

Exponential Stabilization of the Complete Attitude of an Underactuated Spacecraft

John Matthew Brewer Panagiotis Tsiotras

Abstract—In this paper, two kinematic controllers capable of stabilizing the complete attitude of an underactuated satellite are presented. Previous controllers in the literature impose limitations on the satellite’s initial conditions or give rise to state manifolds which can lead to undesired behavior. We utilize averaging theory to construct two controllers that exponentially stabilize the satellite’s complete attitude without the previous limitations. Both control laws are shown to yield exponential stability, with the second control law having potentially a larger region of attraction.

I. INTRODUCTION

The problem of developing control laws for an underactuated spacecraft capable of stabilizing the complete attitude has been given much attention in the literature, but the ability to achieve a globally stabilizing control law largely remains open, except for some limiting situations. Previous work in this area is quite rich. Control of an underactuated spacecraft was probably first investigated by Crouch in [1], where the accessibility of $SO(3)$ in terms of attitude is shown for a spacecraft with only two control torques. In [2], [3], it was shown that time-invariant continuous control laws cannot stabilize the complete attitude of an underactuated spacecraft; however, discontinuous, time-varying, or hybrid control laws may still be used [4].

In [5]–[7], partial attitude stabilization of underactuated spacecraft was shown to be possible via the use of the w - z parameterization of Tsiotras and Longuski [8]. In [9]–[17], full attitude stabilization was proven for the case where either the uncontrolled axis of the spacecraft has initial zero angular velocity [11], [12], [15], [17] or the spacecraft has zero total angular momentum [13], [15]–[17]. In addition, [11] proposed bounded control laws and also introduced tracking controllers along with a feasible trajectory generation procedure. Ref. [12] presented a discontinuous control law that was conjectured to globally asymptotically stabilize the complete attitude, and [18] investigated the controllability of a vector fixed in the frame of an underactuated satellite. Ref. [16] presented a dual-action control law for nonsymmetric underactuated satellites capable of full attitude stabilization as long as the initial conditions do not lie along a certain manifold.

Following the above literature, the paper by Tsiotras and Doumchenko [19] investigated various approaches for

controlling an underactuated spacecraft along with their limitations. The papers [20], [21] introduced controllers capable of local asymptotic stabilization, and [22] provided a locally exponentially stabilizing control law that is undefined at the origin. Ref. [23] proved that a modified version of the controller conjectured in [19] is capable of complete attitude stabilization, and [24] showed a controllability analysis result for underactuated spacecraft, along with a discontinuous control law similar to what was proposed in [19] and [23]. However, all these controllers are discontinuous or undefined along a manifold passing through the origin.

In this paper, we present two control laws for a general underactuated spacecraft which, in contrast to the current literature, are well defined, do not result in large control inputs, and can be applied without assumptions about the spacecraft’s momentum or the spacecraft initial conditions. Both controllers achieve full attitude stabilization about the origin. One of these control laws locally exponentially stabilizes the complete attitude, while the second one also achieves the same result but with potentially a larger region of attraction.

The paper is organized as follows: first, some preliminary material is provided concerning the attitude parameterization and the spacecraft dynamical model. Then, the dynamics to be controlled is given and the control laws are introduced along with their corresponding stabilization properties. Finally, simulation results are provided that illustrate the effectiveness of the two proposed control laws.

II. PRELIMINARY MATERIAL

A. Attitude Parameterization

The attitude of the spacecraft used in this work is described in terms of the w - z parameterization introduced in [8] by Tsiotras and Longuski. This parameterization is particularly useful when studying underactuated spacecraft. This is because this parameterization decouples the motion about the underactuated axis from the motion of the underactuated axis itself (the latter being directly unaffected by the available control inputs). Examples of controllers utilizing the w - z parameterization include those presented in [5], [7], [10], [11], [25], [26].

To introduce the w - z attitude parameterization used in this work, consider two reference frames: $\hat{\mathbf{b}} = \{\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3\}$, being some frame of interest fixed on the spacecraft, and $\hat{\mathbf{i}} = \{\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\}$ representing the inertial frame. The rotation matrix transforming vectors from frame $\hat{\mathbf{i}}$ to $\hat{\mathbf{b}}$ can be

J. M. Brewer is a Ph.D. Candidate at the School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA 30332. jbrewer30@gatech.edu

P. Tsiotras is the David and Andrew Lewis Chair Professor at the School of Aerospace Engineering and the Institute for Robotics & Intelligent Machines, Georgia Institute of Technology, Atlanta, GA 30332. tsiotras@gatech.edu

decomposed into two successive rotations as follows:

$$R_i^b(w, z) = R_{i'}^b(w)R_i^{i'}(z),$$

where the initial rotation $R_i^{i'}(z)$ transforms vectors first to an intermediate frame $\hat{\mathbf{i}}'$ by way of a rotation about the initial axis, assumed to be the underactuated axis, by the angle z , and the second rotation $R_{i'}^b(w)$ transforms vectors from the intermediate frame $\hat{\mathbf{i}}'$ to frame $\hat{\mathbf{b}}$ by a rotation about the vector $\hat{\mathbf{h}} = \hat{\mathbf{i}}_3 \times \hat{\mathbf{b}}_3$ through some angle $\theta(w)$, with $w = [w_1, w_2]^\top \in \mathbb{R}^2$. The vector w results from the stereographic projection of the $\hat{\mathbf{i}}_3$ axis (equivalently, the $\hat{\mathbf{i}}_3$ axis) onto the $\hat{\mathbf{b}}_1$ - $\hat{\mathbf{b}}_2$ plane. That is, if $\hat{\mathbf{i}}_3 = a\hat{\mathbf{b}}_1 + b\hat{\mathbf{b}}_2 + c\hat{\mathbf{b}}_3$, then the w parameters are defined as

$$w_1 = \frac{b}{1+c}, \quad w_2 = \frac{-a}{1+c}, \quad (1)$$

leading to the rotation matrices

$$R_i^{i'}(z) = \begin{bmatrix} \cos(z) & \sin(z) & 0 \\ -\sin(z) & \cos(z) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (2)$$

and

$$R_{i'}^b(w) = \frac{1}{1 + \|w\|^2} \begin{bmatrix} 1 + w_1^2 - w_2^2 & 2w_1w_2 & -2w_2 \\ 2w_1w_2 & 1 - w_1^2 + w_2^2 & 2w_1 \\ 2w_2 & -2w_1 & 1 - \|w\|^2 \end{bmatrix}.$$

B. Kinematics

Next, we introduce the kinematics in terms of the w - z parameters. Let $\omega = [\omega_1, \omega_2, \omega_3]^\top \in \mathbb{R}^3$ represent the angular velocity of the $\hat{\mathbf{b}}$ frame with respect to the inertial frame, expressed in the $\hat{\mathbf{b}}$ frame. In addition, let $S(\cdot)$ represent the skew symmetric matrix representing the cross product between two vectors, that is, $S(x_1)x_2 = -x_1 \times x_2$. Then, the kinematics of the unit vector $[a, b, c]^\top \in S^2$ is

$$\begin{bmatrix} \dot{a} \\ \dot{b} \\ \dot{c} \end{bmatrix} = S(\omega) \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad (3)$$

where

$$S(\omega) = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}, \quad (4)$$

leading to the kinematics of the w - z parameters being expressed as follows

$$\dot{w}_1 = \omega_3 w_2 + w_1 w_2 \omega_2 + \frac{1}{2}(1 + w_1^2 - w_2^2), \quad (5a)$$

$$\dot{w}_2 = -\omega_3 w_1 + w_1 w_2 \omega_1 + \frac{1}{2}(1 - w_1^2 + w_2^2), \quad (5b)$$

$$\dot{z} = \omega_3 + w_1 \omega_2 - w_2 \omega_1. \quad (5c)$$

C. Spacecraft Dynamics

Consider the frame $\hat{\mathbf{b}}$ attached to a spacecraft having principal moments of inertia $J = \text{diag}(J_1, J_2, J_3)$, where the axes of the $\hat{\mathbf{b}}$ frame align with the principle inertia axes of the spacecraft. As before, let $\omega = [\omega_1, \omega_2, \omega_3]^\top$ represent the angular velocity of the $\hat{\mathbf{b}}$ frame with respect to the inertial frame, expressed in the $\hat{\mathbf{b}}$ frame. Then, the dynamics of the spacecraft follow the equation

$$J\dot{\omega} = S(\omega)J\omega + \tau, \quad (6)$$

where $\tau = [\tau_1, \tau_2, \tau_3]^\top \in \mathbb{R}^3$ is the torque input vector. The dynamics of the satellite can be further simplified so that instead of controlling torques, the angular velocities can be controlled. In addition, we focus exclusively on the case when the spacecraft is underactuated, that is, only two control torques are available, namely, about the $\hat{\mathbf{b}}_1$ and $\hat{\mathbf{b}}_2$ axes leaving (without loss of generality) the $\hat{\mathbf{b}}_3$ axis uncontrolled (i.e., $\tau_3 \equiv 0$). The complete dynamics of the system may then be expressed with the equations

$$\dot{w}_1 = \omega_3 w_2 + w_1 w_2 \omega_2 + \frac{1}{2}(1 + w_1^2 - w_2^2)\omega_1, \quad (7a)$$

$$\dot{w}_2 = -\omega_3 w_1 + w_1 w_2 \omega_1 + \frac{1}{2}(1 - w_1^2 + w_2^2)\omega_2, \quad (7b)$$

$$\dot{z} = \omega_3 + w_1 \omega_2 - w_2 \omega_1, \quad (7c)$$

$$\dot{\omega}_3 = \alpha \omega_1 \omega_2, \quad (7d)$$

and

$$\dot{\omega}_1 = \frac{J_2 - J_3}{J_1} \omega_2 \omega_3 + \tau_1, \quad (8a)$$

$$\dot{\omega}_2 = \frac{J_3 - J_1}{J_2} \omega_3 \omega_1 + \tau_2, \quad (8b)$$

where $\alpha = (J_1 - J_2)/J_3 \neq 0$ is a necessary condition for accessibility [1]. Henceforth, and without loss of generality, we assume $\alpha > 0$. Since the subsystem (8) is separable from the remaining dynamics (7), ω_1 and ω_2 can be thought of as the new “control inputs” for the reduced system (7), which omits the equations for $\dot{\omega}_1$ and $\dot{\omega}_2$, as these can be directly controlled via τ_1 and τ_2 .

III. MAIN RESULT

A. Averaged Spacecraft Model

Consider the reduced underactuated spacecraft (7) with inputs ω_1 , and ω_2 . Since a time-invariant continuous stabilizing control is impossible [2], [19], a time-varying controller is proposed instead. To this end, consider the control input [19]

$$\omega_1 = \nu_1 + |\nu|^{1/2} \cos(t/\varepsilon), \quad (9a)$$

$$\omega_2 = \nu_2 + \text{sgn}(\nu)|\nu|^{1/2} \cos(t/\varepsilon), \quad (9b)$$

where $0 < \varepsilon \ll 1$ and where the new inputs to the system are now ν_1 , ν_2 , and ν . With (9) as control inputs, system (7)

yields

$$\begin{aligned}\dot{w}_1 &= \omega_3 w_2 + w_1 w_2 \nu_2 + \frac{1}{2}(1 + w_1^2 - w_2^2)\nu_1 \\ &\quad + (w_1 w_2 \operatorname{sgn}(\nu)|\nu|^{1/2} + \frac{1}{2}(1 + w_1^2 - w_2^2)|\nu|^{1/2}) \cos(t/\varepsilon),\end{aligned}\quad (10a)$$

$$\begin{aligned}\dot{w}_2 &= -\omega_3 w_1 + w_1 w_2 \nu_1 + \frac{1}{2}(1 - w_1^2 + w_2^2)\nu_2 \\ &\quad + (w_1 w_2 |\nu|^{1/2} + \frac{1}{2}(1 - w_1^2 + w_2^2)\operatorname{sgn}(\nu)|\nu|^{1/2}) \cos(t/\varepsilon),\end{aligned}\quad (10b)$$

$$\begin{aligned}\dot{z} &= \omega_3 + w_1 \nu_2 - w_2 \nu_1 \\ &\quad + (w_1 \operatorname{sgn}(\nu)|\nu|^{1/2} - w_2 |\nu|^{1/2}) \cos(t/\varepsilon),\end{aligned}\quad (10c)$$

$$\begin{aligned}\dot{\omega}_3 &= \alpha(\nu_1 \nu_2 + (\nu_1 \operatorname{sgn}(\nu)|\nu|^{1/2} + \nu_2 |\nu|^{1/2}) \cos(t/\varepsilon) \\ &\quad + \nu \cos^2(t/\varepsilon)),\end{aligned}\quad (10d)$$

which is T -periodic, and possesses continuous and bounded first and second order partial derivatives with respect to (w_1, w_2, z, ω_3) . Using the new time variable $\tau = t/\varepsilon$, the averaged system takes the form

$$\dot{\bar{w}}_1 = \bar{\omega}_3 \bar{w}_2 + \bar{w}_1 \bar{w}_2 \nu_2 + \frac{1}{2}(1 + \bar{w}_1^2 - \bar{w}_2^2)\nu_1, \quad (11a)$$

$$\dot{\bar{w}}_2 = -\bar{\omega}_3 \bar{w}_1 + \bar{w}_1 \bar{w}_2 \nu_1 + \frac{1}{2}(1 - \bar{w}_1^2 + \bar{w}_2^2)\nu_2, \quad (11b)$$

$$\dot{\bar{z}} = \bar{\omega}_3 + \bar{w}_1 \nu_2 - \bar{w}_2 \nu_1, \quad (11c)$$

$$\dot{\bar{\omega}}_3 = \alpha(\nu_1 \nu_2 + \frac{\nu}{2}), \quad (11d)$$

where $[\bar{w}_1, \bar{w}_2, \bar{z}, \bar{\omega}_3]^\top$ represents the state of the averaged system with inputs ν_1 , ν_2 , and ν . Henceforth, and for notational simplicity, we will drop the bars and denote the average state as $[w_1, w_2, z, \omega_3]^\top$ since the remainder of this paper is concerned exclusively with the study of the averaged system.

B. Local Exponential Stability of the Averaged System

In this section, a control law is proposed which exponentially stabilizes the averaged system (11).

Theorem 1: Consider a spacecraft having dynamics (10) and let the control law for the spacecraft be given by

$$\nu_1 = -kw_1 + \mu zw_2, \quad (12a)$$

$$\nu_2 = -kw_2 - \mu zw_1, \quad (12b)$$

$$\nu = -2\nu_1 \nu_2 - \frac{2}{\alpha}z - 2\lambda\omega_3, \quad (12c)$$

where $k > 0$, $\mu \in \mathbb{R}$, $\lambda > 0$. Then, the origin $(w_1, w_2, z, \omega_3) = (0, 0, 0, 0)$ is an exponentially stable equilibrium.

Proof: Consider the time-averaged dynamics of the spacecraft (11). Using the control law (12), the averaged

controlled system then takes the form

$$\begin{aligned}\dot{w}_1 &= \omega_3 w_2 + w_1 w_2 (-kw_2 - \mu zw_1) \\ &\quad + \frac{1}{2}(1 + w_1^2 - w_2^2)(-kw_1 + \mu zw_2),\end{aligned}\quad (13a)$$

$$\begin{aligned}\dot{w}_2 &= -\omega_3 w_1 + w_1 w_2 (-kw_1 + \mu zw_2) \\ &\quad + \frac{1}{2}(1 - w_1^2 + w_2^2)(-kw_2 - \mu zw_1),\end{aligned}\quad (13b)$$

$$\dot{z} = \omega_3 + w_1(-kw_2 - \mu zw_1) - w_2(-kw_1 + \mu zw_2), \quad (13c)$$

$$\dot{\omega}_3 = \alpha(\nu_1 \nu_2 + \frac{1}{2}(-2\nu_1 \nu_2 - \frac{2}{\alpha}z - 2\lambda\omega_3)), \quad (13d)$$

which simplifies to

$$\begin{aligned}\dot{w}_1 &= \omega_3 w_2 + w_1 w_2 (-kw_2 - \mu zw_1) \\ &\quad + \frac{1}{2}(1 + w_1^2 - w_2^2)(-kw_1 + \mu zw_2),\end{aligned}\quad (14a)$$

$$\begin{aligned}\dot{w}_2 &= -\omega_3 w_1 + w_1 w_2 (-kw_1 + \mu zw_2) \\ &\quad + \frac{1}{2}(1 - w_1^2 + w_2^2)(-kw_2 - \mu zw_1),\end{aligned}\quad (14b)$$

$$\dot{z} = \omega_3 - \mu z(w_1^2 + w_2^2), \quad (14c)$$

$$\dot{\omega}_3 = -z - \lambda\alpha\omega_3. \quad (14d)$$

The system matrix corresponding to the linearized system of (14) about the origin is

$$\begin{bmatrix} -k/2 & 0 & 0 & 0 \\ 0 & -k/2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -\lambda\alpha \end{bmatrix},$$

which has eigenvalues $-k/2$ and $(-\lambda\alpha \pm \sqrt{\lambda^2\alpha^2 - 4})/2$ and is thus Hurwitz for $k > 0$ and $\lambda > 0$. Therefore, under the proposed control law (12), the origin $(w_1, w_2, z, \omega_3) = (0, 0, 0, 0)$ is an exponentially stable equilibrium for the averaged system. It follows that $(w_1, w_2, z, \omega_3) = (0, 0, 0, 0)$ is also an exponentially stable equilibrium for the original system using the control law (9) and (12) [27]. ■

C. Almost Global Exponential Stability of the Averaged System

In this section, the previous controller is modified to also exponentially stabilize the averaged system (11), with a potentially larger region of attraction.

Theorem 2: Consider a spacecraft having dynamics (10) and let the control law for the spacecraft be given by

$$\nu_1 = -kw_1 + \mu(z + \omega_3)w_2, \quad (15a)$$

$$\nu_2 = -kw_2 - \mu(z + \omega_3)w_1, \quad (15b)$$

$$\nu = -2\nu_1 \nu_2 - \frac{2}{\alpha}z - \frac{4}{\alpha}\omega_3, \quad (15c)$$

where $k \geq 2$ and $\mu \geq 0$. Then, the origin $(w_1, w_2, z, \omega_3) = (0, 0, 0, 0)$ is an exponentially stable equilibrium.

Proof: Consider the time-averaged dynamics of the spacecraft (11). Using the control law (12), the averaged

system then takes the form

$$\begin{aligned}\dot{w}_1 &= \omega_3 w_2 + w_1 w_2 (-k w_2 - \mu(z + \omega_3) w_1) \\ &\quad + \frac{1}{2}(1 + w_1^2 - w_2^2)(-k w_1 + \mu(z + \omega_3) w_2),\end{aligned}\quad (16a)$$

$$\begin{aligned}\dot{w}_2 &= -\omega_3 w_1 + w_1 w_2 (-k w_1 + \mu(z + \omega_3) w_2) \\ &\quad + \frac{1}{2}(1 - w_1^2 + w_2^2)(-k w_2 - \mu(z + \omega_3) w_1),\end{aligned}\quad (16b)$$

$$\dot{z} = \omega_3 - \mu z (w_1^2 + w_2^2), \quad (16c)$$

$$\dot{\omega}_3 = -z - 2\omega_3. \quad (16d)$$

The system matrix corresponding to the linearized form of (16) about the origin is

$$\begin{bmatrix} -k/2 & 0 & 0 & 0 \\ 0 & -k/2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -2 \end{bmatrix},$$

which has eigenvalues $-k/2$ and -1 and is thus Hurwitz for $k > 0$, proving local exponential stability.

Now, consider the Lyapunov function candidate

$$V = \frac{1}{2}(w_1^2 + w_2^2 + (z + \omega_3)^2). \quad (17)$$

Taking its derivative along the trajectories of the system (16), yields

$$\begin{aligned}\dot{V} &= w_1 \dot{w}_1 + w_2 \dot{w}_2 + (z + \omega_3)(\dot{z} + \dot{\omega}_3) \\ &= \frac{1}{2}(1 + w_1^2 + w_2^2)(w_1 \nu_1 + w_2 \nu_2) + z \omega_3 + \omega_3^2 \\ &\quad + (z + \omega_3)(w_1 \nu_2 - w_2 \nu_1) + \alpha(z + \omega_3)(\nu_1 \nu_2 + \frac{\nu}{2}).\end{aligned}$$

Using (16), along with the facts that

$$\begin{aligned}(w_1 \nu_1 + w_2 \nu_2) &= -k w_1^2 + \mu(z + \omega_3) w_1 w_2 \\ &\quad - k w_2^2 - \mu(z + \omega_3) w_1 w_2 \\ &= -k(w_1^2 + w_2^2),\end{aligned}$$

$$\begin{aligned}(w_1 \nu_2 - w_2 \nu_1) &= -k w_1 w_2 - \mu(z + \omega_3) w_1^2 + k w_1 w_2 \\ &\quad - \mu(z + \omega_3) w_2^2 \\ &= -\mu(z + \omega_3)(w_1^2 + w_2^2),\end{aligned}$$

$$\begin{aligned}\alpha(\nu_1 \nu_2 + \frac{\nu}{2}) &= \alpha(\nu_1 \nu_2 + \frac{1}{2}(-2\nu_1 \nu_2 - 2\frac{z}{\alpha} - 2\lambda \omega_3)) \\ &= -z - 2\omega_3.\end{aligned}$$

and substituting the above into $\dot{V}$, yields

$$\begin{aligned}\dot{V} &= -\frac{k}{2}(1 + w_1^2 + w_2^2)(w_1^2 + w_2^2) + z \omega_3 + \omega_3^2 \\ &\quad - \mu(z + \omega_3)^2(w_1^2 + w_2^2) - (z + \omega_3)(z + 2\omega_3) \\ &= \left(1 - \frac{k}{2}\right) \|w\|_2^2 - \frac{k}{2} \|w\|_2^4 - \mu \|w\|_2^2 (z + \omega_3)^2 \\ &\quad - (w_1^2 + w_2^2 + (z + \omega_3)^2) \\ &\leq -(w_1^2 + w_2^2 + (z + \omega_3)^2) \\ &= -2V,\end{aligned}$$

where $k \geq 2$ and $\mu \geq 0$ are utilized to generate the inequality bound. It follows that $V \rightarrow 0$ as $t \rightarrow \infty$ and thus $z + \omega_3 \rightarrow 0$.

Thus, the manifold $\mathcal{M} = \{(w_1, w_2, z, \omega_3) : w_1 = w_2 = 0, \omega_3 = -z\}$ is globally exponentially stable. It is obvious that the manifold $\mathcal{M}$ is invariant. The system dynamics on the manifold $\mathcal{M}$, are given as

$$\dot{z} = \omega_3 = -z, \quad (18a)$$

$$\dot{\omega}_3 = -z - 2\omega_3 = z = -\omega_3. \quad (18b)$$

The dynamics (18) implies that $z \rightarrow 0$ and $\omega_3 \rightarrow 0$ as $t \rightarrow 0$. Thus, the origin is globally exponentially stable on $\mathcal{M}$.

Therefore, under the proposed control law, $(w_1, w_2, z, \omega_3) = (0, 0, 0, 0)$ is a globally exponentially stable equilibrium for the averaged system. It follows that $(w_1, w_2, z, \omega_3) = (0, 0, 0, 0)$ is also an exponentially stable equilibrium for the original system using the control law (9) and (15) [27]. ■

Remark 1: The control law proposed in Theorem 2 provides almost global exponential stability for the averaged system, with the lack of global stability being due to the inherent singularity of the w parameters (attitude when $\hat{\mathbf{b}}_3 = -\hat{\mathbf{i}}_3$ and thus $w = \infty$). It is therefore conjectured that (15) also provides a larger region of attraction compared to (12).

IV. SIMULATIONS

We demonstrate the effectiveness of the proposed control laws (12) and (15) via numerical simulations.

For the first control law (12), the averaged system and the original system were simulated using the parameters $\varepsilon = 0.1$, $\alpha = 0.2$, $k = 2.5$, $\mu = 2$, $\lambda = 3$, $w_1(0) = 2$, $w_2(0) = -2$, $z(0) = 1$, and $\omega_3(0) = 1$. The results of the states and the control inputs for the averaged system are plotted in Fig. 1 and for the original system are plotted in Fig. 2. Snapshots of the simulation for the original system are plotted in Fig. 3, where the bottom left transparent spacecraft provides the constant reference attitude which the underactuated spacecraft was to stabilize with respect to.

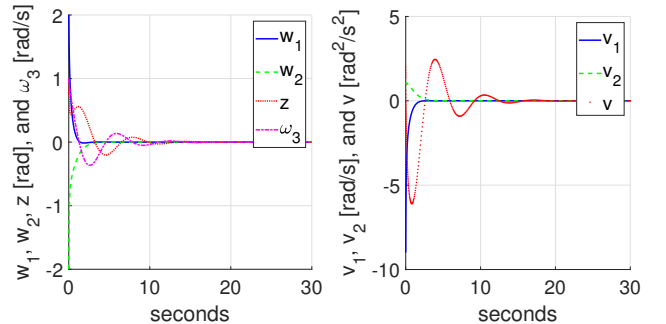


Fig. 1: State and control input history of the averaged system.

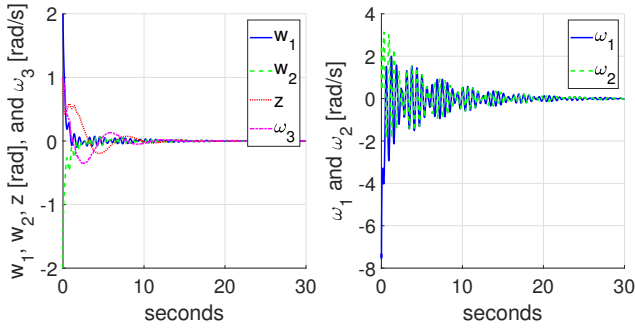


Fig. 2: State and control input history of the original system.

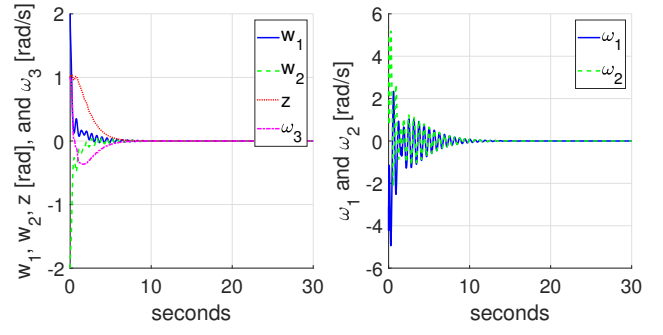


Fig. 5: State and control input history of the original system.

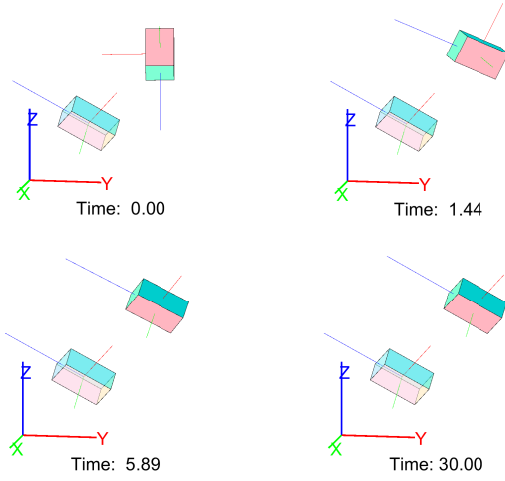


Fig. 3: Simulation snapshots of the original system utilizing (12).

For the second control law (15), the averaged system and the original system were simulated using the parameters $\varepsilon = 0.1$, $\alpha = 0.2$, $k = 0.2$, and $\mu = 0.1$. The initial conditions were the same as the preceding simulation for comparison purposes. The results of the states and the control inputs for the averaged system are plotted in Fig. 4 and for the original system are plotted in Fig. 5.

To test the conjecture that the second control law has a larger region of attraction, the averaged and original system were again simulated using the same parameters as before, but with the initial conditions $w_1(0) = 20$, $w_2(0) = -16$, $z(0) = 3\pi/4$, and $\omega_3(0) = 1$. The results of the states and the control inputs for the averaged system are plotted in Fig. 6 and for the original system are plotted in Fig. 7. Snapshots of the simulation for the original system are plotted in Fig. 8.

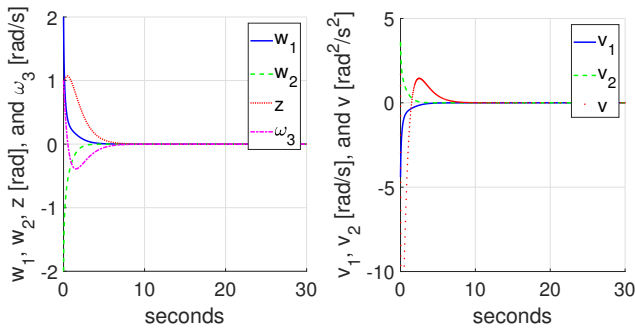


Fig. 4: State and control input history of the averaged system.

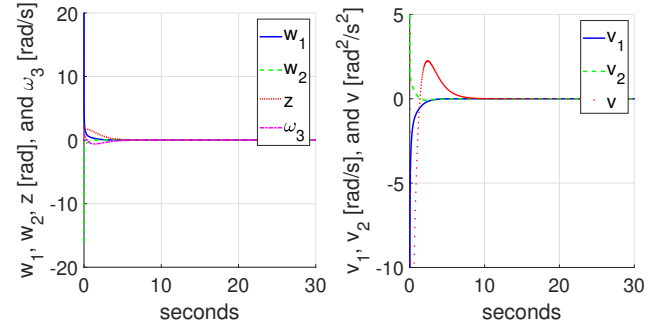


Fig. 6: State and control input history of the averaged system.

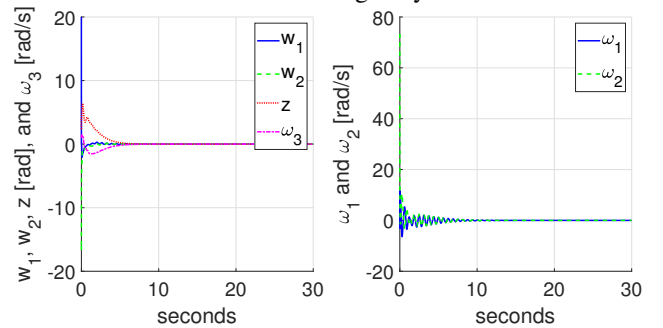


Fig. 7: State and control input history of the original system.

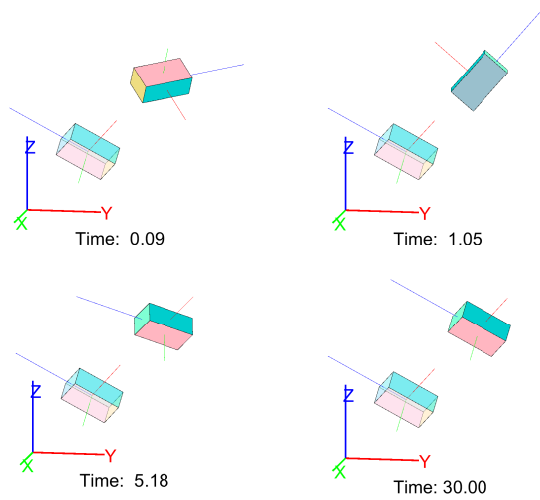


Fig. 8: Simulation snapshots of the original system utilizing (15).

Remark 2: In the method of averaging, the closeness of the behavior of the averaged system (and thus the stability of the averaged system) to the original system, is related to the relative speed of the system perturbations to the speed of the perturbation's average [27]. Therefore, the apparent oscillations seen in the simulations arising from the inputs ω_1 and ω_2 may be reduced in practice by increasing ε so long as the speed of the dynamics (7) and (8) are suitably reduced. This is consistent with actual practice where stabilization may take several orbits to achieve.

V. CONCLUSION

The ability to control the complete attitude of an underactuated spacecraft has many advantages. One of them is the possibility to consider having three input axes as redundant. In this way, if a satellite operates using only magnetorquers as actuation devices (which are unable to provide an input along the local magnetic field lines) or it experiences an actuator failure rendering one of the spacecraft axes uncontrollable, then complete attitude stabilization is still possible. In this paper, two control laws are proposed, which provide exponential stability of the complete attitude of a spacecraft without torque authority about one of its axes. These controllers are well defined throughout the state space, do not result in large input values, and can be applied without any special restrictive requirements on the initial spacecraft momentum.

REFERENCES

- [1] P. Crouch, "Spacecraft attitude control and stabilization: Applications of geometric control theory to rigid body models," *IEEE Transactions on Automatic Control*, vol. 29, no. 4, pp. 321–331, Apr 1984.
- [2] R. W. Brockett, "Asymptotic stability and feedback stabilization," in *Differential Geometric Control Theory*. Birkhäuser, 1983, pp. 181–191.
- [3] C. I. Byrnes and A. Isidori, "On the attitude stabilization of rigid spacecraft," *Automatica*, vol. 27, no. 1, pp. 87 – 95, 1991.
- [4] N. Khaneja and R. Brockett, "Dynamic feedback stabilization of nonholonomic systems," in *Proceedings of the 38th IEEE Conference on Decision and Control*, vol. 2, Phoenix, AZ, 1999, pp. 1640–1645 vol.2.

- [5] P. Tsiotras and J. M. Longuski, "Spin-axis stabilization of symmetric spacecraft with two control torques," *Systems & Control Letters*, vol. 23, no. 6, pp. 395 – 402, 1994.
- [6] P. Tsiotras and A. Schleicher, "Detumbling and partial attitude stabilization of a rigid spacecraft under actuator failure," in *AIAA Guidance, Navigation, and Control Conference*, Denver, CO, August 14–17, 2000.
- [7] H. Shen and C. M. Roithmayr, "Co-spin with symmetry axis stabilization, and de-spin for asteroid capture," in *American Control Conference*, Portland, OR, June 4–6, 2014, pp. 1599–1604.
- [8] P. Tsiotras and J. M. Longuski, "A new parameterization of the attitude kinematics," *The Journal of the Astronautical Sciences*, vol. 43, no. 3, pp. 243–262, 1995.
- [9] H. Krishnan, N. H. McClamroch, and M. Reyhanoglu, "Attitude stabilization of a rigid spacecraft using two momentum wheel actuators," *Journal of Guidance, Control, and Dynamics*, vol. 18, no. 2, pp. 256–263, 1995.
- [10] P. Tsiotras, M. J. Corless, and J. M. Longuski, "A novel approach to the attitude control of axisymmetric spacecraft," *Automatica*, vol. 31, no. 8, pp. 1099 – 1112, 1995.
- [11] P. Tsiotras and J. Luo, "Control of underactuated spacecraft with bounded inputs," *Automatica*, vol. 36, no. 8, pp. 1153 – 1169, 2000.
- [12] S. Kim and Y. Kim, "Sliding mode stabilizing control law of underactuated spacecraft," in *AIAA Guidance, Navigation, and Control Conference and Exhibit*, Denver, CO, August 14–17, 2000.
- [13] N. M. Horri and S. Hodgart, "Attitude stabilization of an underactuated satellite using two wheels," in *2003 IEEE Aerospace Conference*, vol. 6, Big Sky, MT, March 8–15 2003, pp. 2629 – 2635.
- [14] N. M. Horri and P. Palmer, "Practical implementation of attitude-control algorithms for an underactuated satellite," *Journal of Guidance, Control, and Dynamics*, vol. 35, no. 1, pp. 40 – 50, 2012.
- [15] J. R. Chaurais, H. C. Ferreira, J. Y. Ishihara, and R. A. Borges, "Attitude control of an underactuated satellite using two reaction wheels," *Journal of Guidance, Control, and Dynamics*, vol. 38, no. 10, pp. 2010 – 2017, 2015.
- [16] A.-M. Zou, K. Dev Kumar, and A. H. de Ruiter, "Spacecraft attitude control using two control torques," *Information Sciences*, vol. 408, pp. 23–40, 2017.
- [17] R. Nadafi and M. Kabgani, "Robust nonlinear attitude tracking control of an underactuated spacecraft under saturation and time-varying uncertainties," *European Journal of Control*, in-press 2021.
- [18] Y. Katsuyama, T. Ibuki, K. Sekiguchi, and M. Sampei, "Attitude controllability analysis of an underactuated satellite with two reaction wheels and its control," in *54th IEEE Conference on Decision and Control*, Osaka, Japan, December 15–18 2015, pp. 3421–3426.
- [19] P. Tsiotras and V. Doumchenko, "Control of spacecraft subject to actuator failures: State-of-the-art and open problems," *Journal of the Astronautical Sciences*, vol. 48, no. 2, pp. 337–358, 2000.
- [20] P. Morin, C. Samson, J.-B. Pomet, and Z.-P. Jiang, "Time-varying feedback stabilization of the attitude of a rigid spacecraft with two controls," *Systems & Control Letters*, vol. 25, no. 5, pp. 375–385, Aug. 1995.
- [21] J.-M. Coron and E.-Y. Keraï, "Explicit feedbacks stabilizing the attitude of a rigid spacecraft with two control torques," *Automatica*, vol. 32, no. 5, pp. 669–667, 1996.
- [22] P. Morin and C. Samson, "Time-varying exponential stabilization of a rigid spacecraft with two control torques," *IEEE Transactions on Automatic Control*, vol. 42, no. 4, pp. 528–534, Apr 1997.
- [23] Y. Yoshimura, T. Matsuno, and S. Hokamoto, "Three dimensional attitude control of an underactuated satellite with thrusters," *International Journal of Automation Technology*, vol. 5, no. 6, pp. 892 – 899, 2011.
- [24] D. Wang, Y. Jia, L. Jin, and S. Xu, "Control analysis of an underactuated spacecraft under disturbance," *Acta Astronautica*, vol. 83, pp. 44 – 53, 2013.
- [25] J. M. Brewer and P. Tsiotras, "Partial attitude consensus for underactuated satellite clusters," in *IEEE 55th Conference on Decision and Control*, Dec 2016, pp. 3524–3529.
- [26] —, "Partial attitude synchronization for networks of underactuated spacecraft," *Automatica*, vol. 97, pp. 27–37, 2018.
- [27] H. K. Khalil, *Nonlinear Systems*, Third ed. Upper Saddle River, New Jersey: Prentice Hall, 2002.