{s(t_0)}, \bar{\rho}_{s(t_0)}^2 q_{s(t_0)}), \quad (15)$$

where (w_t) is a Brownian motion.

Proof. Let s be the sink distance of the vehicle with dynamics (1). This implies that $s = s(t)$ is a smooth function of time with derivative

$$\dot{s} = -V_t \sin \gamma_t. \quad (16)$$

By the fundamental theorem of calculus we have that $ds(t) = \dot{s}(t) dt$.[†] Lemma III.1 provides a coordinate change for the Brownian motion $(\tilde{w}_s)$, which we write in differential form as

$$d\tilde{w}_{s(t)} = \sqrt{\dot{s}(t)} dw_t. \quad (17)$$

Substituting the vehicle kinematics (16) into the coordinate change expressions, and then substituting into the density process (12), we obtain the SDE in (15). Finally, at the initial time t_0 the density is equal to the value of the density process at the initial vehicle sink distance $s(t_0)$. Thus, the density ρ_{t_0} is normally distributed with mean $\bar{\rho}_{s(t_0)}$ and variance

$$\mathbb{E}[(\rho_{s(t_0)} - \bar{\rho}_{s(t_0)})^2] = \bar{\rho}_{s(t_0)}^2 \mathbb{E}[\delta \rho_{s(t_0)}^2] = \bar{\rho}_{s(t_0)}^2 q_{s(t_0)}. \quad (18)$$

$\square$

IV. Uncertainty Quantification and Control of the Vehicle State

The vehicle dynamics depend on the vehicle states, the control input, and the density. By Theorem III.2, the density, as seen by the vehicle, changes in time according to an SDE with coefficients determined by the vehicle states. In summary, we can describe the combined spacecraft-density state as the solution to the coupled SDE

$$d \begin{bmatrix} z_t \\ \rho_t \end{bmatrix} = \begin{bmatrix} f_z(z_t, \rho_t, u) \\ f_\rho(z_t, \rho_t) \end{bmatrix} dt + \begin{bmatrix} 0 \\ g_\rho(z_t) \end{bmatrix} dw_t, \quad \begin{bmatrix} z_{t_0} \\ \rho_{t_0} \end{bmatrix} \sim \mathcal{N} \left(\begin{bmatrix} \bar{z}_0 \\ \bar{\rho}_{s(t_0)} \end{bmatrix}, \begin{bmatrix} \Sigma_0 & 0 \\ 0 & \bar{\rho}_{s(t_0)}^2 q_{s(t_0)} \end{bmatrix} \right), \quad (19)$$

where the functions f_ρ and g_ρ are the drift and diffusion terms, respectively, from (15), and where f_z is from the vehicle dynamics (1). Introducing the combined spacecraft-density state $x = (z, \rho)$ we may compactly rewrite (19) as

$$dx_t = f(x_t, u_t) dt + g(x_t) dw_t, \quad x_{t_0} \sim \mathcal{N}(\bar{x}_0, P_0), \quad (20)$$

where the functions f , g , the vector $\bar{x}_0$, and the positive semi-definite matrix P_0 are defined as in (19).

The probability distribution of the state x for any time $t \geq t_0$ is uniquely determined as the solution to the Fokker-Planck partial differential equation (PDE) corresponding to by the SDE (20). The authors of [20] applied the

*For a definition of quadratic variation see [18, Theorem 4.9].

[†]We are using differential notation as a shorthand for integral equations. That is, $ds(t) = \dot{s}(t) dt$ is shorthand for $s(t) = s(t_0) + \int_{t_0}^t \dot{s}(a) da$.

method of characteristics to solve the probability density evolution for entry flight in the case when the uncertainties are parametric. In general, however, the Fokker-Planck PDE is difficult to solve and is not suitable for onboard implementation. Therefore, we will instead proceed by linearly approximating the SDE (20) about a provided reference trajectory. The solution to the Fokker-Planck PDE for a linear SDE is provided as the solution of an ODE for the state covariance matrix, and therefore the uncertainty quantification (UQ) problem may be solved by numerical integration.

A. Linear Covariance Approximation

Suppose we are provided with a reference control sequence $(\hat{u}_t)$ over the time interval $[t_0, t_f]$. We assume that the mean state process $(\mathbb{E}(x_t))$ is equal to the reference trajectory $(\hat{x}_t)$ corresponding to the reference control, which is the solution to the deterministic system

$$\dot{\hat{x}}_t = f(\hat{x}_t, \hat{u}_t), \quad \hat{x}_{t_0} = \bar{x}_0. \quad (21)$$

Furthermore, we approximate the dynamics of the state error $\tilde{x}_t = x_t - \hat{x}_t$ to first-order. That is, the state error process $(\tilde{x}_t)$ is assumed to be the solution to the linear SDE

$$d\tilde{x}_t = (A_t \tilde{x}_t + B_t \tilde{u}_t) dt + G_t dw_t, \quad \tilde{x}_{t_0} \sim \mathcal{N}(0, P_0), \quad (22)$$

where the coefficients are evaluated along the reference trajectory as

$$A_t = \frac{\partial f}{\partial x_t}(\hat{x}_t, \hat{u}_t), \quad B_t = \frac{\partial f}{\partial u_t}(\hat{x}_t, \hat{u}_t), \quad G_t = g(\hat{x}_t), \quad (23)$$

and where $\tilde{u}_t = u_t - \hat{u}_t$ is the control deviation, which we fix to be a linear function of the state error. The control is thus given by

$$u_t = \hat{u}_t + K_t \tilde{x}_t, \quad (24)$$

where K_t is a feedback gain matrix. It follows from the linear approximation that the state error remains Gaussian for the entire problem horizon. Therefore, for any time t in the interval $[t_0, t_f]$, the state is distributed as

$$x_t \sim \mathcal{N}(\hat{x}_t, P_t), \quad (25)$$

where $P_t = \mathbb{E}(\tilde{x}_t \tilde{x}_t^\top)$ is the solution to the differential Riccati equation

$$\dot{P}_t = (A_t + B_t K_t) P_t + P_t (A_t + B_t K_t)^\top + G_t G_t^\top, \quad P_{t_0} = P_0. \quad (26)$$

In summary, for a fixed reference control $(\hat{u}_t)$ and feedback gain (K_t) , the state probability distribution over the time horizon $[t_0, t_f]$ is obtained by integrating the ODEs (21) and (26).

Finally, we consider a discrete-time approximation of the covariance evolution on the partition $t_0 = t_0^p < \dots < t_N^p = t_f$ of the time interval $[t_0, t_f]$, given by

$$A_k = \exp(A(t_k^p) \Delta t_k), \quad B_k = A(t_k^p) B(t_k^p) \frac{\Delta t_k^2}{2} + B(t_k^p) \Delta t_k, \quad G_k = G(t_k^p) \sqrt{\Delta t_k}, \quad (27)$$

where $\Delta t_k = t_{k+1}^p - t_k^p$ for $k = 0, \dots, N-1$, and where we write $A(t) = A_t$, $B(t) = B_t$, and $G(t) = G_t$ for clarity. The approximate state covariance evolution is then given by the difference equation

$$P_{k+1} = (A_k + B_k K_k) P_k (A_k + B_k K_k)^\top + G_k G_k^\top, \quad P_{k=0} = P_0, \quad (28)$$

for $k = 0, \dots, N-1$. Numerically evaluating this discrete approximation is often significantly less computationally expensive than the continuous system (26).

B. Uncertainty Control

Consider now the problem of minimizing the range error using feedback control. We assume that the control is discretized to be piecewise constant on the time intervals $[t_k^p, t_{k+1}^p]$ for $k = 0, \dots, N-1$ so that the state covariance evolves according to the discrete model (28). Let $K = (K_k)_{k=0}^{N-1}$ be the sequence of feedback gains as in (24). The range-error cost is a function of this sequence of feedback gains, that is,

$$J_1(K) = \text{Var}(R_{t_f}). \quad (29)$$

Alternately, we can include the effect of the final altitude error on the projected total range error as is done in the derivation of the Apollo direct entry phase guidance [1, 3]. In this case, the cost function is given as

$$J_2(K) = \text{Var}(R_{t_f} + \cot(\hat{\gamma}_{t_f})r_{t_f}) = \text{Var}(R_{t_f}) + \cot^2(\hat{\gamma}_{t_f})\text{Var}(r_{t_f}). \quad (30)$$

For generality, we let C be a 5×5 matrix and define the cost function to be

$$J(K) = \text{tr} CP_N. \quad (31)$$

Hence, the cost functions (29) and (30) are equal to (31) for particular values of the matrix C . We assume that the bank angle must remain within a time-varying interval $[\sigma_{\min}(t), \sigma_{\max}(t)]$ for the entire trajectory, where $\sigma_{\min}(t)$ and $\sigma_{\max}(t)$ are given functions defined on $[t_0, t_f]$. The bank angle σ_t at time t is determined by the control input u_t by

$$\sigma_t = \cos^{-1}(u_t) = \cos^{-1}(\hat{u}_t + K_t \tilde{x}_t), \quad (32)$$

where the state error $\tilde{x}_t$ is a Gaussian random vector, and therefore, when the feedback gain is nonzero, the bank angle is a random variable with nonzero probability of taking values outside the constraint interval. For this reason, we instead require that bank angle σ_t is in the interval $[\sigma_{\min}(t), \sigma_{\max}(t)]$ with high probability. That is, we enforce the probabilistic (chance) constraint

$$\mathbb{P}(\sigma_{\min}(t) \leq \sigma_t \leq \sigma_{\max}(t)) \geq 1 - \beta, \quad (33)$$

where β is a given maximum probability of constraint violation. Then, during flight, the bank angle commands are saturated in the low-probability event that the commanded bank angle is out of the prescribed bounds. However, since the control input is the cosine of the bank angle rather than the bank angle itself, we cannot directly enforce (33). Instead, we determine a maximum variance limit M_k^u for the control so that if

$$\text{Var}(u_{t_k^p}) \leq M_k^u, \quad k = 0, \dots, N-1, \quad (34)$$

then (33) holds for times t_k^p for $k = 0, \dots, N-1$. This constraint is enforced at the discrete time values t_k^p since we will use the discrete covariance model (28) for the purposes of control design. The limit M_k^u is a function of the reference control $\hat{u}_{t_k^p}$, the bank angle bounds $\sigma_{\min}(t_k^p)$ and $\sigma_{\max}(t_k^p)$, and on the constant β . Specifically, we approximated the distribution of σ_t by taking (32) to first order, and then we numerically maximized M_k^u subject to the constraint (33).

While our objective is to minimize the range error, we also require that the other states do not deviate too significantly from the final reference state. Thus, we enforce the constraints

$$\text{Var}(r_{t_f}) \leq M_r^x, \quad \text{Var}(V_{t_f}) \leq M_V^x, \quad \text{Var}(\gamma_{t_f}) \leq M_\gamma^x, \quad (35)$$

where M_r^x, M_V^x, M_γ^x are given upper bounds. We will compactly write the constraints (34) and (35) as $\Phi(K) \leq 0$, where $\Phi : \mathbb{R}^{5N} \rightarrow \mathbb{R}^{N+3}$ is a function determined by the covariance evolution (28). The minimum range-error problem is then stated as the optimization problem:

$$\text{minimize } J(K) \text{ subject to } \Phi(K) \leq 0 \text{ over } K = (K_k)_{k=0}^{N-1}. \quad (36)$$

Theorem IV.1. *Let Q be a positive semi-definite matrix, let $a = (a_k)_{k=0}^{N-1}$ be a sequence of positive numbers, and let the linear quadratic (LQ) cost be given as*

$$J_{LQ}(K; Q, a) = \mathbb{E} \left(\tilde{x}_N^\top Q \tilde{x}_N + \sum_{k=0}^{N-1} a_k \tilde{u}_k^2 \right) = \text{tr} Q P_N + \sum_{k=0}^{N-1} a_k K_k P_k K_k^\top. \quad (37)$$

Then the feedback gain sequence K^ solving problem (36) minimizes the LQ cost for particular weights Q^* and a^* subject to the constraint $\Phi(K) \leq 0$.*

Proof. The Lagrangian for problem (36) is given as

$$L(K, \lambda) = J(K) + \lambda^\top \Phi(K), \quad (38)$$

where $\lambda = (\lambda^x, \lambda^u) \geq 0$ with $\lambda^x \in \mathbb{R}^3$ and $\lambda^u \in \mathbb{R}^N$. Let $M^x = \text{diag}(M_r^x, M_v^x, M_\lambda^x, 0, 0)$, let $e_i \in \mathbb{R}^5$ be a unit vector in the i^{th} coordinate, and let $E_i = e_i e_i^\top$. We expand the Lagrangian as

$$L(K, \lambda) = \text{tr} C P_N + \sum_{i=0}^3 \lambda_i^x (e_i^\top P_N e_i - e_i^\top M^x e_i) + \sum_{k=0}^{N-1} \lambda_k^u (K_k P_N K_k^\top - M_k^u). \quad (39)$$

The second term can be rewritten as

$$\sum_{i=0}^3 \lambda_i^x (e_i^\top P_N e_i - e_i^\top M^x e_i) = \sum_{i=0}^3 \lambda_i^x \text{tr} E_i (P_N - M^x) = \text{tr} \Lambda^x (P_N - M^x), \quad (40)$$

where $\Lambda^x = \text{diag}(\lambda^x, 0, 0)$. Substituting (40) into the Lagrangian (39) and simplifying, we obtain

$$L(K, \lambda) = \text{tr}(C + \Lambda^x) P_N - \text{tr} \Lambda^x M^x + \sum_{k=0}^{N-1} \lambda_k^u K_k P_N K_k^\top - \sum_{k=0}^{N-1} \lambda_k^u M_k^u \quad (41)$$

$$= J_{LQ}(K; Q, a) + Z(\lambda), \quad (42)$$

where $Q = C + \Lambda^x$ and $a = \lambda^u$, and the function Z is a collection of terms that do not depend on K . We use the fact that if (K^*, λ^*) is a solution pair to problem (36), then

$$L(K^*, \lambda^*) \leq L(K, \lambda^*) \quad (43)$$

for all sequences of feedback gains K that satisfy the constraints $\Phi(K) \leq 0$. If we write $J_{LQ}(K; Q, a)$ as $J_{LQ}(K, \lambda)$, using the fact that $Q = C + \Lambda^x$ and $a = \lambda^u$, then (43) implies that

$$J_{LQ}(K^*, \lambda^*) + Z(\lambda^*) \leq J_{LQ}(K, \lambda^*) + Z(\lambda^*) \implies J_{LQ}(K^*, \lambda^*) \leq J_{LQ}(K, \lambda^*) \quad (44)$$

for all K that satisfy the constraints $\Phi(K) \leq 0$. It follows that the optimal feedback gains K^* minimize the LQ cost, subject to the constraints $\Phi(K) \leq 0$, for weights $Q^* = C + \Lambda^{x*}$ and $a^* = \lambda^{u*}$ corresponding to λ^* . $\square$

As a consequence of this theorem, we may solve problem (36) by optimizing over the $N + 3$ Lagrange multipliers corresponding to the constraints (34) and (35) in place of the $5N$ coordinates of the feedback gain sequence K . Furthermore, the resulting optimization can leverage the structure of the LQ solution, which is obtained by the backwards matrix equations [21]

$$S_k = A_k^\top S_{k+1} A_k - K_k^\top (a_k + B_k^\top S_{k+1} B_k) K_k, \quad S_N = Q, \quad (45a)$$

$$K_k = - (B_k^\top S_{k+1} B_k + a_k) B_k^\top S_{k+1} A_k, \quad (45b)$$

for $k = N - 1, \dots, 0$. Denoting $K_{Q,a}$ as the solution to the system (45) for LQ weights Q and a , we now restate problem (36) as the following optimization over the LQ weights:

$$\text{minimize } J(K_{Q,a}) \text{ subject to } \Phi(K_{Q,a}) \leq 0 \text{ over } Q, a. \quad (46)$$

We solve this problem as a nonlinear program with the initial guess provided as a hand-tuned LQ solution. For details on variance-constrained LQ control, see Ref. [22].

V. Numerical Example

A. Martian Atmosphere Model

In this section, we model the Martian atmosphere for altitudes between zero and 140 km. The temperature T and the molar mass M are both assumed to be given as a function of the sink distance, both supplied by MarsGRAM [19]. The gravitational acceleration g_{accel} is a function of the sink distance per the inverse square law. Let $0 = s_0^p < \dots < s_{N_s}^p = s_f$ be a partition of the interval $[0, s_f]$, where $s_f = 140$ km and $N_s = 1,000$, and where we define the scale height H_s as the piecewise-constant function of the sink distance given by

$$H_s = \frac{R_{\text{gas}} T(s_k^m)}{M(s_k^m) g_{\text{accel}}(s_k^m)}, \quad s_k^m = \frac{s_k^p + s_{k+1}^p}{2}, \quad s \in [s_k^p, s_{k+1}^p), \quad k = 0, \dots, N_s - 1, \quad (47)$$

Table 1 Entry vehicle parameters

Reference area S , m ²	Mass m , kg	Aerodynamic efficiency C_L/C_D	Drag coefficient C_L
$4.5^2\pi/4$	3152.5	0.25	1.5

where $R_{\text{gas}} = 8.314 \text{ kg m}^2 \text{ K}^{-1} \text{ mol}^{-1} \text{ s}^{-2}$ is the universal gas constant. We assume that the variance of the density variation process is given as a function of the sink distance, which we denote by $q_{\text{target}}(s)$. This function is assumed to be piecewise constant on the partition intervals $[s_k^p, s_{k+1}^p]$. Per Remark 3, we set the function ψ_s in the definition of the density variation process (7) to be

$$\psi_s = \frac{4q_{\text{target}}(s)}{H_s}. \quad (48)$$

Samples of the density and density variation processes, with 2σ bounds, and a comparison of the target variance q_{target} and the variance computed by 5,000 sample Monte Carlo are all shown in Figure 2. We see that the approximation (48) results in the density variation process having variance nearly equal to the provided target variance.

Since the coefficients of the density and the density variation processes have been assumed to be piecewise constant, these processes may be integrated exactly. For details, see Appendix B.

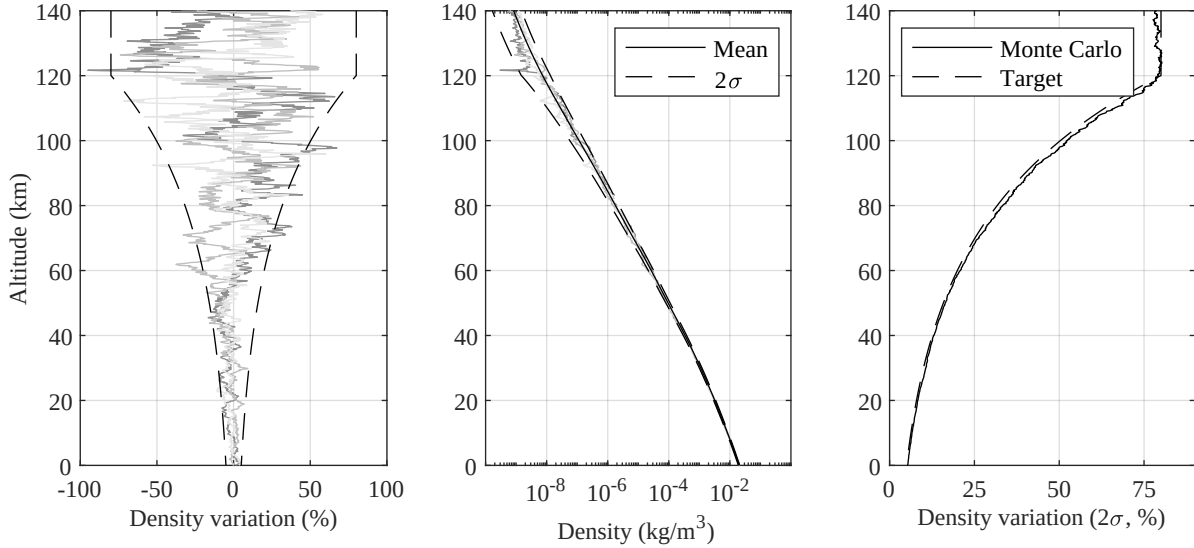


Fig. 2 (Left:) samples of the density variation process, with the 2σ values of the target variance shown as dotted lines, and (center:) corresponding samples of the density process. (Right:) target variance of $\delta\rho_s$ compared to the variance computed from 5,000 trial Monte Carlo.

B. Martian Entry Example

Consider a Mars Science Laboratory-like vehicle, with parameters given in Table 1, descending into the Martian atmosphere with stochastic variations as described in Section V.A. The nominal trajectory is generated by fixing the bank angle as a function of the velocity, as shown in Figure 3, and the corresponding reference trajectory, shown in Figure 4, was solved by setting the density variations to zero with the initial states equal to the mean states in Table 2. The simulation time span was set to $t_f = 200$ seconds, and we used the discrete covariance model (28) with $N = 100$ steps. Lateral guidance, and hence the impact of bank reversals, was not considered. All computations were performed using Matlab on a laptop computer. [‡]

For the dispersion analysis, the initial states were assumed to be normally distributed with distributions given in Table 2. The distribution of trajectory variations were computed by both the proposed linearization method developed

[‡]Simulations were run using Matlab 2018a with a dual-core, 2.9 GHz processor.

Table 2 Initial state distributions

	h_{t_0} , km	V_{t_0} , km/s	γ_{t_0} , deg	R_{t_0} , km
Mean	126	5.9	-14.5	0
3σ	1	0.02	0.2	1

in Section IV.A and using a 5,000 trial Monte Carlo scheme. Each sample trajectory in the Monte Carlo was generated by sampling both an initial vehicle state and a density variation function over the integral $[0, s_f]$. Since the coefficients of the density process are assumed to be piecewise constant, as described in Section V.A, the exact density at each altitude is computed by log-space linear interpolation of values stored in a lookup table. More details on the method of sampling the density variation process are given in Appendix B. The partial derivatives used to construct the linear model are given in Appendix C.

The 3σ bounds on the state dispersions from the reference trajectory, as computed by both the linear method and by Monte Carlo, are shown in Figure 5. In this example, the proposed method closely approximates the Monte Carlo results, and the required computation time to integrate the covariance equations was 1 ms compared to 392 s for the Monte Carlo. [§]

Next, we applied the feedback control method developed in Section IV.B to minimize the final range error given by (29) subject to the maximum variance constraints on of the final altitude, velocity, and flight path given by

$$M_r^x = (2,500 \text{ m}/3)^2, \quad M_V^x = (100 \text{ ms}^{-1}/3)^2, \quad M_\gamma^x = (1.5 \text{ deg} \times 180/3\pi)^2. \quad (49)$$

The control variance was constrained so that the bank angle constraints shown in Figure 3 are met with 3σ confidence (i.e. $\beta = 0.0027$). The LQ weights were optimized using MATLAB's `fmincon`, and the total execution time was 20 seconds. The resulting closed-loop state distributions are shown in Figure 6. In this example, only the final altitude variance constraint was active. Furthermore, by varying the final altitude variance limit, we observed that there is a trade-off between final altitude error and final range error. In particular, if the maximum allowed final altitude variance is lowered, then the minimum range-error will increase.

Finally, we compared the proposed method to the Apollo direct entry guidance as adopted for MSL and derived in Ref. [3]. We solved the range control problem (46) for both the simplified range cost (29) and the range control cost (30), which was used in the derivation of the Apollo guidance; we refer to the solutions of these problems as LQ (a) and LQ (b), respectively. Both the LQ solutions and the Apollo guidance solution were simulated with a 5,000 trial Monte Carlo, and the resulting state covariance evolution is shown in Figure 7. Sample trajectories of the bank angle inputs for each feedback law are shown in Figure 8. Both LQ control laws resulted in a lower final range error compared to the Apollo guidance control law. Furthermore, the Apollo control law resulted in a violation of the final altitude and flight path angle maximum variance constraints.

VI. Conclusion

In this paper, we modeled variations in the atmospheric density as a random process given as the solution to an SDE. The density process model was used to derive an SDE describing entry vehicle trajectories while explicitly modeling the effect of atmospheric disturbances. From this model we compute the approximate covariance of the vehicle states along a nominal trajectory by solving a set of ODEs. Since the required computation is limited to the evaluation of a small number of ODEs for the elements of the state covariance matrix, this method enables onboard calculation of the probability distribution of future vehicle states for a given control law, which, in turn, can enable more sophisticated and robust onboard decision making during atmospheric flight. Furthermore, by explicitly considering the atmospheric disturbances in the system model, we are able to solve for bank angle corrections to control the uncertainty in the vehicle state during entry. In a numerical example, the proposed method performed better than the Apollo entry guidance as adapted for MSL. However, the presented example only included atmosphere uncertainty and did not account for the effects of bank reversals.

The time-change result provided in Lemma III.1, which allowed us to describe density variations along vehicle trajectories, requires that the entry trajectory is non-climbing. It follows that the results presented in this paper cannot be immediately applied to skip-to-entry or aerocapture missions [23]. In future work, we plan to extend the proposed

[§]In this example, the Monte Carlo used Matlab's `ode45` with linear interpolation, and the code was not optimized.

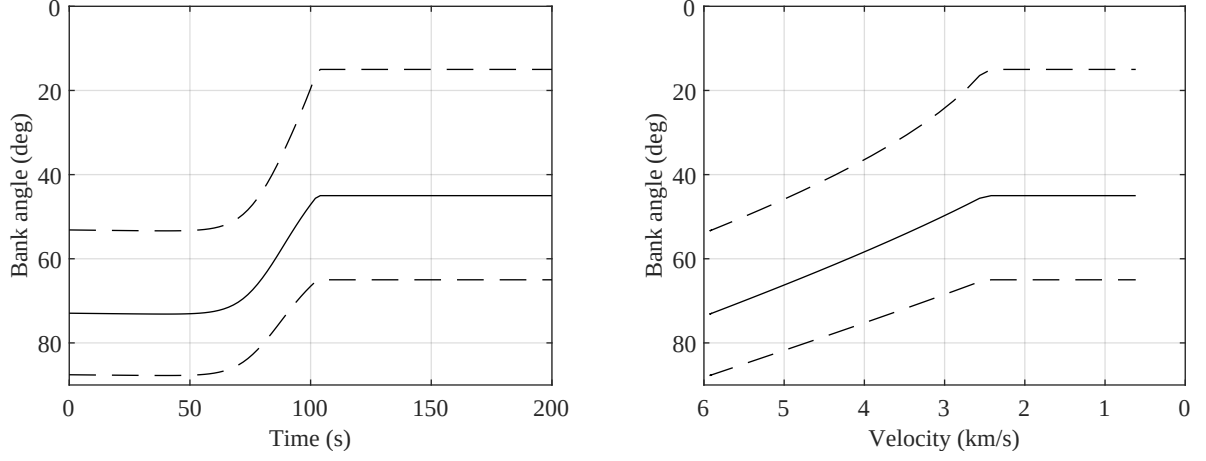


Fig. 3 The nominal bank angle trajectory (solid) and bank angle limits (dashed) as a function of time (left) and as a function of velocity (right).

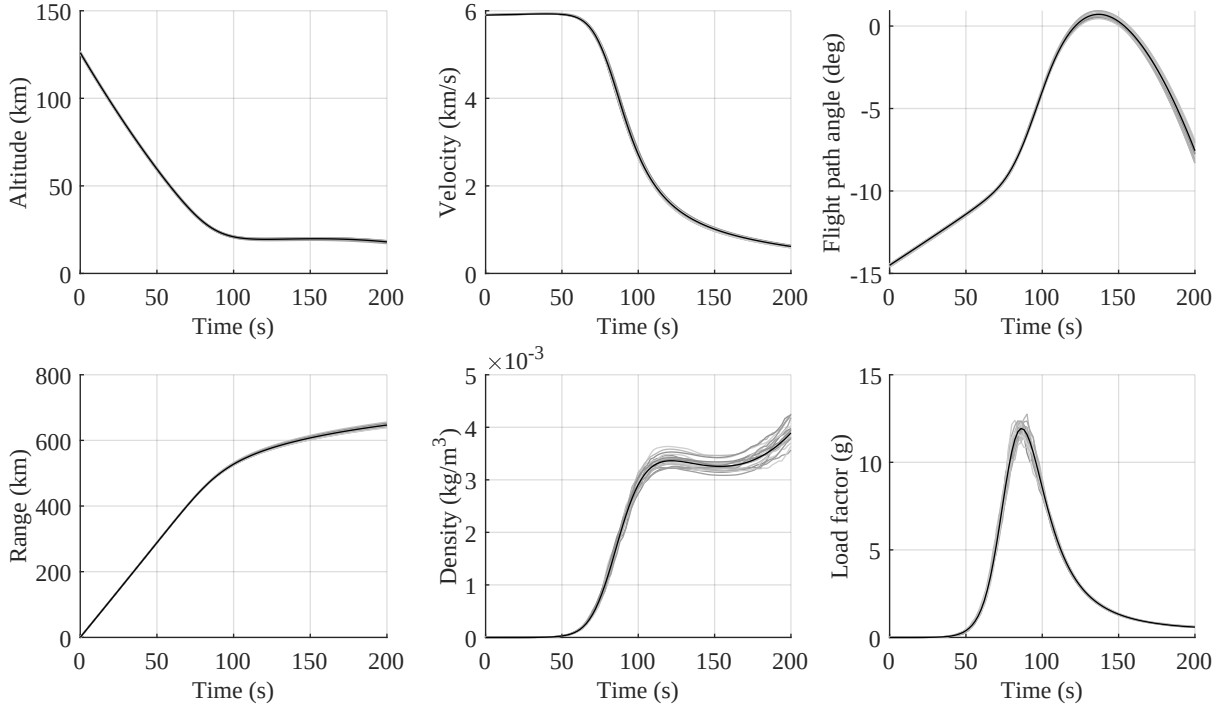


Fig. 4 The nominal entry trajectory (black) and open-loop Monte Carlo samples (gray).

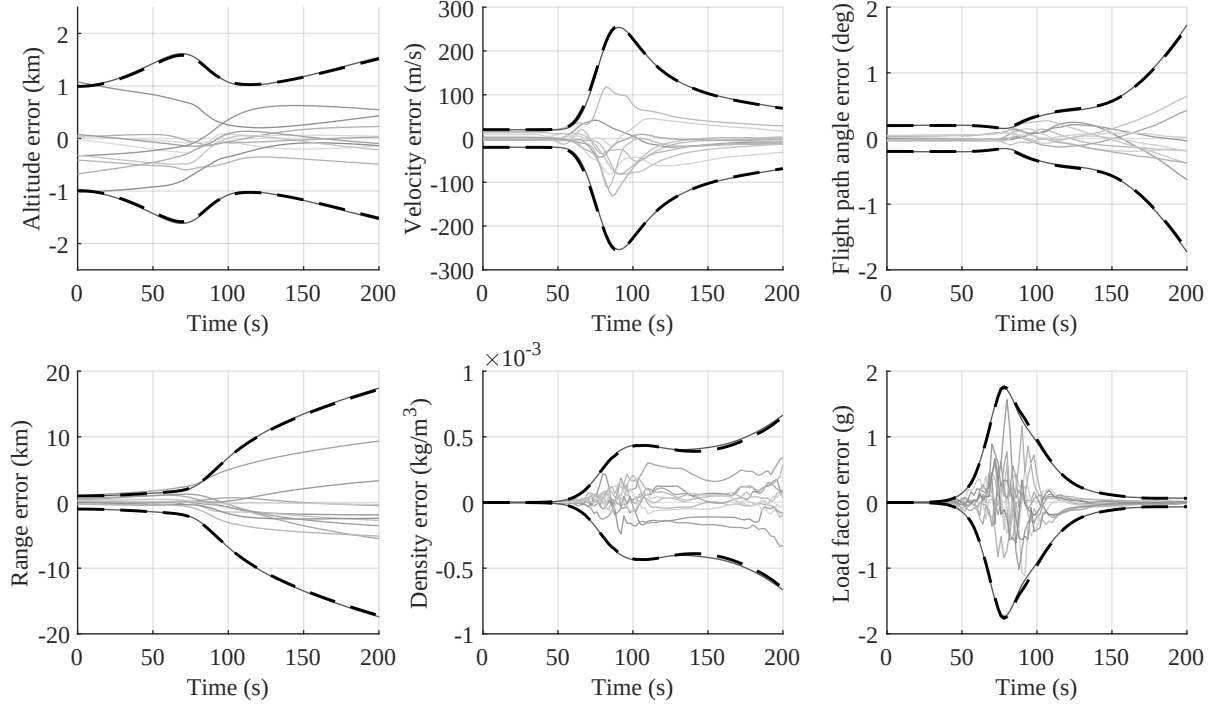


Fig. 5 Open-loop error trajectories with 3σ bounds from 5,000 trial Monte Carlo (dashed) and from the linear model (solid gray).

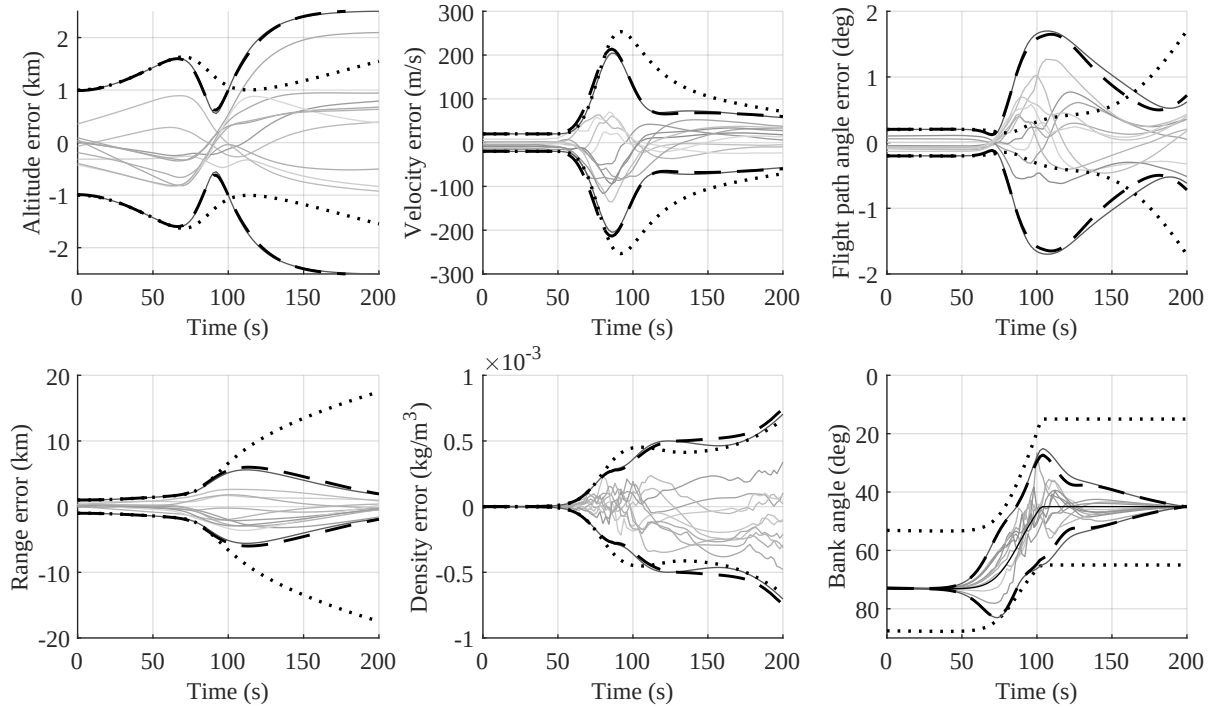


Fig. 6 Closed-loop error trajectories with 3σ bounds from 5,000 trial Monte Carlo (dashed) and from the linear model (solid gray). The open loop 3σ bounds are also shown (dotted) for comparison. The bank angle trajectories are shown with the nominal bank angle (solid black) and the bank limits (dotted).

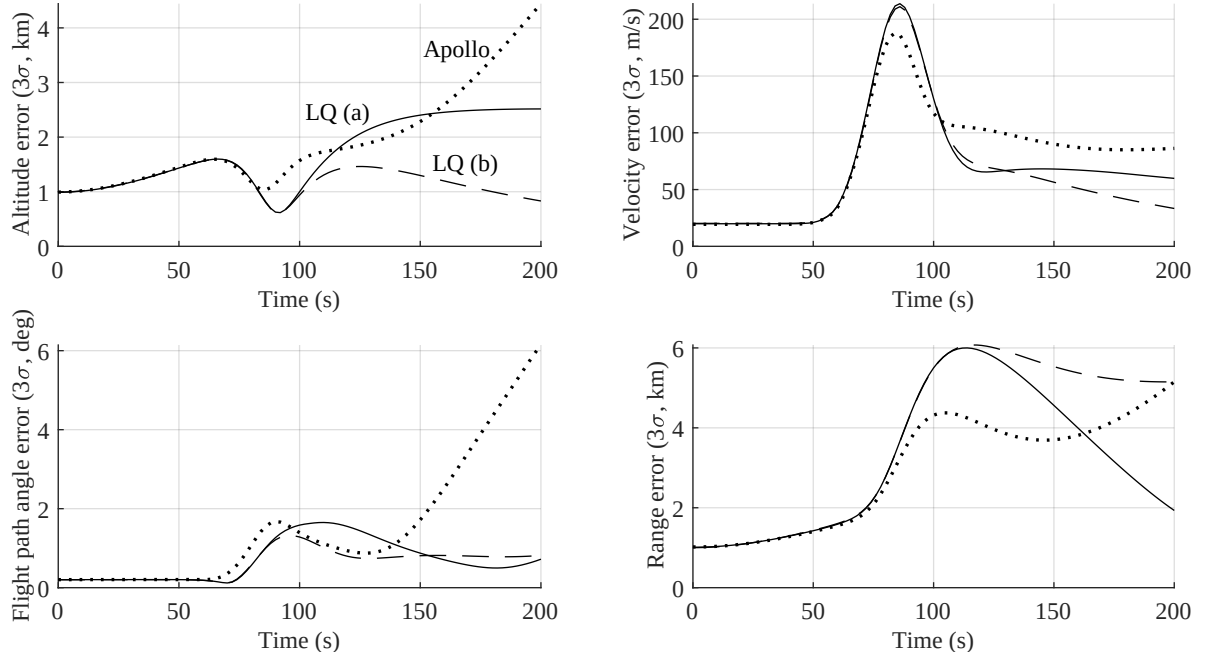


Fig. 7 Comparison of closed-loop state distributions along the reference trajectory computed from 5,000 trial Monte Carlo. LQ (a) and LQ (b) used the proposed method for the cost functions (29) and (30), respectively.

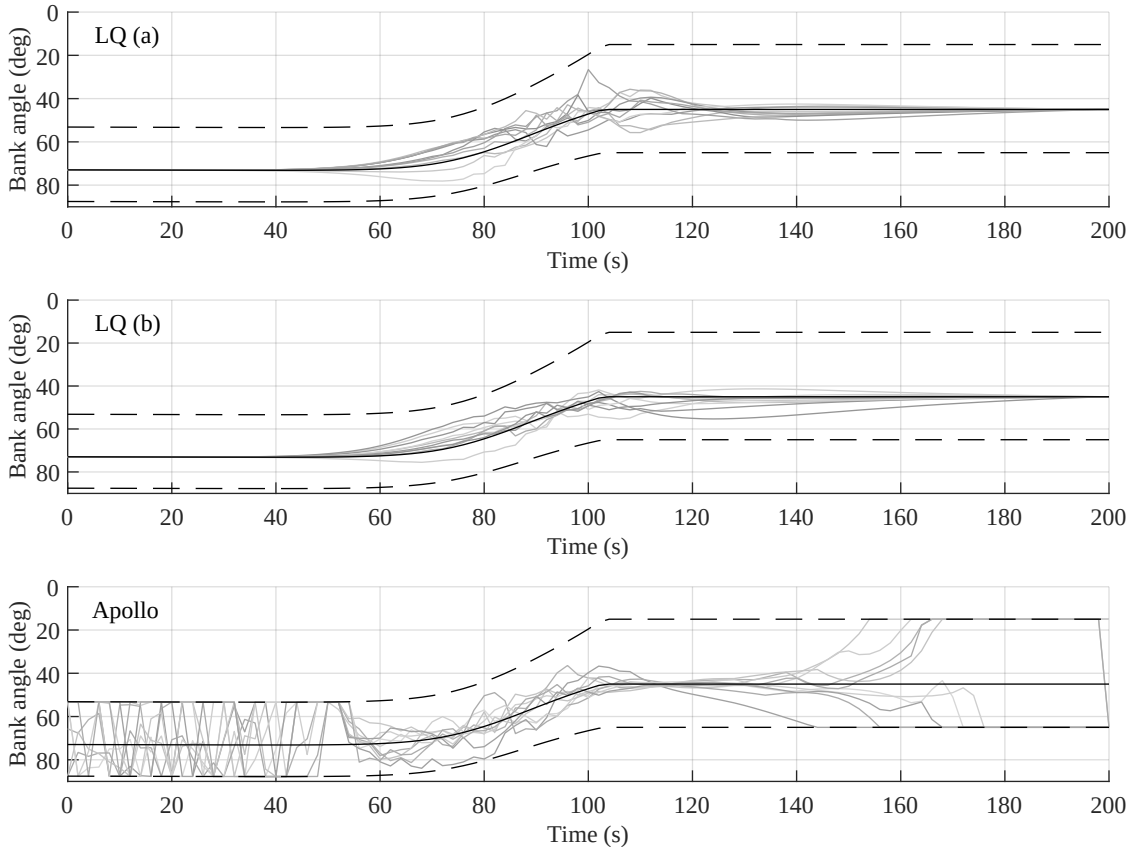


Fig. 8 Sample closed-loop control inputs for the LQ (a), LQ (b), and Apollo control laws. LQ (a) and LQ (b) used the proposed method for the cost functions (29) and (30), respectively.

method to atmospheric flight which is not required to strictly descending. With this added generality, the proposed method could also be applied to predictor-corrector aerocapture guidance.

Appendix

A. Covariance of the Density Process

Since the density process is Gaussian, it is completely characterized by its mean function $\bar{\rho}_s = \mathbb{E}(\rho_s)$ and its covariance function

$$\Sigma(s, a) = \mathbb{E}((\rho_s - \bar{\rho}_s)(\rho_a - \bar{\rho}_a)) = \bar{\rho}_s \bar{\rho}_a \mathbb{E}(\delta\rho_s \delta\rho_a). \quad (50)$$

The covariance of the density process is thus a function of the mean density $\bar{\rho}_s$ and by the covariance of the density variation process. We will therefore proceed to compute $\mathbb{E}(\delta\rho_s \delta\rho_a)$, since the mean density can be easily computed as the solution to an ODE. To this end, let φ be the unique solution to

$$\dot{\varphi}_s = -2\varphi_s/H_s, \quad s \in [0, s_f], \quad \varphi_0 = 1. \quad (51)$$

Applying Itô's formula to the function $F(t, x) = x/\varphi_t$, we obtain the solution to the density variation process as

$$\delta\rho_s = \varphi_s \delta\rho_0 + \int_0^s \frac{\varphi_s}{\varphi_c} \sqrt{\psi_c} dw_c. \quad (52)$$

For $s, a \in [0, s_f]$, we have that

$$\mathbb{E}(\delta\rho_s \delta\rho_a) = \varphi_s \varphi_a \mathbb{E}(\delta\rho_0^2) + \mathbb{E} \left(\int_0^s \frac{\varphi_s}{\varphi_c} \sqrt{\psi_c} dw_c \int_0^a \frac{\varphi_a}{\varphi_c} \sqrt{\psi_c} dw_c \right). \quad (53)$$

By applying both the semigroup property $\varphi_s \varphi_a = \varphi_{s+a}$ and Itô's isometry, we obtain the covariance function for the density variation process as follows

$$\mathbb{E}(\delta\rho_s \delta\rho_a) = \varphi_{s+a} q_0 + \int_0^{\min\{s, a\}} \left(\frac{\varphi_{\min\{s, a\}}}{\varphi_c} \right)^2 \psi_c dc. \quad (54)$$

Finally, the variance $q(s)$ of the density variation is given by setting $a = s$ in (54).

B. Numerical Integration of the Density Process

In this section we provide a method to exactly integrate the density variation process in the case when the coefficients are piecewise constant on subintervals of the partition $0 = s_0^p < \dots < s_{N_s}^p = s_f$ of the interval $[0, s_f]$. Thus, the scale height satisfies $H_s = H_{s_k^p}$ for all $s \in [s_k^p, s_{k+1}^p]$ and $k = 0, \dots, N_s - 1$. We write $H_k = H_{s_k^p}$ and $\psi_k = \psi_{s_k^p}$ for notational simplicity. Then, for $s \in [s_k^p, s_{k+1}^p]$, the mean density is given by

$$\bar{\rho}_s = \exp \left(\frac{s - s_k^p}{H_k} \right) \bar{\rho}_{s_k^p} = \exp \left(\frac{s - s_k^p}{H_k} \right) \prod_{i=1}^k \exp \left(\frac{s_i^p - s_{i-1}^p}{H_k} \right). \quad (55)$$

On the same interval, the density variation process (7) is given by

$$\delta\rho_s = e^{-2(s-s_k^p)/H_k} \delta\rho_{s_k^p} + \sqrt{\psi_k} \int_{s_k^p}^s e^{-2(s-a)/H_k} d\tilde{w}_a, \quad (56)$$

where the integral term is a zero-mean Gaussian random variable, independent from $\delta\rho_{s_k^p}$, with variance

$$\text{Var} \left(\sqrt{\psi_k} \int_{s_k^p}^s e^{-2(s-a)/H_k} d\tilde{w}_a \right) = \psi_k H_k (1 - e^{-4(s-s_k^p)/H_k}). \quad (57)$$

Sample paths of the density variation are computed by, successively adding, for each step k , the exponential drift term to a sample from a Gaussian random variable with variance as in (57). The density process is then given by (5).

C. Partial Derivatives of the Combined Vehicle-Density System

The vehicle dynamics (1) and the drift term in density process (15) are given by the following functions:

$$f_r(x, u) = V \sin \gamma, \quad f_V(x, u) = -\frac{\rho V^2 S C_D}{2m} - \frac{\mu \sin \gamma}{r^2}, \quad f_\gamma(x, u) = \frac{\rho V S C_L}{2m} u - \left(\frac{\mu}{r^2} - \frac{V^2}{r} \right) \frac{\cos \gamma}{V},$$

$$f_R(x, u) = V \cos \gamma, \quad f_\rho(x, u) = \frac{V \sin \gamma}{H_s} (\rho - 2\bar{\rho}_s).$$

The partial derivatives of these functions, evaluated at the nominal values, are given below. Note that when evaluating the partials along a nominal trajectory, we have that $\rho = \bar{\rho}$. For simplicity, we assume that C_L and C_D are constant. However, it is not difficult to add dependence on the Mach number in the following calculations.

$$\begin{array}{llll} \frac{\partial f_r}{\partial r} = 0, & \frac{\partial f_V}{\partial r} = \frac{2\mu \sin \gamma}{r^3}, & \frac{\partial f_R}{\partial r} = 0, & \frac{\partial f_\rho}{\partial r} = \left(\frac{\partial H_s}{\partial r} + 2 \right) \frac{\rho V \sin \gamma}{H_s^2}, \\ \frac{\partial f_r}{\partial V} = \sin \gamma, & \frac{\partial f_V}{\partial V} = -\frac{\rho V S C_D}{m}, & \frac{\partial f_R}{\partial V} = \cos \gamma, & \frac{\partial f_\rho}{\partial V} = -\frac{\rho \sin \gamma}{H_s}, \\ \frac{\partial f_r}{\partial \gamma} = V \cos \gamma, & \frac{\partial f_V}{\partial \gamma} = -\frac{\mu \cos \gamma}{r^2}, & \frac{\partial f_R}{\partial \gamma} = -V \sin \gamma, & \frac{\partial f_\rho}{\partial \gamma} = -\frac{\rho V \cos \gamma}{H_s}, \\ \frac{\partial f_r}{\partial R} = 0, & \frac{\partial f_V}{\partial R} = 0, & \frac{\partial f_R}{\partial R} = 0, & \frac{\partial f_\rho}{\partial R} = 0, \\ \frac{\partial f_r}{\partial \rho} = 0, & \frac{\partial f_V}{\partial \rho} = -\frac{V^2 S C_D}{2m}, & \frac{\partial f_R}{\partial \rho} = 0, & \frac{\partial f_\rho}{\partial \rho} = \frac{V \sin \gamma}{H_s}, \\ \frac{\partial f_r}{\partial u} = 0, & \frac{\partial f_V}{\partial u} = 0, & \frac{\partial f_R}{\partial u} = 0, & \frac{\partial f_\rho}{\partial u} = 0, \end{array}$$

$$\begin{array}{lll} \frac{\partial f_\gamma}{\partial r} = \left(\frac{2\mu}{r^3} - \frac{V^2}{r^2} \right) \frac{\cos \gamma}{V}, & \frac{\partial f_\gamma}{\partial V} = \frac{\rho S C_L}{2m} u + \frac{2 \cos \gamma}{r} + \left(\frac{\mu}{r^2} - \frac{V^2}{r} \right) \frac{\cos \gamma}{V^2}, & \frac{\partial f_\gamma}{\partial \gamma} = \left(\frac{\mu}{r^2} - \frac{V^2}{r} \right) \frac{\sin \gamma}{V}, \\ \frac{\partial f_\gamma}{\partial R} = 0, & \frac{\partial f_\gamma}{\partial \rho} = \frac{V S C_L}{2m} u, & \frac{\partial f_\gamma}{\partial u} = \frac{\rho V S C_L}{2m}. \end{array}$$

Acknowledgments

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References

- [1] Moseley, P. E., "The Apollo Entry Guidance: A Review of the Mathematical Development and its Operational Characteristics," TRW Note No. 69-FMT-791, December 1969.
- [2] Mendeck, G. F., and Craig McGrew, L., "Entry Guidance Design and Postflight Performance for 2011 Mars Science Laboratory Mission," *Journal of Spacecraft and Rockets*, Vol. 51, No. 4, 2014, pp. 1094–1105.
- [3] Carman, G. L., Ives, D. G., and Geller, D. K., "Apollo-Derived Precision Lander Guidance," *23rd Atmospheric Flight Mechanics Conference*, Boston, MA, 1998.
- [4] Kluever, C., "Entry Guidance Performance for Mars Precision Landing," *Journal of Guidance, Control, and Dynamics*, Vol. 31, No. 6, 2008, pp. 1537–1544.
- [5] Wingrove, R. C., "Survey of Atmosphere Re-Entry Guidance and Control Methods," *AIAA Journal*, Vol. 1, No. 9, 1963, pp. 2019–2029.
- [6] Lu, P., "Entry Guidance: A Unified Method," *Journal of Guidance, Control, and Dynamics*, Vol. 37, No. 3, 2014, pp. 713–728.

- [7] Putnam, Z. R., Neave, M. D., and Barton, G. H., "PredGuid Entry Guidance for Orion Return from Low Earth Orbit," *2010 IEEE Aerospace Conference*, Big Sky, MT, 2010.
- [8] Xue, S., and Lu, P., "Constrained Predictor-Corrector Entry Guidance," *Journal of Guidance, Control, and Dynamics*, Vol. 33, No. 4, 2010, pp. 1273–1281.
- [9] Harpold, J. C., and Graves, C. A. J., "Shuttle Entry Guidance," *Journal of the Astronautical Sciences*, Vol. 37, No. 3, 1979, pp. 239–268.
- [10] Harpold, J. C., and Gavert, D. E., "Space Shuttle Entry Guidance Performance Results," *Journal of Guidance, Control, and Dynamics*, Vol. 6, No. 6, 1983, pp. 442–447.
- [11] Lu, P., "Entry Guidance and Trajectory Control for Reusable Launch Vehicle," *Journal of Guidance, Control, and Dynamics*, Vol. 20, No. 1, 1997, pp. 143–149.
- [12] Mease, K. D., and Kremer, J.-P., "Shuttle Entry Guidance Revisited Using Nonlinear Geometric Methods," *Journal of Guidance, Control, and Dynamics*, Vol. 17, No. 6, 1994, pp. 1350–1356.
- [13] Tu, K.-Y., Munir, M. S., Mease, K. D., and Bayard, D. S., "Drag-based Predictive Tracking Guidance for Mars Precision Landing," *Journal of Guidance, Control, and Dynamics*, Vol. 23, No. 4, 2000, pp. 620–628.
- [14] Alabau-Boussouira, F., Brockett, R., Glass, O., Le Rousseau, J., and Zuazua, E., *Control of Partial Differential Equations*, Lecture Notes in Mathematics, Vol. 2048, Springer, Cetraro, Italy, 2012.
- [15] Chen, Y., Georgiou, T. T., and Pavon, M., "Optimal Steering of a Linear Stochastic System to a Final Probability Distribution, Part I," *IEEE Transactions on Automatic Control*, Vol. 61, No. 5, 2016, pp. 1158–1169.
- [16] Lu, P., Cerimele, C. J., Tigges, M. A., and Matz, D. A., "Optimal Aerocapture Guidance," *Journal of Guidance, Control, and Dynamics*, Vol. 38, No. 4, 2015, pp. 553–565.
- [17] Ridderhof, J., and Tsiotras, P., "Minimum-fuel Powered Descent in the Presence of Random Disturbances," *2019 AIAA Guidance, Navigation, and Control Conference*, San Diego, CA, 2019.
- [18] Le Gall, J.-F., "Brownian Motion, Martingales, and Stochastic Calculus," *Graduate Texts in Mathematics*, 2016. <https://doi.org/10.1007/978-3-319-31089-3>.
- [19] Justus, C., James, B., Bougher, S., Bridger, A., Haberle, R., Murphy, J., and Engel, S., "Mars-GRAM 2000: A Mars Atmospheric Model for Engineering Applications," *Advances in Space Research*, Vol. 29, No. 2, 2002, pp. 193–202.
- [20] Halder, A., and Bhattacharya, R., "Dispersion Analysis in Hypersonic Flight During Planetary Entry Using Stochastic Liouville Equation," *Journal of Guidance, Control, and Dynamics*, Vol. 34, No. 2, 2011, pp. 459–474.
- [21] Bryson, A. E., and Ho, Y.-C., *Applied Optimal Control: Optimization, Estimation and Control*, Ginn and Company, 1969.
- [22] Mäkilä, P. M., Westerlund, T., and Toivonen, H. T., "Constrained Linear Quadratic Gaussian Control with Process Applications," *Automatica*, Vol. 20, No. 1, 1984, pp. 15–29.
- [23] Ridderhof, J., and Tsiotras, P., "Stochastic Atmosphere Modeling for Risk Adverse Aerocapture Guidance," *2020 IEEE Aerospace Conference*, Big Sky, MT, 2020.