

CHANCE-CONSTRAINED COVARIANCE CONTROL FOR LOW-THRUST MINIMUM-FUEL TRAJECTORY OPTIMIZATION

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The propellant mass required for a space mission can be greatly reduced by using low-thrust propulsion. Designing low-thrust minimum-fuel trajectories that are robust to disturbances, however, is a challenging optimal control problem. In this paper, we model the spacecraft dynamics as a stochastic system and then solve for both the optimal nominal trajectory and a corresponding feedback law, subject to probabilistic constraints on the thrust commands and on the target state mean and covariance. The optimal control is found by successively linearizing the dynamics and solving for the updated controls as a convex program. The proposed method is demonstrated for a low-thrust interplanetary trajectory from Earth to Mars.

INTRODUCTION

Consider the problem of designing the trajectory for a spacecraft to intercept and rendezvous with a planet or satellite using low-thrust propulsion. Techniques from optimal control theory and numerical optimization can be used to identify the sequence of thrust commands that will bring the spacecraft to its target while using the least amount of propellant. However, as there will be uncertainty in the system that is not considered in this optimization, such as perturbing external accelerations and inaccuracies in the system model, the identified sequence of thrust commands must be supplemented with a feedback control strategy. In this standard construction, we observe that the optimization over the nominal thrust commands is performed independently from the both the model of uncertainty and from the feedback strategy designed to mitigate this uncertainty.

Decoupling the optimization over the nominal control from the closed-loop control has particular implications for low-thrust rendezvous trajectories: since the minimum-fuel thrust magnitude is equal to the maximum allowed value during the final approach to rendezvous, a control margin must be included in the nominal thrust profile to allow for corrective maneuvering.¹⁻³ In general, however, the minimum-fuel guidance system may require some coupling between the nominal trajectory and the feedback strategy in order to most efficiently set the control margins. If the uncertainties are modeled as random variables, then the control margins determine the probability that the required control corrections saturate.

This work seeks to develop a methodology for which the nominal thrust commands are solved in tandem with the feedback strategy, with the objective of improving the overall system performance, while allowing a system designer to explicitly consider constraints in the presence of uncertainty.

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To this end, the uncertain perturbations on the spacecraft trajectory are modeled as a Brownian process, the spacecraft state is considered to be a random vector, and the spacecraft trajectory is considered to be random process. The trajectory optimization problem then becomes a stochastic optimal control problem, which includes the control margin allocation problem via probabilistic (i.e., chance) constraints.

Previous works have addressed low-thrust trajectory design as a stochastic optimal control problem and have presented a solution via stochastic differential dynamic programming (SDDP).^{4,5} The SDDP formulation, which computes the solution to a nonlinear stochastic optimal control problem by successive backward and forward evaluations of the value function and of the dynamics approximated to second order, has been shown to be very successful for a wide variety of problems including low-thrust spacecraft trajectory control.^{6,7} Furthermore, SDDP has been extended to handle state and control chance constraints.⁸

The procedure described in the present paper follows recent works on stochastic control theory wherein the feedback law is designed to explicitly control the distribution of the uncertain state. Indeed, for linear systems — or, as will be done in this paper, for nonlinear systems in a local neighborhood around a trajectory — the nominal control affects the mean of the state distribution while the feedback gain affects the covariance. This method is accordingly referred to in the literature as covariance steering.^{9–14} The present paper extends the method introduced in Ref. [13] to handle constraints on the magnitude of the closed-loop control, which is necessary for modeling the maximum thrust constraint for applications to spacecraft guidance. Similar to SDDP, the nonlinear dynamics are successively approximated about a reference trajectory, but, in contrast to SDDP, the dynamics are only approximated to first order, the state distribution is assumed to be Gaussian, and the linearized dynamics are used to model the evolution of the state covariance. These simplifying approximations allow for the iterative updates to the control law to be solved as a convex program, which can include both chance constraints on the trajectory and constraints on the final state mean and covariance. This approach has the benefit that any convex path constraints on either the state or the control can be added to the problem formulation, and the solution after each iteration is guaranteed to satisfy all of these convex constraints.

In the following sections, the solution method, which we refer to as iterative covariance steering (iCS), is developed for a general nonlinear stochastic system. Following this development, the method is applied first to a discrete-time double integrator system and then to a simplified numerical example for the transfer of a spacecraft from Earth to Mars using low-thrust propulsion, and the resulting properties of the obtained closed-loop trajectory are discussed.

CONSTRAINED STOCHASTIC OPTIMAL CONTROL

Consider the nonlinear stochastic system

$$dx = f(x, u, t)dt + g(x, u)dw, \quad (1)$$

on the time interval $[t_0, t_f]$, where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control, and $(w(t))$ is an n -dimensional Brownian motion. The state at the initial time $x(t_0) = x_0$ is Gaussian distributed with known mean $\bar{x}_0$ and covariance P_0 :

$$x(t_0) = x_0 \sim \mathcal{N}(\bar{x}_0, P_0). \quad (2)$$

In the context of stochastic systems, hard state and control constraints are difficult to impose because the noise can be unbounded, as is the case with Brownian noise, and instead chance constraints are

used to deal with this problem by allowing a small, but finite, probability of violating the constraints. The control magnitude is therefore constrained *in probability* to be no more than a given positive constant ρ , as given by

$$\mathbb{P}(\|u(t)\| \leq \rho) \geq 1 - \beta, \quad (3)$$

where β is a prescribed maximum probability of control saturation. Additionally, we impose the following joint chance constraint that limits the probability of state violation to be less than a pre-specified threshold, i.e., the state is constrained to lie within a convex polytope χ as in

$$\mathbb{P}(x(t) \in \chi) \geq 1 - \Gamma, \quad \chi = \bigcap_{i=1}^{N_\chi} \{x : a_i^\top x \leq b_i\}, \quad (4)$$

where Γ is the prescribed maximum probability of constraint violation, and where the n -dimensional vectors (a_i) and scalars (b_i) characterize the N_χ half-planes which define the convex polytope χ ; see Figure 1. The constraints (3) and (4) are referred to as *chance constraints*, and optimization with respect to these constraints is referred to as chance-constrained optimization.¹⁵ Finally, we require that the mean and the covariance of the state satisfy the terminal constraints

$$\mathbb{E}(x_{t_f}) = \bar{x}_f, \quad (5a)$$

$$\text{Cov}(x_{t_f}) \leq P_{x_f}, \quad (5b)$$

where the final state mean $\bar{x}_f$ is a given vector and the maximum final state covariance P_{x_f} is a given positive-definite matrix.

The control is assumed to follow the causal state history feedback law

$$u(t_k^p) = \sum_{i=0}^k K_{k,i} (x(t_i^p) - \mathbb{E}(x(t_i^p))) + v_k, \quad (6)$$

with the zero-order hold (ZOH) discretization

$$u(t) = u(t_k^p) \quad \text{for } t \in [t_k^p, t_{k+1}^p), \quad k = 0, \dots, N-1, \quad (7)$$

where $(K_{k,i})$ are feedback gain matrices, (v_k) are feed-forward controls, and where the sequence of times (t_k^p) is a partition of the interval $[t_0, t_f]$, that is,

$$t_0 = t_0^p < \dots < t_N^p = t_f. \quad (8)$$

Henceforth, we will write the mean state as $\bar{x} = \mathbb{E}(x)$ and the deviation from the mean as $\tilde{x} = x - \mathbb{E}(x)$ (similarly, $\bar{u} = \mathbb{E}(u)$ and $\tilde{u} = u - \mathbb{E}(u)$).

Our objective is to find the feedback control law, which is parameterized by the gain matrices $(K_{k,i})$ and the feed-forward controls (v_k) , that minimizes the cost function

$$J((K_{k,i}), (v_k)) = \int_{t_0}^{t_f} \|\bar{u}(t)\| dt + \mathbb{E} \left(\int_{t_0}^{t_f} \tilde{u}^\top(t) R(t) \tilde{u}(t) + \tilde{x}^\top(t) Q(t) \tilde{x}(t) dt \right), \quad (9)$$

where $R(t)$ is positive definite and $Q(t)$ is positive semi-definite, subject to the control magnitude constraint (3), the path constraint (4), and the final state distribution constraints (5). The first term of the cost function (9) is the mean-fuel cost and the second term is a penalty on the control and state covariances.

Note that in this formulation, the final state error is constrained by (5b) rather than penalized in the cost.

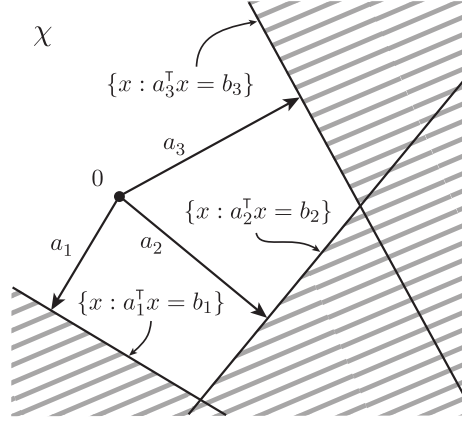


Figure 1. State chance-constrained region χ .

SOLUTION BY ITERATIVE COVARIANCE STEERING

Approximation by a Linear Discrete-Time System

Suppose we are provided with a reference control sequence $(\hat{u}(t))$ over the time interval $[t_0, t_f]$. The corresponding reference state process $(\hat{x}(t))$ is obtained by solving the deterministic system

$$\dot{\hat{x}} = f(\hat{x}, \hat{u}, t), \quad (10)$$

with the initial condition $\hat{x}(t_0) = \bar{x}_0$. The nonlinear system (1) is then linearized along the trajectory $(\hat{u}(t), \hat{x}(t))$ to obtain the linear stochastic system

$$dx = (A(t)x + B(t)u + c(t))dt + G(t)dw, \quad (11)$$

where the system matrices are partial derivatives evaluated along the reference as

$$A(t) = \frac{\partial f}{\partial x}(\hat{x}(t), \hat{u}(t)), \quad B(t) = \frac{\partial f}{\partial u}(\hat{x}(t), \hat{u}(t)), \quad G(t) = g(\hat{x}(t), \hat{u}(t)), \quad (12)$$

$$c(t) = f(\hat{x}(t), \hat{u}(t), t) - A(t)\hat{x}(t) - B(t)\hat{u}(t). \quad (13)$$

Let $x_k = x(t_k^p)$ and $u_k = u(t_k^p)$ denote the state and control vectors at the discrete times (t_k^p) . Since, by construction, the control $u(t)$ is constant on the time intervals $[t_k^p, t_{k+1}^p)$, the discrete-time states x_k satisfy the difference equation

$$x_{k+1} = A_k x_k + B_k u_k + c_k + G_k w_k, \quad (14)$$

for $k = 0, \dots, N-1$, where (w_k) is a sequence of independent and identically distributed standard, n -dimensional, Gaussian random vectors. See the appendix for details for how this discretization is obtained. A computationally efficient procedure to approximate the matrices A_k , B_k , G_k and the vector c_k is also provided in the appendix.

Block Matrix Formulation

Let X be a column vector constructed by stacking the states x_k for $k = 0, 1, \dots, N$, and, similarly, let U and W be the column vectors constructed by stacking the controls u_k and disturbances w_k for $k = 0, 1, \dots, N - 1$. The difference equation (14) may be equivalently written in block-matrix notation as

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} I \\ A_0 \\ A_1 A_0 \\ \vdots \end{bmatrix} x_0 + \begin{bmatrix} 0 & 0 \\ B_0 & 0 \\ A_1 B_0 & B_1 \\ & \ddots \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ I & 0 \\ A_1 I & I \\ & \ddots \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ G_0 & 0 \\ A_1 G_0 & G_1 \\ & \ddots \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ \vdots \end{bmatrix}, \quad (15)$$

and therefore, for appropriately constructed block matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{G}$ as in (15), and with the column vector $\mathbf{C}$ resulting from the matrix multiplication involving the (c_k) terms in (15), the filtered state process can be written as the linear matrix equation

$$X = \mathbf{A}x_0 + \mathbf{B}U + \mathbf{C} + \mathbf{G}W. \quad (16)$$

See Refs. [11, 12] for details on this construction. Furthermore, let

$$\bar{X} = \mathbb{E}(X), \quad \tilde{X} = X - \bar{X}, \quad \bar{U} = \mathbb{E}(U), \quad \tilde{U} = U - \bar{U}. \quad (17)$$

The cost function (9) is given in terms of the state X and control U by

$$J((K_{k,i}), (v_k)) = \frac{t_f - t_0}{N} \left(\sum_{k=0}^{N-1} \|E_k^u \tilde{U}\| + \mathbb{E}(\tilde{X}^\top \mathbf{Q} \tilde{X} + \tilde{U}^\top \mathbf{R} \tilde{U}) \right), \quad (18)$$

where $\mathbf{Q} = \text{blkdiag}(Q_0, \dots, Q_{N-1}, 0) \geq 0$, $\mathbf{R} = \text{blkdiag}(R_0, \dots, R_{N-1}) > 0$, and E_k^u is a matrix defined so that $E_k^u U = u_k$. Similarly, we will define E_k to be the matrix such that $E_k X = x_k$.

We remark that the expression for the cost (18) is an approximation, since, despite the control being piecewise constant, the state is not constant. The control cost terms in (18), however, are exactly equal to the corresponding terms in the original cost (9).

Convex Formulation

Letting the block lower-triangular matrix $\mathbf{K}$ be given by

$$\mathbf{K} = \begin{bmatrix} K_{0,0} & 0 & \cdots & 0 \\ K_{1,0} & K_{1,1} & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ K_{N-1,0} & K_{N-1,1} & K_{N-1,2} & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{Nm \times (N+1)n}, \quad (19)$$

and letting $V \in \mathbb{R}^{Nm}$ be the vector obtained by stacking the feed-forward controls (v_k) , the control law (6) is given in block-matrix notation as

$$U = \mathbf{K}\tilde{X} + V. \quad (20)$$

The mean control is therefore $\bar{U} = V$ and the control deviation is $\tilde{U} = \mathbf{K}\tilde{X}$. The mean state process is obtained by substituting the control expression (20) into the state equation (16) and taking the expectation, which yields

$$\bar{X} = \mathbf{A}\bar{x}_0 + \mathbf{B}V + C. \quad (21)$$

Similarly, we obtain the state deviation process

$$\tilde{X} = \mathbf{A}\tilde{x}_0 + \mathbf{B}\mathbf{K}\tilde{X} + \mathbf{G}W, \quad (22)$$

which, after solving for $\tilde{X}$, we write as

$$\tilde{X} = (\mathbf{I} - \mathbf{B}\mathbf{K})^{-1}(\mathbf{A}\tilde{x}_0 + \mathbf{G}W). \quad (23)$$

Since $\mathbf{K}$ is block lower-triangular and $\mathbf{B}$ is strictly block lower-triangular, it follows that the matrix $\mathbf{I} - \mathbf{B}\mathbf{K}$ is invertible. Following Refs. [10, 12, 16], we let

$$\mathbf{L} = \mathbf{K}(\mathbf{I} - \mathbf{B}\mathbf{K})^{-1} \in \mathbb{R}^{Nm \times (N+1)n} \quad (24)$$

be a new decision variable. It can be shown that $\mathbf{L}$ is block lower-triangular and satisfies the equality

$$\mathbf{I} + \mathbf{B}\mathbf{L} = (\mathbf{I} - \mathbf{B}\mathbf{K})^{-1}. \quad (25)$$

Furthermore, the feedback gain matrix $\mathbf{K}$ is provided as a function of $\mathbf{L}$ as

$$\mathbf{K} = \mathbf{L}(\mathbf{I} + \mathbf{B}\mathbf{L})^{-1}, \quad (26)$$

where, following the same logic as above, the matrix $\mathbf{I} + \mathbf{B}\mathbf{L}$ is invertible since $\mathbf{B}$ and $\mathbf{L}$ are both block lower-triangular.

Substituting the expression (25) into both the state deviation dynamics (23) and into the control deviation equation $\tilde{U} = \mathbf{K}\tilde{X}$, we obtain the state and control deviation sequences as

$$\tilde{X} = (\mathbf{I} + \mathbf{B}\mathbf{L})(\mathbf{A}\tilde{x}_0 + \mathbf{G}W), \quad (27)$$

$$\tilde{U} = \mathbf{L}(\mathbf{A}\tilde{x}_0 + \mathbf{G}W). \quad (28)$$

Since the noise process (w_k) is both white and independent from initial state deviation $\tilde{x}_0$, the vector $\mathbf{A}\tilde{x}_0 + \mathbf{G}W$ has covariance

$$\mathbf{S} = \text{Cov}(\mathbf{A}\tilde{x}_0 + \mathbf{G}W) = \mathbf{A}P_0\mathbf{A}^\top + \mathbf{G}\mathbf{G}^\top. \quad (29)$$

The covariances of the state process X and the control process U are therefore given as

$$\mathbf{P}_X = \text{Cov}(X) = (\mathbf{I} + \mathbf{B}\mathbf{L})\mathbf{S}(\mathbf{I} + \mathbf{B}\mathbf{L})^\top, \quad (30)$$

$$\mathbf{P}_U = \text{Cov}(U) = \mathbf{L}\mathbf{S}\mathbf{L}^\top. \quad (31)$$

The cost (18) may now be rewritten in terms of the block-covariance matrices $\mathbf{P}_X$ and $\mathbf{P}_U$ as a function of the feedforward control sequence V and the gain matrix $\mathbf{K}$ as

$$J(\mathbf{L}, V) = \frac{t_f - t_0}{N} \left(\sum_{k=0}^{N-1} \|E_k^u V\| + \text{tr } \mathbf{P}_X \mathbf{Q} + \text{tr } \mathbf{P}_U \mathbf{R} \right). \quad (32)$$

Substituting the control and state covariance expressions (30) and (31) into the cost function (32), and then simplifying, we obtain the cost function

$$J(\mathbf{L}, V) = \frac{t_f - t_0}{N} \left(\sum_{k=0}^{N-1} \|E_k^u V\| + \text{tr} [(I + \mathbf{B}\mathbf{L})^\top \mathbf{Q} (I + \mathbf{B}\mathbf{L}) + \mathbf{L}^\top \mathbf{R} \mathbf{L}] \mathbf{S} \right). \quad (33)$$

Since the matrix $\mathbf{Q}$ is positive semi-definite and the matrix $\mathbf{R}$ is positive definite, it follows that this cost is convex in $\mathbf{L}$ and V .

Terminal State Constraints

The constraints (5) on the terminal state mean and covariance values are expressed in terms of the block matrix formulation as

$$\bar{x}_N = E_N(\mathbf{A}\bar{x}_0 + \mathbf{B}V + C) = \bar{x}_f, \quad (34)$$

and

$$\text{Cov}(x_N) = E_N(I + \mathbf{B}\mathbf{L})\mathbf{S}(I + \mathbf{B}\mathbf{L})^\top E_N^\top \leq P_{x_f}. \quad (35)$$

The final covariance constraint (35) can be rewritten as the linear matrix inequality (LMI)¹⁰

$$\begin{bmatrix} P_{x_f} & E_N(I + \mathbf{B}\mathbf{L})\mathbf{S}^{1/2} \\ \mathbf{S}^{1/2}(I + \mathbf{B}\mathbf{L})^\top E_N^\top & I \end{bmatrix} \geq 0, \quad (36)$$

or, equivalently, as the second-order cone constraint¹²

$$\|\mathbf{S}^{1/2}(I + \mathbf{B}\mathbf{L})^\top E_N^\top P_{x_f}^{-1/2}\| - 1 \leq 0, \quad (37)$$

where $\mathbf{S}^{1/2}$ is a matrix satisfying $(\mathbf{S}^{1/2})^\top \mathbf{S}^{1/2} = \mathbf{S}$ (and similarly for $P_{x_f}^{1/2}$).

Path Constraints

The chance constraints (4) on the state, enforced at the discrete times (t_k^p) , are given in terms of the state process X as

$$\mathbb{P} \left(\bigwedge_{k=1}^N \bigwedge_{j=1}^{N_x} a_j^\top E_k X > b_k \right) \leq \Gamma. \quad (38)$$

We remark that in this work, since we only compute the state at discrete times, the constraints (4) are only considered at discrete times, and so constraint satisfaction for other times cannot be guaranteed. However, following Ref. [17], one can extend the present formulation to ensure constraint satisfaction at all times.

Using Boole's Inequality,^{18,19} one can decompose the *joint* chance constraint (38) into the *individual* chance constraints

$$\mathbb{P}\left(a_j^\top E_k X \leq b_j\right) \geq 1 - \gamma_k^j, \quad k = 1, \dots, N, \quad j = 1, \dots, N_\chi, \quad (39)$$

for any set of scalars (γ_k^j) that satisfy the inequality

$$\sum_{k=1}^N \sum_{j=1}^{N_\chi} \gamma_k^j \leq \Gamma. \quad (40)$$

That is, if both (39) and (40) hold, then the original constraint (38) holds. Each individual value γ_k^j represents the probability of violating the j th constraint at time step k .

The random variable $a_j^\top E_k X$ is Gaussian with mean $a_j^\top E_k \bar{X}$ and covariance $a_j^\top E_k \mathbf{P}_X E_k^\top a_j$, and therefore the inequality (39) can be equivalently written as

$$\Phi\left(\frac{b_j - a_j^\top E_k \bar{X}}{\sqrt{a_j^\top E_k \mathbf{P}_X E_k^\top a_j}}\right) \geq 1 - \gamma_k^j, \quad k = 1, \dots, N, \quad j = 1, \dots, N_\chi, \quad (41)$$

where $\Phi(\cdot)$ denotes the cumulative distribution function of the standard normal distribution. Simplifying (41) and rearranging terms yields the second-order cone constraint

$$a_j^\top E_k (\mathbf{A}\bar{x}_0 + \mathbf{B}\mathbf{V} + \mathbf{C}) + \|\mathbf{S}^{1/2}(\mathbf{I} + \mathbf{B}\mathbf{L})^\top E_k^\top a_j\| \Phi^{-1}(1 - \gamma_k^j) \leq b_j, \quad (42)$$

where Φ^{-1} denotes the inverse normal cumulative distribution function.

Thrust Magnitude Constraint

Finally, we consider the constraint (3) on the probability that the control magnitude exceeds a prescribed maximum value. The following results will be useful for developing a relaxed convex formulation of this control chance constraint.

Theorem 1. *Let $y \sim \mathcal{N}(\mu, \Sigma)$ be an m -dimensional random vector for $m \geq 1$, let $\sigma = \sqrt{\lambda_{\max}(\Sigma)}$, let $\rho > 0$, and let $1 > \beta > 0$. Then*

$$\|\mu\| + \sigma \left(\sqrt{2 \log \frac{1}{\beta}} + \sqrt{m} \right) \leq \rho \implies \mathbb{P}(\|y\| \leq \rho) \geq 1 - \beta. \quad (43)$$

Proof. See the appendix. □

In the case that control dimension is $m = 1$ or $m = 2$, the following results provide a sharper bound on the allowable controls.

Lemma 2. *Let $y \sim \mathcal{N}(0, \Sigma)$ be an m -dimensional random vector where $m = 1$ or $m = 2$, let $\sigma = \sqrt{\lambda_{\max}(\Sigma)}$, and let $\rho > 0$. Then*

$$\mathbb{P}(\|y\| \leq \rho) \geq 1 - e^{-\rho^2/2\sigma^2}. \quad (44)$$

Proof. See the appendix. □

Theorem 3. Let $y \sim \mathcal{N}(\mu, \Sigma)$ be an m -dimensional random vector where $m = 1$ or $m = 2$, let $\sigma = \sqrt{\lambda_{\max}(\Sigma)}$, let $\rho > 0$, and let $1 > \beta > 0$. Then

$$\|\mu\| + \sigma \sqrt{2 \log \frac{1}{\beta}} \leq \rho \implies \mathbb{P}(\|y\| \leq \rho) \geq 1 - \beta. \quad (45)$$

Proof. See the appendix. □

With these results, we are now prepared to derive a convex relaxation of the control chance constraint (3). First, recall that the norm of a matrix M , defined as

$$\|M\| = \sup_{\|x\|=1} \|Mx\|, \quad (46)$$

satisfies the equality

$$\|M\|^2 = \lambda_{\max}(M^\top M). \quad (47)$$

Since the control $u_k = E_k^u U$ is normally distributed with mean $E_k^u V$ and covariance

$$E_k^u P_U E_k^{u\top} = E_k^u L S L^\top E_k^{u\top}, \quad (48)$$

it follows, by substituting $M = S^{1/2} L^\top E_k^{u\top}$ and $\sigma_{u_k}^2 = \lambda_{\max}(\text{Cov}(u_k))$ into (47), that

$$\sigma_{u_k} = \sqrt{\lambda_{\max}(E_k^u L S L^\top E_k^{u\top})} = \|S^{1/2} L^\top E_k^{u\top}\|. \quad (49)$$

Suppose that the control dimension is $m = 1$ or $m = 2$. In this case, it follows from Theorem 3, with (49) substituted into (45), that if

$$\|E_k^u V\| + \|S^{1/2} L^\top E_k^{u\top}\| \sqrt{2 \log \frac{1}{\beta}} \leq \rho, \quad (50)$$

is satisfied at time t_k^p , then the original constraint (3) is also satisfied at the time t_k^p . Since the control is constant on the intervals $[t_k^p, t_{k+1}^p)$, it follows that if (50) holds for $k = 0, \dots, N-1$, then (3) holds for all $t \in [t_0, t_f]$.

In the case that $m \geq 3$, the same result follows from Theorem 1, except that now the constraint (50) is replaced with

$$\|E_k^u V\| + \|S^{1/2} L^\top E_k^{u\top}\| \left(\sqrt{2 \log \frac{1}{\beta}} + \sqrt{m} \right) \leq \rho. \quad (51)$$

We remark that this bound for $m \geq 3$ is conservative — in practice, the bound (50) may reasonably be substituted for (51) with a corresponding decrease to β for approximate, but less conservative, results.

Iterative Covariance Steering Algorithm

In summary, the original continuous-time, stochastic optimal control problem has been approximated in the neighborhood of a provided reference trajectory as the following deterministic convex optimization problem:

$$\text{minimize (33) subject to (34), (37), (42), (50) over } L \in \mathcal{L}, V \in \mathbb{R}^{Nm}, \quad (52)$$

where $\mathcal{L}$ denotes the class of real-valued, block lower-triangular matrices of dimension $Nm \times (N+1)n$. After solving this problem, the gain matrices $(K_{k,i})$ are computed from $\mathbf{L}$ per (26) and the feedforward controls v_k and expected states $\mathbb{E}(x(t_k^p))$ are given in V and $\bar{X}$, respectively, thus obtaining a feedback law of the desired form (6).

Since the decision variable $\mathbf{L}$ is block lower-triangular, this problem scales as $\mathcal{O}(N^2 n_x n_u)$, which makes the solution computationally challenging for large values of N . For this reason, we will often instead only optimize over a subset of $\mathcal{L}$. Indeed, restricting $\mathbf{L}$ to be block-diagonal reduces the problem to scale as $\mathcal{O}(N n_x n_u)$.^{11,12} More generally, a select number of diagonals may be optimized over, which strikes a balance between the full block lower-triangular structure and a block-diagonal structure.¹⁶ In applications to spacecraft trajectory design, control decisions may be reasonably spaced on intervals of the order of days or weeks, which reduces the horizon number N .

The solution to the original optimal control problem is obtained by successively solving the approximate convex problem, where the optimal control obtained during each iteration is used to propagate the nonlinear system, as in (10), in order to obtain the reference trajectory for the following iteration.¹³ In the context of deterministic optimal control, this method is often referred to as successive linearization or successive convexification;^{20,21} in the absence of constraints, this procedure reduces to iterative linear quadratic Gaussian control (iLQG).²²

The resulting iterative solution procedure, which we refer to as iterative covariance steering (iCS),¹³ is summarized in Algorithm 1.

NUMERICAL EXAMPLES

Double Integrator

Before applying the developed theory to low-thrust trajectory design, we first consider a simple double integrator example described by the discrete-time system

$$x_{k+1} = Ax_k + Bu_k + Gw_k, \quad (53)$$

with the system matrices

$$A = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} \Delta t^2/2 & 0 \\ 0 & \Delta t^2/2 \\ \Delta t & 0 \\ 0 & \Delta t \end{bmatrix}, \quad G = 0.1 \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (54)$$

and time step size $\Delta t = 0.25$. The final time is $t_f = 5$, and thus the horizon size is $N = 20$. The initial state distribution is given by

$$\bar{x}_0 = [2 \quad 4 \quad 3 \quad 2]^\top, \quad P_0 = \text{diag}(0.1, 0.1, 0.02, 0.02), \quad (55)$$

the final state mean must be equal to

$$\bar{x}_f = [8 \quad 2 \quad 0 \quad 0]^\top, \quad (56)$$

and the final state covariance must be less than

$$P_{x_f} = \text{diag}(0.06, 0.06, 0.006, 0.006). \quad (57)$$

Algorithm 1: Iterative covariance steering (iCS)

Input: System model f and g , initial state mean $\bar{x}_0$, initial state covariance P_0 , target state mean $\bar{x}_f$, maximum final state covariance P_{x_f} , times t_0 and t_f , initial control guess $(\hat{u}^1(t) : t \in [t_0, t_f])$, time partition $(t_k^p)_{k=0}^N$, convergence tolerance tol , maximum control magnitude ρ , maximum control saturation probability β , trajectory chance constraints $((a_i)_{i=1}^{N_\chi}, (b_i)_{i=1}^{N_\chi}, \Gamma)$, maximum iterations $i_{\max}$.

Output: Feed-forward controls (v_k) , feedback gains $(K_{k,i})$, and mean states $(\bar{x}(t_k^p))$ for control law (6).

```
1  $j \leftarrow 1$ ;  
2 while true do  
3   Integrate nominal system (10) with reference control  $\hat{u}^j$  and initial condition  
    $\hat{x}^j(t_0) = \bar{x}_0$  to obtain reference trajectory  $(\hat{x}^j(t) : t \in [t_0, t_f])$ ;  
4   Linearize about  $(\hat{u}^j, \hat{x}^j)$  as in (11);  
5   Discretize with respect to partition  $(t_k^p)_{k=0}^N$  and construct block matrix system (16);  
6   Solve convex program (52) to obtain  $\mathbf{L}$ ,  $V$ , and  $\bar{X}$ ;  
7   Obtain  $(v_k)$  and  $(\bar{x}_k)$  from  $V$  and  $\bar{X}$ ;  
8   Obtain  $(K_{k,i})$  from  $\mathbf{L}$  per (26);  
9   if  $\max_k \|\bar{x}_k - \hat{x}_k^i\| \leq tol$  then  
10    return  $(v_k)$ ,  $(K_{k,i})$ , and  $(\bar{x}(t_k^p)) = (\bar{x}_k)$ ;  
11  else  
12    if  $i = i_{\max}$  then  
13      Set flag for convergence not met;  
14      return  $(v_k)$ ,  $(K_{k,i})$ , and  $(\bar{x}(t_k^p)) = (\bar{x}_k)$ ;  
15    else  
16      Set  $\hat{u}^{j+1}$  to be piecewise-constant with  $\hat{u}^{j+1}(t) = v_k$  for  $t \in [t_k^p, t_{k+1}^p)$ ;  
17       $j \leftarrow j + 1$ ;  
18    end  
19  end  
20 end
```

The control $u_k \in \mathbb{R}^2$ is constrained to be inside a disk of radius $\rho = 2$ with a maximum probability of constraint violation of $\beta = 1 - 0.9997$. The running cost matrices are constant for all k :

$$Q_k = I \times 10^{-2}, \quad R_k = I. \quad (58)$$

Finally, the state is required to remain in the region χ with a probability of at least 99.97% (i.e., $\Gamma = 1 - 0.9997$), where χ is defined as in (4) by

$$a_1 = [1 \quad 1 \quad 0 \quad 0]^\top, \quad b_1 = 12.75, \quad a_2 = [1 \quad 0.1 \quad 0 \quad 0]^\top, \quad b_2 = 8.75. \quad (59)$$

The individual chance constraints are defined by assigning a uniform risk: $\gamma_k^j = \Gamma/(2N)$ for all $k = 1, \dots, N$ and $j = 1, 2$. The convex program (52), with $\mathbf{L}$ restricted to be block-diagonal, was solved using YALMIP²³ with MOSEK.²⁴ The computational time was 2.82 s.

The resulting closed-loop trajectory is shown in Figure 2. The initial covariance is indicated by a dashed 3σ ellipse, the maximum final covariance is indicated by a weighted, solid 3σ ellipse, and

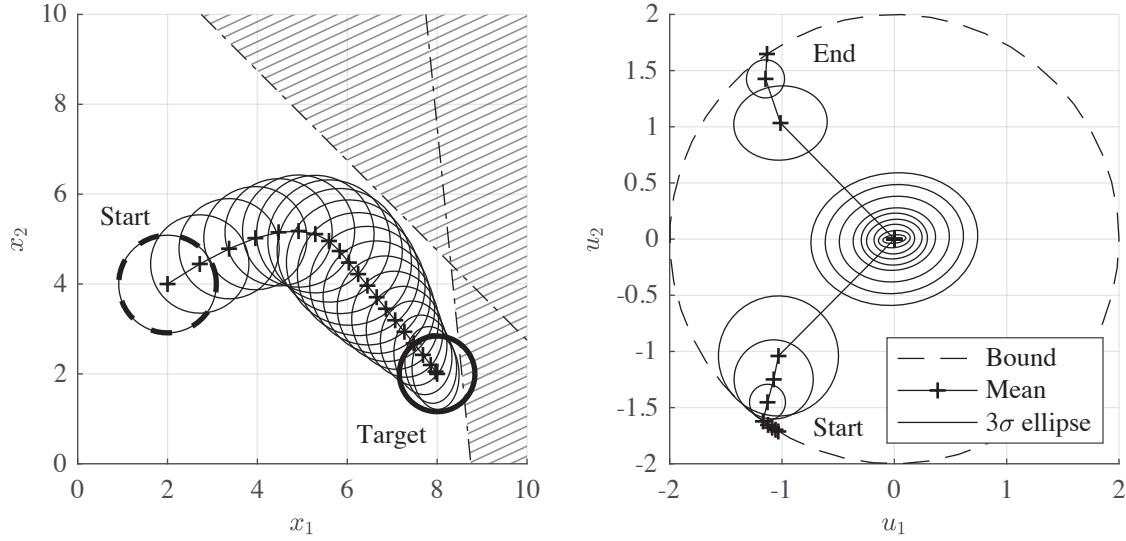


Figure 2. Position and control covariance evolution for the closed-loop double integrator system.

the complement of the chance constrained region χ^c is indicated by diagonal lines. At the start of the trajectory the feedback gains are balanced with the feedforward control so that the state distribution both shrinks and shifts to meet the chance constraint, and the mean of the control is distanced from the limiting magnitude accordingly. The feedback gain is increased while the nominal control is equal to zero in order to shrink the state covariance to meet the endpoint constraint, and then the feedback and feedforward controls are again balanced during the final segment of the trajectory.

Low-Thrust Trajectory Design

In this section, we apply the previously developed iCS method to the problem of guiding a spacecraft on an Earth-to-Mars transfer with a low-thrust engine. For simplicity, the orbits of Earth and Mars are assumed to be co-planar and the spacecraft mass-change dynamics are neglected. The position $r \in \mathbb{R}^2$ and velocity $v \in \mathbb{R}^2$ of the spacecraft are given in a Sun-centered inertial frame, and external perturbing accelerations are modeled as a two-dimensional Brownian motion ($w(t)$). At the initial time t_0 , the spacecraft state $x = (r, v)$ is assumed to be normally distributed as

$$x(t_0) = x_0 \sim \mathcal{N}(\bar{x}_0, P_0), \quad (60)$$

where the vector $\bar{x}_0 = (\bar{r}_0, \bar{v}_0)$ and the positive semi-definite matrix P_0 are both fixed and known. From this initial time, the spacecraft trajectory is described by the SDE

$$d \begin{bmatrix} r \\ v \end{bmatrix} = \begin{bmatrix} v \\ -\mu r / \|r\|^3 + u \end{bmatrix} dt + \begin{bmatrix} 0 \\ \sigma I \end{bmatrix} dw, \quad (61)$$

where μ is the gravitational parameter of the Sun, σ is a positive scalar indicating the disturbance intensity, and $u \in \mathbb{R}^2$ is the commanded thrust acceleration, which is constrained in magnitude by

$$\|u\| \leq u_{\max}. \quad (62)$$

The initial state covariance is given as

$$P_0 = \text{diag}(10^2 \text{ km}^2, 10^2 \text{ km}^2, 0.1^2 \text{ km}^2/\text{s}^2, 0.1^2 \text{ km}^2/\text{s}^2), \quad (63)$$

Table 1. Parameters for Earth-to-Mars transfer example⁴

Property	Value	Unit	Property	Value	Unit
$\bar{r}_0$	$(-0.94052, -0.34502)$	AU	$t_f - t_0$	348.79	days
$\bar{v}_0$	$(9.7746, -28.078)$	km/s	$u_{\max}$	1×10^{-3}	m/s ²
$\bar{r}_f$	$(-1.1543, 1.1829)$	AU	σ	$0.2 \times u_{\max}$	-
$\bar{v}_f$	$(-16.427, -14.861)$	km/s	μ	1.3271×10^{20}	m ³ /s ²
			N	40	-

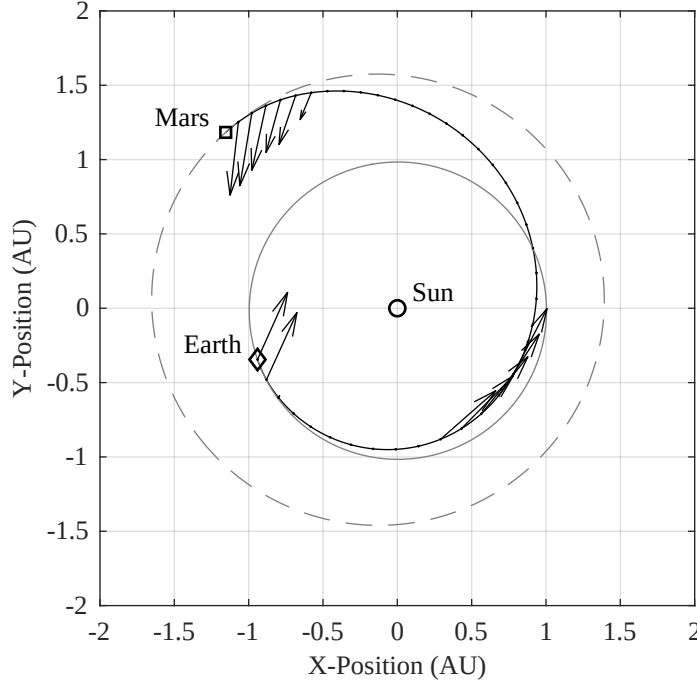


Figure 3. Nominal trajectory and controls with discrete times indicated as dots along the trajectory.

and the final covariance is constrained by (5b) to be less than

$$P_{x_f} = \text{diag}(0.0447 \text{ AU}^2, 0.0447 \text{ AU}^2, 0.3336 \text{ AU}^2/\text{day}^2, 0.3336 \text{ AU}^2/\text{day}^2) \times 10^{-4}. \quad (64)$$

The running cost weights were set as:

$$Q = \text{diag}(0.1 \text{ AU}^{-2}, 0.1 \text{ AU}^{-2}, 0.001 \text{ day}^2/\text{AU}^2, 0.001 \text{ day}^2/\text{AU}^2), \quad (65)$$

$$R = 0.1 I \text{ day}^2/\text{AU}. \quad (66)$$

Following the initial covariance provided in km and km/s, the problem was converted into astronomical units and days, and hence the final covariance and running costs are provided in these units. The remainder of the problem parameters are listed in Table 1. We remark that these parameters are approximately equal to the parameters from the numerical example presented in Ref. [4]. The disturbance intensity, final covariance, and running cost weights, however, are arbitrarily set for the purposes of numerical demonstration.

The initial control guess $\hat{u}^1(t)$ was obtained as the solution to the nominal optimal control problem using an existing solver with a fixed margin on the maximum throttle *. The algorithm ran for five iterations, and the resulting nominal trajectory is shown in Figure 3. The nominal control with Monte Carlo samples is shown in Figure 4 and the evolution of the position covariance for both open and closed-loop control simulations is shown in Figure 5. The final state covariance is

$$P_x(t_f) = \begin{bmatrix} 0.4468 & 0.0000 & 0.0128 & 0.0078 \\ 0.0000 & 0.4468 & 0.0051 & 0.0102 \\ 0.0128 & 0.0051 & 0.0022 & 0.0014 \\ 0.0078 & 0.0102 & 0.0014 & 0.0011 \end{bmatrix} \times 10^{-5}, \quad (67)$$

in units of AU for position values and AU/day for velocity values, which is less than the maximum allowed covariance (64).

Due to the chance constraint (3), the controller plans for more aggressive feedback when the optimal open-loop control is not near the maximum value. Furthermore, as we can see in Figure 4, the nominal control does not immediately increase to its maximum value during the final approach segment as would be the case for an optimal deterministic controller. Instead, the nominal control is set so as to reserve a margin for the corrective actions that are necessary to meet the final covariance constraint. If the final covariance constraint is tightened then we expect that a larger margin will be left on the nominal control. This behavior demonstrates the benefit of coupling the design of the reference trajectory to the feedback gains.

CONCLUSION

In this paper, a method was developed to solve chance-constrained stochastic optimal control problems by successively solving approximate convex subproblems. The formulation includes a maximum control magnitude chance constraint, which makes the method suitable for application to low-thrust spacecraft trajectory design.

By formulating the trajectory optimization as a stochastic optimal control problem, the nominal trajectory was obtained in conjunction with a corresponding feedback law in such a way that the combined closed-loop system satisfies certain chance constraints. In particular, for the case of low-thrust spacecraft trajectory optimization, it was shown that the stochastic optimal control approach implicitly includes optimization over the nominal thrust margin. In the numerical Earth-to-Mars transfer example, the control margin was non-constant during the final approach to Mars, which allowed for the spacecraft to make necessary course corrections. While not demonstrated in the transfer example, the proposed method also allows for the state and control to be constrained within a convex polytope with a prescribed probability — an addition that may be valuable to enforce pointing or maximum radius constraints.

In contrast to SDDP and other stochastic control methods, the proposed method approximates the dynamics to first order and assumes that the state is Gaussian distributed. This allows for each optimization to be solved as a convex program, which, in turn, allows for the inclusion of state and control constraints as well as providing guarantees on each iterative solution. On the other hand, many systems are not adequately represented by this linear-Gaussian approximation, and so trajectory design for these systems would be better handled by the SDDP approach using a second order approximation of the dynamics and the unscented transform for the evolution of the state

*GPOPS-II²⁵ was used to solve for the initial trajectory.

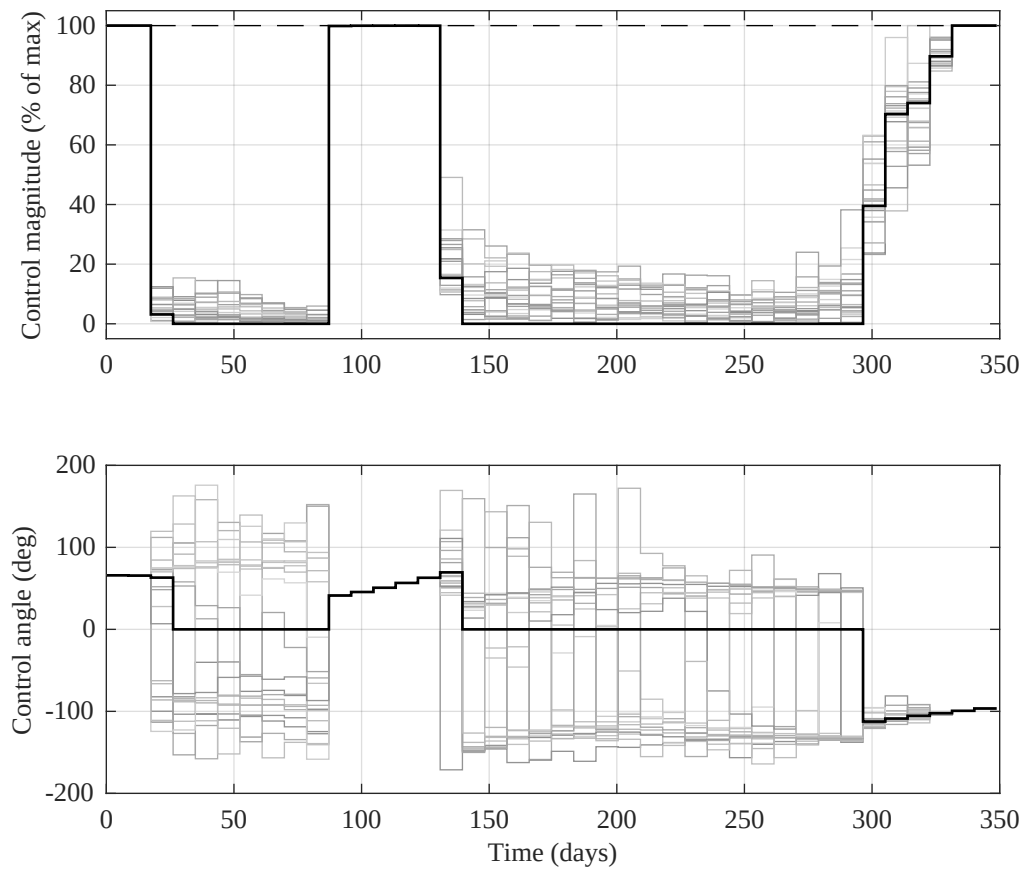


Figure 4. Nominal control (black) with sample control trajectories (gray).

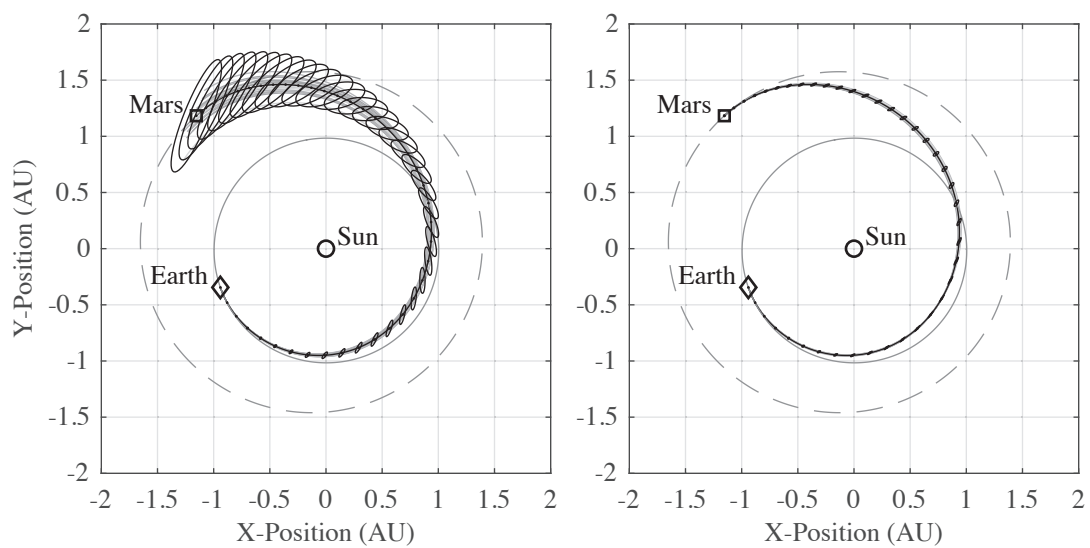


Figure 5. Open-loop (left) and closed-loop (right) position covariances at each time step.

distribution. A comparison between these two methods for a range of trajectory control problems would be an interesting area of future research.

ACKNOWLEDGMENT

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APPENDIX: PROOFS

Proof. (of Theorem 1) Let $z \sim \mathcal{N}(0, I)$ be an m -dimensional standard Gaussian random vector. Then $\|z\|^2$ is a $\chi^2(m)$ -random variable, and thus satisfies the concentration inequality (Ref. [26], Example 2.28):

$$\mathbb{P}(\|z\|^2 \leq m(1 + \delta)^2) \geq 1 - e^{-m\delta^2/2} \quad \text{for all } \delta \geq 0. \quad (68)$$

Writing $y = \mu + \Sigma^{1/2}z \sim \mathcal{N}(\mu, \Sigma)$, it follows that $\|y\| \leq \|\mu\| + \sigma\|z\|$, and thus

$$\mathbb{P}(\|y\| \leq \rho) \geq \mathbb{P}(\|\mu\| + \sigma\|z\| \leq \rho) = \mathbb{P}(\|z\| \leq (\rho - \|\mu\|)/\sigma). \quad (69)$$

Set $\delta = (\rho - \|\mu\|)/\sigma\sqrt{m} - 1$. Since $\sigma\sqrt{2\log(1/\beta)} \geq 0$, the hypothesis implies that $\|\mu\| + \sigma\sqrt{m} \leq \rho$, and hence $\delta \geq 0$. Next, substitute (68) into (69) to obtain

$$\mathbb{P}(\|y\| \leq \rho) \geq 1 - \exp \left[-\frac{m}{2} \left(\frac{\rho - \|\mu\|}{\sigma\sqrt{m}} - 1 \right)^2 \right]. \quad (70)$$

The hypothesis yields:

$$\|\mu\| + \sigma \left(\sqrt{2\log \frac{1}{\beta}} + \sqrt{m} \right) \leq \rho \implies \sqrt{\frac{2}{m} \log \frac{1}{\beta}} + 1 \leq \frac{\rho - \|\mu\|}{\sigma\sqrt{m}} \quad (71)$$

$$\implies \beta \geq \exp \left[-\frac{m}{2} \left(\frac{\rho - \|\mu\|}{\sigma\sqrt{m}} - 1 \right)^2 \right] \quad (72)$$

$$\implies 1 - \beta \leq 1 - \exp \left[-\frac{m}{2} \left(\frac{\rho - \|\mu\|}{\sigma\sqrt{m}} - 1 \right)^2 \right]. \quad (73)$$

Finally, substituting (73) into (70), we obtain the desired result. $\square$

Proof. (of Lemma 2) For the case $m = 1$, the result immediately follows from the exponential bound of the error function:²⁷

$$\mathbb{P}(\|y\| \leq \rho) = \text{erf} \left(\frac{\rho}{\sigma\sqrt{2}} \right) \geq 1 - e^{-\rho^2/2\sigma^2}. \quad (74)$$

Next, consider the case $m = 2$. Let $a \geq 0$, and let $z = \Sigma^{-1/2}y = (r \cos \phi, r \sin \phi)$. The following integral is evaluated by change of variables:

$$\mathbb{P}(y^\top \Sigma^{-1}y \leq a^2) = \frac{1}{2\pi\sqrt{\det(\Sigma)}} \int_{\{y^\top \Sigma^{-1}y \leq a^2\}} \exp \left(-\frac{y^\top \Sigma^{-1}y}{2} \right) dy \quad (75)$$

$$= \frac{1}{2\pi} \int_{\{z^\top z \leq a^2\}} e^{-z^\top z/2} dz \quad (76)$$

$$= \frac{1}{2\pi} \int_0^{a^2} \int_0^{2\pi} r e^{-r^2/2} dr d\phi \quad (77)$$

$$= 1 - e^{-a^2/2}. \quad (78)$$

Since $\|y\|^2 / \sigma^2 \leq y^\top \Sigma^{-1} y$, it follows that

$$\mathbb{P}(\|y\|^2 \leq \sigma^2 a^2) \geq \mathbb{P}(y^\top \Sigma^{-1} y \leq a^2) = 1 - e^{-a^2/2}. \quad (79)$$

Setting $\rho^2 = \sigma^2 a^2$ gives the desired result. Intuitively, the level sets $y^\top \Sigma^{-1} y = a^2$ define the contours of ellipses having probability $1 - e^{-a^2/2}$, and the level sets $\|y\|^2 = \sigma^2 a^2$ are the smallest circles that contain these ellipses. $\square$

Proof. (of Theorem 3) Since $\sigma \geq 0$, the hypothesis implies that $\|\mu\| \leq \rho$. We can therefore rearrange the hypothesis as follows:

$$\|\mu\| + \sigma \sqrt{2 \log \frac{1}{\beta}} \leq \rho \implies \sigma \sqrt{2 \log \frac{1}{\beta}} \leq \rho - \|\mu\| \quad (80a)$$

$$\implies 2\sigma^2 \log \frac{1}{\beta} \leq (\rho - \|\mu\|)^2 \quad (80b)$$

$$\implies \beta \geq \exp \left(- \frac{(\rho - \|\mu\|)^2}{2\sigma^2} \right) \quad (80c)$$

$$\implies 1 - \beta \leq 1 - \exp \left(- \frac{(\rho - \|\mu\|)^2}{2\sigma^2} \right). \quad (80d)$$

Let $\tilde{y} = y - \mu$. Since $\|y\| = \|\mu + \tilde{y}\| \leq \|\mu\| + \|\tilde{y}\|$, it follows that

$$\mathbb{P}(\|y\| \leq \rho) \geq \mathbb{P}(\|\mu\| + \|\tilde{y}\| \leq \rho) = \mathbb{P}(\|\tilde{y}\| \leq \rho - \|\mu\|). \quad (81)$$

Substituting the result of Lemma 2 into (81), with $\rho - \|\mu\|$ in place of ρ , and since $\tilde{y} \sim \mathcal{N}(0, \Sigma)$, we obtain

$$\mathbb{P}(\|y\| \leq \rho) \geq 1 - \exp \left(- \frac{(\rho - \|\mu\|)^2}{2\sigma^2} \right). \quad (82)$$

From (80d), we finally obtain the desired result:

$$\mathbb{P}(\|y\| \leq \rho) \geq 1 - \exp \left(- \frac{(\rho - \|\mu\|)^2}{2\sigma^2} \right) \geq 1 - \beta. \quad (83)$$

$\square$

APPENDIX: EXACT DISCRETIZATION

In this section, we will review the discretization process used to derive the discrete-time system (14) from the continuous-time system (11).

Let $t_0 = t_0^p < \dots, t_N^p = t_f$ be a partition of the interval $[t_0, t_f]$, and assume that the control is piecewise-constant (i.e., ZOH) on this partition; that is,

$$u(t) = u(t_k^p) = u_k \quad \text{for } t \in [t_k^p, t_{k+1}^p), \quad k = 0, \dots, N-1. \quad (84)$$

Then, for any $k = 0, \dots, N-1$ and $t \in [t_k^p, t_{k+1}^p)$, the linear SDE (11) has the solution

$$x(t) = \Phi(t, t_k^p) x(t_k^p) + \int_{t_k^p}^t \Phi(t, s) B(s) ds u_k + \int_{t_k^p}^t \Phi(t, s) c(s) ds + \int_{t_k^p}^t \Phi(t, s) G(s) dw_s, \quad (85)$$

in terms of the state transition matrix $\Phi(t, s)$ corresponding to the state matrix $A(t)$. Next, we let $t = t_{k+1}^p$, let w_k be a standard n_x -dimensional Gaussian random vector, define the matrices

$$A_k = \Phi(t_{k+1}^p, t_k^p), \quad B_k = \int_{t_k^p}^{t_{k+1}^p} \Phi(t_{k+1}^p, s) B(s) ds, \quad c_k = \int_{t_k^p}^{t_{k+1}^p} \Phi(t_{k+1}^p, s) c(s) ds, \quad (86)$$

and set G_k to be any matrix such that $G_k w_k$ has covariance

$$\int_{t_k^p}^{t_{k+1}^p} \Phi(t_{k+1}^p, s) G(s) G(s)^\top \Phi^\top(t_{k+1}^p, s) ds. \quad (87)$$

It follows that the solution (x_k) to the stochastic difference equation

$$x_{k+1} = A_k x_k + B_k u_k + c_k + G_k w_k, \quad x_{k=0} = x(t_0), \quad (88)$$

for $k = 0, \dots, N-1$, satisfies $x_k = x(t_k^p)$ for all $k = 0, \dots, N-1$. In other words, the discrete system (88) is an exact discretization of the continuous SDE (11).

APPENDIX: APPROXIMATE DISCRETIZATION

The evaluation of the integrals (86) and (87) is computationally expensive, which can be problematic when the discrete system must be iteratively computed as in Algorithm 1. In this section, we present a computationally efficient approximate discretization method based on the assumption that the matrices representing the linearized dynamics (11) are piecewise constant on a time partition $(t_{k,\ell}^q)_{k=0, \ell=0}^{N, N_q}$ which satisfies

$$t_0 = t_0^p = t_{0,0}^q < t_{0,1}^q < \dots < t_{0,N_q}^q = t_1^p = t_{1,0}^q < \dots < t_{N,N_q}^q = t_N^p = t_f. \quad (89)$$

In practice, the time steps for the partition may be considerably smaller than the partition (t_k^p) , since the following solution method only requires the evaluation of the matrix exponential for each sub-interval $[t_{k,\ell}^q, t_{k,\ell+1}^q)$.

Provided the assumption that the system matrices are constant on the partition sub-intervals, then, following the analysis in the preceding appendix, the linear system (11) has the solution

$$x(t_{k,\ell+1}^q) = A_{k,\ell} x(t_{k,\ell}^q) + B_{k,\ell} u_k + c_{k,\ell} + G_{k,\ell} w_{k,\ell}, \quad (90)$$

where

$$A_{k,\ell} = \exp(A(t_{k,\ell}^q)(t_{k,\ell+1}^q - t_{k,\ell}^q)), \quad B_{k,\ell} = \left(\int_{t_{k,\ell}^q}^{t_{k,\ell+1}^q} \exp(A(t_{k,\ell}^q)(t_{k,\ell+1}^q - s)) ds \right) B(t_{k,\ell}^q), \quad (91)$$

$$c_{k,\ell} = \left(\int_{t_{k,\ell}^q}^{t_{k,\ell+1}^q} \exp(A(t_{k,\ell}^q)(t_{k,\ell+1}^q - s)) ds \right) c(t_{k,\ell}^q), \quad (92)$$

and where $G_{k,\ell} w_{k,\ell}$ is a zero-mean Gaussian random vector with covariance

$$W_{k,\ell} = \int_{t_{k,\ell}^q}^{t_{k,\ell+1}^q} \exp(A(t_{k,\ell}^q)(t_{k,\ell+1}^q - s)) G(t_{k,\ell}^q) G^\top(t_{k,\ell}^q) \exp(A(t_{k,\ell}^q)(t_{k,\ell+1}^q - s)) ds. \quad (93)$$

Since, by assumption, the system matrices are constant over the integration domain $[t_{k,\ell}^q, t_{k,\ell+1}^q]$, the integrals in (91), (92), and (93) can be computed in terms of the matrix exponential. Following Ref. [28], we construct the square matrix

$$Y = \begin{bmatrix} A(t_{k,\ell}^q) & G(t_{k,\ell}^q)G^\top(t_{k,\ell}^q) & 0 & 0 \\ 0 & -A^\top(t_{k,\ell}^q) & 0 & 0 \\ 0 & 0 & A(t_{k,\ell}^q) & B(t_{k,\ell}^q) \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (94)$$

Then the system matrices are given as

$$B_{k,\ell} = \mathcal{G}_3 B(t_{k,\ell}^q), \quad c_{k,\ell} = \mathcal{G}_3 c(t_{k,\ell}^q), \quad W_{k,\ell} = \mathcal{G}_1 \exp(A^\top(t_{k,\ell}^q)(t_{k,\ell+1}^q - t_{k,\ell}^q)), \quad (95)$$

where the matrices $\mathcal{G}_3$ and $\mathcal{G}_1$ are computed as

$$\exp(Y(t_{k,\ell+1}^q - t_{k,\ell}^q)) = \begin{bmatrix} \cdot & \mathcal{G}_1 & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \mathcal{G}_3 \\ 0 & 0 & 0 & \cdot \end{bmatrix}. \quad (96)$$

The state transition from time t_k^p to time t_{k+1}^p can therefore be computed by evaluating (96) for each interval $[t_{k,\ell}^q, t_{k,\ell+1}^q]$. Indeed, when writing the state as $x_k = x(t_k^p)$, the system evolution is given as

$$x_{k+1} = A_k x_k + B_k u_k + c_k + G_k w_k, \quad (97)$$

where w_k is a standard n -dimensional Gaussian vector, and where the state matrices

$$A_k = \Phi_k(N_q, 0), \quad B_k = \sum_{\ell=0}^{N_q} \Phi_k(N_q, \ell) B_{k,\ell}, \quad c_k = \sum_{\ell=0}^{N_q} \Phi_k(N_q, \ell) c_{k,\ell}, \quad (98)$$

are given in terms of the state transition matrix $\Phi_k(r, \ell) = A_{k,r-1} A_{k,r-2} \cdots A_{k,\ell}$. The noise coefficient matrix G_k is any matrix such that $G_k G_k^\top = W_k$ (i.e., $G_k w_k$ has covariance W_k), which is given as

$$W_k = \text{Cov} \left(\sum_{\ell=0}^{N_q} \Phi_k(N_q, \ell) G_{k,\ell} w_{k,\ell} \right) = \sum_{\ell=0}^{N_q} \Phi_k(N_q, \ell) W_{k,\ell} \Phi_k^\top(N_q, \ell). \quad (99)$$

REFERENCES

- [1] J. Ridderhof and P. Tsiotras, “Minimum-fuel Powered Descent in the Presence of Random Disturbances,” *AIAA Guidance, Navigation, and Control Conference*, No. AIAA 2019-0646, San Diego, CA, Jan. 2019, 10.2514/6.2019-0646.
- [2] M. D. Rayman, T. C. Fraschetti, C. A. Raymond, and C. T. Russell, “Coupling of System Resource Margins Through the Use of Electric Propulsion: Implications in Preparing for the Dawn Mission to Ceres and Vesta,” *Acta Astronautica*, Vol. 60, No. 10–11, 2007, pp. 930–938.
- [3] F. E. Laipert and J. M. Longuski, “Automated Missed-Thrust Propellant Margin Analysis for Low-Thrust Trajectories,” *Journal of Spacecraft and Rockets*, Vol. 52, No. 4, 2015, pp. 1135–1143.
- [4] N. Ozaki, S. Campagnola, R. Funase, and C. H. Yam, “Stochastic Differential Dynamic Programming with Unscented Transform for Low-Thrust Trajectory Design,” *Journal of Guidance, Control, and Dynamics*, Vol. 41, No. 2, 2018, pp. 377–387.
- [5] N. Ozaki, S. Campagnola, and R. Funase, “Tube Stochastic Optimal Control for Nonlinear Constrained Trajectory Optimization Problems,” *Journal of Guidance, Control, and Dynamics*, Vol. 43, No. 4, 2020, pp. 645–655.

- [6] E. Theodorou, Y. Tassa, and E. Todorov, “Stochastic Differential Dynamic Programming,” *American Control Conference*, Baltimore, MD, June 2010, pp. 1125–1132, 10.1109/ACC.2010.5530971.
- [7] Z. Yi, Z. Cao, E. Theodorou, and Y. Chen, “Nonlinear Covariance Control via Differential Dynamic Programming,” *arXiv preprint 1911.09283*, 2019.
- [8] M. Ono, M. Pavone, Y. Kuwata, and J. Balaram, “Chance-Constrained Dynamic Programming with Application to Risk-Aware Robotic Space Exploration,” *Autonomous Robots*, Vol. 39, Aug 2015, pp. 555–571, 10.1007/s10514-015-9467-7.
- [9] Y. Chen, T. T. Georgiou, and M. Pavon, “Optimal Steering of a Linear Stochastic System to a Final Probability Distribution, Part I,” *IEEE Transactions on Automatic Control*, Vol. 61, May 2016, pp. 1158–1169.
- [10] E. Bakolas, “Optimal Covariance Control for Discrete-time Stochastic Linear Systems Subject to Constraints,” *IEEE 55th Conference on Decision and Control*, Las Vegas, NV, December 2016, pp. 1153–1158.
- [11] K. Okamoto and P. Tsiotras, “Optimal Stochastic Vehicle Path Planning Using Covariance Steering,” *IEEE Robotics and Automation Letters*, Vol. 4, No. 3, 2019, pp. 2276–2281, 10.1109/LRA.2019.2901546.
- [12] K. Okamoto, M. Goldshtein, and P. Tsiotras, “Optimal Covariance Control for Stochastic Systems Under Chance Constraints,” *IEEE Control Systems Letters*, Vol. 2, April 2018, pp. 266–271, 10.1109/LC-SYS.2018.2826038.
- [13] J. Ridderhof, K. Okamoto, and P. Tsiotras, “Nonlinear Uncertainty Control with Iterative Covariance Steering,” *IEEE 58th Conference on Decision and Control*, Nice, France, Dec 11 – 13, 2019, pp. 3484–3490.
- [14] J. Ridderhof, K. Okamoto, and P. Tsiotras, “Chance Constrained Covariance Control for Linear Stochastic Systems With Output Feedback,” *IEEE 59th Conference on Decision and Control*, Jeju Island, Republic of Korea, Dec. 2020.
- [15] A. Mesbah, “Stochastic Model Predictive Control: An Overview and Perspectives for Future Research,” *IEEE Control Systems*, Vol. 36, No. 6, 2016, pp. 30–44.
- [16] J. Skaf and S. P. Boyd, “Design of Affine Controllers via Convex Optimization,” *IEEE Transactions on Automatic Control*, Vol. 55, No. 11, 2010, pp. 2476–2487.
- [17] K. Oguri, M. Ono, and J. W. McMahon, “Convex Optimization over Sequential Linear Feedback Policies with Continuous-time Chance Constraints,” *IEEE 58th Conference on Decision and Control*, Nice, France, 2019, pp. 6325–6331, 10.1109/CDC40024.2019.9029604.
- [18] A. Prékopa, “Boole-Bonferroni Inequalities and Linear Programming,” *Operations Research*, Vol. 36, No. 1, 1988, pp. 145–162.
- [19] L. Blackmore, H. X. Li, and B. C. Williams, “A Probabilistic Approach to Optimal Robust Path Planning with Obstacles,” *American Control Conference*, Minneapolis, MN, June 2006, 10.1109/ACC.2006.1656653.
- [20] M. Szmuk, B. Açıkmeşe, and A. W. Berning, “Successive Convexification for Fuel-Optimal Powered Landing with Aerodynamic Drag and Non-Convex Constraints,” *AIAA Guidance, Navigation, and Control Conference*, No. AIAA 2016-0378, 2016.
- [21] Y. Mao, M. Szmuk, and B. Açıkmeşe, “Successive Convexification of Non-convex Optimal Control Problems and its Convergence Properties,” *IEEE 55th Conference on Decision and Control*, Las Vegas, NV, Dec. 2016, pp. 3636–3641.
- [22] E. Todorov and W. Li, “A Generalized Iterative LQG Method for Locally-optimal Feedback Control of Constrained Nonlinear Stochastic Systems,” *American Control Conference*, Portland, OR, June 2005, pp. 300–306.
- [23] J. Lofberg, “YALMIP: A toolbox for modeling and optimization in MATLAB,” *IEEE International Conference on Robotics and Automation*, New Orleans, LA, 2004, pp. 284–289.
- [24] Mosek ApS, “The MOSEK Optimization Toolbox for MATLAB Manual,” Version 9.1, 2019.
- [25] M. A. Patterson and A. V. Rao, “GPOPS-II: A MATLAB Software for Solving Multiple-Phase Optimal Control Problems Using Hp-Adaptive Gaussian Quadrature Collocation Methods and Sparse Nonlinear Programming,” *ACM Trans. Math. Softw.*, Vol. 41, Oct. 2014, 10.1145/2558904.
- [26] M. J. Wainwright, *High-Dimensional Statistics: A Non-Asymptotic Viewpoint*. Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press, 2019, 10.1017/9781108627771.
- [27] M. Chiani, D. Dardari, and M. K. Simon, “New Exponential Bounds and Approximations for the Computation of Error Probability in Fading Channels,” *IEEE Transactions on Wireless Communications*, Vol. 2, July 2003, pp. 840–845, 10.1109/TWC.2003.814350.
- [28] C. Van Loan, “Computing Integrals Involving the Matrix Exponential,” *IEEE Transactions on Automatic Control*, Vol. 23, No. 3, 1978, pp. 395–404.