Control Design for Chained-Form Systems with Bounded Inputs ⋆

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Abstract

Discontinuous, time-invariant controllers have been recently proposed in the literature as an alternative method to stabilize nonholonomic systems. These control laws are not continuous at the origin and although they provide exponential rates of convergence, they may use significant amount of control effort, especially if the initial conditions are close to an equilibrium manifold. We seek to remedy this situation by constructing *bounded* controllers (with exponential convergence rates) for nonholonomic systems in chained form.

Key words: Nonholonomic Systems, Chained Form, Bounded Control, Exponential Convergence.

1 Introduction

In this paper we focus on designing feedback control laws for a nonholonomic system in chained form using inputs bounded by an *a priori* specified upper bound. It is well known that nonholonomic systems may not satisfy Brockett's necessary condition [3], thus no time-invariant smooth, static stabilizing controller exists. One possible avenue to deal with the difficulties implied by Brockett's theorem is to use time-varying controllers. This approach

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has been extensively investigated during the last few years with great success [17,14,15,9,4,7]. It can be shown that time-varying smooth control laws for driftless systems have necessarily algebraic (not exponential) convergence rates [12]. This may hinder the use of these control laws in cases where the speed of response is important. Coron recently showed the existence of finite-time, time-varying continuous control laws for locally controllable systems. The construction of such control laws is, however, nontrivial [5]. Time-varying, exponentially stabilizing [2] control laws have been reported in [13] and [18,6]. In particular, in [13] the authors developed time-varying, non-Lipschitz (at the origin), homogeneous feedback control laws by modifying the control laws of [15]. In [18] non-smooth, time-varying stabilizers were developed for nonholonomic systems in chained form. Similar results were presented in [6] for systems in power form.

More recently, another group of researchers concentrated on the design of time-invariant discontinuous controllers which achieve exponential convergence rates. Based on a nonlinear transformation, an exponentially convergent controller (which, however, may not necessarily achieve stability in the sense of Lyapunov) is constructed in [1] for chained form systems. A non-smooth controller for attitude stabilization of an underactuated spacecraft was proposed in [21]. Using the fact that the underactuated spacecraft problem is equivalent to a 3-dimensional system in power (or chained) form, this idea was later expanded upon and used in [8] to construct exponentially stabilizing control laws for a 3-dimensional system in power form. Similar results appeared in [16] and [2]. Recently, time-invariant discontinuous controllers for n-dimensional power form systems were reported in [11] using an iterative algorithm, which utilized a series of invariant manifolds as new coordinates.

A common characteristic of all these discontinuous controllers is that the control input may become excessively large, especially for initial conditions close to a certain singular manifold which includes the origin. In [22] the non-smooth controller proposed in [21] was modified, to remedy the problem of large control inputs. The procedure in [22] consists of dividing the spate space into two regions. The so-called "good region" in [22] contains all the initial conditions away from the singular manifold which result in moderate control inputs. For initial conditions in the complement of this set (the "bad region") the control inputs are typically large. The control law drives the trajectories of the system away from the singular equilibrium manifold and into the "good region". In this paper we generalize this idea to general nonholonomic systems in chained form.

It should be pointed out that the idea of dividing the state space into two sep-

[2] This definition of stability, often called $\mathcal{K}$- or ρ-exponential stability is a weaker form of stability than the usual concept of exponential stability.

arate regions to construct a switching controller for nonholonomic systems has also been used by Bloch and Drakunov in [2]. In that reference the authors developed several stabilizing controllers for the three-dimensional nonholonomic integrator (which is equivalent to a three-dimensional system in chained form). The main difference with the controllers in [21] or [8] is that in [2] the trajectories reach the manifold in finite time (thus introducing a sliding mode), whereas in [21] the trajectories reach the manifold only asymptotically. Thus, no sliding takes place. For a related discussion one may also consult [20].

The paper is organized as follows. In Section 2, we introduce a nonlinear coordinate transformation (called the σ-process) first presented in [1], and a linear feedback to transform an n-dimensional chained form system to a linear system. This allows a straightforward characterization of the "good" and "bad" regions of the state space. In Section 4 we construct a control law such that, for all initial conditions in the "good" region the trajectories approach the origin exponentially fast. Moreover, the control law remains bounded by an *a priori* specified bound and the domain of attraction of the closed-loop system contains any a priori given set. In Section 5 we complete the controller design by constructing a bounded controller such that for all initial conditions outside the "good" region, the trajectories of the closed-loop system converge to this region in finite time, while keeping the control input bounded. Finally, in Section 6 a numerical example is provided to illustrate the theory.

The notation used in the paper is standard. For a vector $x \in \mathbb{R}^n$, $|x|$ denotes the euclidean norm, for a square matrix A, $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote its maximum and minimum eigenvalues respectively, $sp(A)$ denotes its spectrum, and A^T denotes its transpose. I denotes the identity matrix. Finally, the notation $f \in \mathcal{L}_2$ implies that $\int_0^\infty |f(t)|^2 \, dt < \infty$.

2 The σ-process

Several nonholonomic systems, after appropriate state and input transformations [14], can be put in the so-called chained canonical form. The 1-chain single generator system with two inputs is given by

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_i &= x_{i-1} u_1 \qquad i = 3, \ldots n \end{aligned} \tag{1}$$

The following transformation, valid for all $x_1 \neq 0$,

$$\xi_1 = x_1$$

$$
\begin{aligned}
\xi_2 &= x_2 \\
\xi_i &= \frac{x_i}{x_1^{i-2}} \qquad i = 3, \ldots n
\end{aligned}
\tag{2}
$$

applied to Eq. (1) yields

$$
\begin{aligned}
\dot{\xi}_1 &= u_1 \\
\dot{\xi}_2 &= u_2 \\
\dot{\xi}_i &= (\xi_{i-1} - (i-2)\xi_i)\,\frac{u_1}{\xi_1} \qquad i = 3, \ldots n
\end{aligned}
\tag{3}
$$

If we let

$$u_1 = -k\,\xi_1 \tag{4}$$

the ξ-system becomes

$$
\dot{\xi} = \begin{bmatrix} -k & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & -k & k & \ldots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & (n-2)k \end{bmatrix} \xi + \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u_2 \tag{5}
$$

This is a linear system with u_2 as the new input. For more details on this transformation, the reader is referred to [1]. Since the system in Eq. (5) is stabilizable, one can choose a linear control law [1]

$$u_2 = p_2\,\xi_2 + p_3\,\xi_3 + \ldots + p_n\,\xi_n \tag{6}$$

to place the eigenvalues in the left-half of the complex plane, and make the closed-loop system (in ξ-coordinates) globally exponentially stable. The previous linear control law is not defined on the set

$$\mathcal{S} = \{x \in \mathbb{R}^n : x_1 = 0\ \} \tag{7}$$

Moreover, one cannot conclude that the original closed-loop system in Eq. (1) is asymptotically stable, since the transformation in Eq. (2) is not a diffeomorphism. It can be shown, however, that in terms of the original coordinates, the control law in Eqs. (4) and (6) ensures exponential *convergence* from all initial conditions in the open and dense set $\mathcal{D} = \mathbb{R}^n \backslash \mathcal{S}$ [1].

Although the control law in Eq. (6) is well defined for all initial conditions such that $x_1(0) \neq 0$, it is clear from Eq. (2) that the control input u_2 may take excessively large values when the initial conditions are close to the singular manifold $\mathcal{S}$. Similar problems are encountered with the discontinuous control laws proposed in [21,8,11]. In [22] the problem of avoiding the singular manifold was addressed by decomposing the state space into two regions and by designing a control law which drives all trajectories in a "safe" region away from the singular manifold. It is the objective of this paper to generalize this idea to the system in Eq. (1) in order to derive feedback control laws defined on $\mathcal{D}$ which are bounded.

3 Statement of the problem and approach

We wish to derive a globally valid control law for the system in Eq. (1) such that the following two properties hold.

(1) For all initial conditions $x(0) \in \mathbb{R}^n$, we have that $\lim_{t\to\infty} x(t) = 0$.
(2) For every initial condition $x(0) \in \mathbb{R}^n$, there exists a positive number $\alpha > 0$ such that $\lim_{t\to\infty} e^{\alpha t} x(t) = 0$.
(3) The control law u_i is bounded as $|u_i| \leq \bar{u}_i, (i = 1, 2)$, where $\bar{u}_i$ are any *a priori* given positive numbers.

The first property implies convergence of all trajectories to the origin. The second property implies that the convergence should be (asymptotically) exponential. We only impose convergence of the closed-loop trajectories of the system in Eq. (1) to the origin. Attractivity to the origin for the system in Eq. (1) can be easily deduced if the linear system in Eq. (5) is asymptotically stable or even convergent [1,11]. Moreover, since $\xi_1 = x_1$ and $\xi_2 = x_2$, the control inputs u_1 and u_2 are the same for both systems. If the system in Eq. (5) is asymptotically stable (or even convergent) with inputs bounded by $\bar{u}_i$, then the trajectories of the system in Eq. (1) will converge to the origin and the control inputs will also be bounded by $\bar{u}_i$.

The nonlinear transformation in Eq. (2) and the control input in Eq. (4) have resulted in a linear system with input u_2. Recently, numerous results have appeared in the literature dealing with the problem of global or semi-global stabilization of linear systems with bounded inputs [10,19]. Unfortunately, the open-loop system in Eq. (5) has positive eigenvalues, so it is not asymptotically null-controllable with bounded controls [10]. Asymptotic null-controllability with bounded controls is a necessary condition for the existence of global or semi-global bounded controllers for linear systems. Thus, we cannot use directly the results of [10,19] to derive (globally or semi-globally) bounded controllers for Eq. (5). However, a simple observation shows that the eigenvalues of the uncontrolled linear system in Eq. (5) can be moved arbitrarily close to the imaginary axis by appropriate choice of the control gain k. This allows the construction of exponentially stabilizing controllers for the system in Eq. (5) which are bounded by an arbitrarily small upper bound. In this paper we will use this idea to design bounded stabilizing control laws for the system in Eq. (1).

4 A semi-global controller

In this section we design a controller such that, if the initial conditions are in a given set, the trajectories of the system in Eq. (1) tend asymptotically to the origin and the control inputs are bounded by $\bar{u}_i$. In addition, this set can be chosen arbitrarily large. We call this the "semi-global" controller following the customary terminology from the nonlinear control literature [10].

To proceed with our analysis, we first decompose the system in Eq. (5) as

$$\dot{\xi}_1 = -k\,\xi_1 \tag{8a}$$

$$\dot{\tilde{\xi}} = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ -k & k & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -k & (n-2)k \end{bmatrix} \tilde{\xi} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u_2 \tag{8b}$$

with $\tilde{\xi} = [\xi_2,\ \xi_3,\ \dots,\ \xi_n]^T$. Define the constant matrices A and B as follows

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ -1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & (n-2) \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{9}$$

Then the $\tilde{\xi}$-subsystem can be rewritten as

$$\dot{\tilde{\xi}} = k\,A\,\tilde{\xi} + B\,u_2 \tag{10}$$

Definition 1 *A continuous function $\phi(x)$ will be called a linear-dominant function (l.d.f for short) if it satisfies the following three properties:*

(1) It is monotonically increasing for $x \geq 0$ and $\phi(0) = 1$.

(2) It is an even function, i.e., $\phi(x) = \phi(-x) \quad \forall x \in \mathbb{R}^n$.
(3) $|x| \leq \phi(x)$ *for all* $x \in \mathbb{R}^n$.

From the definition it follows immediately that $\lim_{|x|\to\infty} \phi(x) = \infty$. For example, the functions $\phi(x) = 1 + |x|$, $\phi(x) = \sqrt{1+x^2}$ and $\phi(x) = 1 + x^2$ are all l.d.f. In particular, any function of the from $\phi(x) = (1 + x^{2p})^{\frac{1+\epsilon}{2p}}$ with $\epsilon > 0$ and $p = 1, 2, \dots$ is l.d.f.

Let the set given by $\mathcal{D} = \mathbb{R}^n \backslash \{\mathcal{S}\}$. The following theorem provides a controller for the system in Eq. (3) which is bounded by $\bar{u}_i$ $(i = 1, 2)$ inside a certain subset of $\mathcal{D}$.

Theorem 2 *Consider the system described by Eq. (3) and the region given by* $\mathcal{D}^g_\delta = \{\, \xi \in \mathcal{D} : |\tilde{\xi}| \leq \delta \,\}$. *Let* $\bar{u}_i$ $(i = 1, 2)$ *be given positive numbers and let* P *be the positive definite symmetric matrix which solves the equation*

$$(A + I)P + P(A + I)^T = BB^T \tag{11}$$

Define the matrix $A_c = A - BB^T P^{-1}$ *and let* $k = \min\{\bar{u}_1, \bar{u}_2/(\mu\delta)\}$ *where*

$$\mu = \lambda_{\max}^{\frac{1}{2}}(P^{-1}) \sqrt{B^T P^{-1} B}$$

Then, the control law

$$u_1 = -k\, \xi_1/\phi(\xi_1) \tag{12a}$$
$$u_2 = -k\, B^T P^{-1} \tilde{\xi}/\phi(\xi_1) \tag{12b}$$

with $\phi(\cdot)$ *an l.d.f as in Definition 1, renders the system in Eq. (3) asymptotically stable. The trajectories converge exponentially, in the sense that* $\lim_{t\to\infty} e^{\alpha t}|\xi_i(t)| = 0$ *for all* $i = 1, 2, \dots, n$ *and* $0 < \alpha < k$. *In addition, for all initial conditions* $\xi(0) \in \mathcal{D}^g_\delta$, *we have* $|u_i| \leq \bar{u}_i$ $(i = 1, 2)$.

PROOF. The equation for ξ_1 is given by $\dot{\xi}_1 = -k\, \xi_1/\phi(\xi_1)$. All solutions of this system converge to the origin with (asympotic) exponential rate of convergence, and the control law u_1 is bounded by $|u_1| = k\,|\xi_1|/\phi(\xi_1) \leq \bar{u}_1$.

Define a new independent variable,

$$\tau = \int_0^t \frac{d\sigma}{\phi(\xi_1(\sigma))} \tag{13}$$

Note that τ is monotonically increasing and $\lim_{t\to\infty} \tau = \infty$. Denoting differentiation with respect to τ by ($'$), one obtains that,

$$\tilde{\xi}' = kA\,\tilde{\xi} + B\tilde{u}_2 \tag{14}$$

where $\tilde{u}_2 = u_2\,\phi(\xi_1)$. Since the pair (A, B) in Eq. (14) is controllable, it can be easily shown that the pair $((A+I), B)$ is also controllable. Moreover, all the eigenvalues of $-(A+I)$ are negative. Therefore, there exists a unique $P > 0$ which satisfies Eq. (11). From Eq. (11) we have,

$$(A+I)kQ + Qk(A+I)^T = BB^T \tag{15}$$

where $Q = P/k$. It is now easy to check that,

$$\begin{aligned}&(Ak + Ik - BB^TQ^{-1})Q + Q(Ak + Ik - BB^TQ^{-1})^T\\ &= (A+I)kQ + Qk(A+I)^T - 2BB^T = -BB^T\end{aligned} \tag{16}$$

Since the pair $((Ak + Ik - BB^TQ^{-1}), B)$ is controllable and Q is positive definite, the matrix $Ak + Ik - BB^TQ^{-1}$ is Hurwitz. In particular, the matrix $Ak - BB^TQ^{-1} = k\,A_c$ is Hurwitz and $Re(\lambda) < -k$ for all $\lambda \in sp(kA_c)$. With u_2 as in Eq. (12b) one obtains $\tilde{\xi}' = k\,A_c\,\tilde{\xi}$ and the $\tilde{\xi}$-subsystem is exponentially stable (in τ) with rate k. Since ξ_1 decreases monotonically to zero one obtains that $\phi(\xi_1(t)) \leq \phi(\xi_1(0))$ for all $t \geq 0$. Hence

$$\tau = \int_0^t \frac{d\sigma}{\phi(\xi_1(\sigma))} \geq \int_0^t \frac{d\sigma}{\phi(\xi_1(0))} = \frac{t}{\phi(\xi_1(0))} \tag{17}$$

From the exponential stability of the system $\tilde{\xi}' = k\,A_c\,\tilde{\xi}$ one obtains that there exists some positive constant $c_0 > 0$ such that

$$\begin{aligned}|\tilde{\xi}(t)| = |\tilde{\xi}(\tau(t))| &\leq c_0\,|\tilde{\xi}(0)|\,\exp(-k\tau)\\ &\leq c_0\,|\tilde{\xi}(0)|\exp(-\bar{k}t)\end{aligned} \tag{18}$$

where $\bar{k} = k/\phi(\xi_1(0))$. Hence, the $\tilde{\xi}$-subsystem with control (12) is exponentially convergent [3] (in t) with rate of convergence $\bar{k}$.

Note that A_c is Hurwitz and satisfies the matrix inequality $A_cP + PA_c^T < 0$. Therefore, if $|\tilde{\xi}(0)| \leq \delta$ one obtains that,

[3] We avoid using the term *stable* here, since the rate of convergence depends on the initial condition $\xi_1(0)$.

$$\begin{aligned}\tilde{\xi}^T(t)P^{-1}\tilde{\xi}(t) &\le \tilde{\xi}^T(0)P^{-1}\tilde{\xi}(0) \\ &\le \lambda_{\max}(P^{-1})|\tilde{\xi}(0)|^2 \\ &\le \lambda_{\max}(P^{-1})\,\delta^2 \quad \forall t \ge 0\end{aligned} \tag{19}$$

A straightforward calculation shows that,

$$\max_{\tilde{\xi}^T P^{-1}\tilde{\xi} \le \sigma^2} |B^T P^{-1}\tilde{\xi}| = \sigma\sqrt{B^T P^{-1} B} \tag{20}$$

For all initial conditions $\xi(0) \in \mathcal{D}^g_\delta$ we finally have that,

$$\begin{aligned}|u_2| &= k\,|B^T P^{-1}\tilde{\xi}/\phi(\xi_1)| \le k\,|B^T P^{-1}\tilde{\xi}| \\ &\le k\,\delta\,\lambda^{\frac{1}{2}}_{\max}(P^{-1})\sqrt{B^T P^{-1} B} = k\,\delta\,\mu \le \bar{u}_2\end{aligned} \tag{21}$$

Let $\epsilon^* = (k-\alpha)/\alpha > 0$. From the continuity of $\phi(\cdot)$ and the fact that $\lim_{t\to\infty}\xi_1(t) = 0$ we have that for any $\epsilon \in (0,\epsilon^*)$ there exists $T > 0$ such that $\phi(\xi_1(t)) < 1+\epsilon$ for all $t \ge T$. Without loss of generality, assume that $\xi_1(t) \ge 0$ for all $t \ge 0$. The argument for $\xi_1(t) < 0$ is similar. Hence,

$$\dot{\xi}_1 \le -\frac{k}{1+\epsilon}\,\xi_1 < -\alpha\,\xi_1 \qquad \forall\; t \ge T \tag{22}$$

The last equation implies that $\lim_{t\to\infty} e^{\alpha t}\xi_1(t) = 0$.

From Eq. (18) we have that $|\tilde{\xi}(t)| \le c_0|\tilde{\xi}(0)|\exp(-k\tau)$. From the definition of τ,

$$\tau = \int_0^T \frac{d\sigma}{\phi(\xi_1(\sigma))} + \int_T^t \frac{d\sigma}{\phi(\xi_1(\sigma))} = \tau^* + \int_T^t \frac{d\sigma}{\phi(\xi_1(\sigma))} \tag{23}$$

Since $\phi(\xi_1(t)) < 1+\epsilon$ for all $t \ge T$, one obtains

$$\tau > \tau^* + \int_T^t \frac{d\sigma}{1+\epsilon} = \tau^* + \frac{t-T}{1+\epsilon} = \beta + \frac{t}{1+\epsilon} \tag{24}$$

where $\beta = \tau^* - T/(1+\epsilon)$. The equation for $\tilde{\xi}$ thus gives

$$|\tilde{\xi}(t)| \le c_0|\tilde{\xi(0)}|\exp(-k\beta)\exp(-kt/(1+\epsilon)) = C_0|\tilde{\xi(0)}|\exp(-kt/(1+\epsilon)) \tag{25}$$

where $C_0 = c_0 \exp(-k\beta)$. Since $\epsilon \in (0, \epsilon^*)$ one obtains $k/(1+\epsilon) > \alpha$ and thus, $\lim_{t\to\infty} e^{\alpha t}|\xi_i(t)| = 0$ for $i = 2, \ldots, n$. This completes the proof of the Theorem.

The following Corollary follows immediately from Theorem 2.

Corollary 3 *The trajectories of the closed-loop system (1) with the control law in Eq. (12) satisfy*

$$\lim_{t\to\infty} e^{\alpha_i t} x_i(t) = 0, \qquad i = 1, \ldots, n \tag{26}$$

where $\alpha_1 = \alpha$ *and* $\alpha_i = \alpha(i-1)$ *for* $i = 2, \ldots, n$. *Also, since* $\alpha = \min\{\alpha_i\}$ *we obtain*

$$\lim_{t\to\infty} e^{\alpha t}|x(t)| = 0 \tag{27}$$

for all $0 < \alpha < k$.

Theorem 2 shows that for all initial conditions in the "good" region $\mathcal{D}_\delta^g$ the trajectories of the closed-loop system with control law as in Eq. (12) tend exponentially to zero. The set $\mathcal{D}_\delta^g$ can be made arbitrarily large by appropriate choice of the parameter δ. As $\delta \to \infty$ the region $\mathcal{D}_\delta^g$ increases and tends to the region $\mathcal{D}$. The value of the design parameter δ should be dictated by the bound on the control inputs $\bar{u}_i$ $(i = 1, 2)$. Typically, the larger the bounds $\bar{u}_i$ the larger the δ one can choose. The sets $\mathcal{D}_\delta^g$ and $\mathcal{D}_\delta^b$ for the case of a 3-dimensional chained form system are shown in Fig. 1 ($\delta = 1$).

Remark 4 *Theorem 2 makes no claim that the trajectories have to stay in* $\mathcal{D}_\delta^g$. *Nonetheless, from the proof of Theorem 2 one immediately obtains that for all initial conditions in the set* $\tilde{\mathcal{D}}_\delta^g = \{\, \xi \in \mathcal{D} : \tilde{\xi}^T P^{-1} \tilde{\xi} \leq \delta^2 \,\}$, *the trajectories of the closed-loop system remain in* $\tilde{\mathcal{D}}_\delta^g$ *(i.e.,* $\tilde{\mathcal{D}}_\delta^g$ *is a positively invariant set) and they tend exponentially to the origin.*

It is worth noticing that the matrix P *in Eq. (11) is independent of* k *and thus the sets* $\tilde{\mathcal{D}}_\delta^g$ *and* $\tilde{\mathcal{D}}_\delta^b$ *do not depend on the choice of* k.

From Theorem 2 and Remark 4 we have the following corollary.

Corollary 5 *Consider the system in Eq. (1) with the control*

$$u_1 = -k\, x_1/\phi(x_1) \tag{28a}$$
$$u_2 = -k\, B^T P^{-1} \tilde{\xi}/\phi(x_1) \tag{28b}$$

where $\mu = \sqrt{B^T P^{-1} B}$ and k, B, P, $\tilde{\xi}$ as in Theorem 2 and $\phi(\cdot)$ an l.d.f. as in Definition 1. Then, for all initial conditions $\xi(0) \in \tilde{\mathcal{D}}_\delta^g$, the trajectories remain in $\tilde{\mathcal{D}}_\delta^g$ for all $t \geq 0$ and satisfy the property $\lim_{t\to\infty} x(t) = 0$. In addition, the control law is bounded by $|u_i| \leq \bar{u}_i$ $(i = 1, 2)$.

5 A global controller

To complete the construction of the controller, we need to force all trajectories starting in the "bad" region $\tilde{\mathcal{D}}_\delta^b = \mathcal{D} \backslash \tilde{\mathcal{D}}_\delta^g$ to enter the region $\tilde{\mathcal{D}}_\delta^g$ in finite time.

Proposition 6 *Consider the system in Eq. (3) and the control law*

$$u_1 = k\,\xi_1/\phi(\xi_1) \tag{29a}$$

$$u_2 = -k\,\frac{\xi_2}{\phi(\xi_1)\phi(\xi_2)} \tag{29b}$$

with $k > 0$. Then, for every $\delta > 0$ and $\xi_1(0) \neq 0$, there exists a time $t^* > 0$ such that $\tilde{\xi}^T(t) P^{-1} \tilde{\xi}(t) \leq \delta^2$ for all $t \geq t^*$. Moreover, if $k = \min\{\bar{u}_i\}$ then $|u_i| \leq \bar{u}_i$ $(i = 1, 2)$.

PROOF. The differential equation for ξ_1 is given by

$$\dot{\xi}_1 = k\,\xi_1/\phi(\xi_1) \tag{30}$$

Clearly, $\lim_{t\to\infty} \xi_1(t) = \infty$ for $\xi_1(0) \neq 0$. Consider again the change of independent variable introduced in Eq. (13). Since

$$\tau = \int_0^t \frac{d\sigma}{\phi(\xi_1(\sigma))} = \frac{1}{k} \int_{\xi_1(0)}^{\xi_1(t)} \frac{d\eta}{\eta} \tag{31}$$

one obtains that τ is monotonically increasing and $\lim_{t\to\infty} \tau = \infty$.

With the control law as in Eq. (29) the closed-loop system in Eq. (8b) can be written in the form

$$\frac{d\zeta}{d\tau} = \begin{bmatrix} -k & 0 \dots 0 & 0 \\ k & -2k \dots 0 & 0 \\ \vdots & \vdots \ddots \vdots & \vdots \\ 0 & 0 \dots k & -(n-2)k \end{bmatrix} \zeta + \begin{bmatrix} k \\ 0 \\ \vdots \\ 0 \end{bmatrix} \xi_2 \tag{32}$$

where $\zeta = [\xi_3, \ldots, \xi_n]^T$ and where ξ_2 satisfies the equation $\xi_2' = -k\,\xi_2/\phi(\xi_2)$. The last equation implies that $\xi_2 \in \mathcal{L}_2$. Since the matrix in Eq. (32) is Hurwitz, $\zeta \in \mathcal{L}_2$ [23]. Moreover, $\lim_{t\to\infty}\zeta(t) = 0$ and thus $\lim_{t\to\infty}\tilde{\xi}(t) = 0$. Therefore, given any $\gamma > 0$, there exists a time $t^* > 0$ such that $|\tilde{\xi}(t)| \leq \gamma$ for all $t \geq t^*$.

Let now $\gamma = \delta/\lambda_{\max}^{\frac{1}{2}}(P^{-1})$. Then one obtains that $\tilde{\xi}^T(t)P^{-1}\tilde{\xi}(t) \leq \lambda_{\max}(P^{-1})|\tilde{\xi}(t)|^2 \leq \delta^2$ for all $t \geq t^*$. From the definition of $\tilde{\mathcal{D}}_\delta^g$, the last inequality implies that $\xi(t) \in \tilde{\mathcal{D}}_\delta^g$ for all $t \geq t^*$.

For $k = \min\{\bar{u}_i\}$ one obtains $|u_1| = k\,|\xi_1|/\phi(\xi_1) < \bar{u}_1$, and similarly,

$$|u_2| = \frac{k}{\phi(\xi_1)}\frac{|\xi_2|}{\phi(\xi_2)} < \frac{k}{\phi(\xi_1)} < \bar{u}_2 \tag{33}$$

This completes the proof of the proposition.

The following Theorem combines the results of Corollary 5 and Proposition 6 to obtain a global controller bounded by a specified upper bound.

Theorem 7 *Let the system in Eq. (1) and consider the following control law*

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{if} \quad x(0) = 0 \\ \begin{pmatrix} \bar{u} \\ 0 \end{pmatrix} & \text{if} \quad x(0) \in \mathcal{S}\backslash\{0\} \\ \begin{pmatrix} \text{Eq. (29a)} \\ \text{Eq. (29b)} \end{pmatrix} & \text{if} \quad \xi(t) \in \tilde{\mathcal{D}}_\delta^b \\ \begin{pmatrix} \text{Eq. (12a)} \\ \text{Eq. (12b)} \end{pmatrix} & \text{if} \quad \xi(t) \in \tilde{\mathcal{D}}_\delta^g \end{cases} \tag{34}$$

Then, for all initial conditions $x(0) \in \mathbb{R}^n$, the closed-loop system trajectories satisfy the property $\lim_{t\to\infty} x(t) = 0$ and the control law is bounded as $|u_i| \leq \bar{u}_i$ $(i = 1, 2)$.

PROOF. Note that if $x(0) \notin \mathcal{S}$ then $x(t) \notin \mathcal{S}$ for all $t > 0$ and the control law in Eq. (34) is well defined for all $t \geq 0$. If $\xi \in \tilde{\mathcal{D}}_\delta^b$ from Proposition 6 we have that after some finite time t^*, the trajectories will enter $\tilde{\mathcal{D}}_\delta^g$. Since $\tilde{\mathcal{D}}_\delta^g$ is positively invariant (Corollary 5) the trajectories will remain there for all $t \geq t^*$ and will converge asymptotically to the origin.

From the previous discussion, it should be clear that the asymptotic convergence to the origin with the control law in Eq. (34) is exponential.

Remark 8 *Although the control law in Theorem 7 has a switching structure, the behavior at the boundary between the sets $\tilde{\mathcal{D}}_\delta^g$ and $\tilde{\mathcal{D}}_\delta^b$ is well-defined. Existence of trajectories is thus guaranteed and, in particular, there is no possibility of chattering at the boundary between the sets $\tilde{\mathcal{D}}_\delta^g$ and $\tilde{\mathcal{D}}_\delta^b$.*

6 Numerical example

In general, the matrix P from the solution of the Lyapunov equation (11) has a large condition number. This implies that the corresponding invariant sublevel sets are greatly skewed (especially for high dimensional systems). Thus, results obtained using the decomposition of the state space into the sets $\tilde{\mathcal{D}}_\delta^g$ and $\tilde{\mathcal{D}}_\delta^b$ may be overly conservative. From a practical point of view, it makes more sense to use the sets $\mathcal{D}_\delta^g$ and $\mathcal{D}_\delta^b$. The drawback of the latter implementation is that the set $\mathcal{D}_\delta^g$ is not necessarily invariant. In particular, the trajectories may exit $\mathcal{D}_\delta^g$. Nevertheless, by virtue of Theorem 2 the trajectories will tend to the origin and the control law will remain bounded. Implementation of the control law in Theorem 7 using this set decomposition requires that one should make sure that the control law switches no more than once from Eq. (29) to Eq. (12) in order to avoid chattering at the boundary between $\mathcal{D}_\delta^g$ and $\mathcal{D}_\delta^b$. For instance, a "practical" implementation of the controller in Theorem 7 would be the following

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{if } \; x(0) = 0 \\ \begin{pmatrix} \bar{u} \\ 0 \end{pmatrix} & \text{if } \; x(0) \in \mathcal{S} \setminus \{0\} \\ \begin{pmatrix} \text{Eq. (29a)} \\ \text{Eq. (29b)} \end{pmatrix} & \text{if } \; \xi(t) \in \mathcal{D}_\delta^b \text{ and } \xi(\tau) \notin \mathcal{D}_\delta^g \text{ for } \tau \in [0, t) \\ \begin{pmatrix} \text{Eq. (12a)} \\ \text{Eq. (12b)} \end{pmatrix} & \text{otherwise} \end{cases} \tag{35}$$

To demonstrate this approach, we consider a 5-dimensional chained form system. We assume $\bar{u}_i = 10$ $(i = 1, 2)$ and we choose $\delta = 0.125$. Because the minimum eigenvalue of the matrix P in Eq. (11) is typically small, the convergence in the set $\mathcal{D}_\delta^g$ may be slow. To keep the rates of convergence in both regions the same, for the simulations we have chosen $k = 10$ both in $\mathcal{D}_\delta^g$ and $\mathcal{D}_\delta^b$.

Our simulations showed that this value gives a good compromise between the maximum control input attained and the speed of response.

The simulations for an initial condition $x(0) = [1, 1, -2, -1, 3]$ and $\phi(x) = \sqrt{1 + x^2}$ are shown in Fig. 2. The upper plot shows the states and the lower plot in Fig. 2 shows the control inputs. Both control inputs are bounded by $\bar{u}_i$ as required.

For comparison, Fig. 3 shows the state and control histories for the corresponding control law without input constraints. The gains in Fig. 3 were chosen such that the convergence rates are approximately the same as for the bounded input case.

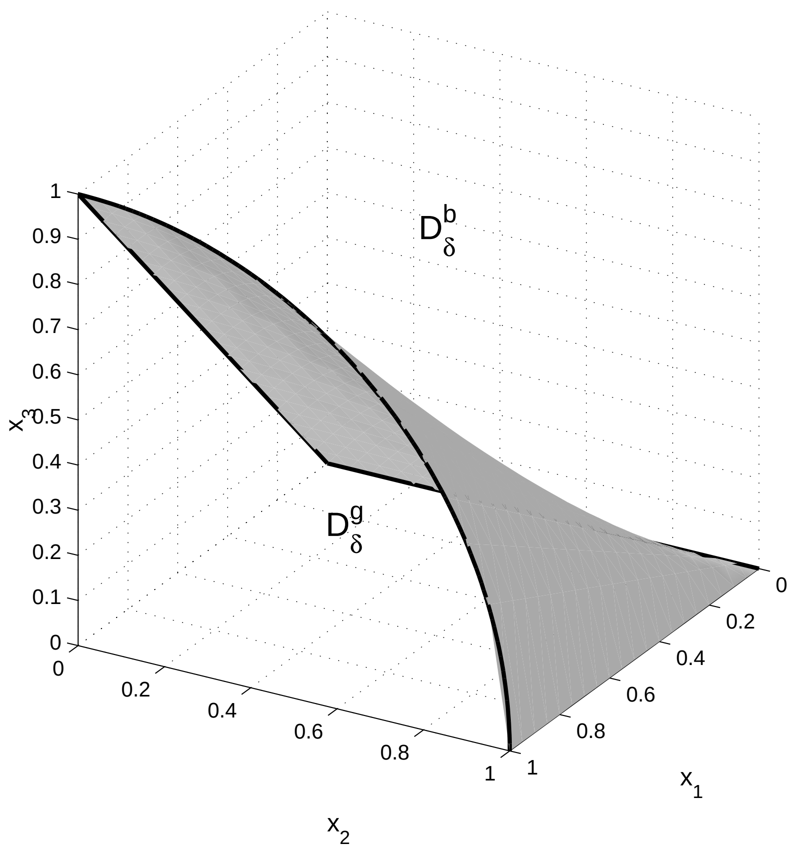

Fig. 1. The regions $\mathcal{D}_\delta^g$ and $\mathcal{D}_\delta^b = \mathcal{D} \backslash \mathcal{D}_\delta^g$ for $\delta = 1$ $(x_1, x_2, x_3 > 0)$.

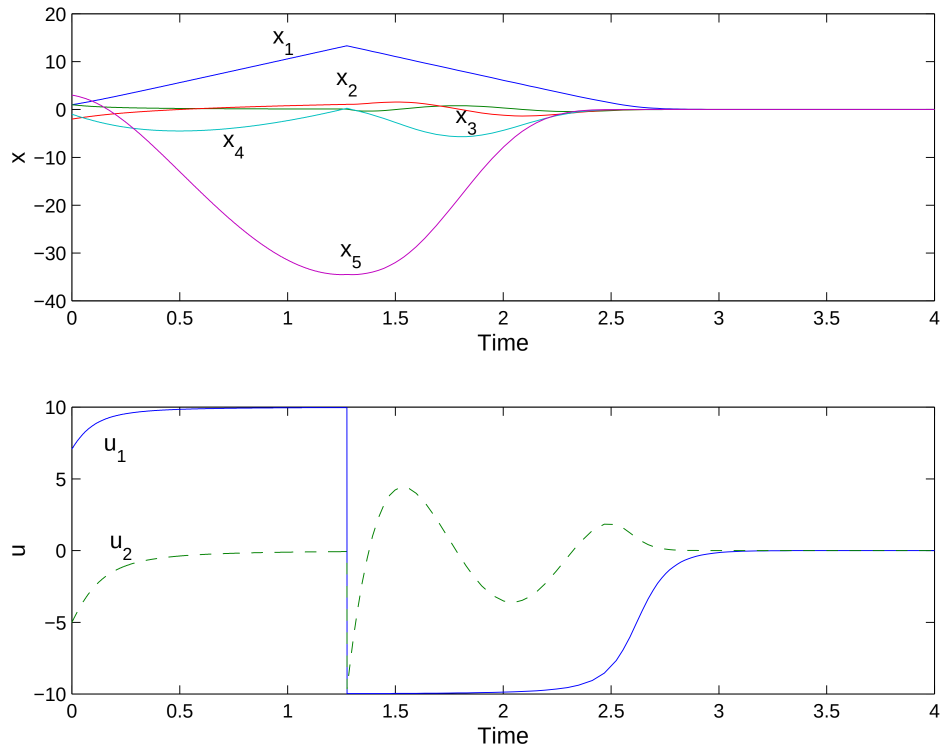

Fig. 2. History of states and control inputs with constraints.

7 Conclusion

In this paper, we describe an approach to address a common problem associated with a class of discontinuous controllers for nonholonomic systems proposed recently in the literature. Namely, these feedback controllers may require very large control inputs if the initial conditions are close to a singular manifold. The proposed methodology decomposes the state space into two separate regions and the controller forces all trajectories into a region of the state space where all control inputs are typically small. The control law guarantees exponential convergence of the closed-loop trajectories to the origin using bounded inputs.

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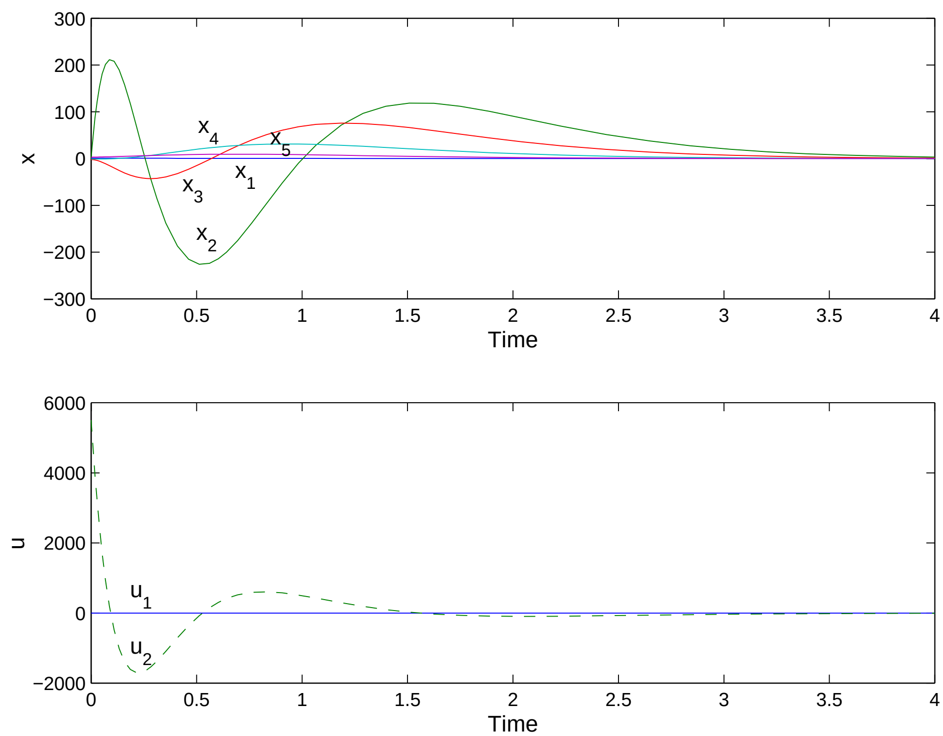

Fig. 3. History of states and control inputs without constraints.

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