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Optimal Synthesis of the Asymmetric Sinistral/Dextral Markov–Dubins Problem

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Abstract We consider a variation of the classical Markov–Dubins problem dealing with curvature-constrained, shortest paths in the plane with prescribed initial and terminal positions and tangents, when the lower and upper bounds of the curvature of the path are not necessarily equal. The motivation for this problem stems from vehicle navigation applications, when a vehicle may be biased in taking turns at a particular direction due to hardware failures or environmental conditions. After formulating the shortest path problem as a minimum-time problem, a family of extremals, which is sufficient for optimality, is characterized, and subsequently the complete analytic solution of the optimal synthesis problem is presented. In addition, the synthesis problem, when the terminal tangent is free, is also considered, leading to the characterization of the set of points that can be reached in the plane by curves satisfying asymmetric curvature constraints.

Keywords Markov–Dubins problem · Curvature constrained paths · Asymmetric steering constraints · Non-holonomic systems

1 Introduction

The origins of the problem dealing with the characterization of curvature-constrained planar paths of minimal length and with prescribed positions and tangents can be

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traced back to the end of the nineteenth century, when the Russian mathematician A.A. Markov posed the problem for the first time. In 1957 L.E. Dubins generalized the original problem formulation by posing the problem "on curves of minimal length with a constraint on average curvature, and with prescribed initial and terminal positions and tangents" in the n-dimensional Euclidean space. Dubins addressed the planar case of this minimization problem by characterizing six families of paths, that were sufficient for optimality for any set of prescribed boundary conditions [1]. We shall refer to the problem of finding the shortest, curvature-constrained planar path as the Markov–Dubins (MD for short) problem, as suggested by Sussmann [2].

The solution of the MD problem is commonly interpreted as the minimum-time path of a vehicle that travels in the plane with constant unit speed, and such that the direction of its velocity vector cannot be changed faster than a given constant. This simple kinematic model is known in the literature as the Dubins' car although, as it is highlighted in [3], Dubins never introduced such a kinematic model in his work. It was actually R. Isaacs, who first introduced the kinematic model that is widely referred as the Dubins car in the formulation of his classic homicidal chauffeur problem [4, 5]. In this paper, we shall refer to this kinematic model as the Isaacs–Dubins (ID) car as suggested by Patsko and Turova [3]. The accessibility/reachability properties of the ID car were first studied by Cockayne and Hall in [6]. In addition, Reeds and Shepp examined a generalization of the MD problem, known as the Reeds–Shepp (RS) problem, when the minimal-length path may contain cusps, or equivalently the ID car is allowed to move both forwards and backwards with constant unit speed (a kinematic model known as the Reeds–Shepp car) [7].

All the aforementioned results were based more or less on constructive proofs and/or ad hoc methods. These approaches, even though sufficient for the examination of each particular optimization problem, are of limited use as tools for addressing other similar problems. A number of authors, during the 1990s, argued that the systematic application of optimal control techniques would provide more rigorous proofs to the MD and RS problems along with a more general framework for addressing similar problems in the future. Following this line of argument, Sussmann and Tang [8] and Boissonnat et al. [9] reformulated the RS and the MD problems as minimum-time problems, and they subsequently solved them by employing standard optimal control tools along with geometric control ideas. They provided more general and rigorous proofs, refining the original results of [1] and [7]. There is a plethora of interesting extensions/variations of the MD problem based on its kinematic interpretation. The reader may refer to [10–19].

In this work, we consider the problem of finding curvature-constrained, shortest paths in the plane with prescribed positions and tangents, when the lower and upper bounds of the curvature are not necessarily equal. The motivation for this problem stems from vehicle navigation problems when the maneuverability of the vehicle taking a left or a right turn is asymmetric. A typical case would be a UAV with a damaged aileron as it is shown in [20]. Henceforth, we shall refer to this generalization of the standard MD problem as the Asymmetric, Sinistral/Dextral[1] Markov–Dubins problem (ASDMD for short) [22]. Following the approach of [2, 9], we formulate the

[1]The term sinistral (dextral) means "inclined to left (right)" [21].

ASDMD problem as a minimum-time problem, and we investigate its (time-) optimal synthesis, that is, (a) we characterize a family of extremal controls that is sufficient for optimality; (b) we provide a state-feedback minimum-time control scheme; and finally (c) we compute the level sets of the minimum time analytically. Different parts of the synthesis of the standard MD problem, which form the complete solution of the problem, when combined appropriately, are presented in [2, 9, 23, 24]. Additionally, the synthesis problem of the ASDMD, when the tangent of the curve at the terminal position is free, is also considered, leading us to the analytic characterization of the set of points that can be reached by curves satisfying asymmetric curvature constraints.

The rest of the paper is organized as follows. In Sect. 2, we formulate the ASDMD as a minimum-time problem, and we subsequently solve the corresponding synthesis problem in Sects. 3 and 4. Furthermore, the solution of the synthesis problem, when the tangent of the path at the terminal position is free, is presented in Sect. 5. Finally, Sect. 6 concludes the paper with a summary of remarks.

2 Kinematic Model and Problem Formulation

In this paper, we are interested in the solution of the curvature-constrained, shortest-path problem in the plane with prescribed initial and final positions and tangents, when the lower and upper bounds of the path curvature are not necessarily equal. Equivalently, this problem can be cast as a minimum-time problem for a vehicle, whose motion is described by the following kinematic equations:

$$\dot{x} = \cos\vartheta, \qquad \dot{y} = \sin\vartheta, \qquad \dot{\vartheta} = u/\rho, \tag{1}$$

where x, y are the Cartesian coordinates of a reference point of the vehicle, ϑ is the direction of motion of the vehicle, u is the control input and ρ is a positive constant. We assume that the set of admissible control inputs, denoted by $\mathcal{U}$, consists of all measurable functions u defined on $[0, T]$, where $T > 0$, taking values in $U_\delta := [-\delta, 1]$, where $\delta \in]0, 1]$. To this end, let $\varrho := \rho/\delta$; then it follows that ρ and ϱ are the minimum turning radii for counterclockwise and clockwise turns, respectively. The case $U_\delta := [-1, \delta]$ can be treated similarly. We call the system described by (1) and with input value set U_δ the asymmetric, sinistral/dextral Isaacs–Dubins (ASDID for short) car.

It is a well-known fact that the standard ID car is completely controllable [8]. Next, it is shown that the ASDID car is also completely controllable. The controllability of the ASDID is established by proving that (1), with input value set $U'_\delta := [-\delta, \delta] \subseteq U$, define a completely controllable system. It suffices to note that the system (1), with input value set U'_δ, is the standard ID car with minimum turning radius ϱ (for both left and right turns), which is a completely controllable system.

It is worth noting that the assumption $\delta \in]0, 1]$, which guarantees that 0 is an interior point of the input value set, can actually be relaxed, and it can be assumed instead that $\delta \in [0, 1]$. In the latter case, $\delta = 0$ implies that the ASDID car cannot take right turns at all. A proof of the complete controllability in this case, which is based on solely geometric arguments, can be found in [22].

Next, we formulate the following minimum-time problem with fixed initial and terminal boundary conditions for the system (1).

Problem 2.1 Given the system described by (1) and the cost functional

$$J(u) = \int_0^{T_\mathsf{f}} 1\,\mathrm{d}t = T_\mathsf{f}, \tag{2}$$

where T_f is the free final time, determine a control input $u^* \in \mathcal{U}$ such that

(i) The trajectory $\mathsf{x}^* : [0, T_\mathsf{f}] \mapsto \mathbb{R}^2 \times \mathbb{S}^1$, generated by the control u^*, satisfies the boundary conditions

$$\mathsf{x}^*(0) = (0, 0, 0), \qquad \mathsf{x}^*(T_\mathsf{f}) = (x_\mathsf{f}, y_\mathsf{f}, \vartheta_\mathsf{f}). \tag{3}$$

(ii) The control u^* minimizes the cost functional $J(u)$ given in (2).

The existence of an optimal solution to Problem 2.1 can be established by means of Filippov's Theorem on minimum-time problems with prescribed initial and terminal states [25], leading to the following proposition.

Proposition 2.1 *The minimum-time Problem* 2.1 *with boundary conditions* (3) *has a solution for all* $(x_\mathsf{f}, y_\mathsf{f}, \vartheta_\mathsf{f}) \in \mathbb{R}^2 \times \mathbb{S}^1$.

3 Analysis of the ASDMD Minimum-Time Problem

In this section, we characterize the structure of the optimal paths using a similar approach as in [8, 26]. To this end, consider the Hamiltonian $\mathcal{H} : \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R}^3 \times U_\delta \mapsto \mathbb{R}$ of Problem 2.1, which is defined as

$$\mathcal{H}(\mathsf{x}, \mathsf{p}, u) := p_0 + p_1 \cos\vartheta + p_2 \sin\vartheta + \frac{p_3 u}{\rho}, \tag{4}$$

where $\mathsf{p} := (p_1, p_2, p_3)$. From Pontryagin's Maximum Principle (PMP) it follows that, if x^* is a minimum-time trajectory generated by the control u^*, then there exists a scalar $p_0^* \in \{0, 1\}$ and an absolutely continuous function $\mathsf{p}^* : [0, T_\mathsf{f}] \mapsto \mathbb{R}^3$, where $\mathsf{p}^* := (p_1^*, p_2^*, p_3^*)$, known as the costate, such that

(i) $\|\mathsf{p}^*(t)\| + |p_0^*|$ does not vanish for all $t \in [0, T_\mathsf{f}]$,

(ii) $\mathsf{p}^*(t)$ satisfies for almost all $t \in [0, T_\mathsf{f}]$ the canonical equation $\dot{\mathsf{p}}^* = -\frac{\partial \mathcal{H}}{\partial \mathsf{x}}(\mathsf{x}^*, \mathsf{p}^*, u^*)$, which for the system (1) reduces to

$$\dot{p}_1^* = 0, \qquad \dot{p}_2^* = 0, \qquad \dot{p}_3^* = p_1^* \sin\vartheta^* - p_2^* \cos\vartheta^*, \tag{5}$$

(iii) $\mathsf{p}^*(T_\mathsf{f})$ satisfies the transversality condition associated with the free final-time Problem 2.1

$$\mathcal{H}\big(\mathsf{x}^*(T_\mathsf{f}), \mathsf{p}^*(T_\mathsf{f}), u^*(T_\mathsf{f})\big) = 0. \tag{6}$$

Because the Hamiltonian does not depend explicitly on time, it follows from (6) that

$$\mathcal{H}\big(\mathsf{x}^*(t), \mathsf{p}^*(t), u^*(t)\big) = 0 \tag{7}$$

for almost all $t \in [0, T_\mathsf{f}]$, which furthermore implies, by virtue of (5), that

$$-p_0^* = p_1^*(0)\cos\vartheta^* + p_2^*(0)\sin\vartheta^* + \frac{p_3^* u^*}{\rho}. \tag{8}$$

Furthermore, the optimal control u^* satisfies

$$\mathcal{H}\big(\mathsf{x}^*(t), \mathsf{p}^*(t), u^*(t)\big) = \min_{v\in[-\delta,1]} \mathcal{H}\big(\mathsf{x}^*(t), \mathsf{p}^*(t), v\big) \tag{9}$$

for almost every $t \in [0, T_\mathsf{f}]$. It follows that

$$u^*(t) = \begin{cases} +1, & \text{if } p_3^*(t) < 0, \\ v \in [-\delta, 1] & \text{if } p_3^*(t) = 0, \\ -\delta, & \text{if } p_3^*(t) > 0. \end{cases} \tag{10}$$

Using similar arguments as in [8, 26] one can show the following proposition.

Proposition 3.1 *The optimal control u^* of Problem 2.1 belongs necessarily to U^*, where*

$$\mathsf{U}^* := \big\{\{\mathsf{u}^\pm, 0, \mathsf{u}^\pm\}, \{\mathsf{u}^\pm, 0, \mathsf{u}^\mp\}, \{\mathsf{u}^\pm, \mathsf{u}^\mp, \mathsf{u}^\pm\}\big\}, \qquad \mathsf{u}^+ := 1, \qquad \mathsf{u}^- := -\delta. \tag{11}$$

Proposition 3.1 implies that a time-optimal path of Problem 2.1 is a concatenation of at most three segments, which are either bang arcs, denoted by b^- (when $u^* = -\delta$) and b^+ (when $u^* = 1$), or a singular arc (when $u^* = 0$), denoted by s. Note that b^- and b^+ arcs correspond to circular arcs of radius ϱ and ρ respectively, whereas a singular arc s corresponds to a straight line segment. It follows that a minimum-time path of Problem 2.1 has necessarily one of the following structures:

(i) $\mathsf{b}_\alpha^- \mathsf{s}_\beta \mathsf{b}_\gamma^-$, $\mathsf{b}_\alpha^+ \mathsf{s}_\beta \mathsf{b}_\gamma^+$, $\mathsf{b}_\alpha^- \mathsf{s}_\beta \mathsf{b}_\gamma^+$ and $\mathsf{b}_\alpha^+ \mathsf{s}_\beta \mathsf{b}_\gamma^-$ (two bang arcs connected via a singular arc),
(ii) or $\mathsf{b}_\alpha^+ \mathsf{b}_\beta^- \mathsf{b}_\gamma^+$ and $\mathsf{b}_\alpha^- \mathsf{b}_\beta^+ \mathsf{b}_\gamma^-$ (no singular arc),

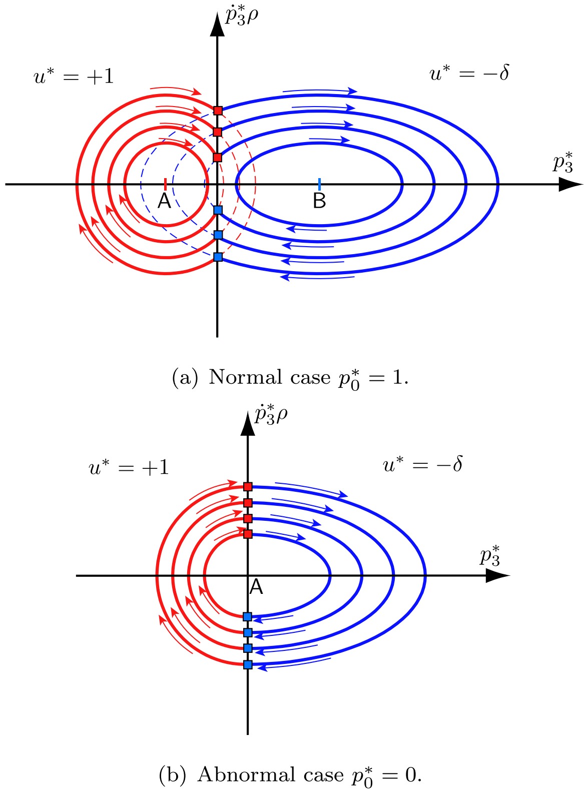
where the subscripts α, β, and γ denote the duration of motion along the first, second, and third path segments, respectively.

Proposition 3.1 provides us with six families of paths, that suffice to connect any pair of prescribed initial and terminal configurations in $\mathbb{R}^2 \times \mathbb{S}^1$ similarly to the solution of the standard MD problem. Although the collection of candidate optimal paths, that solve Problem 2.1, are at this point significantly reduced, it is still possible to refine these families further, as it is demonstrated shortly later, by analyzing the times at which the concatenations between different arcs take place (switching times).

To this end, let us consider an open interval $\mathcal{I} \subset [0, T_\mathsf{f}]$ with $p_3^*(t) \neq 0$ for all $t \in \mathcal{I}$. The restriction of the optimal control u^* on $\mathcal{I}$ is a piecewise constant function, which jumps at most twice, and $u^*(t) \in \{-\delta, +1\}$ for all $t \in \mathcal{I}$. By virtue of (5) and (8), for any subinterval $\mathcal{I}_\mathsf{b}$ of $\mathcal{I}$ where u^* is constant, p_3^* satisfies

$$\ddot{p}_3^* = -\left(\frac{u^*}{\rho}\right)^2 p_3^* - \frac{u^* p_0^*}{\rho} \tag{12}$$

Fig. 1 Phase portrait of $(p_3^*, \dot{p}_3^*\rho)$



for all $t \in \mathcal{I}_{\mathsf{b}}$. The general solution of (12) and its derivative for all $t \in \mathcal{I}$ are given by

$$p_3^*(t) = C_1 \cos \frac{u^* t}{\rho} + C_2 \sin \frac{u^* t}{\rho} - \frac{\rho p_0^*}{u^* t}, \tag{13}$$

$$\dot{p}_3^*(t) = \frac{C_2 u^*}{\rho} \cos \frac{u^* t}{\rho} - \frac{C_1 u^*}{\rho} \sin \frac{u^* t}{\rho}, \tag{14}$$

where C_1, C_2 are real constants. It follows readily that

$$\left(\frac{\rho \dot{p}_3^*(t)}{u^*}\right)^2 + \left(p_3^*(t) + \frac{\rho p_0^*}{u^*}\right)^2 = C_1^2 + C_2^2, \quad t \in \mathcal{I}_{\mathsf{b}}. \tag{15}$$

The phase portrait of $(p_3^*, \dot{p}_3^*\rho)$ is given in Fig. 1. In particular, Fig. 1(a) and Fig. 1(b) illustrate the phase portrait of $(p_3^*, \dot{p}_3^*\rho)$ for the normal case ($p_0^* = 1$) and the abnormal case ($p_0^* = 0$), respectively. Contrary to the standard MD, the phase portrait of $(p_3^*, \dot{p}_3^*\rho)$ is not symmetric with respect to the axis $p_3 = 0$ (compare for example, with [26]).

Proposition 3.2 *A* $\mathsf{b}_\alpha^- \mathsf{b}_\beta^+ \mathsf{b}_\gamma^-$ $[\mathsf{b}_\alpha^+ \mathsf{b}_\beta^- \mathsf{b}_\gamma^+]$ *path with* $\min\{\alpha, \beta, \gamma\} > 0$ *corresponds to an optimal trajectory of Problem 2.1 only if*

(i) $\beta \in]\pi\rho, 2\pi\rho[$ $[\beta \in]\pi\varrho, 2\pi\varrho[]$,
(ii) $\max\{\alpha, \gamma\} \leq \varepsilon(\delta, \beta)$, *where*

$$\varepsilon(\delta, \beta) = 2\pi\varrho + 2\varrho\,\mathrm{atan}\left(\delta \tan\frac{\beta}{2\rho}\right) \quad \left[\varepsilon(\delta, \beta) = 2\pi\rho + 2\rho\,\mathrm{atan}\left(\delta^{-1} \tan\frac{\beta}{2\varrho}\right)\right], \tag{16}$$

(iii) $\min\{\alpha, \gamma\} < \delta^{-1}\beta - \pi\varrho$ $[\min\{\alpha, \gamma\} < \delta\beta - \pi\rho]$.

Proof We consider a $\mathsf{b}_\alpha^-\mathsf{b}_\beta^+\mathsf{b}_\gamma^-$ path. The case of a $\mathsf{b}_\alpha^+\mathsf{b}_\beta^-\mathsf{b}_\gamma^+$ path can be treated similarly. First, we consider the abnormal case $p_0^* = 0$. As it is illustrated in Fig. 1(b), a point in the $(p_3^*, \rho\dot{p}_3^*)$ plane stays in the half plane $p_3 \leq 0$ for exactly time $\beta = \pi\rho$, which is the time required for a particle with coordinates $(p_3^*, \rho\dot{p}_3^*)$ to travel half of the circumference of a circle centered at the origin with constant angular speed $\omega = 1/\rho$. However, using the same geometric argument as in Lemma 23 in [8], we can show that the resulting path with $\beta = \pi\rho$ is not optimal. Hence, all optimal extremals of $\mathsf{b}_\alpha^-\mathsf{b}_\beta^+\mathsf{b}_\gamma^-$ type must be normal.

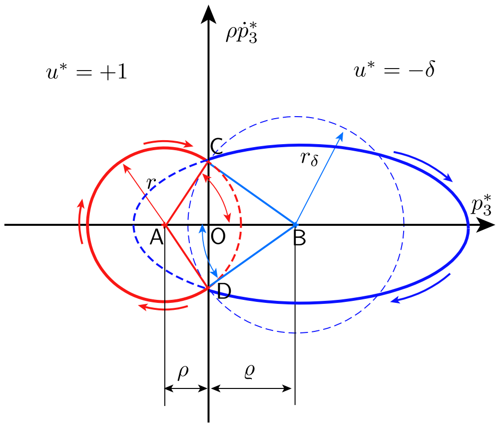
We therefore let $p_0^* = 1$ in (12)–(15). In Fig. 2, we observe that the phase portrait of $(p_3^*, \rho\dot{p}_3^*)$ consists of a circle centered at A, denoted by C_A, and an ellipse centered at B, denoted by E_B. It is assumed that both C_A and E_B are traversed clockwise by a particle with coordinates $(p_3^*, \rho\dot{p}_3^*)$, such that the rate of change of the angular position of the particle is, respectively, equal to $1/\rho$ and $1/\varrho$, when measured from A and B. Note that a jump from $u^* = -\delta$ to $u^* = +1$, and vice versa, occurs only if E_B intersects C_A along the axis $p_3^* = 0$. If this intersection does occur, we denote by C and D the points of intersection. Let r and r_δ denote the distance of either C or D from A and B, respectively. Then E_B and C_A intersect only if $r \geq \rho$ and $r_\delta \geq \varrho$, and furthermore $r_\delta = \sqrt{r^2 + \varrho^2 - \rho^2}$ as shown in Fig. 2.

From Fig. 2 it follows that β corresponds to the travel time of the point $(p_3^*, \rho\dot{p}_3^*)$ from D to C along the circle C_A. Moreover, α and γ are upper bounded by the travel time from C to D along the ellipse E_B. We observe that $\pi\rho$ is a strict lower bound for β since $\rho > 0$ (note that β approaches $\pi\rho$ as A gets closer to O, without reaching it as far as $\rho > 0$). Furthermore, $2\pi\rho$ and $2\pi\varrho$ are strict upper bounds for β and both α and γ, respectively, since a bang arc $\mathsf{b}_{2\pi\varrho}^+$ or $\mathsf{b}_{2\pi\rho}^-$ corresponds to a full circle driving the system (1) to the same state, and thus neither $\mathsf{b}_{2\pi\varrho}^+$ nor $\mathsf{b}_{2\pi\rho}^-$ can be part of an optimal solution.

Next, we improve the upper bound on α, γ. In particular, we observe in Fig. 2 that, given β where $\beta = 2(\pi - \widehat{\mathsf{CAO}})\rho$, then α or γ is maximized if the point $(p_3^*, \rho\dot{p}_3^*)$ coincides with C at $t = 0$ or D at $t = T_\mathsf{f}$, respectively; that is, $\max\{\alpha, \gamma\} \leq 2(\pi - \widehat{\mathsf{DBO}})\varrho$. By using simple geometric arguments, along with the fact that $\delta \in]0, 1]$, it follows that $\widehat{\mathsf{DBO}} = \mathrm{atan}(\delta \tan\widehat{\mathsf{CAO}})$. Thus, $\max\{\alpha, \gamma\} \leq 2(\pi - \mathrm{atan}(\delta \tan\widehat{\mathsf{CAO}}))\varrho$, and $\beta = 2(\pi - \widehat{\mathsf{CAO}})\rho$. Equation (16) follows immediately.

Finally, the third condition of the proposition is proved by means of simple geometric arguments as in Lemma 3 of [24]. □

Fig. 2 Phase portrait $(p_3^*, \rho\dot{p}_3^*)$



Proposition 3.3 *A* $\mathsf{b}_\alpha^-\mathsf{s}_\beta\mathsf{b}_\gamma^-$ *path corresponds to a time-optimal trajectory of Problem 2.1 only if* $\alpha + \gamma \leq 2\pi\varrho$.

Proof See the proof of Lemma 5 of [24]. □

It is worth mentioning that Lemma 5 of [24] does not apply for $\mathsf{b}^+\mathsf{s}\mathsf{b}^+$ paths of the ASDMD problem. In particular, as it is illustrated in Fig. 3, the ASDID car emanating from O reaches the terminal configuration $\mathsf{x}_\mathsf{f} = (x_\mathsf{f}, y_\mathsf{f}, \vartheta_\mathsf{f})$ by traversing a $\mathsf{b}_\alpha^+\mathsf{s}_\beta\mathsf{b}_\gamma^+$ path with $\alpha + \gamma > 2\pi\rho$. The total elapsed time is the same as if the ASDID car had traversed a $\mathsf{b}_\alpha^-\mathsf{s}_\beta\mathsf{b}_\gamma^-$ with $\alpha + \gamma \leq 2\pi\varrho$. Therefore, if the path $\mathsf{b}^-\mathsf{s}\mathsf{b}^-$ is time-optimal, then the $\mathsf{b}_\alpha^+\mathsf{s}_\beta\mathsf{b}_\gamma^+$ path is necessarily time-optimal as well. Thus, we conjecture that there exist $\mathsf{b}_\alpha^+\mathsf{s}_\beta\mathsf{b}_\gamma^+$ paths with $\alpha + \gamma > 2\pi\rho$, that are optimal paths of the ASDMD problem. As it is demonstrated in Sect. 4, our conjecture is indeed correct. Next we provide a conservative bound on the sum of α and γ along $\mathsf{b}_\alpha^+\mathsf{s}_\beta\mathsf{b}_\gamma^+$ paths.

Proposition 3.4 *A* $\mathsf{b}_\alpha^+\mathsf{s}_\beta\mathsf{b}_\gamma^+$ *path corresponds to a time-optimal trajectory of Problem 2.1 only if* $\alpha + \gamma \leq (4\pi - \vartheta_\mathsf{f})\rho$.

Finally, for $\mathsf{b}^-\mathsf{s}\mathsf{b}^+$ and $\mathsf{b}^+\mathsf{s}\mathsf{b}^-$ paths, as in the standard MD, we simply take the most conservative bounds. In particular, we have the following proposition.

Proposition 3.5 *A* $\mathsf{b}_\alpha^+\mathsf{s}_\beta\mathsf{b}_\gamma^-$ *and a* $\mathsf{b}_\alpha^-\mathsf{s}_\beta\mathsf{b}_\gamma^+$ *path correspond to a time-optimal trajectory of Problem 2.1 only if* $\max\{\alpha, \delta\gamma\} < 2\pi\rho$ *and* $\max\{\delta\alpha, \gamma\} < 2\pi\rho$, *respectively.*

4 Time-Optimal Synthesis

In this section, we address the time-optimal synthesis problem for the ASDMD problem, and thus provide a complete characterization of the optimal control that solves Problem 2.1 with boundary conditions (3) for all $(x_\mathsf{f}, y_\mathsf{f}, \vartheta_\mathsf{f}) \in \mathbb{R}^2 \times \mathbb{S}^1$.

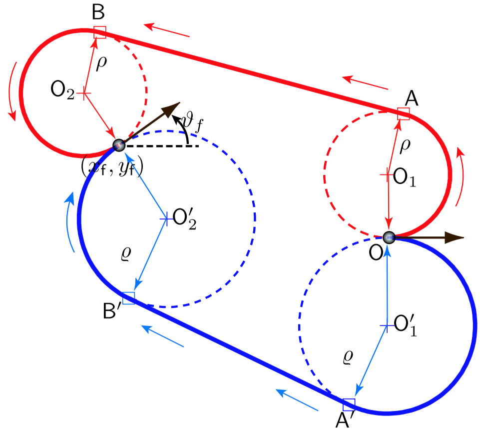
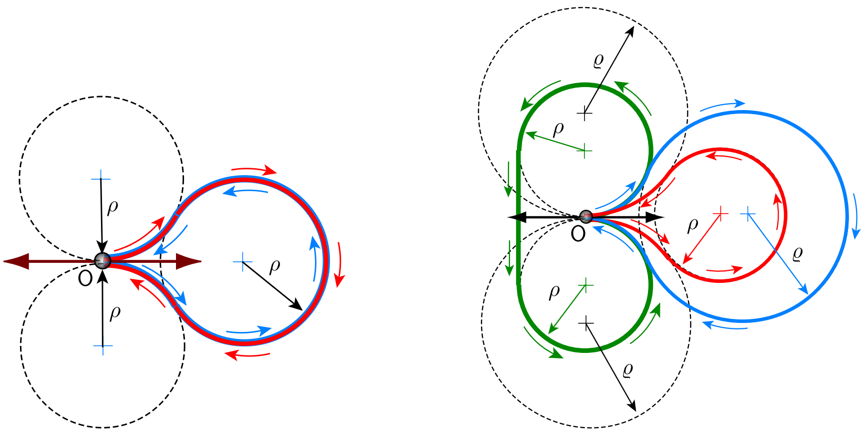


Fig. 3 A $\mathsf{b}_\alpha^+\mathsf{s}_\beta\mathsf{b}_\gamma^+$ path with $\alpha+\gamma>2\pi\rho$ can be an optimal solution of the ASDMD problem, in contrast to the solution of the standard MD problem



(a) Standard MD problem. (b) Asymmetric sinistral/dextral Markov-Dubins problem.

Fig. 4 The minimum-time paths for the steering problem from $(0,0,0)$ to $(0,0,\pi)$ for the ID and the ASDID cars

First, we show by means of an example, that the synthesis of optimal paths for the ASDMD problem may be quite different from that of the MD problem. In particular, let us consider the problem of characterizing the minimum-time path from $(0,0,0)$ to $(0,0,\pi)$ for the ID and the ASDID cars. On the one hand, the optimal solution of the standard MD problem is either a $\mathsf{b}_\alpha^+\mathsf{b}_\beta^-\mathsf{b}_\gamma^+$ path or a $\mathsf{b}_\alpha^-\mathsf{b}_\beta^+\mathsf{b}_\gamma^-$ path, where $\alpha=\gamma=\pi\rho/3$ and $\beta=5\pi\rho/3$, as shown in Fig. 4(a) (these two paths have exactly the same length). On the other hand, as it is illustrated in Fig. 4(b), the optimal path for the ASDMD problem is either a $\mathsf{b}_\alpha^-\mathsf{b}_\beta^+\mathsf{b}_\gamma^-$ path, where $\alpha=\gamma=\varrho\ \mathrm{acos}(1/(1+\delta))$ and $\beta=\pi\rho+2\delta\alpha$ or an $\mathsf{b}_\alpha^+\mathsf{s}_\beta\mathsf{b}_\gamma^+$ path, where $\alpha=\gamma=3\pi\rho/2$ and $\beta=2\rho$. The $\mathsf{b}_\alpha^-\mathsf{b}_\beta^+\mathsf{b}_\gamma^-$ and the $\mathsf{b}_\alpha^+\mathsf{s}_\beta\mathsf{b}_\gamma^+$ paths have exactly the same length when $\delta=\tilde{\delta}$, where $\tilde{\delta}$ is the solution of the equation $1/(1+\delta)+\cos((\pi-\delta)/(1+\delta))=0$. Note that for this specific problem, the $\mathsf{b}_\alpha^+\mathsf{s}_\beta\mathsf{b}_\gamma^+$ path can never be an optimal path of the standard MD problem, in light of Lemma 5 of [24].

To simplify the presentation and without loss in generality, we henceforth consider the minimum trajectories of the ASDID car from $(0,0,0)$ to $(x_\mathsf{f}, y_\mathsf{f}, \vartheta_\mathsf{f}) \in P_{\vartheta_\mathsf{f}}$, where $P_{\vartheta_\mathsf{f}} := \{(x, y, \vartheta) \in \mathbb{R}^2 \times \mathbb{S}^1 \ : \ \vartheta = \vartheta_\mathsf{f}\}$ as suggested in [23, 24]. To this end, let $\mathfrak{R}_{\vartheta_\mathsf{f}}(u)$ denote the reachable set that corresponds to the control sequence $u \in \mathsf{U}^*$. The coordinates of all points in P_{ϑ_f} that can be reached by means of a $\mathsf{b}^+\mathsf{s}\mathsf{b}^+$ control sequence can be expressed as functions of the times of motion along the three arcs of the path, namely α, β, and γ, by simply integrating (1) from $t=0$ to $t=\alpha$ for $u=+1$, and subsequently from $t=\alpha$ to $t=\alpha+\beta$ for $u=0$, and finally from $t=\alpha+\beta$ to the final time $T_\mathsf{f}(\mathsf{b}^+\mathsf{s}\mathsf{b}^+) = \alpha+\beta+\gamma$. Note that γ can always be expressed in terms of the parameters α and β (actually for a $\mathsf{b}^+\mathsf{s}\mathsf{b}^+$ path γ depends only on α as we shall see shortly later). In particular, since the total change of the velocity direction ϑ (initially $\vartheta = 0$) along the path must equal $\vartheta_\mathsf{f} \pmod{2\pi}$, it follows readily that $\alpha/\rho + \gamma/\rho = \vartheta_\mathsf{f} \pmod{2\pi}$, which furthermore implies that

$$\gamma(\alpha) = \begin{cases} \rho\vartheta_\mathsf{f} - \alpha, & \text{if } \vartheta_\mathsf{f} \geq \dfrac{\alpha}{\rho}, \\ \rho(2\pi + \vartheta_\mathsf{f}) - \alpha, & \text{if } \vartheta_\mathsf{f} < \dfrac{\alpha}{\rho}. \end{cases} \tag{17}$$

It follows after routine calculations that

$$x_\mathsf{f}(\alpha,\beta) = \rho\sin\vartheta_\mathsf{f} + \beta\cos\frac{\alpha}{\rho}, \qquad y_\mathsf{f}(\alpha,\beta) = \rho + \beta\sin\frac{\alpha}{\rho} - \rho\cos\vartheta_\mathsf{f}. \tag{18}$$

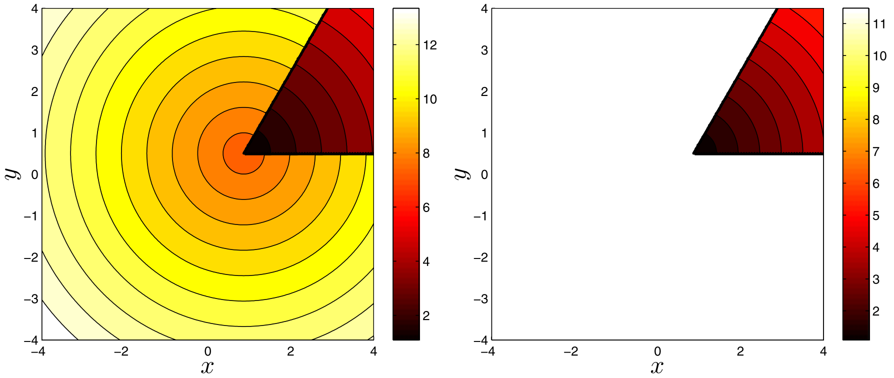
Furthermore, Proposition 3.4 determines the intervals of admissible values of α and β for a $\mathsf{b}^+\mathsf{s}\mathsf{b}^+$ control sequence, denoted by $\mathcal{I}_\alpha(\mathsf{b}^+\mathsf{s}\mathsf{b}^+)$ and $\mathcal{I}_\beta(\mathsf{b}^+\mathsf{s}\mathsf{b}^+)$, respectively. Thus, the reachable set of the control sequence $\mathsf{b}^+\mathsf{s}\mathsf{b}^+$ is constructed by determining all points $(x_\mathsf{f}, y_\mathsf{f}, \vartheta_\mathsf{f}) \in P_{\vartheta_\mathsf{f}}$ for every pairs of $(\alpha,\beta) \in \mathcal{I}_\alpha(\mathsf{b}^+\mathsf{s}\mathsf{b}^+) \times \mathcal{I}_\beta(\mathsf{b}^+\mathsf{s}\mathsf{b}^+)$.

Conversely, given a point $(x_\mathsf{f}, y_\mathsf{f}, \vartheta_\mathsf{f}) \in \mathfrak{R}_{\vartheta_\mathsf{f}}(\mathsf{b}^+\mathsf{s}\mathsf{b}^+)$ one can determine the parameters α and β such that x_f and y_f satisfy (18). In particular, after some calculation it follows from (18) that

$$\alpha(x_\mathsf{f}, y_\mathsf{f}) = \rho\,\mathrm{atan}\,2\big(B(y_\mathsf{f}), A(x_\mathsf{f})\big), \qquad \beta(x_\mathsf{f}, y_\mathsf{f}) = \sqrt{A^2(x_\mathsf{f}) + B^2(y_\mathsf{f})}, \tag{19}$$

where $A(x_\mathsf{f}) = x_\mathsf{f} - \rho\sin\vartheta_\mathsf{f}$, $B(y_\mathsf{f}) = y_\mathsf{f} + \rho\cos\vartheta_\mathsf{f} - \rho$, and $\mathrm{atan}\,2 : \mathbb{R}^2 \mapsto [0, 2\pi[$ is the two-argument arctangent function.

Figure 5(a) illustrates the reachable set $\mathfrak{R}_{\vartheta_\mathsf{f}}(\mathsf{b}^+\mathsf{s}\mathsf{b}^+)$ of the ASDID car (note that for this path family the value of δ does not affect the geometry of the reachable set), whereas the same reachable set for the standard ID car is illustrated in Fig. 5(b). We observe that the former set is a superset of the latter. This is because for the ASDMD problem α satisfies $\alpha + \gamma(\alpha) \leq (4\pi - \vartheta_\mathsf{f})\rho$ (Proposition 3.4), whereas for the standard MD problem it satisfies the stricter condition $\alpha + \gamma(\alpha) \leq 2\pi\rho$ (Lemma 5 of [24]).



(a) $0 < \delta < 1$ (b) $\delta = 1$ (standard MD)

Fig. 5 Reachable set $\mathfrak{R}_{\vartheta_\mathsf{f}}(\mathsf{b}^+\mathsf{sb}^+)$ for $\delta \in]0, 1[$ (ASDMD problem) and $\delta = 1$ (standard MD problem). The *white colored region* corresponds to terminal configurations that cannot be reached in minimum time by means of a $\mathsf{b}^+\mathsf{sb}^+$ control sequence for the standard MD problem

Finally, after having established the connection between (α, β) and $(x_\mathsf{f}, y_\mathsf{f})$, the total time $T_\mathsf{f}(\mathsf{b}^+\mathsf{sb}^+)$ is given, via (17), by

$$T_\mathsf{f}(\mathsf{b}^+\mathsf{sb}^+) = \begin{cases} \beta + \rho\vartheta_\mathsf{f}, & \text{if } \vartheta_\mathsf{f} \geq \alpha/\rho, \\ \beta + \rho(2\pi + \vartheta_\mathsf{f}), & \text{if } \vartheta_\mathsf{f} < \alpha/\rho. \end{cases} \tag{20}$$

The previous procedure can be applied mutatis mutandis for the rest of the control sequences from U^* (although the algebra, especially in the case of $\mathsf{b}^+\mathsf{b}^-\mathsf{b}^+$ or $\mathsf{b}^-\mathsf{b}^+\mathsf{b}^-$ paths, is significantly more evolved). In the Appendix we provide the equations that give α and β as functions of x_f and y_f, and vice versa, as well as the minimum time T_f for all the control sequences $u \in \mathsf{U}^*$.

The next step involves the partitioning of P_{ϑ_f} into at most six domains, denoted by $\mathfrak{R}^*_{\vartheta_\mathsf{f}}(u)$, where $u \in \mathsf{U}^*$, such that if $(x_\mathsf{f}, y_\mathsf{f}, \vartheta_\mathsf{f}) \in \mathrm{int}(\mathfrak{R}^*_{\vartheta_\mathsf{f}}(u))$, then $(x_\mathsf{f}, y_\mathsf{f}, \vartheta_\mathsf{f})$ cannot be reached faster with the application of $v \in \mathsf{U}^*$, where $v \neq u$. We shall refer to this partition of P_{ϑ_f} as the optimal control partition of the ASDMD problem. The number of these domains can be strictly less than six in case the domain associated with a particular control sequence has an empty interior. As we shall see shortly afterwards, such "pathological" cases arise in the time-optimal synthesis of the ASDMD problem in contrast to the optimal synthesis of the standard MD problem. The procedure required for the characterization of the domain over which the control sequence, say $\mathsf{b}^+\mathsf{sb}^+$, is optimal, is summarized below. We denote this domain by $\mathfrak{R}^*_{\vartheta_\mathsf{f}}(\mathsf{b}^+\mathsf{sb}^+)$. In particular, let $(x_\mathsf{f}, y_\mathsf{f}, \vartheta_\mathsf{f}) \in \mathfrak{R}_{\vartheta_\mathsf{f}}(\mathsf{b}^+\mathsf{sb}^+)$, and let $\mathsf{U}^c(\mathsf{b}^+\mathsf{sb}^+) \subset \mathsf{U}^*$ denote the set of control sequences u that are different from $\mathsf{b}^+\mathsf{sb}^+$ and such that $(x_\mathsf{f}, y_\mathsf{f}, \vartheta_\mathsf{f}) \in \mathfrak{R}_{\vartheta_\mathsf{f}}(u)$. Then $(x_\mathsf{f}, y_\mathsf{f}, \vartheta_\mathsf{f}) \in \mathfrak{R}^*_{\vartheta_\mathsf{f}}(\mathsf{b}^+\mathsf{sb}^+)$ if and only if $T_\mathsf{f}(\mathsf{b}^+\mathsf{sb}^+) \leq \min_{u \in \mathsf{U}^c(\mathsf{b}^+\mathsf{sb}^+)} T_\mathsf{f}(u)$, and furthermore $(x_\mathsf{f}, y_\mathsf{f}, \vartheta_\mathsf{f}) \in \mathrm{int}(\mathfrak{R}^*_{\vartheta_\mathsf{f}}(\mathsf{b}^+\mathsf{sb}^+))$ if and only if $T_\mathsf{f}(\mathsf{b}^+\mathsf{sb}^+) < \min_{u \in \mathsf{U}^c(\mathsf{b}^+\mathsf{sb}^+)} T_\mathsf{f}(u)$.

Figure 6 illustrates the optimal control partition of $P_{\pi/3}$ as well as the level sets of the minimum time T_f, for different values of the ratio $\delta^{-1} = \varrho/\rho$. In particular, each

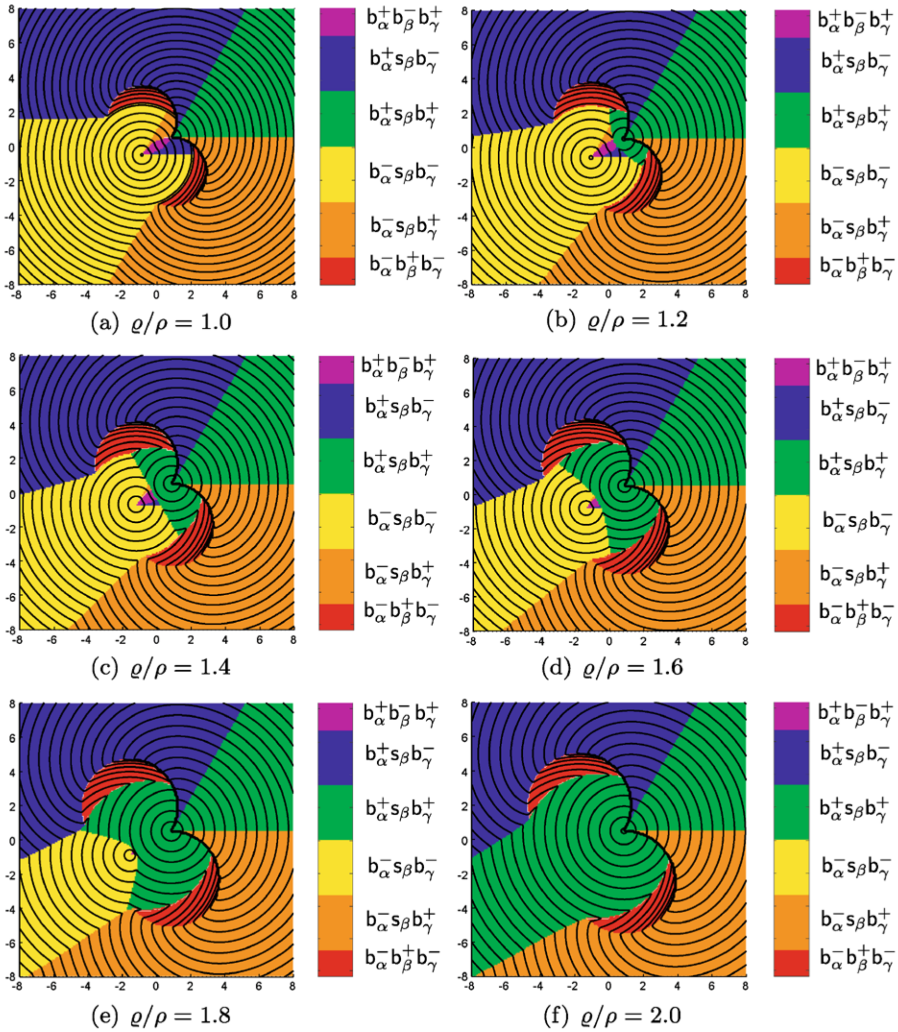


Fig. 6 Partition of $P_{\pi/3}$ and level sets of $T_\mathfrak{f} = T_\mathfrak{f}(x, y)$ for different values of the ratio $\delta^{-1} = \varrho/\rho$

domain of the partition $P_{\pi/3}$ is illustrated by a colored set whereas the level sets of the minimum time are denoted by solid black lines. We observe that as the ratio ϱ/ρ increases, the domains $\mathfrak{R}^*_{\pi/3}(\mathsf{b}^+\mathsf{sb}^+)$, $\mathfrak{R}^*_{\pi/3}(\mathsf{b}^-\mathsf{sb}^+)$ and $\mathfrak{R}^*_{\pi/3}(\mathsf{b}^+\mathsf{sb}^-)$, primarily, and the domain $\mathfrak{R}^*_{\pi/3}(\mathsf{b}^-\mathsf{b}^+\mathsf{b}^-)$, secondarily, expand against the domain $\mathfrak{R}^*_{\pi/3}(\mathsf{b}^-\mathsf{sb}^-)$ as well as the disconnected components of $\mathfrak{R}^*_{\pi/3}(\mathsf{b}^+\mathsf{sb}^-)$ and $\mathfrak{R}^*_{\pi/3}(\mathsf{b}^-\mathsf{sb}^+)$ that are close to the origin. We observe, in particular, that for $\varrho/\rho = 1.8$ (Fig. 6(e)) the partition of $P_{\pi/3}$ consists of five domains since the domain $\mathfrak{R}^*_{\pi/3}(\mathsf{b}^+\mathsf{b}^-\mathsf{b}^+)$ is reduced to the empty set. Similarly, for $\varrho/\rho = 2$ (Fig. 6(f)) only four domains are non-empty since $\mathfrak{R}^*_{\pi/3}(\mathsf{b}^-\mathsf{sb}^-) = \mathfrak{R}^*_{\pi/3}(\mathsf{b}^+\mathsf{b}^-\mathsf{b}^+) = \varnothing$. In addition, we observe in Fig. 6(a)–6(f) that the boundaries of each domain change significantly as the ratio ϱ/ρ varies.

5 Time Optimal Synthesis and Reachable Sets of the ASDMD when the Final Tangent of the Path is Free

In this section, we consider the optimal synthesis of Problem 2.1, when ϑ_{f} is assumed to be free. The solution of this variation of Problem 2.1 will allow us to characterize analytically the set of points in the plane that can be reached by curves with asymmetric curvature constraints. These reachable sets along with the level sets of the minimum time of the ASDMD problem, when ϑ_{f} is free, exhibit a few notable features related to the existence/absence of symmetry planes that are not observed neither in the reachable sets nor the syntheses of the standard MD and the ASDMD, when ϑ_{f} is fixed, problems. Favoring the economy of presentation, we shall not discuss in detail the analysis of this problem, which is similar to the discussion presented in Sects. 3–4, but instead we will present the solution of the time-optimal synthesis problem directly.

First, we discuss briefly the structure of the family of extremal controls, which is sufficient for optimality for Problem 2.1, when ϑ_{f} is free. In particular, the new transversality condition for ϑ is given by $p_3^*(T_{\mathsf{f}}) = 0$. Following the same line of arguments as in [26], where the standard MD, when ϑ_{f} is free, is addressed in detail, we conclude that a composite path whose final arc is either a b^- or a b^+ arc, that is preceded by an s arc, cannot be part of an optimal path. The following proposition gives us the family of candidate optimal controls for Problem 2.1, when ϑ_{f} is free (it follows similarly to [26]).

Proposition 5.1 *The optimal control u^* of Problem 2.1, when ϑ_{f} is free, belongs necessarily to U^*, where*

$$\mathsf{U}^* := \{\{\mathsf{u}^\pm, 0\}, \{\mathsf{u}^\pm, \mathsf{u}^\mp\}\}, \qquad \mathsf{u}^+ := 1, \qquad \mathsf{u}^- := -\delta. \tag{21}$$

Proposition 5.1 implies that the set of candidate optimal controls of Problem 2.1, when ϑ_{f} is free, consists of only four control sequences with at most one switch. It follows that the minimum-time paths of Problem 2.1, when ϑ_{f} is free, have necessarily one of the following structures: (i) $\mathsf{b}_\alpha^+\mathsf{b}_\beta^-$, $\mathsf{b}_\alpha^-\mathsf{b}_\beta^+$, (ii) $\mathsf{b}_\alpha^+\mathsf{s}_\beta$, $\mathsf{b}_\alpha^-\mathsf{s}_\beta$.

By repeating the analysis carried out in Sects. 3 and 4, we can refine the family of candidate optimal controls (this analysis will lead to a number of propositions similar to Propositions 3.2–3.5), and subsequently solve the synthesis problem for Problem 2.1, when ϑ_{f} is free. Favoring the economy of presentation, we show directly the solution of the synthesis problem. In particular, Fig. 7 illustrates the optimal control partition of the plane as well as the level sets of the minimum time T_{f}, when ϑ_{f} is free (assuming that the ID/ASDID car starts from the origin with $\vartheta = 0$) for both the standard ID car (Fig. 7(a)) and the ASDID car (Figs. 7(b)–7(d)). Figures 7(b)–7(d) illustrate that as the agility of the ASDID car to perform right turns, which is measured by the ratio ϱ/ρ, is reduced, the sets $\mathfrak{R}^*(\mathsf{b}^-\mathsf{s})$ and $\mathfrak{R}^*(\mathsf{b}^+\mathsf{b}^-)$ "shrink" in favor of the set $\mathfrak{R}^*(\mathsf{b}^+\mathsf{s})$, whereas the set $\mathfrak{R}^*(\mathsf{b}^-\mathsf{b}^+)$ remains invariant under the variations of the ratio ϱ/ρ.

It is worth noting that contrary to the synthesis of the ASDMD problem, when ϑ_{f} is fixed, where both the level sets of the minimum time and the domains of the optimal control partition are symmetric with respect to some plane of symmetry (also

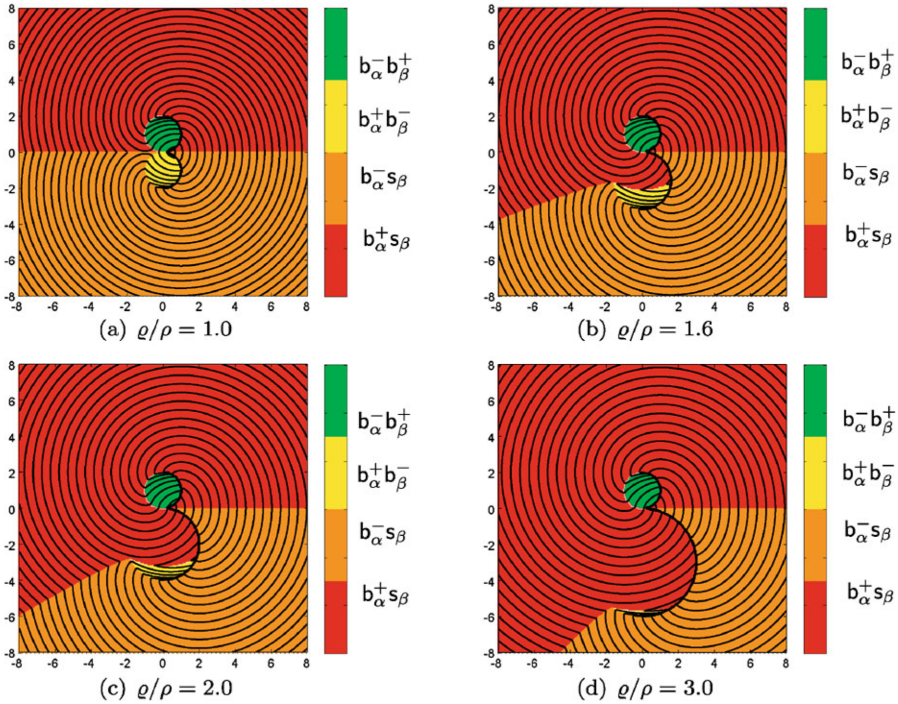


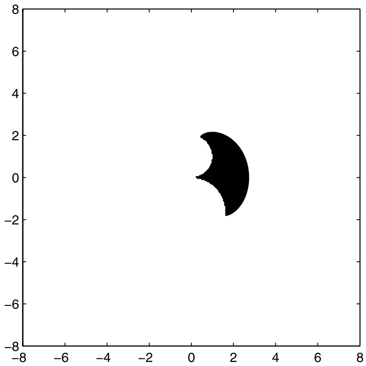
Fig. 7 Partition of P and level sets of $T_{\mathsf{f}} = T_{\mathsf{f}}(x, y)$ for different values of the ratio ϱ/ρ

a characteristic of the optimal synthesis of the standard MD problem), both the level sets and the domains of the optimal control partition of the ASDMD problem, when ϑ_{f} is free, do not enjoy similar symmetry properties. It appears that the term "asymmetric" used in the title of this work is more obviously justified in the case when ϑ_{f} is free rather than when ϑ_{f} is fixed.

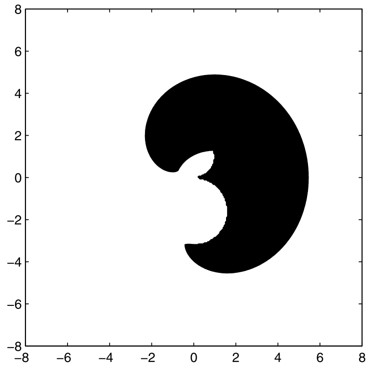
To this end, let $\mathfrak{R}^{\mathrm{s}}_{t\leq\tau}$ and $\mathfrak{R}^{\mathrm{asym}}_{t\leq\tau}$ denote the set of points in the plane that can be reached by the ID and ASDID car in time $t \in [0, \tau]$, respectively (assuming again that the ID/ASDID car starts from the origin with $\vartheta = 0$). The reachable sets $\mathfrak{R}^{\mathrm{asym}}_{t\leq\tau}$ for different values of τ are illustrated in Fig. 8. In Figs. 8(a)–8(d), we observe that the reachable sets $\mathfrak{R}^{\mathrm{asym}}_{t\leq\tau}$ are not symmetric with respect to the x-axis in contrast to the sets $\mathfrak{R}^{\mathrm{s}}_{t\leq\tau}$ (see, for example, [10, 27]). This comes as no surprise, since both $\mathfrak{R}^{\mathrm{s}}_{t\leq\tau}$ and $\mathfrak{R}^{\mathrm{asym}}_{t\leq\tau}$ can be interpreted as the union of all the level sets $\{(x, y) : T_{\mathsf{f}} = t\}$, for $t \in [0, \tau]$, which, as we have already mentioned, are symmetric with respect to x-axis for the standard MD problem but not for the ASDMD problem, when ϑ_{f} is free.

6 Conclusions

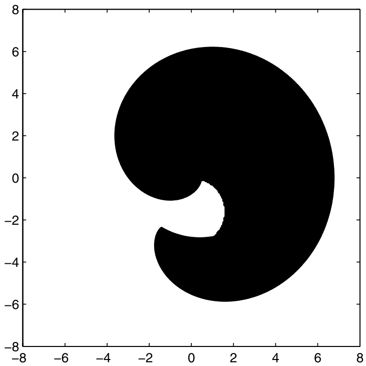
In this article, we have proposed and solved a generalization of the Markov–Dubins problem that deals with the characterization of minimal-length paths with asymmetric curvature constraints. This shortest-path problem is equivalent to the characterization



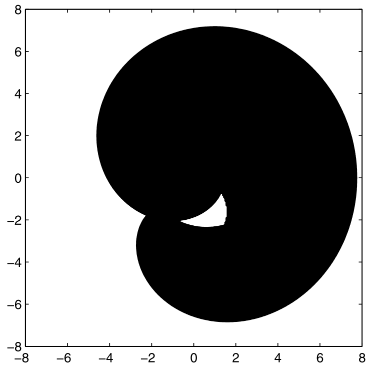
(a) $\mathfrak{R}^{\mathrm{asym}}_{t\leq\tau}$, $\tau = (\rho + \varrho)\pi/3$.



(b) $\mathfrak{R}^{\mathrm{asym}}_{t\leq\tau}$, $\tau = (\rho + \varrho)2\pi/3$.



(c) $\mathfrak{R}^{\mathrm{asym}}_{t\leq\tau}$, $\tau = (\rho + \varrho)(3\pi + 1)/4$.



(d) $\mathfrak{R}^{\mathrm{asym}}_{t\leq\tau}$, $\tau = (\rho + \varrho)(3\pi + 2.5)/4$.

Fig. 8 Reachable sets $\mathfrak{R}^{\mathrm{asym}}_{t\leq\tau}$ for different values of τ and for $\varrho/\rho = 1.6$

of time-optimal trajectories for a vehicle with Isaacs–Dubins' car kinematics, which has a bias towards left (alternatively, right) turns; a situation that may be the result of an actuator failure. In the minimum-time formulation of our problem, the asymmetric constraints over the curvature of the minimal-length path are associated with the minimum radii of a left and a right turn of the Isaacs–Dubins car, which may not be necessarily equal. Our analysis has revealed that while the structure of the optimal control is qualitatively the same with the standard MD problem, the synthesis problem is, nonetheless, significantly different. In addition, we have examined the case when the tangent of the curve at the terminal point is free, and we have subsequently characterized the set of points in the plane that can be reached by curves satisfying asymmetric curvature constraints.

Appendix

In this section, we provide the details for the solution of the equations for the synthesis problem.

A.1 $\mathsf{b}_\alpha^+\mathsf{s}_\beta\mathsf{b}_\gamma^+$ [$\mathsf{b}_\alpha^-\mathsf{s}_\beta\mathsf{b}_\gamma^-$] Paths

The coordinates of a point in $\mathfrak{R}_{\vartheta_\mathsf{f}}(\mathsf{b}^+\mathsf{s}\mathsf{b}^+)$ [$\mathfrak{R}_{\vartheta_\mathsf{f}}(\mathsf{b}^-\mathsf{s}\mathsf{b}^-)$] as a function of the parameters α and β are given by

$$x_\mathsf{f} = \rho[-\varrho]\sin\vartheta_\mathsf{f} + \beta\cos\frac{\alpha}{\rho[\varrho]}, \qquad y_\mathsf{f} = \rho[-\varrho] + [-]\beta\sin\frac{\alpha}{\rho[\varrho]} - \rho[+\varrho]\cos\vartheta_\mathsf{f}. \tag{22}$$

Conversely, the parameters $\alpha \in \mathcal{I}_\alpha(\mathsf{b}^+\mathsf{s}\mathsf{b}^+)$ [$\mathcal{I}_\alpha(\mathsf{b}^-\mathsf{s}\mathsf{b}^-)$] and $\beta \in \mathcal{I}_\beta(\mathsf{b}^+\mathsf{s}\mathsf{b}^+)$ [$\mathcal{I}_\beta(\mathsf{b}^-\mathsf{s}\mathsf{b}^-)$] satisfy

$$\alpha = \rho[\varrho]\,\mathrm{atan}\,2(B, A), \qquad \beta = \sqrt{A^2 + B^2}, \tag{23}$$

where $A = x_\mathsf{f} - \rho[+\varrho]\sin\vartheta_\mathsf{f}$ and $B = [-]y_\mathsf{f} + \rho[\varrho]\cos\vartheta_\mathsf{f} - \rho[\varrho]$. Finally, the final time $T_\mathsf{f}(\mathsf{b}^+\mathsf{s}\mathsf{b}^+)$ [$T_\mathsf{f}(\mathsf{b}^-\mathsf{s}\mathsf{b}^-)$] is given by

$$T_\mathsf{f} = \begin{cases} \beta + \rho\vartheta_\mathsf{f}\ [\varrho(2\pi - \vartheta_\mathsf{f}], & \text{if } \vartheta_\mathsf{f} \geq \dfrac{\alpha}{\rho}\left[\vartheta_\mathsf{f} \leq 2\pi - \dfrac{\alpha}{\varrho}\right], \\ \beta + \rho(\vartheta_\mathsf{f} + 2\pi)\ [\beta + \varrho(4\pi - \vartheta_\mathsf{f})], & \text{if } \vartheta_\mathsf{f} < \dfrac{\alpha}{\rho}\left[\vartheta_\mathsf{f} > 2\pi - \dfrac{\alpha}{\varrho}\right]. \end{cases} \tag{24}$$

A.2 $\mathsf{b}_\alpha^+\mathsf{s}_\beta\mathsf{b}_\gamma^-$ [$\mathsf{b}_\alpha^-\mathsf{s}_\beta\mathsf{b}_\gamma^+$] Paths

The coordinates of a point in $\mathfrak{R}_{\vartheta_\mathsf{f}}(\mathsf{b}^+\mathsf{s}\mathsf{b}^-)$ as a function of the parameters α and β are given by

$$x_\mathsf{f} = (\varrho + \rho)\sin\frac{\alpha}{\rho[\varrho]} + \beta\cos\frac{\alpha}{\rho[\varrho]} - \varrho[+\rho]\sin\vartheta_\mathsf{f}, \tag{25}$$

$$y_\mathsf{f} = \rho[-\varrho] - [+](\varrho + \rho)\cos\frac{\alpha}{\rho[\varrho]} + [-]\beta\sin\frac{\alpha}{\rho[\varrho]} + \varrho[-\rho]\cos\vartheta_\mathsf{f}. \tag{26}$$

Conversely, the parameters $\alpha \in \mathcal{I}_\alpha(\mathsf{b}^+\mathsf{s}\mathsf{b}^-)$ [$\mathcal{I}_\alpha(\mathsf{b}^-\mathsf{s}\mathsf{b}^+)$] and $\beta \in \mathcal{I}_\beta(\mathsf{b}^+\mathsf{s}\mathsf{b}^-)$ [$\mathcal{I}_\beta(\mathsf{b}^-\mathsf{s}\mathsf{b}^+)$] satisfy

$$\alpha = \rho[\varrho]\,\mathrm{atan}\,2\big((\rho + \varrho)A - B\beta, (\rho + \varrho)B + A\beta\big), \quad \beta = \sqrt{A^2 + B^2 - (\varrho + \rho)^2}, \tag{27}$$

where $A = x_\mathsf{f} + \varrho[-\rho]\sin\vartheta_\mathsf{f}$ and $B = \rho[\varrho] - [+]y_\mathsf{f} + \varrho[\rho]\cos\vartheta_\mathsf{f}$.

Finally, the final time $T_\mathsf{f}(\mathsf{b}^+\mathsf{s}\mathsf{b}^-)$ [$T_\mathsf{f}(\mathsf{b}^-\mathsf{s}\mathsf{b}^+)$] is given by

$$T_\mathsf{f} = \begin{cases} \beta + \alpha(1 + \delta^{-1}[\delta]) - \varrho[+\rho]\vartheta_\mathsf{f}, & \text{if } \vartheta_\mathsf{f} \leq \dfrac{\alpha}{\rho}\left[\vartheta_\mathsf{f} + \dfrac{\alpha}{\varrho} \leq 2\pi\right], \\ \beta + \alpha(1 + \delta^{-1}[\delta]) + \varrho[-\rho](2\pi - \vartheta_\mathsf{f}), & \text{if } \vartheta_\mathsf{f} > \dfrac{\alpha}{\rho}\left[\vartheta_\mathsf{f} + \dfrac{\alpha}{\varrho} > 2\pi\right]. \end{cases} \tag{28}$$

A.3 $\mathsf{b}^+_\alpha \mathsf{b}^-_\beta \mathsf{b}^+_\gamma$ $[\mathsf{b}^-_\alpha \mathsf{b}^+_\beta \mathsf{b}^-_\gamma]$ Paths

The coordinates of a point in $\mathfrak{R}_{\vartheta_\mathsf{f}}(\mathsf{b}^+\mathsf{b}^-\mathsf{b}^+)$ as a function of the parameters α and β are given by

$$x_\mathsf{f} = (\varrho + \rho)\sin\frac{\alpha}{\rho[\varrho]} + (\varrho + \rho)\sin\left(\frac{\beta}{\varrho[\rho]} - \frac{\alpha}{\rho[\varrho]}\right) + \rho[-\varrho]\sin\vartheta_\mathsf{f}, \tag{29}$$

$$y_\mathsf{f} = \rho[-\varrho] - [+](\varrho + \rho)\cos\frac{\alpha}{\rho[\varrho]} + [-](\varrho + \rho)\cos\left(\frac{\beta}{\varrho[\rho]} - \frac{\alpha}{\rho[\varrho]}\right) - \rho[+\varrho]\cos\vartheta_\mathsf{f}. \tag{30}$$

Conversely, the parameters $\alpha \in \mathcal{I}_\alpha(\mathsf{b}^+\mathsf{b}^-\mathsf{b}^+)$ $[\mathcal{I}_\alpha(\mathsf{b}^-\mathsf{b}^+\mathsf{b}^-)]$ and $\beta \in \mathcal{I}_\beta(\mathsf{b}^+\mathsf{b}^-\mathsf{b}^+)$ $[\mathcal{I}_\beta(\mathsf{b}^-\mathsf{b}^+\mathsf{b}^-)]$ satisfy

$$\alpha = \rho[\varrho]\,\mathrm{atan}\,2\left(A\left(1 - \cos\frac{\beta}{\varrho[\rho]}\right) + [-]B\sin\frac{\beta}{\varrho[\rho]}, - [+]B\left(1 - \cos\frac{\beta}{\varrho[\rho]}\right) + A\sin\frac{\beta}{\varrho[\rho]}\right), \tag{31}$$

$$\beta = \varrho[\rho]\arccos\left(1 - \frac{A^2 + B^2}{2(\rho + \varrho)^2}\right), \tag{32}$$

where $A = x_\mathsf{f} - \rho[+\varrho]\sin\vartheta_\mathsf{f}$, $B = y_\mathsf{f} - \rho[+\varrho] + \rho[-\varrho]\cos\vartheta_\mathsf{f}$, and where $\arccos : \mathbb{R} \mapsto [\pi, 2\pi]$ is the inverse cosine function.

Finally, the final time $T_\mathsf{f}(\mathsf{b}^+\mathsf{b}^-\mathsf{b}^+)$ $[T_\mathsf{f}(\mathsf{b}^-\mathsf{b}^+\mathsf{b}^-)]$ is given by

$$T_\mathsf{f} = \begin{cases} \beta(1 + \delta[\delta^{-1}]) + \rho(\vartheta_\mathsf{f} + 2\pi)\ [-\varrho\vartheta_\mathsf{f}], \\ \quad \text{if } \vartheta_\mathsf{f} - \dfrac{\alpha}{\rho} + \dfrac{\beta}{\varrho}\left[\vartheta_\mathsf{f} + \dfrac{\alpha}{\varrho} - \dfrac{\beta}{\rho}\right] \in]-2\pi, 0], \\ \beta(1 + \delta[\delta^{-1}]) + \rho\vartheta_\mathsf{f}[\varrho(2\pi - \vartheta_\mathsf{f})], \\ \quad \text{if } \vartheta_\mathsf{f} - \dfrac{\alpha}{\rho} + \dfrac{\beta}{\varrho}\left[\vartheta_\mathsf{f} + \dfrac{\alpha}{\varrho} - \dfrac{\beta}{\rho}\right] \in]0, 2\pi], \\ \beta(1 + \delta[\delta^{-1}]) + \rho(\vartheta_\mathsf{f} - 2\pi)\ [\varrho(4\pi - \vartheta_\mathsf{f})], \\ \quad \text{if } \vartheta_\mathsf{f} - \dfrac{\alpha}{\rho} + \dfrac{\beta}{\varrho}\left[\vartheta_\mathsf{f} + \dfrac{\alpha}{\varrho} - \dfrac{\beta}{\rho}\right] \in]2\pi, 4\pi[. \end{cases} \tag{33}$$

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