

# Chance-Constrained Optimal Covariance Steering with Iterative Risk Allocation

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**Abstract**—This paper extends the optimal covariance steering problem for linear stochastic systems subject to chance constraints to account for optimal risk allocation. Previous works have assumed a *uniform* risk allocation to cast the optimal control problem as a semi-definite program (SDP). We adopt the Iterative Risk Allocation (IRA) formalism from [1], which uses a two-stage approach to solve the optimal risk allocation problem for covariance steering. The upper-stage of IRA optimizes the risk, which is proved to be a convex problem, while the lower-stage optimizes the controller with the new constraints. The proposed framework results in solutions that tend to maximize the terminal covariance, while still satisfying the chance constraints, thus leading to less conservative solutions than previous methodologies.

## I. INTRODUCTION

In this paper we address the problem of finite-horizon stochastic optimal control of a discrete linear time-varying (LTV) system with time-independent white-noise Gaussian diffusion. The control task is to steer the state from an initial Gaussian distribution to a final Gaussian distribution with known statistics. In addition to the boundary conditions, we consider chance constraints that restrict the probability of violating the state constraints to be less than a certain threshold. Hard state constraints are difficult to impose in stochastic systems because the noise can be unbounded, so chance constraints are used to deal with this problem by imposing a small, but finite, probability of violating the constraints. In the literature, there are two kinds of chance constraints; *individual* and *joint* [2]. Individual chance constraints limit the probability of violating each constraint, while joint chance constraints limit the probability of violating *any* constraint over the whole time horizon. In this paper, we consider the case of joint chance constraints, because they are a more natural choice for most applications.

The control of stochastic systems can be best formulated as a problem of controlling the distribution of trajectories over time. Moreover, Gaussian distributions are completely characterized by their first and second moments, so the control problem can be thought of as one of steering the mean and the covariance to their terminal values.

The problem could be thought of as a linear-quadratic Gaussian (LQG) problem with a condition on the terminal covariance [3]. Moreover, it has been shown that the finite-horizon controller can be constructed as a state-feedback

controller and the problem can be formulated as a convex program [3], [4], or as the solution of a pair Lyapunov differential equations coupled through their boundary conditions [5], [6]. Alternatively, for certain special cases one can solve the CS problem directly by solving an LQ stochastic problem with a particular choice of cost weights [7]. Other approaches [8], [9] use an affine disturbance feedback controller having two components, one that steers the mean state and the other that steers the covariance.

In this work, we extend the Covariance Steering Chance Constraint (CSCC) problem, to account for *optimal* risk allocation. By risk allocation we mean allocating the probability of violating each individual chance constraint at each time step. For example, if there are  $M$  chance constraints and  $N$  time steps, there would be  $NM$  total allocations for the whole problem. Previous works [7], [9]–[12] have assumed a constant risk allocation, so that the resulting problem can be turned into a semi-definite program (SDP). Here, however, we adopt a two-stage algorithm that optimizes the risk distribution over all time steps, and subsequently optimizes the controller by solving a SDP. Other works have tried to optimize the risk using techniques such as ellipsoidal relaxation [13] and particle control [14]. However, ellipsoidal relaxation techniques are overly conservative and lead to highly suboptimal solutions. Particle control methods are computationally too demanding, since the number of decision variables grows with the number of samples. The two-stage risk allocation scheme proposed in this paper is computed iteratively until the cost is within a given tolerance of the minimum, from which we get the *optimal* risk allocation for the problem, as well as the optimal controller.

To illustrate the proposed risk allocation algorithm we use as an example a spacecraft rendezvous problem between two spacecraft, in which the approaching spacecraft has to remain within a predetermined line-of-sight (LOS) region during the whole maneuver.

## II. PROBLEM STATEMENT

We consider the following discrete-time stochastic time-varying system subject to noise

$$x_{k+1} = A_k x_k + B_k u_k + D_k w_k, \quad (1)$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$ , with time steps  $k = 0, \dots, N - 1$ , where  $N$  representing the finite horizon. The uncertainty  $w \in \mathbb{R}^r$  is a zero-mean white Gaussian noise with unit covariance, i.e.,  $\mathbb{E}[w_k] = 0$  and  $\mathbb{E}[w_{k_1} w_{k_2}^T] = I_r \delta_{k_1, k_2}$ . Additionally, we assume that  $\mathbb{E}[x_{k_1} w_{k_2}^T] = 0$ , for  $0 \leq k_1 \leq k_2 \leq N$ . The initial state  $x_0$  is a random vector drawn from the normal

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distribution

$$x_0 \sim \mathcal{N}(\mu_0, \Sigma_0), \quad (2)$$

where  $\mu_0 \in \mathbb{R}^n$  is the initial state mean and  $\Sigma_0 \in \mathbb{R}^{n \times n} > 0$  is the initial state covariance. The objective is to steer the trajectories of (1) from the initial distribution (2) to the terminal distribution

$$x_f \sim \mathcal{N}(\mu_f, \Sigma_f), \quad (3)$$

where  $\mu_f \in \mathbb{R}^n$  and  $\Sigma_f > 0$  are the state mean and covariance at time  $N$ , respectively. The cost function to be minimized is

$$J(u_0, \dots, u_{N-1}) := \mathbb{E} \left[ \sum_{k=0}^{N-1} x_k^\top Q_k x_k + u_k^\top R_k u_k \right], \quad (4)$$

where  $Q_k \geq 0$  and  $R_k > 0$  for all  $k = 0, \dots, N-1$ . Additionally, and over the *whole* horizon, we impose the following joint chance constraint that limits the probability of state violation to be less than a pre-specified threshold, i.e.,

$$\mathbb{P} \left( \bigcap_{k=0}^N x_k \notin \mathcal{X} \right) \leq \Delta, \quad (5)$$

where  $\mathbb{P}(\cdot)$  denotes the probability of an event,  $\mathcal{X} \subset \mathbb{R}^n$  is the state constraint set, and  $\Delta \in (0, 0.5]$ .

*Remark 1:* We assume that the system (1) is controllable, that is, for any  $x_0, x_f \in \mathbb{R}^n$ , and no noise ( $w_k \equiv 0$ ,  $k = 0, \dots, N-1$ ), there exists a sequence of control inputs  $\{u_k\}_{k=0}^{N-1}$  that steer the system from  $x_0$  to  $x_f$ .

First, we provide an alternative description of the system (1) in order to solve the problem at hand. Using [7], [9]–[12], we can reformulate (1) as

$$X = \mathcal{A}x_0 + \mathcal{B}U + \mathcal{D}W, \quad (6)$$

where  $X := [x_0^\top, \dots, x_N^\top]^\top \in \mathbb{R}^{(N+1)n}$ ,  $U := [u_0^\top, \dots, u_{N-1}^\top]^\top \in \mathbb{R}^{Nm}$ , and  $W := [w_0^\top, \dots, w_{N-1}^\top]^\top \in \mathbb{R}^{Nr}$  are the state, input, and disturbance sequences, respectively. The matrices  $\mathcal{A}$ ,  $\mathcal{B}$ , and  $\mathcal{D}$  are defined accordingly [7]. Using this notation, we can write the cost function compactly as

$$J(U) = \mathbb{E}[X^\top \bar{Q} X + U^\top \bar{R} U], \quad (7)$$

where  $\bar{Q}$  and  $\bar{R}$  are defined accordingly. Note that since  $Q_k \geq 0$  and  $R_k > 0$  for all  $k = 0, \dots, N-1$ , it follows that  $\bar{Q} \geq 0$  and  $\bar{R} > 0$ . The initial and terminal conditions (2) and (3) can be written as

$$\mu_0 = E_0 \mathbb{E}[X], \quad \Sigma_0 = E_0 \Sigma_X E_0, \quad (8)$$

and

$$\mu_f = E_N \mathbb{E}[X], \quad \Sigma_f = E_N \Sigma_X E_N, \quad (9)$$

where  $\Sigma_X := \mathbb{E}[XX^\top] - \mathbb{E}[X]\mathbb{E}[X]^\top$ , and  $E_k := [0_{n, kn}, I_n, 0_{n, (N-k)n}]$  picks out the  $k$ th component of a vector. Consequently, the state chance constraints (5) can be written as

$$\mathbb{P} \left( \bigcap_{k=1}^N E_k X \notin \mathcal{X} \right) \leq \Delta. \quad (10)$$

In summary, we wish to solve the following stochastic optimal control problem.

*Problem 1 :* Given the system (6), find the optimal control sequence  $U^* := U_{N-1}^*$  that minimizes the objective function (7), subject to the initial state (8), terminal state (9), and the state chance constraints (10).

### III. CHANCE CONSTRAINED COVARIANCE STEERING WITH RISK ALLOCATION

#### A. Lower-Stage Covariance Steering

Borrowing from the work in [9], we adopt the control policy

$$u_k = v_k + K_k y_k, \quad (11)$$

where  $v_k \in \mathbb{R}^m$ ,  $K_k \in \mathbb{R}^{m \times n}$ , and  $y_k \in \mathbb{R}^n$  is given by

$$y_{k+1} = A_k y_k + D_k w_k, \quad (12a)$$

$$y_0 = x_0 - \mu_0. \quad (12b)$$

*Remark 2:* The proposed control scheme (11)–(12) leads to a convex programming formulation of Problem 1 as follows.

Using (11)–(12), we can write the control sequence as  $U = V + KY$ , where  $Y := \mathcal{A}y_0 + \mathcal{D}W \in \mathbb{R}^{(N+1)n}$ ,  $V := [v_0^\top, \dots, v_{N-1}^\top]^\top \in \mathbb{R}^{Nm}$  and  $K \in \mathbb{R}^{Nm \times (N+1)n}$  a matrix containing the gains  $K_k$ . It follows that the dynamics can be decoupled into a mean and error state as follows

$$\bar{X} = \mathbb{E}[X] = \mathcal{A}\mu_0 + \mathcal{B}V, \quad (13)$$

$$\tilde{X} = X - \bar{X} = (I + \mathcal{B}K)Y. \quad (14)$$

Additionally, by noting that the cost function (7) can be decoupled into a mean cost  $J_\mu = \bar{X}^\top \bar{Q} \bar{X} + \bar{U}^\top \bar{R} \bar{U}$  and a covariance cost  $J_\Sigma = \text{tr}(Q \Sigma_X) + \text{tr}(Q \Sigma_U)$ , the cost function takes the form [9]

$$J(V, K) = (\mathcal{A}\mu_0 + \mathcal{B}V)^\top \bar{Q} (\mathcal{A}\mu_0 + \mathcal{B}V) + V^\top \bar{R} V + \text{tr} \left[ ((I + \mathcal{B}K)^\top \bar{Q} (I + \mathcal{B}K) + K^\top \bar{R} K) \Sigma_Y \right], \quad (15)$$

where  $\Sigma_Y := \mathcal{A} \Sigma_0 \mathcal{A}^\top + \mathcal{D} \mathcal{D}^\top$ . The terminal constraints can be reformulated as

$$\mu_f = E_N (\mathcal{A}\mu_0 + \mathcal{B}V), \quad (16a)$$

$$\Sigma_f = E_N (I + \mathcal{B}K) \Sigma_Y (I + \mathcal{B}K)^\top E_N^\top. \quad (16b)$$

Qualitatively speaking,  $V$  steers the mean of the system to  $\mu_f$ , while  $K$  steers the covariance to  $\Sigma_f$ . In order to make the problem convex, we relax the terminal covariance constraint (16b) to  $\Sigma_f \geq E_N (I + \mathcal{B}K) \Sigma_Y (I + \mathcal{B}K)^\top E_N^\top$ , which can be written as the linear matrix inequality (LMI)

$$\begin{bmatrix} \Sigma_f & E_N (I + \mathcal{B}K) \Sigma_Y^{1/2} \\ \Sigma_Y^{1/2} (I + \mathcal{B}K)^\top E_N^\top & I \end{bmatrix} \geq 0. \quad (17)$$

### B. Polyhedral Chance Constraints

When dealing with the risk allocation problem, it is customary to assume that the state constraint set  $\mathcal{X}$  is a convex polytope  $\mathcal{X}^p$ , so that

$$\mathcal{X}^p := \bigcap_{j=1}^M \{x : \alpha_j^\top x \leq \beta_j\}, \quad (18)$$

where  $\alpha_j \in \mathbb{R}^n$  and  $\beta_j \in \mathbb{R}$ . Under this assumption, the probability of violating the state constraints (10) can be written as

$$\mathbb{P}\left(\bigwedge_{k=1}^N \bigwedge_{j=1}^M \alpha_j^\top E_k X > \beta_j\right) \leq \Delta. \quad (19)$$

Equation (19) represents the objective that the joint probability of violating any of the  $M$  state constraints over the horizon  $N$  is less than or equal to  $\Delta$ . Using Boole's Inequality [15], [16], one can decompose a *joint* chance constraint into *individual* chance constraints as follows

$$\mathbb{P}(\alpha_j^\top E_k X \leq \beta_j) \geq 1 - \delta_k^j, \quad k = 1, \dots, N, \quad j = 1, \dots, M. \quad (20)$$

with

$$\sum_{k=1}^N \sum_{j=1}^M \delta_k^j \leq \Delta, \quad (21)$$

where each  $\delta_k^j$  represents the probability of violating the  $j$ th constraint at time step  $k$ . Notice that the probability in (20) is of a random variable with mean  $\alpha_j^\top E_k \bar{X}$  and covariance  $\alpha_j^\top E_k \Sigma_X E_k^\top \alpha_j$ . Thus, (20) can be equivalently written as

$$\Phi\left(\frac{\beta_j - \alpha_j^\top E_k \bar{X}}{\sqrt{\alpha_j^\top E_k \Sigma_X E_k^\top \alpha_j}}\right) \geq 1 - \delta_k^j, \quad (22)$$

where  $\Phi(\cdot)$  denotes the cumulative distribution function of the standard normal distribution. Simplifying (22) and noting that  $\Sigma_X = (I + \mathcal{B}K)\Sigma_Y(I + \mathcal{B}K)^\top$  yields

$$\alpha_j^\top E_k (\mathcal{A}\mu_0 + \mathcal{B}V) + \|\Sigma_Y^{1/2}(I + \mathcal{B}K)^\top E_k^\top \alpha_j\| \Phi^{-1}(1 - \delta_k^j) \leq \beta_j. \quad (23)$$

*Remark 3:* Since  $\Sigma_0 > 0$ , it follows that  $\Sigma_Y > 0$ , and  $\Sigma_Y^{1/2}$  in (23) can be computed from its Cholesky decomposition.

The expression in (23) gives  $NM$  inequality constraints for the optimization problem. In summary, Problem 1 is converted into a convex programming problem.

**Problem 2 :** Given the system (13) and (14), find the optimal control sequences  $V^*$  and  $K^*$  that minimize the cost function (15) subject to the terminal state constraints (16a) and (17), and the *individual* chance constraints (23).

### C. Risk Allocation Optimization

Since  $\delta_k^j$  are decision variables in (23), the constraints are *bilinear*, which makes it difficult to solve this problem. As mentioned previously, in order to transform Problem 2 to a more tractable form, the allocation of the risk levels  $\delta_k^j$  may be assumed to be fixed to some pre-specified values, usually

uniformly. In this case,  $\delta_k^j$  are no longer decision variables and the problem can be efficiently solved as an SDP. However, a better approach is to allocate  $\delta_k^j$  concurrently when solving the optimization Problem 2, so as to minimize the total cost. This gives rise to a natural two-stage optimization framework [1].

Following the approach in [1], the upper stage optimization finds the optimal risk allocation  $\delta := [\delta_1^1, \delta_1^2, \dots, \delta_N^{M-1}, \delta_N^M] \in \mathbb{R}^{NM}$ , and the lower stage solves the CS problem for the optimal controller  $U^* = U_{N-1}^*$  given the risk allocation  $\delta$  from the upper-stage.

Let the value of the objective function after the lower-stage optimization for a given risk allocation  $\delta$  be  $J^*$ , that is,

$$J^*(\delta) = \min_{V, K} J(V, K), \quad (24)$$

where  $J(V, K)$  is given in (15). The upper-stage optimization problem can then be formulated as follows.

**Problem 3 (Risk Allocation):**

$$\min_{\delta} J^*(\delta), \quad (25)$$

$$\text{such that } \sum_{k=1}^N \sum_{j=1}^M \delta_k^j \leq \Delta, \quad (26)$$

$$\delta_k^j > 0, \quad (27)$$

As shown in [1], Problem 3 is a convex optimization problem, given that the objective function  $J(V, K)$  is convex, and  $\Delta \in (0, 0.5]$ .

### D. Iterative Risk Allocation Motivation

Even though we have formulated the solution of Problem 2 as a two-stage optimization problem, it is not clear yet how to solve Problem 3 efficiently in order to determine the optimal risk allocation. To gain insight into the solution, we first state a theorem from [1] about the monotonicity of  $J^*(\delta)$ .

**Theorem 1.** *The optimal cost from solving Problem 2 is a monotonically decreasing function in  $\delta_k^j$ , that is,*

$$\frac{\partial J^*}{\partial \delta_k^j} \leq 0, \quad k = 1, \dots, N, \quad j = 1, \dots, M. \quad (28)$$

*Proof.* See [1]. □

In the following section, we will use this theorem to create an iterative algorithm that solves Problem 3 and lowers the cost at each iteration. The main idea is by carefully increasing the risk allocations  $\delta_k^j$ , Theorem 1 guarantees the optimal cost to be smaller with each successive iteration. As such, we will lower the risk, or tighten the constraints, for the constraints that are too conservative, and increase the risk, or loosen the constraints, for the constraints that are already active. The following remark introduces the idea and defines what active and inactive constraints are in the context of risk allocation.

*Remark 5:* The chance constraints can be written in yet

another form that will prove useful below. Starting from (23), notice that we can write the chance constraints as

$$\delta_k^j \geq 1 - \Phi\left(\frac{\beta_j - \alpha_j^\top E_k \bar{X}^*}{\|\Sigma_Y^{1/2}(I + \mathcal{B}K^*)^\top E_k^\top \alpha_j\|}\right) =: \bar{\delta}_k^j. \quad (29)$$

The quantity  $\bar{\delta}_k^j$  represents the *true* risk experienced by the optimal trajectories, i.e., when using  $(V^*, K^*)$ . Clearly, the risk we have selected does not need to be equal to the actual risk once the optimization is completed. When these values are equal we will say that the constraint is active, and is inactive otherwise. Good solutions correspond to cases when the true risk is within a small margin of the allocated risk. Many  $\delta_k^j$  having values smaller than their true counterparts would imply an overly conservative solution.

#### E. Iterative Risk Allocation Algorithm

We can exploit Theorem 1 in the context of CS to create an iterative risk allocation algorithm that simultaneously finds the optimal risk allocation  $\delta^*$  and the optimal control pair  $(V^*, K^*)$ . To this end, suppose we start with some feasible risk allocation  $\delta_{k(i)}^j$ , for all  $k, j$ , where  $i$  denotes the iteration number. Using this risk allocation, we then solve Problem 2 to get the optimal controller  $(V_{(i)}^*, K_{(i)}^*)$ , which corresponds to the optimal mean trajectory  $\bar{X}_{(i)}^*$  at iteration  $i$ . Next, we construct a new risk allocation  $\delta_{k(i)}^{j'}$  as follows: for all  $k, j$  such that  $\delta_{k(i)}^j$  is active, we keep the corresponding allocation the same, i.e.,  $\delta_{k(i)}^{j'} = \delta_{k(i)}^j$ . However, for all  $k, j$  such that  $\delta_{k(i)}^j$  is inactive we let  $\delta_{k(i)}^{j'} < \delta_{k(i)}^j$ . This corresponds to tightening the constraints. Since this new risk allocation is smaller, and since  $\Phi^{-1}(z)$  is a monotonically increasing function, it follows that  $\Phi^{-1}(1 - \delta_{k(i)}^{j'}) > \Phi^{-1}(1 - \delta_{k(i)}^j)$ . Furthermore, this implies that

$$\begin{aligned} \alpha_j^\top E_k \bar{X}_{(i)}^* &< \beta_j - \|\Sigma_Y^{1/2}(I + \mathcal{B}K_{(i)}^*)^\top E_k^\top \alpha_j\| \Phi^{-1}(1 - \delta_{k(i)}^{j'}) \\ &< \beta_j - \|\Sigma_Y^{1/2}(I + \mathcal{B}K_{(i)}^*)^\top E_k^\top \alpha_j\| \Phi^{-1}(1 - \delta_{k(i)}^j). \end{aligned} \quad (30)$$

The constraint (30) ensures that the optimal solution for  $\delta_{(i)}$  is feasible for  $\delta'_{(i)}$ . Furthermore, since  $\delta_{k(i)}^{j'} < \delta_{k(i)}^j$ , it follows that  $\mathcal{R}(\delta') \subseteq \mathcal{R}(\delta)$ , so the optimal solution for  $\delta_{(i)}$  is also the optimal solution for  $\delta'_{(i)}$  as well, hence  $J^*(\delta') = J^*(\delta)$ .

Next, we construct a new risk allocation  $\delta_{k(i+1)}^j$  from  $\delta_{k(i)}^{j'}$  as follows. For all  $k, j$  such that  $\delta_{k(i)}^{j'}$  is inactive, leave the new risk allocation the same. For all  $k, j$  such that  $\delta_{k(i)}^{j'}$  is active, let  $\delta_{k(i+1)}^j > \delta_{k(i)}^{j'}$ , which corresponds to relaxing the constraints. Following the same logic, Theorem 1 implies that  $J^*(\delta'_{(i)}) \geq J^*(\delta_{(i+1)})$ . Thus, we have laid out an iterative scheme for a sequence of risk allocations  $\{\delta_{(0)}, \delta_{(1)}, \dots, \delta_{(i)}\}$  that continually lowers the optimal cost.

This leads to Algorithm 1 that solves the optimal risk allocation for the CS problem subject to chance constraints. To tighten the inactive constraints in Line 9, the corresponding risk is scaled by a parameter  $0 < \rho < 1$  that weighs the current risk with the *true* risk from that solution. Additionally, to loosen the active constraints in Line 13, the corresponding risk is increased proportionally to the residual risk remaining.

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#### Algorithm 1: Iterative Risk Allocation CS

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1 Input:  $\delta_k^j \leftarrow \Delta/(NM), \epsilon, \rho$ 
2 Output:  $\delta^*, J^*, V^*, K^*$ 
3 while  $|J^* - J_{\text{prev}}^*| > \epsilon$  do
4    $J_{\text{prev}}^* \leftarrow J^*$ 
5   Solve Problem 2 with current  $\delta$  to obtain  $\bar{\delta}$ 
6    $\hat{N} \leftarrow$  number of indices where constraint is active
7   if  $\hat{N} = 0$  or  $\hat{N} = MN$  then
8     break;
9   end
10  foreach  $(k, j)$  such that  $j$ th constraint at  $k$ th time
    step is inactive do
11     $\delta_k^j \leftarrow \rho \delta_k^j + (1 - \rho) \bar{\delta}_k^j$ 
12  end
13   $\delta_{\text{res}} \leftarrow \Delta - \sum_{k=1}^N \sum_{j=1}^M \delta_k^j$ 
14  foreach  $(k, j)$  such that  $j$ th constraint at  $k$ th time
    step is active do
15     $\delta_k^j \leftarrow \delta_k^j + \delta_{\text{res}}/\hat{N}$ 
16  end
17 end

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## IV. SPACECRAFT RENDEZVOUS EXAMPLE

### A. IRA-CS with Polytopic Chance Constraints

In this section, we implement the previous theory of CS with optimal risk allocation to the problem of spacecraft proximity operations in orbit. We consider the problem where one of the spacecraft, called the Deputy, approaches and docks with the second spacecraft, called the Chief, such that in the process, the Deputy remains within the line-of-sight (LOS) of the Chief, defined initially to be the polytopic region shown in Figure 1.

It is assumed that the two spacecraft are in the LVLH frame, that is, a rotating reference frame where the  $z$  axis is oriented in the direction of the center of the Earth, the  $y$  axis negative to the orbit normal, and the  $x$  axis to complete the right hand rule. Moreover, assuming that the Chief is in a circular orbit (at an altitude  $h = 800$  km), the relative dynamics of the motion between the two spacecraft are given by the Clohessy-Wiltshire-Hill Equations [17], which when written in discrete form, are given as

$$x_{k+1} = A_d x_k + B_d u_k + G w_k, \quad (31)$$

where  $A_d = e^{A\Delta t}$ ,  $B_d = \int_0^{\Delta t} e^{A\tau} B \, d\tau$ , and

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3\omega^2 & 0 & 0 & 0 & 2\omega & 0 \\ 0 & 0 & 0 & -2\omega & 0 & 0 \\ 0 & 0 & -\omega^2 & 0 & 0 & 0 \end{bmatrix}, \quad B = m_c^{-1} [0_3, I_3]^\top, \quad (32)$$

where  $m_c = 300$  kg,  $\omega = \sqrt{\mu/R_0^3}$  is the orbital frequency, the state is  $\mathbf{x} := [x, y, z, \dot{x}, \dot{y}, \dot{z}]^\top \in \mathbb{R}^6$  and  $\mathbf{u} := [F_x, F_y, F_z]^\top \in \mathbb{R}^3$  represents the thrust input components to the spacecraft.

We choose the associated noise characteristics as  $G = \text{diag}(10^{-4}, 10^{-4}, 5 \times 10^{-8}, 5 \times 10^{-8})$  [18]. We assume that the initial state mean and covariance are  $\mu_0 = [90, -120, 90, 0, 0, 0]^T$  and  $\Sigma_0 = \text{diag}(10, 10, 10, 1, 1, 1)$ , respectively. We wish to steer the distribution from the above initial state to the final mean  $\mu_f = 0$  with final covariance  $\Sigma_f = \frac{1}{4}\Sigma_0$ , while minimizing the cost function (4) with weight matrices  $Q = \text{diag}(10, 10, 10, 1, 1, 1)$  and  $R = 10^3 I_3$ . We impose the joint probability of failure over the whole horizon to be  $\Delta = 0.03$ , which implies that the probability of violating any state constraint over the whole horizon is less than 3%. The control inputs are bounded as  $\mathbb{P}(\|u_k\|_\infty \leq 30 \text{ N}) \geq 1 - \Delta_u$ , where  $\Delta_u = 0.05$ . Note that these bounds are (*soft*) chance constraints as opposed to hard constraints, and as such can be formulated as second order cone constraints, similar to the state chance constraints.

Lastly, in the iterative risk allocation algorithm, we use a scaling parameter  $\rho_{(i)} = (0.7)(0.98)^i$  in Line 10 of the algorithm to ensure convergence of the method, where  $i$  represents the current iteration. More details may be found in [1]. The SDP in Problem 2 was implemented in MATLAB using YALMIP [19] along with MOSEK [20] to solve the relevant optimization problems.

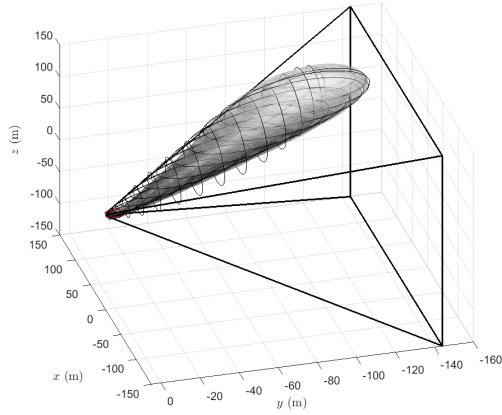


Fig. 1: Optimal trajectories using IRA-CS, with  $3 - \sigma$  covariance ellipsoids.

Figures 1 and 2 show the trajectories with optimal risk allocation, and Figure 2 shows the two-dimensional planar motion. Figure 3 compares the terminal trajectories of CS with a uniform risk allocation with the proposed method. The two solutions look similar and both satisfy the terminal constraints on the mean and the covariance. However, due to the relaxation  $\Sigma_N \leq \Sigma_f$ , the uniform risk allocation scheme leads to more conservative solutions, as shown in Figure 3. The volume of the final covariance ellipsoid,  $V_N \propto \log \det \Sigma_N$  is smaller for the uniform allocation solution compared to the optimal allocation solution (see Table I). In fact, we see that a consequence of optimal risk allocation is that it *maximizes* the final covariance given all the constraints, while still being bounded by  $\Sigma_f$ .

Figure 4 shows the state trajectories and the optimal controls for the polyhedral chance constraints. The control

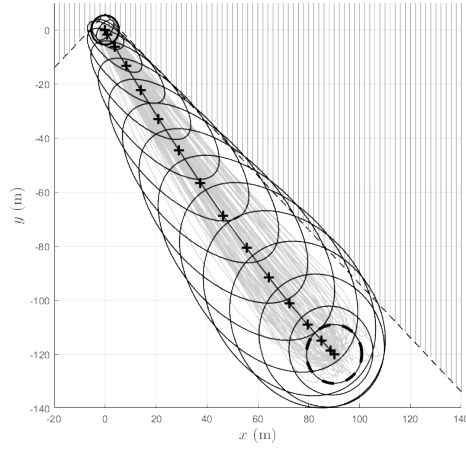


Fig. 2: Optimal planar trajectories using IRA-CS, with  $3 - \sigma$  covariance ellipses.

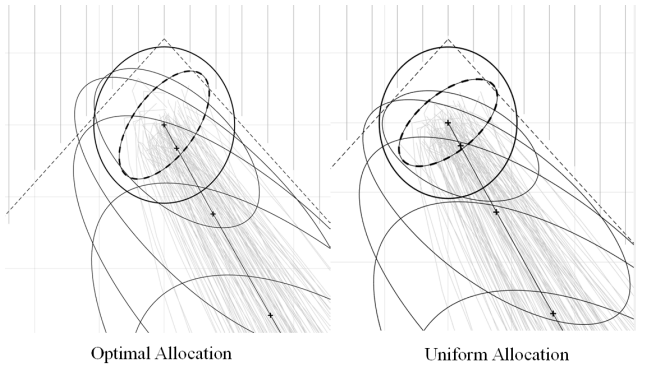


Fig. 3: Comparison of terminal covariance steering using a uniform and the optimal risk allocation.

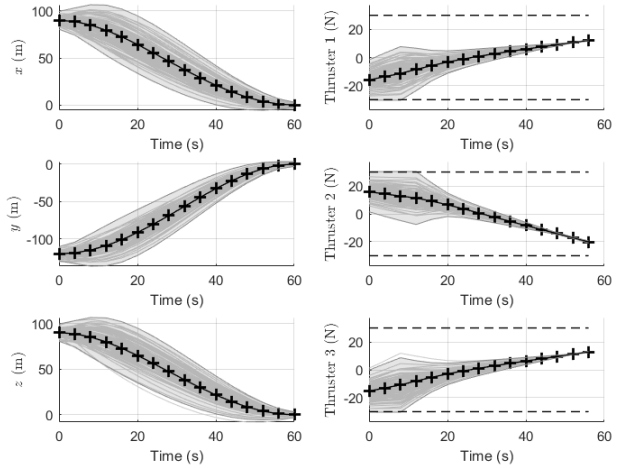


Fig. 4: Trajectories of controlled system and their associated standard deviations using iterative risk allocation.

is almost linear but has a large variance in the first 10 time steps, where it may saturate due to the disturbances.

The iterative risk allocation algorithm is robust in the sense that the algorithm will assign risk proportionately to how close the solution trajectories are to the boundaries of

the state space. Since solutions are close to the right and top boundaries of the allowable LOS region for most of the horizon, the optimal allocation weighs these respective risks more than those of the left and bottom boundaries. Thus, IRA purposefully steers the trajectories away from the boundaries proportionally to the amount of risk that is allocated to violating those respective boundaries. Table I shows the true joint probability of failure, defined as

$$\bar{\Delta} := 1 - \mathbb{P} \left[ \bigwedge_{k=1}^N \bigwedge_{j=1}^M \alpha_j^T E_k X^* \leq \beta_j \right]. \quad (33)$$

It is clear that the uniform risk allocation does not even come close to the desired design of  $\Delta = 0.03$ , while the IRA gives a true probability of failure very close to the desired one.

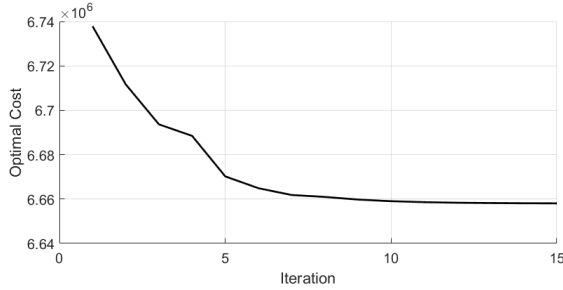


Fig. 5: Optimal cost  $J^*(\delta)$  after every IRA iteration.

TABLE I: Comparison of total true risk and terminal volume between uniform and optimal allocations.

| -              | Uniform | IRA Polyhedron |
|----------------|---------|----------------|
| $\bar{\Delta}$ | 0.0123  | 0.02998        |
| $V_N$          | 0.5546  | 0.6279         |

Finally, we looked at the optimal cost function over each IRA iteration, as in Figure 5. The convergence criterion set in this example is  $\epsilon = 10^{-5}$ , or when all of the constraints are active or inactive, which can be proved in [1] to be a sufficient condition for optimality for Problem 3. We see that indeed (28) holds, and the optimization resulted even in a slight decrease of the objective function, converging in 15 iterations. Thus, the iterative risk allocation algorithm optimizes the risk allocation at each time step without increasing the cost.

## V. CONCLUSION

In this paper, we have incorporated an iterative risk allocation (IRA) strategy to optimize the probability of violating the state constraints at every time step within the covariance steering problem of a linear stochastic system subject to chance constraints. For the covariance steering problem, we showed that employing IRA not only leads to less conservative solutions that are more practical, but also tends to maximize the final covariance. Additionally, the use of IRA in the context of CS with chance constraints results in optimal solutions that have a true risk much closer to the intended design requirements, compared to the use of

a uniform risk allocation. Future work will aim to include “hard” bounds on the input into the problem formulation, using the approach of Kazuhide et al. [11].

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