STABILIZATION AND TRACKING OF UNDERACTUATED AXISYMMETRIC SPACECRAFT WITH BOUNDED CONTROL [1]

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Abstract: We provide stabilizing and tracking feedback control laws for the kinematic system of an underactuated axisymmetric spacecraft subject to input constraints. As a special case we also provide a feedback control to track a specified direction in inertial space. All proposed control laws achieve asymptotic stability with exponential convergence. One of the novelties of the proposed control design is the use of a new, non-standard description of the attitude motion, which allows the decomposition of the general motion into two rotations. This attitude description is especially useful for analyzing axisymmetric bodies, where the motion of the symmetry axis maybe of prime importance.

Keywords: Attitude control, satellite control, saturation, tracking, asymptotic stability.

1. INTRODUCTION

The problem of attitude stabilization has been the subject of numerous research articles in the last decade (Crouch, 1984; Byrnes and Isidori, 1991; Wen and Kreutz-Delgado, 1991; Krishnan *et al.*, 1992; Tsiotras *et al.*, 1995; Bach and Paielli, 1993). Most of these results deal with the case of complete control actuation. A complete mathematical description of the attitude stabilization problem was presented as early as 1984 by Crouch (1984), where he provided the necessary and sufficient conditions for the controllability of a rigid body in the case of one, two and three independent control torques. This sparked a renewed interest in the area of control of rigid spacecraft with less than three control torques. Stabilization of the angular velocity equations was addressed, for example, in (Aeyels and Szafranski, 1988; Sontag and Sussmann, 1988) and (Outbib and Sallet, 1992). The complete set of attitude equations (including the kinematics) was addressed in (Byrnes and Isidori, 1991) where they established that a rigid spacecraft controlled by two pairs of gas jet actuators cannot be asymptotically stabilized to an equilibrium using a smooth feedback control law. Subsequently, in (Krishnan *et al.*, 1992) and later in (Tsiotras *et al.*, 1995), nonsmooth controllers were established to stabilize an axisymmetric spacecraft. This is an interesting control problem because, as for the nonsymmetric case (Coron and Kerai, 1996; Morin and Samson, 1997), any stabilizing control law has to be necessarily nonsmooth. In addition, as was shown in (Sørdalen *et al.*, 1992), this problem is equivalent to a well-studied benchmark problem in the area of nonholonomic systems, namely, that of the nonholonomic integrator or, equivalently, of a three-wheel mobile robot. Khennouf and Canudas de Wit (1995) have shown how to construct discontinuous controllers for this problem by extending the results of (Tsiotras *et al.*, 1995). The controller in (Tsiotras *et al.*, 1995), in particular, is not Lipschitz continuous at the equilibrium, and may require significant amounts of control effort, especially if the initial conditions are close to an equilibrium manifold. In (Tsiotras and Luo, 1996) this controller was modified, to remedy the problem of large control inputs. The procedure in (Tsiotras and Luo, 1996) consists of dividing the state space into two regions. The control law drives the trajectories of the close-loop system away from the singular equilibrium manifold (which gives rise to high control inputs) and into the region in the state space where the high authority part of the control input remains small.

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In this paper, we continue the approach initiated in (Tsiotras and Luo, 1996) and derive a controller for the kinematics of an axisymmetric spacecraft with two inputs (and zero spin rate) which remains bounded by an *a priori* specified bound. We make use of the formulation for the attitude kinematics developed in (Tsiotras and Longuski, 1995). This attitude description allows one to isolate and describe the motion of the symmetry axis of the body using a single complex variable. We also solve the problem of tracking an attitude trajectory for an axisymmetric spacecraft with two control inputs. Finally, as a special case, we present a feedback control law to track a specified direction in inertial space. Numerical examples demonstrate the theoretical developments.

2. THE (w,z) ATTITUDE PARAMETERIZATION

The orientation of a rigid spacecraft can be specified using various parameterizations, for example, Eulerian Angles, Euler Parameters, Cayley-Rodrigues Parameters, etc; see, for instance, the recent survey article by Shuster (1993). Recently, a new parameterization using a pair of a complex and a real coordinate was introduced based on an extension of an old result by Darboux (Darboux, 1887; Tsiotras and Longuski, 1995). According to the results of (Tsiotras *et al.*, 1995) the relative orientation between two reference frames can be represented by *two successive rotations*. The first rotation is about the inertial $\hat{\imath}_3$-axis at an angle z. The second rotation is about the unit vector

$$\hat{h} = \left(\frac{w+\bar{w}}{2|w|}\right)\hat{\imath}'_1 + \left(\frac{\mathrm{i}(\bar{w}-w)}{2|w|}\right)\hat{\imath}'_2 \qquad (1)$$

and has magnitude

$$\theta = \arccos\left(\frac{1-|w|^2}{1+|w|^2}\right) \qquad (2)$$

In Eq. (1) $\hat{\mathbf{i}}' = (\hat{\imath}'_1, \hat{\imath}'_2, \hat{\imath}'_3)$ is the intermediate reference frame resulting from the rotation z about the inertial $\hat{\imath}_3$-axis. The situation is depicted in Fig. 1, where (a,b,c) denote the coordinates of the unit vector $\hat{\imath}'_3$ in the body frame, $\hat{\imath}'_3 = a\hat{b}_1 + b\hat{b}_2 + c\hat{b}_3$. It can be shown (Tsiotras and Longuski, 1995) that the location of the body $\hat{b}_3$-axis in the $\hat{\mathbf{i}}'$ frame is also determined by a,b,c from $\hat{b}_3 = -a\hat{\imath}'_1 - b\hat{\imath}'_2 + c\hat{\imath}'_3$ (Fig. 1). With this notation, the w coordinate is defined by

$$w = \frac{b - \mathrm{i}a}{1+c} \qquad (3)$$

We note here that in Eqs. (1) and (2) $\mathrm{i} = \sqrt{-1}$, bar denotes the complex conjugate, and $|w|^2 = w\bar{w}$ denotes the absolute value of the complex number w. Conversely, from w one can compute (a,b,c) from $a = \mathrm{i}(w-\bar{w})/(1+|w|^2), b = (w+\bar{w})/(1+|w|^2)$ and $c = (1-|w|^2)/(1+|w|^2)$.

The rotation matrix corresponding to the (w,z) kinematic description has been calculated in (Tsiotras and Longuski, 1995). Conversely, given a proper rotation matrix R, one can compute w and z as follows.

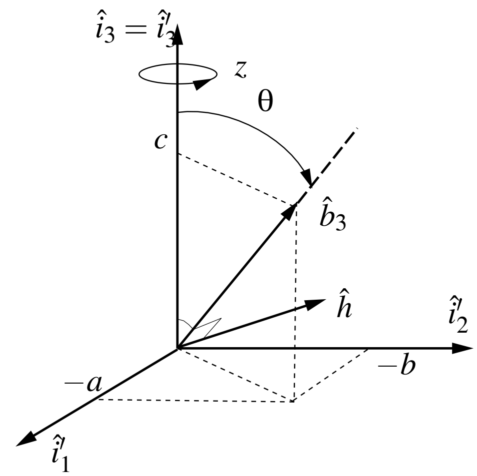


Fig. 1. Attitude description in terms of (w,z) coordinates.

Lemma 2.1. For any rotation matrix $R \in SO(3)$, let

$$w = \frac{R_{23} - \mathrm{i}R_{13}}{1+R_{33}} \qquad (4)$$

and

$$\cos z = \tfrac{1}{2}\left((1+|w|^2)\,trace(R) + |w|^2 - 1\right) \qquad (5a)$$

$$\sin z = \frac{1}{1+|w|^2}\left[(1+Re(w^2))R_{12} + Im(w^2)R_{22} - 2\,Im(w)R_{32}\right] \qquad (5b)$$

Then (w,z) are the corresponding attitude coordinates for the matrix R.

The kinematic equations in terms of w and z can be written as follows (Tsiotras *et al.*, 1995; Tsiotras and Longuski, 1995)

$$\dot{w} = -\mathrm{i}\omega_3 w + \frac{\omega}{2} + \frac{\bar{\omega}}{2}w^2 \qquad (6a)$$

$$\dot{z} = \omega_3 + Im(\omega\bar{w}) \qquad (6b)$$

where $\omega = \omega_1 + \mathrm{i}\omega_2$ and $w = w_1 + \mathrm{i}w_2$.

In this paper we assume that only the angular velocity ω (equivalently, ω_1 and ω_2) can be manipulated. The angular velocity component about the body $\hat{b}_3$-axis ω_3 cannot be changed due to, say, a thruster failure. In this case, three-axis stabilization and pointing is possible only if, in addition, $\omega_3 \equiv 0$.

Letting $\omega_3 = 0$ the rigid spacecraft kinematic equations become

$$\dot{w} = \frac{\omega}{2} + \frac{\bar{\omega}}{2}w^2 \qquad (7a)$$

$$\dot{z} = Im(\omega\bar{w}) \qquad (7b)$$

In (Tsiotras *et al.*, 1995) the following feedback control law was proposed in order to stabilize (7)

$$\omega = -kw - \mathrm{i}\mu\frac{z}{\bar{w}}, \qquad \mu > k/2 \qquad (8)$$

3. STABILIZATION WITH BOUNDED CONTROL

Without any further modification, the domain of validity of the system in Eqs. (7)-(8) is the set of pairs $(w,z) \in (\mathbb{C}\backslash\{0\}) \times S^1$. Equation (8) suggests that the control inputs may become very large for initial conditions close to the manifold $w = 0$ (and $z \neq 0$). In addition, Eq. (8) suggests that the control input ω will remain "small" if the trajectories belong to the set

$$\mathcal{D}_g = \{(w,z) \in \mathbb{C} \times S^1 : |z|/|w| \leq 1\} \qquad (9)$$

We seek to construct a control law that will keep all trajectories in $\mathcal{D}_g$ and force the trajectories outside $\mathcal{D}_g$ to enter this set in finite time.

Before we state the main result in this section we need the following definition.

Definition 1. Given two scalars $z \in \mathbb{R}$ and $w \in \mathbb{C}$, we define the *complex saturating function* $\mathrm{sat_c}(\cdot)$ by

$$\mathrm{sat_c}(z,w) = \begin{cases} 0 & \text{if } z = 0,\, w = 0 \\ \mathrm{sat}\left(\dfrac{z}{|w|}\right) e^{\mathrm{i}\phi} & \text{if } w \neq 0 \\ \mathrm{sgn}(z) & \text{if } z \neq 0,\, w = 0 \end{cases} \qquad (10)$$

and $\phi = \arg(w)$ is the argument of w, i.e, $w = |w|e^{\mathrm{i}\phi}$.

The function $\mathrm{sat_c}$ is defined for all $(w,z) \in \mathcal{D} := \mathbb{C} \times S^1$. The following proposition provides a stabilizing control law which is bounded by a specified constant.

Proposition 3.1. Consider the system in Eq. (7) and the following control law

$$\omega = -k\frac{w}{\sqrt{1+|w|^2}} - \mathrm{i}\mu\,\mathrm{sat_c}(z,w) \qquad (11)$$

where $\mathrm{sat_c}(z,w)$ as in Definition 1, and where k and μ are constants satisfying

$$\mu > k/2 > 0 \quad \text{if} \quad (w,z) \in \mathcal{D}_g \qquad (12a)$$
$$\mu > -k > 0 \quad \text{if} \quad (w,z) \in \mathcal{D}_b := \mathcal{D}\backslash\mathcal{D}_g \qquad (12b)$$

Then, for all initial conditions $(w(0),z(0)) \in \mathcal{D}$, the control law (11) is well-defined and the corresponding closed-loop trajectories satisfy $\lim_{t\to\infty}(w(t),z(t)) = 0$. In addition, the control law is bounded as $|\omega(t)| \leq \max\{|k|\} + \mu$ for all $t \geq 0$, where $\max\{|k|\}$ denotes the maximum of the absolute values of k in $\mathcal{D}_b$ and $\mathcal{D}_g$.

Proof. Consider the positive definite, radially unbounded function $V : \mathbb{C} \times \mathbb{R} \to \mathbb{R}_+$ defined by $V(w,z) = 2(\sqrt{1+|w|^2} - 1) + \frac{1}{2}z^2$. The derivative of V along the closed-loop trajectories yields

$$\begin{aligned}\dot{V} &= \frac{1}{\sqrt{1+|w|^2}}(1+|w|^2)\,Re(\omega\,\bar{w}) + z\,Im(\omega\,\bar{w}) \\ &= -k\,|w|^2 - \mu z\,\mathrm{sat}\left(\frac{z}{|w|}\right)|w| \end{aligned} \qquad (13)$$

If $(w,z) \in \mathcal{D}_b$ then $|z|/|w| > 1$ and $z\,\mathrm{sat}(z/|w|) = z\,\mathrm{sgn}(z) = |z|$. Since $\mu > -k > 0$ one obtains from Eq. (13)

$$\dot{V} = -|w|^2(k+\mu|z|/|w|) < -|w|^2(k+\mu) < 0 \quad (14)$$

for all $(w,z) \in \mathcal{D}_b$. Notice also that in $\mathcal{D}_b$, $V \geq 2(\sqrt{1+|w|^2} - 1) + \frac{1}{2}|w|^2$. The last equation, along with Eq. (14) imply that if the trajectories remain in $\mathcal{D}_b$, then $\lim_{t\to\infty}|w(t)| = 0$. This leads to a contradiction, since $d|w|^2/dt = -k\,|w|^2\sqrt{1+|w|^2}$ with $k < 0$, and $|w|$ is monotonically increasing in $\mathcal{D}_b$. Therefore, the trajectories leave $\mathcal{D}_b$ and enter the region $\mathcal{D}_g$ in finite time. Moreover, for $(w,z) \in \mathcal{D}_g$ we have that $|z|/|w| \leq 1$ and hence from Eq. (13)

$$\dot{V} = -k\,|w|^2 - \mu z^2 < 0 \qquad \forall (w,z) \in \mathcal{D}_g \quad (15)$$

since $k > 0$ for $(w,z) \in \mathcal{D}_g$.

We have shown that $\dot{V} < 0$ for all $(w,z) \in \mathcal{D}$ and hence $\lim_{t\to\infty} V(t) = 0$. In particular, $\lim_{t\to}(w(t),z(t)) = 0$. The asymptotic convergence to the origin is exponential, as can be easily seen by checking the the closed-loop system for $(w,z) \in \mathcal{D}_g$,

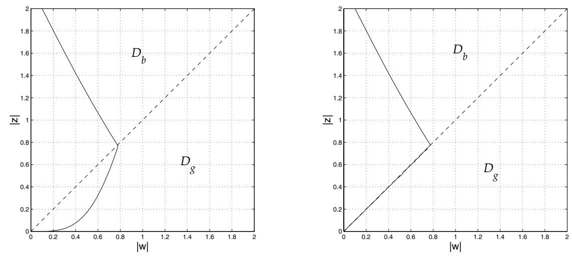
$$\frac{d|w|^2}{dt} = -k\,|w|^2\sqrt{1+|w|^2} \leq -k\,|w|^2 \qquad (16a)$$
$$\dot{z} = -\mu z \qquad (16b)$$

Moreover, a straightforward calculation shows that $|\omega(t)| \leq |k||w(t)|/\sqrt{1+|w(t)|^2} + \mu \leq \max\{|k|\} + \mu$ for all $t \geq 0$ and the control law is bounded. ■

The motivation behind the proposed control law is simple. It forces all trajectories to the "good" region $\mathcal{D}_g$ where the potentially bothersome term $z/\bar{w}$ in Eq. (8) is bounded by a known constant. Moreover, Eqs. (16) show that if $\mu > k/2$ the vector fields on $|z| = |w|$ point in the interior of $\mathcal{D}_g$ and thus, $\mathcal{D}_g$ is a positively invariant set of the closed-loop trajectories. Once in $\mathcal{D}_g$, Eqs. (16) ensure that trajectories go to the origin with exponential rate of decay. As a result, there is at most one switching as the control law crosses the boundary $|z| = |w|$ and there is no possibility of chattering. Although the control law in Eq. (11) is discontinuous, the solutions of the closed-loop system are well-defined and unique.

Remark 3.1. We can use Eq. (16) to introduce a sliding mode defined by the equation $|z| = |w|$ by making the vector field on the boundary of $\mathcal{D}_g$ point to the interior of $\mathcal{D}_b$. This can be achieved by choosing, for example, $k > 2\mu$ in Eq. (12b).

Figure 2(a) shows the sets $\mathcal{D}_b$ and $\mathcal{D}_g$ in the $(|w|,|z|)$ space, along with typical trajectories for the closed-loop system in Eq. (7) with the control law in Eq. (11). Figure 2(b) shows the corresponding trajectories when choosing $k > 2\mu$ in $\mathcal{D}_g$. The trajectories tend to the origin along the sliding mode described by the boundary of the sets $\mathcal{D}_g$ and $\mathcal{D}_b$, i.e., along $|z| = |w|$ (see also Remark 3.1 above).



(a) Closed-loop trajectories with control in Eq. (11).

(b) Sliding mode for the case $k > 2\mu$ in $\mathcal{D}_g$.

Fig. 2. Typical closed-loop trajectories for the system of Eqs. (7)-(11) and the sets $\mathcal{D}_b$ and $\mathcal{D}_g$.

4. TRACKING OF AN UNDERACTUATED SPACECRAFT

In this section we derive a controller for an underactuated spacecraft to track a desired attitude. The desired attitude history is given in terms of the complex/real parameters of Section 2 as $w_d(t)$ and $z_d(t)$. These parameters represent the orientation of a "virtual" spacecraft in *inertial* space. The governing kinematic equations for this "virtual" spacecraft are of the same form as Eqs. (7)

$$\dot{w}_d = \frac{\omega_d}{2} + \frac{\bar{\omega}_d}{2} w_d^2 \tag{17a}$$

$$\dot{z}_d = Im(\omega_d \bar{w}_d) \tag{17b}$$

where $\omega_d = \omega_{d_1} + \mathrm{i}\,\omega_{d_2}$ is the complex variable of the known angular velocities expressed in the "virtual" frame. They are assumed to be bounded by $|\omega_{d_i}(t)| \leq \beta_i$ for $i = 1, 2$.

We wish to design a control law $\omega = \omega(w, z, \omega_d, w_d, z_d)$ such that it satisfies the following two requirements:

(R1) If $w(0) = w_d(0)$ and $z(0) = z_d(0)$ then $w(t) = w_d(t)$ and $z(t) = z_d(t)$ for all $t \geq 0$.
(R2) For all initial conditions $(w(0), z(0)) \in \mathbb{C} \times S^1$ we have that $\lim_{t\to\infty}(w(t), z(t)) = (w_d(t), z_d(t))$.

Let the inertial frame be $\hat{\mathbf{i}} = (\hat{\imath}_1, \hat{\imath}_2, \hat{\imath}_3)$, the body frame of the spacecraft be $\hat{\mathbf{b}} = (\hat{b}_1, \hat{b}_2, \hat{b}_3)$, and the reference frame on the "virtual" spacecraft be $\hat{\mathbf{v}} = (\hat{v}_1, \hat{v}_2, \hat{v}_3)$. We can then express the body frame of the spacecraft in the reference frame of the "virtual" spacecraft as follows $\hat{\mathbf{b}} = R(w, z) R^T(w_d, z_d)\, \hat{\mathbf{v}} := R_r(w_r, z_r)\, \hat{\mathbf{v}}$ where $R_r(w_r, z_r)$ is the rotation matrix from $\hat{\mathbf{v}}$ to $\hat{\mathbf{b}}$ and where w_r and z_r are the corresponding attitude coordinates. Lemma 2.1 shows how to compute (w_r, z_r) from (w, z) and (w_d, z_d), which can then serve as a coordinate description of the relative orientation between the $\hat{\mathbf{b}}$ and $\hat{\mathbf{v}}$ frames.

The angular velocity between these two frames (expressed in the $\hat{\mathbf{b}}$ frame) is given by

$$\begin{bmatrix} \omega_{r_1} \\ \omega_{r_2} \\ \omega_{r_3} \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ 0 \end{bmatrix} - R_r(w_r, z_r) \begin{bmatrix} \omega_{d_1} \\ \omega_{d_2} \\ 0 \end{bmatrix} \tag{18}$$

The kinematic equations of the target frame (as seen from $\hat{\mathbf{b}}$) are therefore given by

$$\dot{w}_r = -\mathrm{i}\,\omega_{r_3} w_r + \frac{\omega_r}{2} + \frac{\bar{\omega}_r}{2} w_r^2 \tag{19a}$$

$$\dot{z}_r = \omega_{r_3} + Im(\omega_r \bar{w}_r) \tag{19b}$$

Proposition 4.1. Let the kinematics of the spacecraft described by Eqs. (7), and the kinematics of the target attitude trajectory generated by Eqs. (17) for some known $\omega_d(t)$. Consider the controller

$$\omega = -k\, w_r - \mathrm{i}\left(\frac{\mu z_r + \omega_{r_3}}{\bar{w}_r}\right) + \eta(R_r, \omega_d) \tag{20}$$

where w_r and z_r as in Eqs. (4)-(5), R_r is the rotation matrix from the target to the body frame of the spacecraft, $k > 0$ and $\mu > k/2$ are constants, and

$$\omega_{r3} = -R_{r_{31}}\,\omega_{d_1} - R_{r_{32}}\,\omega_{d_2} \tag{21a}$$

$$\eta(R_r, \omega_d) = R_{r_{11}}\,\omega_{d_1} + R_{r_{12}}\,\omega_{d_2} + \mathrm{i}\,(R_{r_{21}}\,\omega_{d_1} + R_{r_{22}}\,\omega_{d_2}) \tag{21b}$$

Then this kinematic controller is well-defined for all $t \geq 0$. Moreover, for all initial conditions such that $w(0) \neq w_d(0)$ we have that $\lim_{t\to\infty}(w(t), z(t)) = (w_d(t), z_d(t))$. In addition, this controller is bounded along the closed-loop trajectories.

Proof. First notice that the relative angular velocity between $\hat{\mathbf{b}}$ and $\hat{\mathbf{v}}$ is given by

$$\omega_r := \omega_{r_1} + \mathrm{i}\,\omega_{r_2} = -k\, w_r - \mathrm{i}\left(\frac{\mu z_r + \omega_{r_3}}{\bar{w}_r}\right) \tag{22}$$

Substituting the previous equation in Eqs. (19) one obtains

$$\frac{d}{dt}|w_r|^2 = -k\,|w_r|^2\,(1 + |w_r|^2) \tag{23a}$$

$$\dot{z}_r = -\mu z_r \tag{23b}$$

and thus, $\lim_{t\to\infty}(w_r(t), z_r(t)) = 0$ with exponential rate of decay for all $(w_r, z_r) \in \mathcal{D}$.

The control law in Eq. (22) is well defined for all initial conditions $(w_r, z_r) \in (\mathbb{C}\backslash\{0\}) \times S^1$ since if $w_r(0) \neq 0$ Eq. (23a) implies that $w_r(t) \neq 0$ for all $t \geq 0$.

It remains to show that the control law in Eq. (22) is bounded. From Eq. (23a) one readily obtains that w_r is bounded. Moreover, using Eqs. (23) a direct calculation shows that $z_r/\bar{w}_r$ is bounded if $\mu > k/2$. In addition, from Eq. (21a) one obtains that

$$\frac{|\omega_{r_3}|}{|w_r|} \leq \frac{|R_{r_{31}}|}{|w_r|}|\omega_{d_1}| + \frac{|R_{r_{32}}|}{|w_r|}|\omega_{d_2}| \leq \frac{2}{1 + |w_r|^2}(|\omega_{d_1}| + |\omega_{d_2}|) \leq 2\,(\beta_1 + \beta_2) \tag{24}$$

where we have used the fact that $|Re(we^{\mathrm{i}z})| \leq |w|$ and $|Im(we^{\mathrm{i}z})| \leq |w|$ for any $w \in \mathbb{C}$. Also, since R_r is a rotation matrix, a direct calculation shows that $|\eta(R_r, \omega_d)| \leq |\omega_d| \leq |\omega_{d_1}| + |\omega_{d_2}| = \beta_1 + \beta_2$ and $\eta(R_r, \omega_d)$ is bounded. Thus, ω is bounded. This completes the proof of the proposition. ■

A tracking controller bounded by a given upper bound can be obtained simply by combining the results of Propositions 3.1 and 4.1.

Theorem 4.1. Let the kinematics of a spacecraft described by Eqs. (7), and the kinematics of a target attitude trajectory generated by Eqs. (17) where $|\omega_{d_i}(t)| \leq \beta_i$, for $i = 1,2$. Consider a constant $\beta_3 > 3(\beta_1 + \beta_2)$. Let the feedback control law

$$\omega = -k \frac{w_r}{\sqrt{1+|w_r|^2}} - \mathrm{i}\,\mu\,\mathrm{sat_c}(z_r, w_r) - \mathrm{i}\,\frac{\omega_{r_3}}{\bar{w}_r} + \eta(R_r, \omega_d) \tag{25}$$

where $w_r, z_r, R_r, \omega_{r_3}$, and $\eta(R_r, \omega_d)$ as in Proposition 4.1. Assume that the gains k and μ are as in Eq. (12) and that satisfy $\max\{|k|\} + \mu < \beta_3 - 3(\beta_1 + \beta_2)$. Then the control law in Eq. (25) is well-defined for all $(w,z) \in \mathcal{D}$, satisfies the requirements *(R1)* and *(R2)* and it is bounded by $|\omega(t)| \leq \beta_3$ for all $t \geq 0$.

Proof. The proof is straightforward and it is left to the interested reader. ∎

5. SPECIAL CASE: TRACKING OF THE SYMMETRY AXIS

The results of the previous section can also be used in the special case of tracking a specific *direction* in inertial space with the body $\hat{b}_3$-axis (which we assume to be symmetry axis of the axisymmetric spacecraft). This would be the case when, for example, the symmetry axis is the axis of a communications antenna, the line-of-sight of an onboard telescope or camera, etc. In all these case, the relative rotation about the symmetry axis is irrelevant. In particular, the body is now allowed to rotate about its $\hat{b}_3$-axis at a constant angular rate ω_{30}.

It is assumed that the desired pointing direction with respect to the inertial frame is given as $w_d(t)$. Consulting Fig. 1 this implies that the desired direction in inertial frame is given by the unit vector $\hat{v}_3 = -a_d\,\hat{\imath}_1 - b_d\,\hat{\imath}_2 + c_d\,\hat{\imath}_3$ where $w_d = (b_d - \mathrm{i}\,a_d)/(1 + c_d)$. A tracking controller's objective is then to make $\hat{b}_3$ track $\hat{v}_3$ as $t \to \infty$.

Proposition 5.1. Consider the system of Eqs. (6) describing the orientation of a rigid spacecraft in inertial frame. Let the direction along the unit vector in inertial frame given by $\hat{v}_3$ where $|\omega_{d_i}(t)| \leq \beta_i$ for $i = 1,2$. Let the control law

$$\omega = -k \frac{w_r}{\sqrt{1+|w_r|^2}} + \eta(R, \omega_d) \tag{26}$$

where $k > 0$, and where $\eta(R, \omega_d) = R_{11}\,\omega_{d_1} + R_{12}\,\omega_{d_2} + \mathrm{i}\,(R_{21}\,\omega_{d_1} + R_{22}\,\omega_{d_2})$ with $R = R(w,z)$. Then with this control law the body $\hat{b}_3$-axis will track exponentially the direction along the unit vector $\hat{v}_3$ from all initial conditions. Moreover, the control law is bounded by $|\omega| \leq k + \beta_1 + \beta_2$ for all $t \geq 0$.

Remark 5.1. The complex variable w_r serves the purpose of an "error" between the $\hat{v}_3$ and $\hat{b}_3$ unit axes. However, notice that $w_r = 0$ does *not* necessarily imply that $w = w_d$. This is due to our specific definitions for w and w_d.

6. NUMERICAL EXAMPLE

In this section we provide a numerical example to demonstrate the control laws of Sections 4 and 5. For the attitude tracking problem, we consider the kinematic equations of a rigid body, described by Eqs. (7). We let the trajectory to be tracked generated by the system in Eqs. (17) where $\omega_d(t) = 0.5\sin(0.5t) + \mathrm{i}\cos(0.25t)$. The initial conditions are given by $(w(0), z(0)) = (5+\mathrm{i}, 3)$ and $(w_d(0), z_d(0)) = (\mathrm{i}, 2.5)$.

Figure 3 shows a series of "snapshots" of the actual orientation of the body and the target reference frames. The solid parallelepiped in the figure represents the rigid spacecraft while the wire frame represents the "virtual" spacecraft along the desired attitude history. Figure 3 shows clearly that tracking of the target frame has been achieved after approximately 5 sec.

The next example demonstrates tracking of a desired *direction* in inertial space. The body is assumed axisymmetric having a constant velocity component about the $\hat{b}_3$ axis equal to $\omega_{30} = -0.5$ r/s. The control law in Eq. (26) is used with $k = 2$. The reference trajectory for the unit vector $\hat{v}_3$ is generated by the system in Eqs. (17a) with $\omega_d(t) = t\,\sin(0.5t) + \mathrm{i}\,1.5\cos(t)$.

The actual orientation of the spacecraft during the tracking maneuver is shown in Fig. 4. The solid line in Fig. 4 represents the desired reference direction $\hat{v}_3$. Figure 4 shows that tracking of $\hat{v}_3$ has been achieved after approximately 4 sec.

7. CONCLUSIONS

In this paper we solve the problems of stabilization and tracking of an underactuated rigid spacecraft. An example of this situation is the case of an axisymmetric rigid spacecraft with a thruster failure along the symmetry axis. For the restricted case of zero spin rate, stabilization is possible but any stabilizing control laws has to be nonsmooth. We present such a control law which, in addition, remains bounded by an *a priori* specified bound. We then extend these stabilization results to develop controllers which are able to track a given attitude trajectory. As a special case, we also present a control law to track an arbitrary direction in the inertial space using two bounded control inputs. The proposed control laws achieve asymptotically stability and tracking with (asymptotic) exponential convergence rates for all initial conditions. One of the novelties of the proposed approach is the use of a recently developed, non-standard coordinate attitude parameterization.

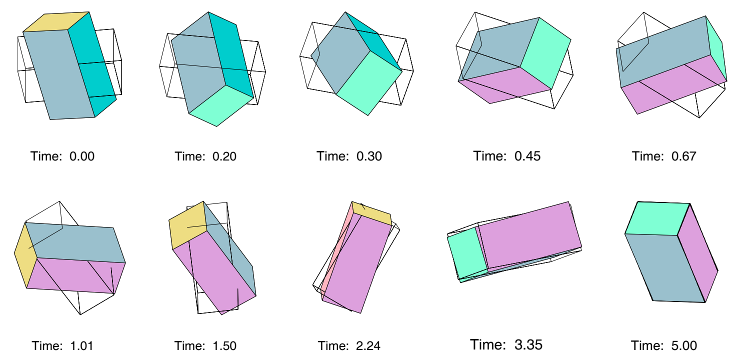


Fig. 3. Snapshots of the attitude orientation history. The wire frame represents the "virtual" spacecraft which furnishes the reference attitude to be tracked.

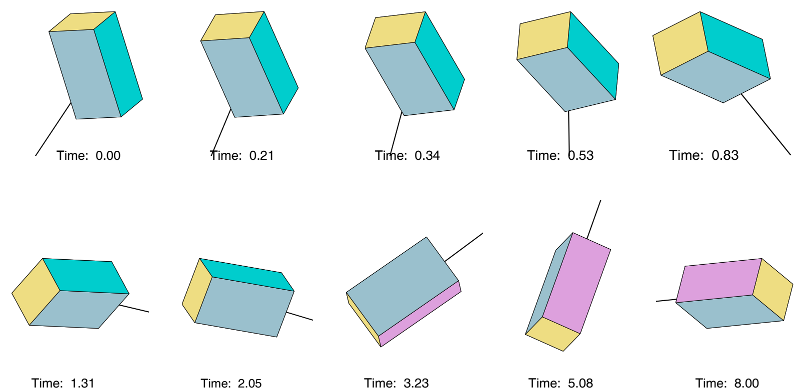


Fig. 4. Snapshots of the attitude orientation history for the reference direction tracking problem. The solid line represents the desired direction in inertial frame.

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