Optimal Covariance Control for Stochastic Systems Under Chance Constraints

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***Abstract*—This letter addresses the optimal covariance control problem for stochastic discrete-time linear systems subject to chance constraints. To the best of our knowledge, covariance steering problems with probabilistic chance constraints have not been discussed previously in the literature, although their treatment seems to be a natural extension. In this letter, we first show that, unlike the case with no chance constraints, the covariance steering problem with chance constraints cannot be decoupled to mean and covariance steering sub-problems. We then propose an approach to solve the covariance steering problem with chance constraints by converting it to a convex programming problem. The proposed algorithm is verified using a numerical example.**

***Index Terms*—Stochastic systems, stochastic optimal control, uncertain systems.**

I. Introduction

In this letter we address the problem of finite-horizon stochastic optimal control for a discrete-time linear time-varying stochastic system with a fully-observable state, a given Gaussian distribution of the initial state, and a state and input-independent white-noise Gaussian diffusion with given statistics. The control task is to steer the system state to the target Gaussian distribution, while minimizing a state and control expectation-dependent cost. In addition to the boundary condition, in the aim of adding robustness to the controller under stochastic uncertainty, we consider *chance constraints*, restricting the probability of violating the state constraints to be less than a pre-specified threshold.

Since the Gaussian distribution can be fully defined by its first two moments, this problem can be described as a finite-time optimal mean and covariance steering problem of a stochastic time-varying discrete linear system, with a boundary conditions in the form of given initial and final mean and covariance, and with constraints on the trajectory in the form of a probability function.

The chance-constrained optimal covariance control problem is relevant to a wide range of control and planning tasks, such as decentralized control of swarm robots [1], closed-loop cooling [2], and others, in which the state is more naturally described by its distribution, rather than a fixed set of values. In addition, this approach is readily-applicable to a stochastic MPC framework [3].

The problem of controlling the state covariance of a linear system goes back to the late 80s. The so-called covariance steering (or "Covariance Assignment") problem was first introduced by Hotz and Skelton [4], where they computed the state feedback gains of a linear time-invariant system, such that the state covariance converges to a pre-specified value. Since then, many works have been devoted to this problem of infinite-horizon covariance assignment, both for continuous and discrete time systems [5]–[9]. Recently, the finite-horizon covariance control problem has been investigated by a number of researchers [10]–[13], relating to the problems of Shrödinger bridges [14] and Optimal Mass Transfer [15]. Others, including our previous work [16], showed that the finite covariance control problem solution can be seen as a LQG with a particular terminal weights [16], [17], which can be also formulated (and solved) as an LMI problem [18]–[20].

The chance-constrained optimization has been extensively studied since 50's, with the purpose of system design with guaranteed performance under uncertainty [21]. A stochastic model-predictive control design with a chance-constraints has been solved using various techniques (see [22] for an extensive review).

This letter contributes to this line of work by adding chance state constraints to the underlying stochastic optimal covariance steering problem. The covariance control problem is reformulated as a convex optimization problem, with a decision variable that is quadratic in the cost function. To the best of the authors' knowledge, this letter is the first that solves the covariance-steering problem with chance constraints.

II. Problem Statement

A. Problem Formulation

We consider the following discrete-time stochastic linear system (possibly time-varying) with additive uncertainty,

$$x_{k+1} = A_k x_k + B_k u_k + D_k w_k, \tag{1}$$

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where $k = 0, 1, \ldots, N-1$ is the time step, $x \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ is the control input, and $w \in \mathbb{R}^{n_w}$ is a zero-mean white Gaussian noise with unit covariance, that is, $\mathbb{E}[w_k] = 0$ and $\mathbb{E}[w_{k_1} w_{k_2}^\top] = I_{n_w}\delta_{k_1,k_2}$. We assume that $\mathbb{E}[x_{k_1} w_{k_2}^\top] = 0$ for $0 \le k_1 \le k_2 \le N$. The initial state x_0 is a random vector drawn from the normal distribution

$$x_0 \sim \mathcal{N}(\mu_0, \Sigma_0), \tag{2}$$

where $\mu_0 \in \mathbb{R}^{n_x}$ is the initial state mean and $\Sigma_0 \in \mathbb{R}^{n_x \times n_x}$ is the initial state covariance. We assume that $\Sigma_0 \succeq 0$. Our objective is to steer the trajectories of the system (1) from this initial distribution to the terminal Gaussian distribution

$$x_N \sim \mathcal{N}(\mu_N, \Sigma_N), \tag{3}$$

where $\mu_N \in \mathbb{R}^{n_x}$ and $\Sigma_N \in \mathbb{R}^{n_x \times n_x}$ with $\Sigma_N \succ 0$, at a given time N, while minimizing the cost function

$$\begin{aligned} &J(x_0, \ldots, x_{N-1}, u_0, \ldots, u_{N-1}) \\ &\quad = \mathbb{E}\left[\sum_{k=0}^{N-1} x_k^\top Q_k x_k + u_k^\top R_k u_k\right], \end{aligned} \tag{4}$$

where $Q_k \succeq 0$ and $R_k \succ 0$ for all $k = 0, 1, \ldots, N-1$.

The objective is to compute the optimal control input, which ensures that the probability of the state violation at any given time is below a pre-specified threshold, say,

$$\texttt{Pr}(x_k \notin \chi) \le P_{\text{fail}}, \qquad k = 1, \ldots, N, \tag{5}$$

where $\texttt{Pr}()$ denotes the probability of an event, $\chi \subset \mathbb{R}^{n_x}$ is the state constraint set, and $P_{\text{fail}} \in [0, 1]$ is the threshold for the probability of failure. Optimization problems with these types of constraints are known as *chance-constrained* optimization problems [23]. In this letter, we assume for simplicity that χ is convex, but chance-constraints with non-convex constraints are also possible (see [24]).

It is assumed that the system (1) is controllable, that is, given any $x_N \in \mathbb{R}^{n_x}$ and $x_0 \in \mathbb{R}^{n_x}$, and provided that $w_k = 0$ for $k = 0, \ldots N-1$, there exists a sequence of control inputs $\{u_k\}_{k=0}^{N-1}$ that steers x_0 to x_N.

B. Preliminaries

We provide an alternative description of the system dynamics in (1) that will be instrumental for solving the covariance sterring problem. The notation below is borrowed from [16]. Let A_{k_1,k_0}, B_{k_1,k_0}, and D_{k_1,k_0}, where $k_1 > k_0$, denote the transition matrices of the state, input, and the noise term from step k_0 to step k_1, respectively, as follows $A_{k_1,k_0} = A_{k_1}A_{k_1-1}\cdots A_{k_0}$, $B_{k_1,k_0} = A_{k_1,k_0+1}B_{k_0}$, $D_{k_1,k_0} = A_{k_1,k_0+1}D_{k_0}$. We define the augmented vectors $U_k \in \mathbb{R}^{(k+1)n_u}$ and $W_k \in \mathbb{R}^{(k+1)n_w}$ as $U_k = [u_0, u_1, \ldots, u_k]^\top$ and $W_k = [w_0, w_1, \ldots, w_k]^\top$. Then x_k can be, equivalently, computed from

$$x_k = \bar{A}_k x_0 + \bar{B}_k U_k + \bar{D}_k W_k, \tag{6}$$

where $\bar{A}_k = A_{k-1,0}$, $\bar{B}_k = [B_{k-1,0}, B_{k-1,1}, \ldots, B_{k-1}]$, $\bar{D}_k = [D_{k-1,0}, D_{k-1,1}, \ldots, D_{k-1}]$. Furthermore, we introduce the augmented state vector $X \in \mathbb{R}^{(k+1)n_x}$ as follows $X_k = [x_0, x_1, \ldots, x_k]^\top$. It follows that the system dynamics (1) take the equivalent form

$$X = \mathcal{A}x_0 + \mathcal{B}U + \mathcal{D}W, \tag{7}$$

where $X = X_N \in \mathbb{R}^{(N+1)n_x}$, $U = U_{N-1} \in \mathbb{R}^{Nn_u}$, and $W = W_{N-1} \in \mathbb{R}^{Nn_w}$, and the matrices $\mathcal{A} \in \mathbb{R}^{(N+1)n_x \times n_x}$, $\mathcal{B} \in \mathbb{R}^{(N+1)n_x \times Nn_u}$, and $\mathcal{D} \in \mathbb{R}^{(N+1)n_x \times Nn_w}$ are defined in [16].

Note that $\mathbb{E}[x_0 x_0^\top] = \Sigma_0 + \mu_0\mu_0^\top$, $\mathbb{E}[x_0 W^\top] = 0$, and $\mathbb{E}[WW^\top] = I_{Nn_w}$. Using the previous expressions for X and U, we may rewrite the objective function in (4) as follows

$$J(X, U) = \mathbb{E}\Big[X^\top \bar{Q} X + U^\top \bar{R} U\Big], \tag{8}$$

where $\bar{Q} = \text{blkdiag}(Q_0, Q_1, \ldots, Q_{N-1}, 0)$ and $\bar{R} = \text{blkdiag}(R_0, R_1, \ldots, R_{N-1})$. Note that, since $Q_k \succeq 0$ and $R_k \succ 0$ for all $k = 0, 1, \ldots, N-1$, it follows that $\bar{Q} \succeq 0$ and $\bar{R} \succ 0$.

The boundary conditions (2) and (3) take the form

$$\mu_0 = E_0\mathbb{E}[X], \tag{9a}$$
$$\Sigma_0 = E_0\Big(\mathbb{E}[XX^\top] - \mathbb{E}[X]\mathbb{E}[X]^\top\Big)E_0^\top, \tag{9b}$$

and

$$\mu_N = E_N\mathbb{E}[X], \tag{10a}$$
$$\Sigma_N = E_N\Big(\mathbb{E}[XX^\top] - \mathbb{E}[X]\mathbb{E}[X]^\top\Big)E_N^\top, \tag{10b}$$

where $E_0 \triangleq [I_{n_x}, 0, \ldots, 0] \in \mathbb{R}^{n_x \times (N+1)n_x}$ and $E_N \triangleq [0, \ldots, 0, I_{n_x}] \in \mathbb{R}^{n_x \times (N+1)n_x}$, respectively. Finally, the chance constraints (5) can be rewritten as

$$\texttt{Pr}(X \notin \mathcal{X}) \le P_{\text{fail}}, \tag{11}$$

where $\mathcal{X} \subset \mathbb{R}^{(N+1)n_x}$ is a convex set.

The objective of this letter is to solve the following problem.

Problem 1: Given the system (7), find the control sequence U^* that minimizes the cost function Eq. (8) subject to the initial state constraints (9), the terminal state constraints (10), and the chance constraint (11).

In Section IV we show how to solve Problem 1 by converting it to a convex programming problem. Before doing that, we first investigate the case without chance constraints.

III. No Chance Constraint Case

Before discussing the general case with chance constraints, in this section we briefly revisit the case without chance constraints and show that, similarly to the work by Goldshtein and Tsiotras [16], where the authors considered the case with minimal control effort ($\bar{Q} = 0, \bar{R} = I$), it is possible to separately solve the mean and the covariance steering optimization problems, even with the more general ℓ_2-norm objective function of equation (4).

A. Separation of Mean and Covariance Problems

It follows immediately from Eq. (6) that

$$\mu_k \triangleq \mathbb{E}[x_k] = \bar{A}_k\mu_0 + \bar{B}_k\bar{U}_k, \tag{12}$$

where $\bar{U}_k = \mathbb{E}[U_k]$. Furthermore, by defining $\tilde{U}_k \triangleq U_k - \bar{U}_k$, $\tilde{x}_k \triangleq x_k - \mu_k$, and using (6), we have that

$$\tilde{x}_k = \bar{A}_k\tilde{x}_0 + \bar{B}_k\tilde{U}_k + \bar{D}_k W_k. \tag{13}$$

Furthermore,

$$\begin{aligned} \Sigma_k \triangleq \mathbb{E}[\tilde{x}_k\tilde{x}_k^\top] = {} & \bar{A}_k\mathbb{E}[\tilde{x}_0\tilde{x}_0^\top]\bar{A}_k^\top + \bar{A}_k\mathbb{E}[\tilde{x}_0\tilde{U}_k^\top]\bar{B}_k^\top \\ & + \bar{B}_k\mathbb{E}[\tilde{U}_k\tilde{x}_0^\top]\bar{A}_k^\top + \bar{B}_k\mathbb{E}[\tilde{U}_k\tilde{U}_k^\top]\bar{B}_k^\top \\ & + \bar{D}_k\mathbb{E}[W_kW_k^\top]\bar{D}_k^\top + \bar{D}_{k-1}\mathbb{E}[W_{k-1}\tilde{U}_k^\top]\bar{B}_k^\top \\ & + \bar{B}_k\mathbb{E}[\tilde{U}_kW_{k-1}^\top]\bar{D}_{k-1}^\top. \end{aligned} \tag{14}$$

Note that the evolution of the mean μ_k from (12) depends only on $\bar{U}_k$, whereas the evolution of $\tilde{x}_k$ and Σ_k depend solely on $\tilde{U}_k$ and W_k. It follows from Eqs. (7) and (12) that

$$\bar{X} \triangleq \mathbb{E}[X] = \mathcal{A}\mu_0 + \mathcal{B}\bar{U}, \tag{15}$$

and from (13) that

$$\tilde{X} \triangleq X - \mathbb{E}[X] = \mathcal{A}\tilde{x}_0 + \mathcal{B}\tilde{U} + \mathcal{D}W. \tag{16}$$

The objective function (8) can also be rewritten as $J(X, U) = \mathbb{E}[X^\top \bar{Q}X + U^\top \bar{R}U] = \mathrm{tr}(\bar{Q}\,\mathbb{E}[\tilde{X}\tilde{X}^\top]) + \bar{X}^\top \bar{Q}\bar{X} + \mathrm{tr}(\bar{R}\mathbb{E}[\tilde{U}\tilde{U}^\top]) + \bar{U}^\top \bar{R}\bar{U}, = J_\mu(\bar{X}, \bar{U}) + J_\Sigma(\tilde{X}, \tilde{U})$, where

$$J_\mu(\bar{X}, \bar{U}) = \bar{X}^\top \bar{Q}\bar{X} + \bar{U}^\top \bar{R}\bar{U}, \tag{17}$$

$$J_\Sigma(\tilde{X}, \tilde{U}) = \mathrm{tr}\Big(\bar{Q}\mathbb{E}[\tilde{X}\tilde{X}^\top]\Big) + \mathrm{tr}\Big(\bar{R}\mathbb{E}[\tilde{U}\tilde{U}^\top]\Big), \tag{18}$$

and where tr() denotes the trace of a matrix. It follows that the original optimization problem in terms of (X, U) is equivalent to two separate optimization problems in terms of $(\bar{X}, \bar{U})$ and $(\tilde{X}, \tilde{U})$ with optimization costs (17) and (18), respectively.

We have therefore shown the following result.

Proposition 1: Let the system (7), the initial and terminal state constraints (2) and (3), and the objective function (4). The control sequence U^* that solves this optimization problem is given by $U^* = \bar{U}^* + \tilde{U}^*$, where $\bar{U}^*$ solves the *mean steering* optimization problem

$$\mathrm{MS}\begin{cases} \min_{(\bar{X},\bar{U})} J_\mu(\bar{X}, \bar{U}) = \bar{X}^\top \quad \bar{Q}\bar{X} + \bar{U}^\top \bar{R}\bar{U} \\ \text{subject to} \quad \bar{X} = \mathcal{A}\mu_0 + \mathcal{B}\bar{U}, \quad E_0\bar{X} = \mu_0, \quad E_N\bar{X} = \mu_N, \end{cases} \tag{19}$$

and $\tilde{U}^*$ solves the *covariance steering* optimization problem

$$\mathrm{CS}\begin{cases} \min_{(\tilde{X},\tilde{U})} \quad J_\Sigma(\tilde{X}, \tilde{U}) = \mathrm{tr}\big(\bar{Q}\mathbb{E}[\tilde{X}\tilde{X}^\top]\big) + \mathrm{tr}\big(\bar{R}\mathbb{E}[\tilde{U}\tilde{U}^\top]\big), \\ \text{subject to} \quad \tilde{X} \triangleq X - \mathbb{E}[X] = \mathcal{A}\tilde{x}_0 + \mathcal{B}\tilde{U} + \mathcal{D}W \\ \qquad E_0\tilde{X}\tilde{X}^\top E_0^\top = \Sigma_0, \quad E_N\tilde{X}\tilde{X}^\top E_N^\top = \Sigma_N. \end{cases} \tag{20}$$

The rest of this section introduces the methods to solve these two subproblems.

B. Optimal Mean Steering

The solution to the optimal mean steering subproblem is summarized in the following proposition. Note that we assume a general (nonzero) mean.

Proposition 2: The optimal control sequence that solves the optimization problem (19) is given by

$$\bar{U}^* = \mathcal{R}^{-1}\Big(\mathcal{B}^\top \bar{Q}\mathcal{A}\mu_0 + \bar{B}_N^\top(\bar{B}_N\mathcal{R}^{-1}\bar{B}_N^\top)^{-1} \Big(\mu_N - \bar{A}_N\mu_0 - \bar{B}_N\mathcal{R}^{-1}\mathcal{B}^\top \bar{Q}\mathcal{A}\mu_0\Big)\Big), \tag{21}$$

where $\mathcal{R} = (\mathcal{B}^\top \bar{Q}\mathcal{B} + \bar{R})$.

Proof: Since the terminal constraint is $\mu_N = E_N\bar{X} = \bar{A}_N\mu_0 + \bar{B}_N\bar{U}$ we can write the Lagrangian as $\mathcal{L}(\bar{U}, \lambda) = \bar{X}^\top \bar{Q}\bar{X} + \bar{U}^\top \bar{R}\bar{U} + \lambda^\top(\mu_N - \bar{A}_N\mu_0 - \bar{B}_N\bar{U}) = (\mathcal{A}\mu_0 + \mathcal{B}\bar{U})^\top \bar{Q}(\mathcal{A}\mu_0 + \mathcal{B}\bar{U}) + \bar{U}^\top \bar{R}\bar{U} + \lambda^\top(\mu_N - \bar{A}_N\mu_0 - \bar{B}_N\bar{U})$, where $\lambda \in \mathbb{R}^{n_x}$. The first-order optimality condition yields $\nabla_{\bar{U}}\mathcal{L} = 2(\mathcal{B}^\top \bar{Q}\mathcal{B} + \bar{R})\bar{U} + 2\mathcal{B}^\top \bar{Q}\mathcal{A}\mu_0 - \bar{B}_N^\top\lambda = 0$. Thus,

$$\bar{U}^* = \mathcal{R}^{-1}(\mathcal{B}^\top \bar{Q}\mathcal{A}\mu_0 + \tfrac{1}{2}\bar{B}_N^\top\lambda), \tag{22}$$

where $\mathcal{R} = (\mathcal{B}^\top \bar{Q}\mathcal{B} + \bar{R})$ is invertible because of the second-order optimality condition $\nabla_{\bar{U}\bar{U}}\mathcal{L} = \mathcal{B}^\top \bar{Q}\mathcal{B} + \bar{R} \succ 0$. In order to find the optimal value of λ we substitute equation (22) into the terminal constraint to obtain

$$\tfrac{1}{2}\bar{B}_N\mathcal{R}^{-1}\bar{B}_N^\top\lambda = \mu_N - \bar{A}_N\mu_0 - \bar{B}_N\mathcal{R}^{-1}\mathcal{B}^\top \bar{Q}\mathcal{A}\mu_0. \tag{23}$$

Note that $\mathrm{rank}(\bar{B}_N\mathcal{R}^{-1}\bar{B}_N^\top) = \mathrm{rank}(\mathcal{R}^{-1/2}\bar{B}_N^\top)$. Also, since the system is controllable, it follows that $\mathrm{rank}(\bar{B}_N)$ is full row rank, that is, $\mathrm{rank}(\bar{B}_N) = n_x$ [16]. In addition, since $\mathcal{R}$ is invertible, $\mathrm{rank}(\mathcal{R}^{-1/2}) = Nn_u$. It follows from [25, Corollary 2.5.10] that $\mathrm{rank}(\mathcal{R}^{-1/2}) + \mathrm{rank}(\bar{B}_N^\top) - Nn_u \le \mathrm{rank}(\mathcal{R}^{-1/2}\bar{B}_N^\top) \le \min\{\mathrm{rank}(\mathcal{R}^{-1/2}), \mathrm{rank}(\bar{B}_N^\top)\}$ and $n_x \le \mathrm{rank}(\bar{B}_N\mathcal{R}^{-1}\bar{B}_N^\top) \le \min\{Nn_u, n_x\} = n_x$. Thus, the matrix $(\bar{B}_N\mathcal{R}^{-1}\bar{B}_N^\top)$ is full rank and invertible. Therefore,

$$\lambda = 2(\bar{B}_N\mathcal{R}^{-1}\bar{B}_N^\top)^{-1}\Big(\mu_N - \bar{A}_N\mu_0 - \bar{B}_N\mathcal{R}^{-1}\mathcal{B}^\top \bar{Q}\mathcal{A}\mu_0\Big).$$

By substituting in (22) the expression for the optimal mean steering controller, the expression (21) follows. ■

By comparing (21) with the corresponding controller in [16] we have the following immediate result.

Corollary 1: The minimum-effort mean-steering optimal controller introduced in [16] is a special case of the optimal controller (21) with $\bar{Q} = 0$, $\bar{R} = I$ in (21).

C. Optimal Covariance Steering

While many previous works have attempted to solve the optimal covariance-steering problem, the majority of them solve this problem subject to a minimum effort cost function. Bakolas [19] addressed the case with the more general ℓ_2-norm cost function Eq. (4) (and zero mean). He also introduced a convex relaxation to change the terminal constraint to an inequality as follows

$$E_N\Big(\mathbb{E}[XX^\top] - \mathbb{E}[X]\mathbb{E}[X]^\top\Big)E_N^\top \preceq \Sigma_N. \tag{24}$$

By making the problem convex, it can be efficiently solved using standard convex programming solvers. At the same time, but independently, Halder and Wendel [17] solved a problem with a similar terminal covariance constraint using a soft constraint on the terminal state covariance under continuous-time dynamics.

IV. Chance Constrained Case

This section introduces the proposed approach to solve the covariance steering problem with chance constraints as stated in Problem 1.

A. Proposed Approach

First, we assume that, at each time step, the control input is represented as follows $u_k = \ell_k[1_{n_x}^\top, x_0^\top, x_1^\top, \ldots, x_k^\top]^\top$, where $1_{n_x} = [1, \ldots, 1]^\top \in \mathbb{R}^{n_x}$ and $\ell_k \in \mathbb{R}^{n_u \times n_x(k+2)}$. Thus, we may write the relationship between X and U as follows

$$U = L\mathbf{X}, \tag{25}$$

where $\mathbf{X} = [1_{n_x}^\top, X^\top]^\top \in \mathbb{R}^{(N+2)n_x}$ is the augmented state sequence until step N and $L \in \mathbb{R}^{Nn_u \times (N+2)n_x}$ is the control gain matrix. In order to ensure that the control input at time step k depends only on x_i for $i = 0, 1, \ldots, k$ (so that the control input U is causally related to the state history, that is, it is non-anticipative) the matrix L has to be of the form

$$L = [L_1, L_X], \tag{26}$$

where $L_1 \in \mathbb{R}^{Nn_u \times n_x}$ and $L_X \in \mathbb{R}^{Nn_u \times (N+1)n_x}$ is a lower block triangular matrix. Using L in (26) we convert the problem from finding the optimal control input sequence U^* to one of finding the optimal control gain matrix L^*. It follows from (7) that

$$\mathbf{X} = \begin{bmatrix} I_{n_x} & 0 \\ 0 & \mathcal{A} \end{bmatrix}\begin{bmatrix} 1_{n_x} \\ x_0 \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{B} \end{bmatrix} L\mathbf{X} + \begin{bmatrix} 0 \\ \mathcal{D} \end{bmatrix} W, \tag{27}$$

and hence

$$\mathbf{X} = (I - \boldsymbol{\mathcal{B}}L)^{-1}(\boldsymbol{\mathcal{A}}\mathbf{X}_0 + \boldsymbol{\mathcal{D}}W), \tag{28}$$

where $\mathbf{X}_0 = [1_{n_x}^\top,\ x_0^\top]^\top$, $\boldsymbol{\mathcal{A}} = \mathrm{blkdiag}(I_{n_x}, \mathcal{A})$, $\boldsymbol{\mathcal{B}} = [0,\ \mathcal{B}^\top]^\top$, $\boldsymbol{\mathcal{D}} = [0,\ \mathcal{D}^\top]^\top$. Note that $(I - \boldsymbol{\mathcal{B}}L)$ is invertible because

$$\boldsymbol{\mathcal{B}}L = \begin{bmatrix} 0 \\ \mathcal{B} \end{bmatrix}[L_1, L_X] = \begin{bmatrix} 0 & 0 \\ \mathcal{B}L_1 & \mathcal{B}L_X \end{bmatrix}. \tag{29}$$

Since $\mathcal{B}L_X$ is strictly lower-block triangular, $\boldsymbol{\mathcal{B}}L$ is also strictly lower-block triangular.[1] Using $\mathbf{X}$ from (28), the objective function (8) can be written as

$$J(L) = \mathbb{E}\Big[(\boldsymbol{\mathcal{A}}\mathbf{X}_0 + \boldsymbol{\mathcal{D}}W)^\top (I - \boldsymbol{\mathcal{B}}L)^{-\top}\bar{\boldsymbol{Q}}(I - \boldsymbol{\mathcal{B}}L)^{-1} (\boldsymbol{\mathcal{A}}\mathbf{X}_0 + \boldsymbol{\mathcal{D}}W) + \mathbf{X}^\top L^\top \bar{R} L \mathbf{X}\Big],$$

where $\bar{\boldsymbol{Q}} = \mathrm{blkdiag}(0, \bar{Q})$. Note that $\bar{\boldsymbol{Q}} \succeq 0$. Similarly to [19], we introduce the new decision variable K such that

$$K \triangleq L(I - \boldsymbol{\mathcal{B}}L)^{-1}. \tag{30}$$

It follows that $I + \boldsymbol{\mathcal{B}}K = (I - \boldsymbol{\mathcal{B}}L)^{-1}$. Then, $\mathbf{X}$ and U can be rewritten as

$$\mathbf{X} = (I + \boldsymbol{\mathcal{B}}K)(\boldsymbol{\mathcal{A}}\mathbf{X}_0 + \boldsymbol{\mathcal{D}}W), \tag{31}$$

$$U = K(\boldsymbol{\mathcal{A}}\mathbf{X}_0 + \boldsymbol{\mathcal{D}}W). \tag{32}$$

Before continuing, we show that K defined in (30) is lower block triangular. This ensures that the resulting U is non-anticipative.

Lemma 1: Let L be defined as in Eq. (26), let $\boldsymbol{\mathcal{B}}$ be a strictly lower block triangular matrix, and let I be an identity matrix with proper dimensions. Then, K defined as in Eq. (30) can be represented as $K = \begin{bmatrix} K_1 & K_X \end{bmatrix}$, where $K_1 \in \mathbb{R}^{Nn_u \times n_x}$ and $K_X \in \mathbb{R}^{Nn_u \times (N+1)n_x}$ is lower block triangular.

Proof: The proof is straightforward and thus it is omitted. ∎

We may now prove the following result.

Proposition 3: Let $\mathbf{X}$ and U as in (31), the objective function (8) and the boundary conditions

$$\boldsymbol{\mu}_0 = \begin{bmatrix} 1_{n_x} \\ \mu_0 \end{bmatrix}, \qquad \Sigma_0 = \begin{bmatrix} 0_{n_x} & 0_{n_x} \\ 0_{n_x} & \Sigma_0 \end{bmatrix} \succeq 0. \tag{33}$$

Then, the objective function (8) takes the form

$$J(K) = \mathrm{tr}\Big(\big((I + \boldsymbol{\mathcal{B}}K)^\top \bar{\boldsymbol{Q}}(I + \boldsymbol{\mathcal{B}}K) + K^\top \bar{R} K\big) \big(\boldsymbol{\mathcal{A}}\big(\boldsymbol{\mu_0}\boldsymbol{\mu_0}^\top + \Sigma_0\big)\boldsymbol{\mathcal{A}}^\top + \boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{D}}^\top\big)\Big). \tag{34}$$

which is a quadratic expression in K.

[1] A strictly lower-block triangular matrix is a lower-block triangular matrix with zero matrices on the diagonal elements.

Proof: The proof follows easily by using (31) in the objective function (8), expanding and performing the necessary algebraic manipulations. ∎

B. Conversion of Chance Constraints to Deterministic Inequality Constraints

We assume that the feasible region $\mathcal{X}$ is defined as an intersection of M linear inequality constraints as follows

$$\mathcal{X} \triangleq \bigcap_{j=1}^{M} \{\mathbf{X} : \alpha_j^\top \mathbf{X} \le \beta_j\}, \tag{35}$$

where $\alpha_j \in \mathbb{R}^{(N+2)n_x}$ and $\beta_j \in \mathbb{R}$ with $j = 1, 2, \ldots, M$. Thus, the chance constraint (11) is converted to

$$\mathtt{Pr}(\alpha_j^\top \mathbf{X} > \beta_j) \le p_j, \qquad j = 1, \ldots, M, \tag{36a}$$

$$\sum_{j=1}^{M} p_j \le P_{\mathrm{fail}}. \tag{36b}$$

Using the Boole-Bonferroni inequality [26], Blackmore and Ono [27] showed that a feasible solution to the problem (35)-(36) is a feasible solution to the original chance-constrained problem. Note that the constraint (36a) can also be written as

$$\mathtt{Pr}(\alpha_j^\top \mathbf{X} \le \beta_j) \ge 1 - p_j. \tag{37}$$

As a result, $\alpha_j^\top \mathbf{X}$ is a univariate Gaussian random variable such that $\alpha_j^\top \mathbf{X} \sim \mathcal{N}(\alpha_j^\top \bar{\mathbf{X}}, \alpha_j^\top \Sigma_{\mathbf{X}} \alpha_j)$. where $\bar{\mathbf{X}} = \mathbb{E}[\mathbf{X}] = (I + \boldsymbol{\mathcal{B}}K)\boldsymbol{\mathcal{A}}\boldsymbol{\mu}_0$, and $\Sigma_{\mathbf{X}} = (I + \boldsymbol{\mathcal{B}}K)(\boldsymbol{\mathcal{A}}\Sigma_0\boldsymbol{\mathcal{A}}^\top + \boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{D}}^\top)(I + \boldsymbol{\mathcal{B}}K)^\top$. It follows from inequality (37) that

$$\mathtt{Pr}(\alpha_j^\top \mathbf{X} \le \beta_j) = \Phi\left(\frac{\beta_j - \alpha_j^\top \bar{\mathbf{X}}}{\sqrt{\alpha_j^\top \Sigma_{\mathbf{X}} \alpha_j}}\right) \ge 1 - p_j, \tag{38}$$

where Φ is the cumulative distribution function of the standard normal distribution, which is a monotonically increasing function. Thus,

$$\frac{\beta_j - \alpha_j^\top \bar{\mathbf{X}}}{\sqrt{\alpha_j^\top \Sigma_{\mathbf{X}} \alpha_j}} \ge \Phi^{-1}\big(1 - p_j\big), \tag{39}$$

where Φ^{-1} is the inverse of Φ. Therefore,

$$\alpha_j^\top \bar{\mathbf{X}} - \beta_j + \sqrt{\alpha_j^\top \Sigma_{\mathbf{X}} \alpha_j}\, \Phi^{-1}\big(1 - p_j\big) \le 0. \tag{40}$$

Previous works [27]–[29] assumed some prior knowledge about the covariance $\Sigma_{\mathbf{X}}$, enabling Eq. (40) to be a linear inequality constraint. However, as we are interested in the covariance steering problem, we cannot assume any prior knowledge of $\Sigma_{\mathbf{X}}$.

Theorem 1: Let $\bar{\mathbf{X}}$ and $\Sigma_{\mathbf{X}}$ as before, and let $\boldsymbol{\mu}_0$ and $\boldsymbol{\Sigma}_0$ as in (33). With the assumption, $\Sigma_0 \succeq 0$, the inequality constraint (40) is converted to the inequality constraint $\alpha_j^\top (I + \boldsymbol{\mathcal{B}}K)\boldsymbol{\mathcal{A}}\boldsymbol{\mu}_0 - \beta_j \| (\boldsymbol{\mathcal{A}}\Sigma_0\boldsymbol{\mathcal{A}}^\top + \boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{D}}^\top)^{1/2}(I + \boldsymbol{\mathcal{B}}K)^\top \alpha_j \|\, \Phi^{-1}(1 - p_j) \le 0$.

Proof: Since $\Sigma_0 \succeq 0$, it follows that $\boldsymbol{\Sigma}_0 \succeq 0$ and $\boldsymbol{\mathcal{A}}\Sigma_0\boldsymbol{\mathcal{A}}^\top + \boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{D}}^\top \succeq 0$. Therefore, the expression for $\Sigma_{\mathbf{X}}$ yields $\Sigma_{\mathbf{X}} = (I + \boldsymbol{\mathcal{B}}K)(\boldsymbol{\mathcal{A}}\Sigma_0\boldsymbol{\mathcal{A}}^\top + \boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{D}}^\top)^{1/2}(\boldsymbol{\mathcal{A}}\Sigma_0\boldsymbol{\mathcal{A}}^\top + \boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{D}}^\top)^{1/2}(I + \boldsymbol{\mathcal{B}}K)^\top$, and (40) can be rewritten as $\alpha_j^\top \bar{\mathbf{X}} - \beta_j + \alpha_j^\top (I + \boldsymbol{\mathcal{B}}K)(\boldsymbol{\mathcal{A}}\Sigma_0\boldsymbol{\mathcal{A}}^\top +$

$\mathcal{D}\mathcal{D}^\top)^{1/2}(\mathcal{A}\Sigma_0\mathcal{A}^\top+\mathcal{D}\mathcal{D}^\top)^{1/2}(I+\mathcal{B}K)^\top\alpha_j)^{\frac{1}{2}}\,\Phi^{-1}(1-p_j)\le 0$. Note that since $(\mathcal{A}\Sigma_0\mathcal{A}^\top+\mathcal{D}\mathcal{D}^\top)^{1/2}(I+\mathcal{B}K)^\top\alpha_j$ is a vector, one obtains that $\alpha_j^\top\bar{\mathbf{X}}-\beta_j+\|(\mathcal{A}\Sigma_0\mathcal{A}^\top+\mathcal{D}\mathcal{D}^\top)^{1/2}(I+\mathcal{B}K)^\top\alpha_j\|\,\Phi^{-1}(1-p_j)\le 0$, where $\|\cdot\|$ denotes the 2-norm of a vector. The result then follows easily. ■

The inequality constraint in Theorem 1 is a bilinear constraint, which makes it difficult to efficiently solve this problem. Thus, we convert the chance constraints (36) as follows

$$\mathtt{Pr}(\alpha_j^\top\mathbf{X}>\beta_j)\le p_{j,\mathrm{fail}},\qquad j=1,\dots M, \tag{41a}$$

$$\sum_{j=1}^{M} p_{j,\mathrm{fail}}\le P_{\mathrm{fail}}. \tag{41b}$$

Note that, unlike p_j, the $p_{j,\mathrm{fail}}$ is not a decision variable but a pre-specified value satisfying inequality (41b). This alternative formulation implies the specification of the maximum collision probability with each obstacle at each time step a priori.

In summary, the chance constraints are formulated as follows.

$$\alpha_j^\top(I+\mathcal{B}K)\mathcal{A}\boldsymbol{\mu}_0+\|(\mathcal{A}\Sigma_0\mathcal{A}^\top+\mathcal{D}\mathcal{D}^\top)^{1/2}(I+\mathcal{B}K)^\top\alpha_j\|\Phi^{-1}\big(1-p_{j,\mathrm{fail}}\big)-\beta_j\le 0. \tag{42}$$

Unlike the case described in Section III, where no chance constraints exist, we cannot decouple the mean and covariance steering problems owing to (42).

C. Terminal Gaussian Distribution Constraint

As discussed in Section III-C, the terminal covariance constraint (10b) is not convex. We therefore relax this constraint to the inequality constraint [19]

$$\mathbb{E}[\tilde{x}_N\tilde{x}_N^\top]\preceq\Sigma_N. \tag{43}$$

This condition implies that the covariance of the terminal state is smaller than a pre-specified Σ_N, which is a reasonable assumption in practice. This change of terminal constraint relaxes the chance-constraint requirement for Σ_N as well. Namely, if μ_N is inside the feasible region, Σ_N can have any value as long as it is positive definite. We are now ready to prove the following result.

Proposition 4: The terminal constraints (10a) and (43) can be formulated as

$$\mu_N=\mathbf{E}_N(I+\mathcal{B}K)\mathcal{A}\boldsymbol{\mu}_0,$$
$$1-\|(\mathcal{A}\Sigma_0\mathcal{A}^\top+\mathcal{D}\mathcal{D}^\top)^{1/2}(I+\mathcal{B}K)^\top\mathbf{E}_N^\top\Sigma_N^{-1/2}\|\ge 0,$$

where $\mathbf{E}_N\triangleq\begin{bmatrix}0_{n_x} & E_N\end{bmatrix}$.

Proof: It follows from the expressions of $\bar{\mathbf{X}}$ and $\Sigma_{\mathbf{X}}$ that $\mathbb{E}[x_N]=\mathbf{E}_N(I+\mathcal{B}K)\mathcal{A}\boldsymbol{\mu}_0$, and $\mathbb{E}[\tilde{x}_N\tilde{x}_N^\top]=\mathbf{E}_N(I+\mathcal{B}K)(\mathcal{A}\Sigma_0\mathcal{A}^\top+\mathcal{D}\mathcal{D}^\top)(I+\mathcal{B}K)^\top\mathbf{E}_N^\top$. Using inequality (43), it follows that the previous expression results in the following inequality constraint, which is convex in K

$$\mathbf{E}_N(I+\mathcal{B}K)(\mathcal{A}\Sigma_0\mathcal{A}^\top+\mathcal{D}\mathcal{D}^\top)(I+\mathcal{B}K)^\top\mathbf{E}_N^\top\preceq\Sigma_N. \tag{44}$$

Since by assumption $\Sigma_N\succ 0$, inequality (44) becomes $I_{n_x}-\Sigma_N^{-1/2}\mathbf{E}_N(I+\mathcal{B}K)(\mathcal{A}\Sigma_0\mathcal{A}^\top+\mathcal{D}\mathcal{D}^\top)(I+\mathcal{B}K)^\top\mathbf{E}_N^\top\Sigma_N^{-1/2}\succeq 0$. Being symmetric, the matrix $\Sigma_N^{-1/2}\mathbf{E}_N(I+\mathcal{B}K)(\mathcal{A}\Sigma_0\mathcal{A}^\top+\mathcal{D}\mathcal{D}^\top)(I+\mathcal{B}K)^\top\mathbf{E}_N^\top\Sigma_N^{-1/2}$ is diagonalizable via an orthogonal matrix $S\in\mathbb{R}^{n_x\times n_x}$. Thus, $S(I_{n_x}-\mathrm{diag}(\lambda_1,\dots,\lambda_{n_x}))S^\top\succeq 0$, where $\lambda_1,\dots,\lambda_{n_x}$ are the eigenvalues of $\Sigma_N^{-1/2}\mathbf{E}_N(I+\mathcal{B}K)(\mathcal{A}\Sigma_0\mathcal{A}^\top+\mathcal{D}\mathcal{D}^\top)(I+\mathcal{B}K)^\top\mathbf{E}_N^\top\Sigma_N^{-1/2}$. The last inequality is implied by $1-\lambda_{\max}(\Sigma_N^{-1/2}\mathbf{E}_N(I+\mathcal{B}K)(\mathcal{A}\Sigma_0\mathcal{A}^\top+\mathcal{D}\mathcal{D}^\top)(I+\mathcal{B}K)^\top\mathbf{E}_N^\top\Sigma_N^{-1/2})\ge 0$. An easy calculation shows that this inequality is equivalent to

$$1-\|(\mathcal{A}\Sigma_0\mathcal{A}^\top+\mathcal{D}\mathcal{D}^\top)^{1/2}(I+\mathcal{B}K)^\top\mathbf{E}_N^\top\Sigma_N^{-1/2}\|^2\ge 0, \tag{45}$$

thus completing the proof. ■

V. Numerical Simulations

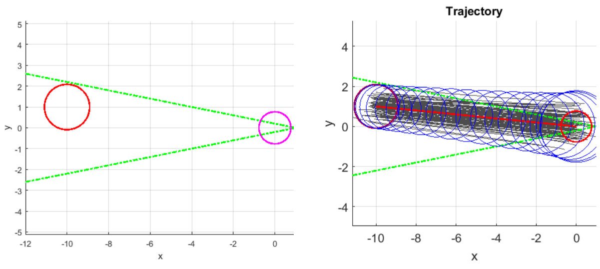
In this section we validate the proposed algorithm using a simple numerical example. We use CVX [30] with MOSEK [31] to solve the relevant optimization problems. Note that the structure of K from Lemma 1 enters as a constraint in the resulting optimization problem.

We consider the path-planning problem for a vehicle under the following time invariant system dynamics with $x_k=[x,y,v_x,v_y]^\top\in\mathbb{R}^4$, $u_k=[a_x,a_y]^\top\in\mathbb{R}^2$, $w_k\in\mathbb{R}^4$ and

$$A=\begin{bmatrix}1&0&\Delta t&0\\0&1&0&\Delta t\\0&0&1&0\\0&0&0&1\end{bmatrix},\quad B=\begin{bmatrix}\Delta t^2&0\\0&\Delta t^2\\\Delta t&0\\0&\Delta t\end{bmatrix}, \tag{46}$$

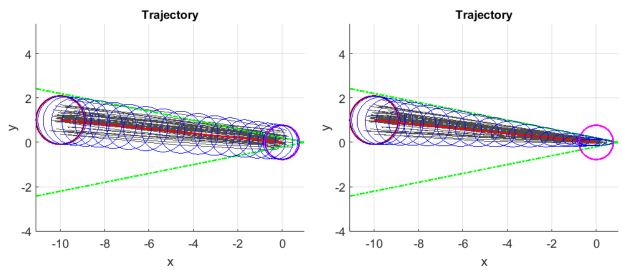
and $D=\mathrm{diag}(0.01,0.01,0.01,0.01)$, where $\Delta t=0.2$ is the time-step size. Figure 1(a) illustrates the problem setup. The red circle denotes the 3σ error of the initial state distribution of x and y coordinates. The magenta circle denotes the 3σ error of the terminal state distribution of x and y coordinates. Specifically, the initial condition is $\mu_0=[-10,1,0,0]$ and $\Sigma_0=\mathrm{diag}(0.1,0.1,0.01,0.01)$, while the terminal constraint is $\mu_N=[0,0,0,0]$ and $\Sigma_N=0.5\Sigma_0$.

The green dotted lines illustrate the state constraints given by $0.2(x-1)\le y\le -0.2(x-1)$. The vehicle has to remain in the region between the two lines while moving from the red to the magenta regions. Such a "cone"-shaped constraint is seen in many engineering applications, e.g., the instrument landing for aircraft, spacecraft rendezvous, and drone-landing on a moving platform. The probabilistic threshold for the violation of the chance constraints was specified a priori, as $p_{j,\mathrm{fail}}=0.0005$ for $j=1,2,\dots,2(N+1)$ with horizon $N=20$. The objective function weights are $Q_k=\mathrm{diag}(10,10,1,1)$ and $R_k=\mathrm{diag}(10^3,10^3)$. This problem is infeasible if we do not control the state covariance. See, for example, Figure 1(b), which shows the results using only the mean steering controller (21). As the covariance grows, it is impossible to find a feasible solution to this problem that will guarantee the satisfaction of chance constraints. The case without chance constraints imposed is illustrated in Fig. 2(a). By introducing covariance steering, the uncertainty of the future trajectory is successfully reduced but, nonetheless, it violates the constraint. Finally, Fig. 2(b) illustrates the results of the proposed chance-constrained covariance steering approach. The error ellipse successfully changed its shape to avoid collision with the constraints while maintaining the terminal covariance constraints to be less than the pre-specified state covariance bound.



(a) (b)

Fig. 1. (a) Problem setup; (b) Mean steering.



(a) (b)

Fig. 2. (a) Covariance steering results without chance constraints; (b) Covariance steering results with chance constraints.

VI. Summary

This letter has addressed the problem of optimal steering of the covariance for a stochastic linear time-varying system subject to chance constraints in discrete time. We showed that if there are no chance constraints, one can independently design the mean and covariance steering controllers. It is shown that the optimal covariance steering problem with chance constraints can be cast as a convex programming problem. The proposed approach was verified using numerical examples. Future work will investigate the applications of the proposed approach to stochastic model predictive controllers.

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