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Optimal Evading Strategies for Two-Pursuer/One-Evader Problems

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Pursuit–evasion problems involving two pursuers and one evader are presented and analyzed. Two distinct scenarios differing in the type of information that the pursuers have about the evader's strategy are analyzed. The first scenario involves pursuers that have access to the evader's position and velocity at each instant of time, and consequently they follow a constant-bearing strategy. In the second scenario, it is assumed that the pursuers have only information about the evader's instantaneous position, and they follow a pure-pursuit strategy. In both these scenarios, the evader has information about the pursuers' location at every instant of time, and in addition, it also knows their pursuit strategy. The evading strategies maximizing capture time in both scenarios are studied under an optimal control framework. The two scenarios are further analyzed assuming that there is only one active pursuer. Such a case arises when the two pursuers follow a relay pursuit strategy.

I. Introduction

THE seminal work of Isaacs on differential games paved the way for the study of pursuit–evasion (PE) problems [1]. In [1], the focus was on two-player PE games, but some techniques to study multiplayer PE games were also briefly discussed. Following [1], many researchers took an interest in multiplayer PE games, motivated by applications such as collision avoidance [2], cooperative surveillance [3], and defense and security systems [4–6]. An extensive amount of literature is now available, and a recent survey on zero-sum PE games with multiple agents is available in [7].

Evasion from a group of pursuers is a subset of the class of multiplayer PE games. Classical results include those of Pshenichnyi [8], who provided a sufficient condition for successful evasion from a group of homogeneous pursuers, Blagodatskikh [9], and Chernous'ko [10], among many others. Specifically, Chernous'ko showed that an evader can avoid point capture from any number of pursuers having a lower speed. The PE differential game involving many pursuers and one evader has also been investigated subject to fixed terminal time, integral constraints, and different payoff models [11,12].

The study of multiplayer PE games has been revitalized in recent years owing to the growing interest in general multi-agent systems. Obtaining closed-form optimal strategies for the players for this class of games using Hamilton–Jacobi–Isaacs equation formulations is elusive, owing to the curse of dimensionality. Jang and Tomlin proposed some control strategies obtained from direct differentiation of a given value function, but these strategies are suboptimal [13]. An extension to this problem, which assumes that the evader is more agile than the pursuers, was studied by Zak [14]. Oyler et al. [15] studied planar PE games in the presence of obstacles by constructing dominant regions for each player. Some limitations of capturing a faster evader were proposed, and a heuristic group pursuit strategy was presented in [16,17]. In another version of the group pursuit problem, a group of faster, yet less agile, pursuers against a slower, but more agile, evader was solved by Bopardikar et al. [18]. Group pursuit problems involving general dynamics were studied in [19–22]. A probabilistic variant of group pursuit problems was investigated in [23] using a greedy policy. The analysis was later used to study some heuristic strategies in the case of games with incomplete information [24]. Last, relay group pursuit using dynamic Voronoi diagrams was studied in [25,26].

The two-pursuer/one-evader problem has been previously discussed by Kelley in [27], where some aspects of two-on-one team tactics were identified. Bhattacharya and Hutchinson analyzed the problem subject to visibility constraints and provided approximate schemes to construct the set of initial pursuer configurations from which capture is guaranteed [28]. The differential game involving two pursuers and one evader with linear dynamics and quadratic cost was investigated for Nash equilibrium solutions in [29]. Partitions of the state space were identified and categorized, which are similar to the degenerate and the nondegenerate regions discussed later on in this paper. A linear differential game formulation restricting the motion of the players to a straight line with the two pursuers coordinating to reduce the miss distance was extensively studied by several researchers [30–32]. An algorithm to numerically construct level sets of the value function with fixed final time was also discussed in [33]. The two-pursuer/one-evader problem was previously analyzed by considering a nonconvex payoff of the distance between the evader and the two pursuers [34].

A case of two identical inertial pursuers (second-order dynamics) pursuing a noninertial evader (first-order dynamics) was studied by Levchenkov and Pashkov [35]. Hagedorn and Breakwell considered the problem of a faster evader that must pass between two pursuers [36]. The stochastic version of the two-pursuer/one-evader differential game was discussed by Yavin [37]. Finally, a version of the relay pursuit problem discussed in this paper was previously presented by Sun and Tsiotras, along with a suboptimal strategy [38].

Most of the existing work on two-pursuer/one-evader problems either deals with linear dynamics and then tries to optimize miss distance (with quadratic cost and fixed final time, [30–32]) or considers formulations involving particular tasks and constraints [28,35,36]. Though time-optimal solutions for these problems were briefly investigated in the book by Isaacs [1], to the best of the authors' knowledge, there is no rigorous work existing beyond that discussion. In this paper, time-optimal evading strategies are investigated for a class of two-pursuer/one-evader problems, which can be scaled to pursuit–evasion problems involving multiple pursuers and evaders, thus providing a framework for multiplayer time-optimal pursuit–evasion games.

A. Motivation

Consider a group of n agents (pursuers) guarding a given area of interest. The objective of the agents is to pursue and intercept m (where typically $m \leq n$) adversary agents (evaders) that may be

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detected in this area. Relevant questions dealing with this problem are as follows.

1) Which pursuer(s) should go after which evader(s)?

2) How many pursuers should chase each intruder (evader) to capture it in the shortest time possible?

3) What is the shortest time to capture, given the fact that the evaders are intelligent and will try to postpone capture indefinitely?

Answering the previous questions in their most general form seems to be an intractable undertaking. Solving exactly a multiplayer dynamic game such as the one considered previously will necessitate the solution of a high-dimensional partial differential equation, whose dimensionality increases with the number of players. To remedy this problem, in this work, we use the following "divide and conquer" approach that simplifies the problem and leads, in addition, to decentralized (although likely suboptimal) solutions. Specifically, we assume that a subset of pursuers is assigned to each evader and none of these pursuers is reassigned to another evader until capture occurs. This means that, instead of solving the full multi-agent PE game involving n pursuers and m evaders, we solve a series of m many/against-one pursuit–evasion games. These problems are much easier to solve than the original problem.

The simplest case of the problem of pursuit–evasion with a single evader is the case of two pursuers against one evader, which is the objective of this work. The generalization to more than two pursuers against one evader can be obtained using ideas similar to the ones introduced in this paper and will be the detailed subject of future investigation. The basic idea of the proposed methodology is to determine which of the pursuers is the most critical one and which pursuer(s) do not affect the outcome of the game and thus may be eliminated from the analysis of the problem. This leads to the notions of degenerate and nondegenerate regions for each PE game. The determination of these regions is a key aspect toward the solution of multiplayer games because they delineate the regions in which one pursuer acting alone can capture the evader without the help of its teammates, from the regions in which some form of cooperation/coordination between the pursuers is necessary for optimal capture.

In this work, we assume that both pursuers are faster than the evader, and they follow simple navigation laws (pure pursuit or constant bearing). The rationale behind this assumption is the following. First, under the assumption that the pursuers are faster than the evader, pure-pursuit or a constant-bearing strategy guarantees capture. Second, these two strategies highlight the information that a pursuer has to capture the evader. A constant-bearing (CB) strategy is known to be efficient when a pursuer knows the instantaneous position and velocity of an evader [39]. On the other hand, an individual pursuer that is able to access only the evader's instantaneous position can, at best, employ a pure-pursuit (PP) strategy [39]. Furthermore, both of these strategies are easy to execute, and they have been implemented in various defense and security systems. For a more detailed discussion on navigation laws based on different information structures, the reader may peruse [40].

Decomposition strategies of multiplayer games as the ones providing the motivation of this work have been previously introduced in [25,26]. In those references, "regions of influence" characterized by Voronoi-like partitions were used to decompose the game into a sequence of one-versus-one pursuit–evasion games. In the resulting so-called "relay–pursuit" strategy, only a single pursuer goes after the target, whereas all other pursuers remain stationary. From an application point of view, these strategies result in a tradeoff between time and resources. For instance, employing more than one pursuer may reduce capture time, but the deployment of more pursuers require additional resources (e.g., fuel, communication bandwidth, etc). In the last part of this paper, we extend our work to such cases, namely, where one of the two pursuers is stationary, motivated by such relay–pursuit scenarios [26,38].

B. Problem Statement and Contributions

Motivated by the previous discussion, consider a PE problem with two pursuers and one evader in the plane. The objective of the pursuers is that at least one of them enters the evader's capture zone, assumed here to be a disk of radius $\epsilon > 0$ centered at the current position of the evader, whereas the objective of the evader is to avoid or delay capture as long as possible. The subscripts 1 and 2 will be used for the two pursuers (P_1 and P_2), whereas the subscript E will be used for the evader. The equations of motion for all the players involved in the game are given next:

$$\dot{x}_1 = u_1 \cos\theta_1, \qquad \dot{y}_1 = u_1 \sin\theta_1 \tag{1}$$

$$\dot{x}_2 = u_2 \cos\theta_2, \qquad \dot{y}_2 = u_2 \sin\theta_2 \tag{2}$$

$$\dot{x}_E = v \cos\theta_E, \qquad \dot{y}_E = v \sin\theta_E \tag{3}$$

where $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$, and $p_E = (x_E, y_E)$ denote the positions of pursuer P_1, pursuer P_2, and the evader E, respectively. Similarly, θ_1, θ_2, $\theta_E \in (-\pi, \pi]$ denote the control inputs of the players, and u_1, u_2 and v are the speeds (constant) of P_1, P_2, and E, respectively, with $\min\{u_1, u_2\} > v$. The game evolves in the six-dimensional state space, $[x_1, y_1, x_2, y_2, x_E, y_E]^T \in \mathbb{R}^6$.

Problem: Find the optimal control input for the evader, $\theta_E \in (-\pi, \pi]$, that maximizes the time of capture t_c in the following cases.

1) *CB*: The two pursuers follow a constant-bearing strategy.

2) *PP*: The two pursuers follow a pure-pursuit strategy.

3) *R-CB*: Two identical ($u_1 = u_2$) pursuers follow a relay pursuit strategy with the active pursuer employing constant bearing.

4) *R-PP*: Two identical ($u_1 = u_2$) pursuers follow a relay pursuit strategy with the active pursuer employing pure pursuit.

In all these cases, note that the control inputs of the pursuers, θ_1 and θ_2, depend solely on the instantaneous states of the players.

The main contributions of this paper are listed next.

1) The regions of nondegeneracy are identified for CB (Sec. II.A), PP (Sec. II.B), and R-CB (Sec. IV.A).

2) The optimal evading strategies for CB and PP are identified, and it is established for the first time that, in both cases, when the problem is nondegenerate, the solution involves simultaneous capture (theorems 1, 2).

3) A competitive suboptimal strategy is suggested for PP, and a comparative study is provided for the case of identical pursuers to demonstrate this claim.

4) An optimal evading strategy is derived for R-CB, along with the corresponding switching condition (proposition 3).

The rest of the paper is organized as follows. Section II contains a discussion on the regions of nondegeneracy observed for both CB and PP cases. Section III analyzes the optimal strategies of the evader, obtained by formulating the CB and PP problems in a reduced state space. A suboptimal evading strategy for PP along with numerical simulations and a comparative study are also presented in this section. Section IV provides the results for relay pursuit strategies (R-CB and R-PP), and Sec. V concludes the paper.

II. Regions of Nondegeneracy

Assuming that each of the pursuers follows either a constant-bearing strategy or a pure-pursuit strategy, the two-pursuer/one-evader problem may result in a degenerate case. A degenerate case is one in which only one of the pursuers is sufficient to capture the evader in minimum time. In degenerate problems, the presence of one of the two pursuers is inconsequential, and the problem can be treated as a one-against-one PE problem. For instance, if one of the pursuers (say P_2) is very far away from the evader (E), then P_2 does not play a role in the solution of the problem, and the evader's optimal strategy is a pure evasion from P_1. Similarly, if P_2 is very close to E then P_2 dominates, and the optimal evading strategy in this case will be a pure evasion from P_2. However, there exists a region of initial positions for P_2 for which a form of coordination with P_1 ensues, and an optimal evading strategy (other than pure evasion from P_1 or P_2) needs to take into consideration the presence of both pursuers.

Because we are interested in studying the effects of adding a second pursuer (P_2) to the problem, it follows from the preceding discussion that, given the initial positions of the first pursuer (P_1) and the evader (E) along with the speed capabilities of all the three players, it is important to find the set of initial positions of P_2 for which 1) the optimal evading strategy is a pure evasion from P_1, and P_2 plays no role in the solution of the problem ($\mathcal{D}_2$: degenerate region with respect to P_2); 2) the optimal evading strategy if the evader follows pure evasion from P_2 and P_1 plays no role in the solution ($\mathcal{D}_1$: degenerate region with respect to P_1); and 3) the optimal evading strategy is not a pure evasion from any of the two pursuers, and both P_1 and P_2 play a role in the problem ($\mathcal{N}$: nondegenerate region). We can compute these regions for the cases CB and PP as follows.

A. Case of Constant-Bearing Strategy

Without of loss of generality, assume that the initial positions of P_1 and E are such that $p_1(0) = (0, 0)$ and $p_E(0) = (d, 0)$, respectively ($d \neq 0$). The capture time, assuming that the evader follows a pure evasion strategy from P_1, is given by

$$t_f = \frac{d}{u_1 - v} \tag{4}$$

The point of capture is $C = (u_1 t_f, 0)$. Define now the circle $\mathcal{C}_1 = \{x \in \mathbb{R}^2\colon \|x - C\| = u_2 t_f\}$. The circle $\mathcal{C}_1$ is an isochrone that contains the set of initial positions for P_2 that guarantees capture exactly at time t_f (at location C) under a constant-bearing strategy for the given initial position of the evader and its heading, assuming that the evader is nonmaneuvering. The center of this circle is the capture point C, and its radius is $u_2 t_f$. If the initial position of P_2 (following a constant-bearing strategy) lies inside $\mathcal{C}_1$, then P_2 can capture the evader in a time less than t_f for the given initial position of the evader and its heading. If the initial position of P_2 lies outside $\mathcal{C}_1$, then it cannot capture the evader in a time less than or equal to t_f, which means that the presence of P_2 is inconsequential to the solution of the problem, and the optimal evading strategy is pure evasion from P_1. As a result, in the case of CB, the circle $\mathcal{C}_1$ and its exterior constitute the degenerate region with respect to P_2, and $\mathcal{C}_1$ is the boundary between $\mathcal{D}_2$ and $\mathcal{N}$.

Next, the degenerate region with respect to P_1 can be obtained by looking at the locus of the initial points of P_2 such that a pure evasion from P_2 would result in simultaneous capture of the evader by both P_1 and P_2. In this regard, consider the Apollonius circle $\mathcal{A}$ corresponding to P_1 and E, whose center is at $(x_a, y_a) = (u_1^2 d/(u_1^2 - v^2), 0)$, and its radius is $r_a = u_1 v d/(u_1^2 - v^2)$ [1]. The circle constitutes capture points for P_1 for a given constant heading of the evader. Clearly, finding the set of initial points of P_2 such that the evader hits a point on the Apollonius circle under pure evasion from P_2 provides the locus of interest; see Fig. 1. This can be achieved in the following manner. Consider a point T on $\mathcal{A}$ in its parametric form $T = (x_a + r_a \cos\phi, r_a \sin\phi)$ (note that $y_a = 0$), and let (x, y) be the initial position of P_2 such that it hits the evader at T. From the geometry of the problem, as defined in Fig. 1, and because the triangles $\Delta ETT'$ and $\Delta EP_2P_2'$ are similar, it follows that

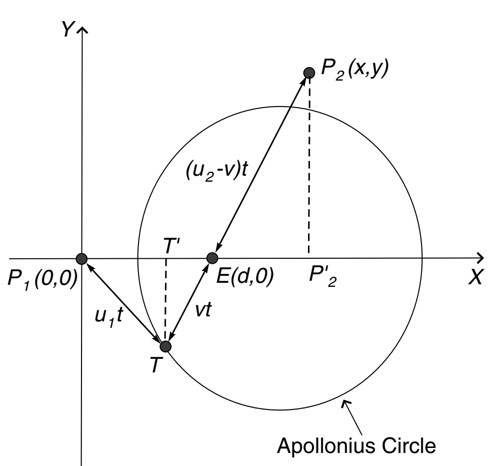


Fig. 1 Limiting case scenario of the degeneracy with respect to P_1 for the case of CB.

$$\frac{x - d}{d - x_a - r_a \cos\phi} = \frac{y}{r_a \sin\phi} = \frac{u_2 - v}{v} \tag{5}$$

The coordinates x, y can then be given as

$$x = \left(d + \frac{(u_2 - v)(d - x_a)}{v}\right) - r_a \left(\frac{u_2 - v}{v}\right)\cos\phi, \qquad y = r_a \left(\frac{u_2 - v}{v}\right)\sin\phi \tag{6}$$

Under this parametric representation, it can be realized that the set of points (x, y) form a circle (call it $\mathcal{C}_2$) with its center at $(d + (u_2 - v)(d - x_a)/v, 0)$ and radius $r_a(u_2 - v)/v$. It is understood that if P_2 lies inside $\mathcal{C}_2$, then the evader will get captured by P_2 under pure evasion, before it hits the Apollonius circle $\mathcal{A}$ (i.e., P_1 does not play a role in the solution of the problem). Hence, in the CB case, the circle $\mathcal{C}_2$ and its interior constitute the degenerate region with respect to P_1, and $\mathcal{C}_2$ acts as the boundary between $\mathcal{D}_1$ and $\mathcal{N}$. Finally, $\mathbb{R}^2 \setminus (\mathcal{D}_1 \cup \mathcal{D}_2)$ constitutes the region of nondegeneracy.

The geometry of these regions can be visualized in Fig. 2, which shows the regions of degeneracy and nondegeneracy for three different cases, where $u_2 = 1.5, 1, 0.5$. As can be seen in this figure, the evader's initial position is at (1,0), with P_1 located at (0,0). The speeds of E and P_1 are $v = 0.5$ and $u_1 = 1$, respectively. In Fig. 2a, P_2 is faster compared to P_1, and P_1 is inside circle $\mathcal{C}_2$. The converse is observed in Fig. 2c, when P_2 is slower than P_1. It can be observed that the region of nondegeneracy increases with the speed of P_2, which suggests that, given the speeds of P_1 and E, adding a second pursuer that has higher speed would enable cooperation among the pursuers in a larger region, and vice versa.

B. Case of Pure-Pursuit Strategy

To compute the nondegenerate region in this case (shown in Fig. 3), first define the ellipse, $\mathcal{E} = \{x \in \mathbb{R}^2\colon \|x - p_E(0)\| + \|x - p_E'\| = 2u_2 t_f\}$, where

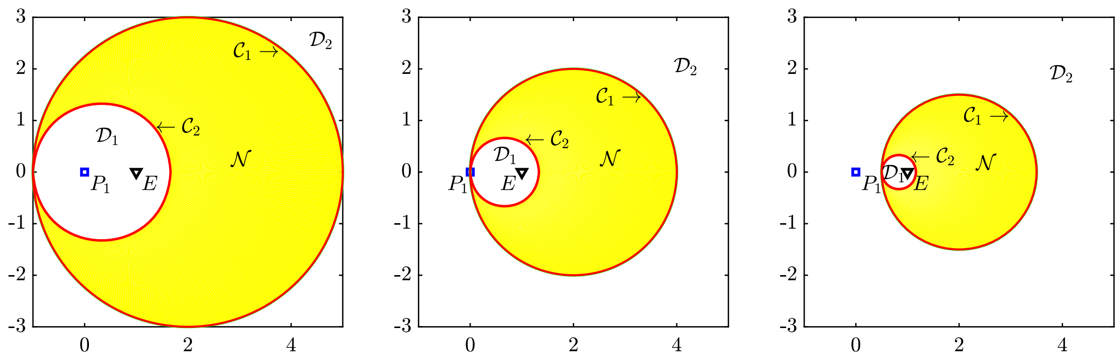
$$p_E' = \left(\left[\frac{u_1 + v}{u_1 - v}\right] d, 0\right)$$

The ellipse $\mathcal{E}$ is an isochrone that contains the set of initial positions for a pursuer that guarantees capture at time t_f (at location C) under a pure-pursuit strategy for a given initial position of the evader and its heading, assuming that the evader is nonmaneuvering. In the literature, $\mathcal{C}_1$ and $\mathcal{E}$ are called t_f isochrones [39]. It can be seen that $\mathcal{E}$ is an ellipse centered at C having the initial position of the evader at one of its foci. For any initial position of a pursuer (following a pure-pursuit strategy) inside $\mathcal{E}$, the capture time is less than t_f. If the initial position of P_2 (following a constant-bearing strategy) lies inside $\mathcal{C}_1$, then it can capture the evader in a time less than t_f. Therefore, if the initial position of P_2 lies outside $\mathcal{E}$, then P_2 cannot capture the evader in a time less than or equal to t_f (i.e., P_2's presence has no strategic significance, and the optimal evading strategy is a pure evasion from P_1). Consequently, in the case of PP, the ellipse $\mathcal{E}$ and its exterior constitutes the degenerate region with respect to P_2, and $\mathcal{E}$ is the boundary between $\mathcal{D}_2$ and $\mathcal{N}$.

The degenerate region with respect to P_1 in this case can be obtained from the relation

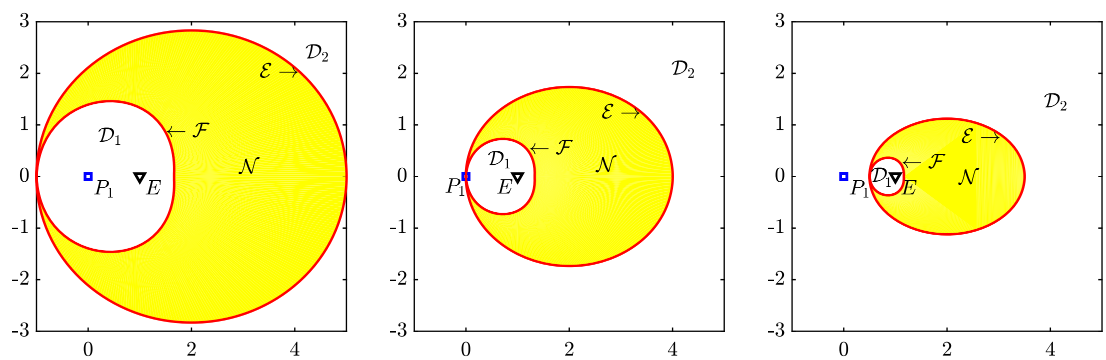
$$t_p = \frac{r_o(u + v\cos\theta)}{u_1^2 - v^2}, \qquad u_1 \neq v \tag{7}$$

which provides the capture time for a pursuer that follows a pure-pursuit strategy, assuming that the evader is nonmaneuvering (constant heading) [39]. Here, r_o is the initial distance between the



a) $u_2 = 1.5$ b) $u_2 = 1$ c) $u_2 = 0.5$

Fig. 2 Regions of degeneracy and nondegeneracy for the case of CB: $u_1 = 1$, $v = 0.5$, and $\epsilon = 0$ (point capture).



a) $u_2 = 1.5$ b) $u_2 = 1$ c) $u_2 = 0.5$

Fig. 3 Regions of degeneracy and nondegeneracy for the case of PP: $u_1 = 1$, $v = 0.5$, and $\epsilon = 0$ (point capture).

pursuer and the evader ($= d$ for P_1 and E), and θ is the evader's heading measured with respect to the line-of-sight from the pursuer to the evader. Using this relation, and following an approach similar to the one in Sec. II.A, the locus of the initial points of P_2, such that a pure evasion from P_2 would result in simultaneous capture of the evader by both P_1 and P_2, can be obtained. Consider now the case of an evader following a pure evasion strategy from P_2 with the heading θ, as shown in Fig. 4. Let the initial position of P_2 be (x, y). Assuming that the evader gets captured by both P_1 and P_2 at T (see Fig. 4), it follows from Eq. (7) that

$$x = d - (u_2 - v)d\cos\theta\left(\frac{u_1 + v\cos\theta}{u_1^2 - v^2}\right),$$
$$y = -(u_2 - v)d\sin\theta\left(\frac{u_1 + v\cos\theta}{u_1^2 - v^2}\right) \tag{8}$$

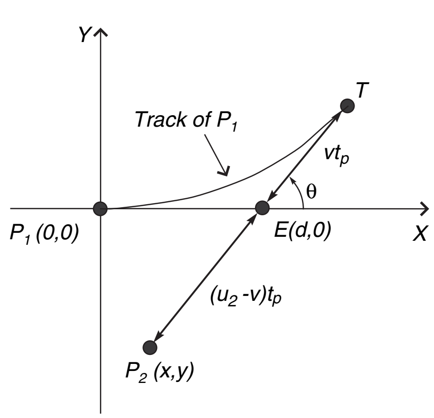


Fig. 4 Limiting case scenario of the degeneracy with respect to P_1 for the case of PP.

Under this parametric representation, the locus of interest can be obtained, which is a closed curve ($\mathcal{F}$) around the initial position of the evader. If P_2 lies inside $\mathcal{F}$, then the evader will get captured by P_2 under pure evasion, before E is captured by P_1. And hence, in the case of PP, the closed curve $\mathcal{F}$ and its interior constitutes the degenerate region with respect to P_1, and $\mathcal{F}$ acts as the boundary between $\mathcal{D}_1$ and $\mathcal{N}$. Finally, in the case of PP, the set $\mathbb{R}^2 \setminus (\mathcal{E} \cup \mathcal{F})$ constitutes the region of nondegeneracy.

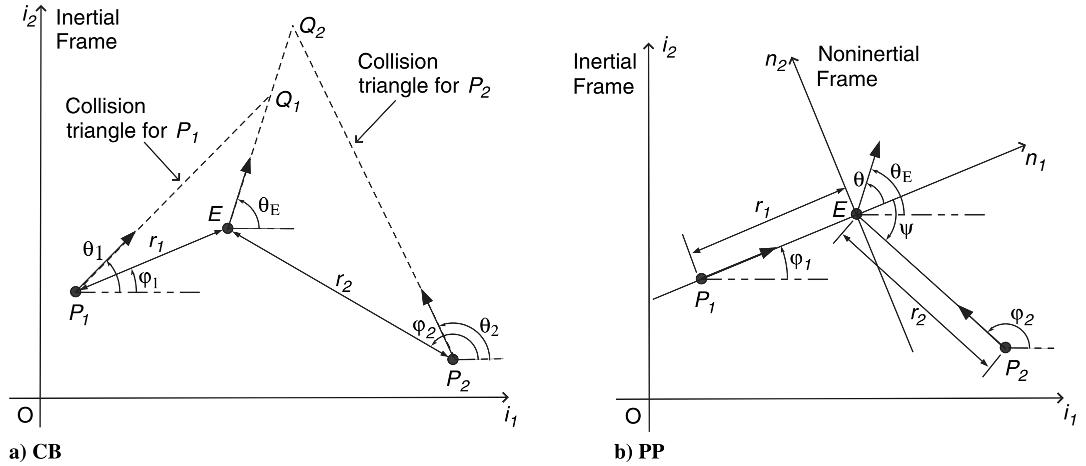
The regions of degeneracy and nondegeneracy observed in these cases are depicted using the previous example in Sec. II.A, see Fig. 3. Note that the ellipse $\mathcal{E}$ is contained in the circle $\mathcal{C}_1$, and on the other hand, the closed curve $\mathcal{F}$ contains $\mathcal{C}_2$. This is a consequence of the fact that the pursuers use pure pursuit because they lack information about the evader's speed. As a result, the regions of nondegeneracy are smaller compared to their counterparts in Sec. II.A. This observation further supports the fact that the information structure plays a crucial role in problems involving cooperation among agents.

III. Optimal Evading Strategies Against Two Pursuers

A. Case of Constant-Bearing Strategy

As per the formulation in Sec. I.B, it can be seen that the game evolves in the six-dimensional state space. However, the problem formulation can be reduced to the two-dimensional state space in the following manner. Consider the relative distances between the evader and each of the pursuers ($r_1 - p_1$, $r_2 - p_2$). The corresponding line-of-sight (LOS) angles (φ_1, φ_2) are shown in Fig. 5a. Using Eqs. (1–3), the equations can be expressed as

$$\dot{r}_1 = v\cos(\theta_E - \varphi_1) - u_1\cos(\theta_1 - \varphi_1),$$
$$\dot{\varphi}_1 = \frac{1}{r_1}[v\sin(\theta_E - \varphi_1) - u_1\sin(\theta_1 - \varphi_1)] \tag{9}$$



a) CB b) PP

Fig. 5 Schematics of the proposed pursuit–evasion problems.

$$\dot{r}_2 = v\cos(\theta_E - \varphi_2) - u_2\cos(\theta_2 - \varphi_2),$$
$$\dot{\varphi}_2 = \frac{1}{r_2}[v\sin(\theta_E - \varphi_2) - u_2\sin(\theta_2 - \varphi_2)] \tag{10}$$

Furthermore, it is assumed that the pursuers follow a constant-bearing strategy, and hence the LOS for a given pursuer does not rotate (i.e., $\dot{\varphi}_1 = 0$ and $\dot{\varphi}_2 = 0$). That is, $\varphi_1(t) = \varphi_{10}$, $\varphi_2(t) = \varphi_{20}$, for all $t \geq 0$, where φ_{10} and φ_{20} are the LOS angles at the initial time $t = 0$. Therefore, r_1 and r_2 are the only states that have to be taken into consideration to solve for the optimal evading strategy.

We are dealing with a time-maximization problem subject to the dynamics

$$\dot{r}_1 = v\cos(\theta_E - \varphi_{10}) - u_1\cos(\theta_1 - \varphi_{10}) \tag{11}$$

$$\dot{r}_2 = v\cos(\theta_E - \varphi_{20}) - u_2\cos(\theta_2 - \varphi_{20}) \tag{12}$$

Note that θ_1, θ_2 are functions of θ_E. They can be determined at each instant of time, given θ_E, using Eqs. (9) and (10) and the fact that $\dot{\varphi}_1 = \dot{\varphi}_2 = 0$. Therefore,

$$v\sin(\theta_E - \varphi_{10}) = u_1\sin(\theta_1 - \varphi_{10}),$$
$$v\sin(\theta_E - \varphi_{20}) = u_2\sin(\theta_2 - \varphi_{20}) \tag{13}$$

Each of the preceding relations has two possible solutions for θ_1 or θ_2, given θ_E, and each pursuer chooses the solution for which $\dot{r}_1 < 0$ or $\dot{r}_2 < 0$, respectively. The initial conditions are $r_1(0) = r_{10} = \|p_E(0) - p_1(0)\|$ and $r_2(0) = r_{20} = \|p_E(0) - p_2(0)\|$. The terminal condition for capture is

$$\Psi(r_1(t_c), r_2(t_c)) = \min\{r_1(t_c), r_2(t_c)\} - \epsilon = 0 \tag{14}$$

The Hamiltonian for this problem can be expressed as

$$H(r_1, r_2, \lambda_1, \lambda_2, \theta_E) = -1 + \lambda_1[v\cos(\theta_E - \varphi_{10}) - u_1\cos(\theta_1 - \varphi_{10})]$$
$$+ \lambda_2[v\cos(\theta_E - \varphi_{20}) - u_2\cos(\theta_2 - \varphi_{20})] \tag{15}$$

where λ_1 and λ_2 are the costates. The corresponding adjoint equations are given by

$$\dot{\lambda}_1 = 0, \qquad \dot{\lambda}_2 = 0 \tag{16}$$

and therefore $\lambda_1(t) = c_1$, $\lambda_2(t) = c_2$, for $t \in [0, t_c]$, where c_1 and c_2 are constants. The transversality conditions are given by

$$\lambda_1(t_c) = \nu\frac{\partial\Psi}{\partial r_1}\bigg|_{t=t_c}, \quad \lambda_2(t_c) = \nu\frac{\partial\Psi}{\partial r_2}\bigg|_{t=t_c}, \quad H(t_c) = 0 \tag{17}$$

where $\nu \in \mathbb{R}$. Because the Hamiltonian has no explicit dependency on time, it follows that $H(t) = 0$ for all $t \in [0, t_c]$. Note that the terminal condition is not fully differentiable, and it can be written as

$$\frac{\partial\Psi}{\partial r_1}\bigg|_{t=t_c} = \frac{r_2 - \min\{r_1, r_2\}}{r_2 - r_1}, \quad \frac{\partial\Psi}{\partial r_2}\bigg|_{t=t_c} = \frac{r_1 - \min\{r_1, r_2\}}{r_1 - r_2} \tag{18}$$

At $r_1(t_c) = r_2(t_c) = \epsilon$, the partial derivatives are undefined. Using Pontryagin's minimum principle, the following expression is obtained:

$$\lambda_1\left[-v\sin(\theta_E - \varphi_{10}) + u_1\sin(\theta_1 - \varphi_{10})\frac{\partial\theta_1}{\partial\theta_E}\right]$$
$$+ \lambda_2\left[-v\sin(\theta_E - \varphi_{20}) + u_2\sin(\theta_2 - \varphi_{20})\frac{\partial\theta_2}{\partial\theta_E}\right] = 0 \tag{19}$$

where, from Eq. (13),

$$\frac{\partial\theta_1}{\partial\theta_E} = \frac{v\cos(\theta_E - \varphi_{10})}{u_1\cos(\theta_1 - \varphi_{10})}, \qquad \cos(\theta_1 - \varphi_{10}) \neq 0 \tag{20}$$

$$\frac{\partial\theta_2}{\partial\theta_E} = \frac{v\cos(\theta_E - \varphi_{20})}{u_2\cos(\theta_2 - \varphi_{20})}, \qquad \cos(\theta_2 - \varphi_{20}) \neq 0 \tag{21}$$

Because λ_1 and λ_2 are constants, θ_1 and θ_2 and their partials from Eqs. (20) and (21) are dependent only on θ_E, we can conclude from Eq. (19) that the optimal heading of the evader θ_E is constant in time, and hence the headings of the pursuers are constant as well.

Theorem 1: For the CB time-optimal control problem, given the initial positions of P_1 and E, if P_2 initially lies in the nondegenerate region ($\mathcal{N}$), then the optimal control strategy of the evader involves simultaneous capture by P_1 and P_2.

Proof: First, consider the case when $r_1(t_c) = \epsilon < r_2(t_c)$ (i.e., only P_1 captures the evader at the final time). In this case, it follows from Eqs. (17) and (18) that $\lambda_2(t_c) = 0$, which from Eq. (16) implies that $c_2 = 0$. From Eqs. (15) and (19), we then have that

$$-1 + c_1[v\cos(\theta_E - \varphi_{10}) - u_1\cos(\theta_1 - \varphi_{10})] = 0 \tag{22}$$

$$c_1\left[-v\sin(\theta_E - \varphi_{10}) + u_1\sin(\theta_1 - \varphi_{10})\frac{\partial\theta_1}{\partial\theta_E}\right] = 0 \tag{23}$$

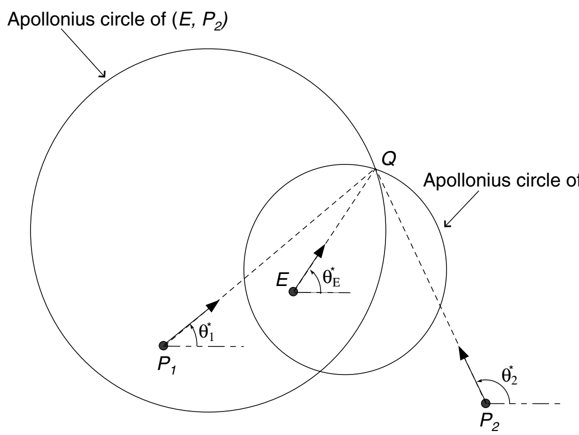


Fig. 6 Schematic of finding the optimal heading of the evader using Apollonius circles.

and $c_1 \neq 0$ because it leads to a contradiction in Eq. (22). Therefore,

$$-v\sin(\theta_E - \varphi_{10}) + u_1\sin(\theta_1 - \varphi_{10})\frac{\partial\theta_1}{\partial\theta_E} = 0 \tag{24}$$

From Eq. (20), $\sin(\theta_1 - \varphi_{10})\cos(\theta_E - \varphi_{10}) - \sin(\theta_E - \varphi_{10}) \times \cos(\theta_1 - \varphi_{10}) = 0$, which implies that $\sin(\theta_E - \theta_1) = 0$. Further analysis leads to $\theta_E^* = \varphi_{10}$ (i.e., the optimal strategy is a pure evasion from P_1). This is the solution for a degenerate case of the problem. It has been proven that, in the nondegenerate case with this strategy, P_2 will capture the evader before P_1, leading to a contradiction. Similarly, the strategy when $r_2(t_c) = \epsilon < r_1(t_c)$ turns out to be a pure evasion from P_2. But because P_1 is closer to the evader, it lies inside the circle of equal time to capture corresponding to P_2, and therefore P_1 reaches the evader before P_2, again leading to a contradiction. Hence, the optimal evading strategy in the nondegenerate case should involve $r_1(t_c) = r_2(t_c) = \epsilon$, namely, simultaneous capture. □

Because the optimal heading θ_E^* is constant and involves simultaneous capture in the nondegenerate case, it is easy to obtain the heading using the well-known Apollonius circles [1]; see Fig. 6. The Apollonius circles of the pairs (E, P_1) and (E, P_2) at the initial time can be constructed from the players' initial positions. If the problem is nondegenerate, then there always exist two intersection points, Q and Q', as shown in Fig. 6. During optimal play, the evader should head toward one of the intersection points, namely, the one that is farther away. If both the points are equidistant, then the evader can choose either point. This completes the analysis on the optimal evading strategy for CB.

B. Case of Pure-Pursuit Strategy

In this case, the problem can be examined in the three-dimensional state space that makes the analysis simpler. A schematic of the geometry of the proposed pursuit–evasion problem is shown in Fig. 5b. First, we translate the problem into a rotating/noninertial frame with the origin fixed on the evader (E) and with the x axis along the line joining P_1 and E. The velocity vector of P_1 is along the x axis because it follows a pure-pursuit strategy. In this frame, P_1 is restricted to move only along the x axis. The positions of the players expressed in polar coordinates are given by $p_1 = (r_1, \pi)$, $p_2 = (r_2, \psi)$, and $p_E = (0, 0)$, $-\pi < \psi \leq \pi$. Because the pursuers follow a pure-pursuit strategy, their headings are along their corresponding LOS directions (i.e., $\theta_1 = \varphi_1$, $\theta_2 = \varphi_2$); see Fig. 5b. The angle between the velocity vectors of P_1 and E is $\theta = \theta_E - \varphi_1$. The rotation rate of the noninertial frame is given by

$$\dot{\varphi}_1 = \frac{v\sin\theta}{\|p_E - p_1\|} = \frac{v\sin\theta}{r_1} \tag{25}$$

In the reduced state space, the number of states is only three, and the corresponding equations of motion are given by

$$\dot{r}_1 = -u_1 + v\cos\theta \tag{26}$$

$$\dot{r}_2 = -u_2 - v\cos(\psi - \theta) \tag{27}$$

$$\dot{\psi} = \frac{v}{r_2}\sin(\psi - \theta) - \frac{v}{r_1}\sin\theta \tag{28}$$

The initial conditions for the states are $r_1(0) = \|p_E(0) - p_1(0)\|$, $r_2(0) = \|p_E(0) - p_2(0)\|$, $\psi(0) = \pi - \varphi_{10} + \varphi_{20}$, where φ_{10} and φ_{20} are now the initial headings of P_1 and P_2, respectively, which can be obtained from the initial positions of the players. The terminal condition remains the same as in Eq. (14).

The problem statement is then to find the optimal control $\theta^*(t)$ that maximizes the capture time t_c given the equations of motion Eqs. (26–28) and the given initial conditions and terminal conditions. It is assumed that the initial conditions are such that the problem is nondegenerate for the given speeds of the players. Otherwise, the pursuit strategy for the evader is pure evasion from either pursuer.

The Hamiltonian for this problem can be written as

$$H(r_1, r_2, \psi, \lambda_1, \lambda_2, \lambda_3, \theta) = -1 + \lambda_1(-u_1 + v\cos\theta) + \lambda_2[-u_2 - v\cos(\psi - \theta)] + \lambda_3\left[\frac{v}{r_2}\sin(\psi - \theta) - \frac{v}{r_1}\sin\theta\right] \tag{29}$$

where λ_1, λ_2, and λ_3 are the costates and satisfy the adjoint equations (dropped for brevity). Because $\psi(t_c)$ is not specified and is free, the transversality conditions are given by

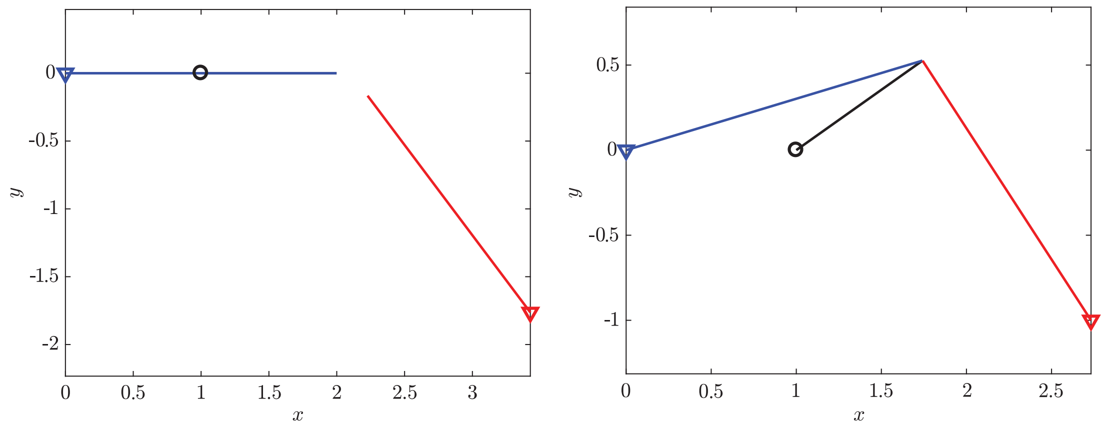
$$\lambda_1(t_c) = \nu\frac{\partial\Psi}{\partial r_1}\bigg|_{t=t_c}, \quad \lambda_2(t_c) = \nu\frac{\partial\Psi}{\partial r_2}\bigg|_{t=t_c}, \quad \lambda_3(t_c) = 0, \quad H(t_c) = 0 \tag{30}$$

where $\nu \in \mathbb{R}$. Because the Hamiltonian has no explicit dependency on time, it follows that $H(t) = 0$ for all $t \in [0, t_c]$. Note that because the terminal condition is the same as in Eq. (14), its derivatives are implicit from Eq. (18). From Pontryagin's minimum principle, it follows that

$$-\lambda_1 v\sin\theta - \lambda_2 v\sin(\psi - \theta) - \lambda_3\left[\frac{v}{r_2}\cos(\psi - \theta) + \frac{v}{r_1}\cos\theta\right] = 0 \tag{31}$$

Theorem 2: For the PP time-optimal control problem, given the initial positions of P_1 and E, if P_2 initially lies in the nondegenerate region $\mathcal{N}$, then the optimal control strategy of the evader involves simultaneous capture.

Proof: Consider the case when $r_1(t_c) = \epsilon < r_2(t_c)$. This implies $\lambda_1(t_c) = \nu$, and $\lambda_2(t_c) = 0$. Note that the adjoint equations are linear in the costates λ_2, λ_3, and because $\lambda_2(t_c) = \lambda_2(t_c) = 0$, the costates are constant in time, i.e., $\lambda_1(t) = \nu$, $\lambda_2(t) = 0$, $\lambda_3(t) = 0$. From Eqs. (29) and (31), it follows that $-1 + \nu(-u_1 + v\cos\theta) = 0$ and $\nu v\sin\theta = 0$. Because $\nu = 0$, these two equations lead to a contradiction. It follows that $\sin\theta = 0$, and thus $\theta^*(t) = 0$, which means that the optimal strategy is pure evasion from P_1. However, this is true only when the problem is degenerate. In a nondegenerate case, this would lead to an early capture by P_2. In this case, when $r_2(t_c) = \epsilon < r_1(t_c)$, we have $\lambda_1(t_c) = 0$ and $\lambda_2(t_c) = \nu$. Furthermore, from Eqs. (29) and (31), at $t = t_c$, it follows that $-1 + \nu(-u_1 - v\cos(\psi(t_c) - \theta(t_c))) = 0$, and $\nu v\sin(\psi(t_c) - \theta(t_c)) = 0$. Because $\nu \neq 0$, it follows that $\sin(\psi(t_c) - \theta(t_c)) = 0$. With this terminal condition, it can be seen that the costates are constant and similarly is the optimal heading, which in this case is given by $\theta^*(t) = \pi + \psi$. This means that the optimal evading strategy is a pure evasion from P_2. However, this strategy is infeasible in the



a) Degenerate case **b) Nondegenerate case**

Fig. 7 Trajectories of the players for optimal control inputs in I1; black: evader, blue: P_1, red: P_2 (Please refer the online version for colors).

nondegenerate case. Hence, the optimal evading strategy in a nondegenerate case would result in $r_1(t_c) = r_2(t_c) = \epsilon$ (i.e., in simultaneous capture). □

Proposition 1: Consider the time-optimal control problem expressed using Eq. (26). Given the initial positions of P_1 and E, the optimal control strategy of the evader can be summarized as follows:

$$\tan\theta^* = \begin{cases} 0, & (x_2(0), y_2(0)) \in \mathcal{D}_2, \\ \Theta(r_1, r_2, \psi, \lambda_1, \lambda_2, \lambda_3), & (x_2(0), y_2(0)) \in \mathcal{N}, \\ \pi + \psi, & (x_2(0), y_2(0)) \in \mathcal{D}_1 \end{cases} \tag{32}$$

where $(x_2(0), y_2(0))$ is the initial position of P_2 in the inertial frame, and

$$\Theta(r_1, r_2, \psi, \lambda_1, \lambda_2, \lambda_3) = -\frac{\lambda_2 \sin\psi + (\lambda_3/r_1) + (\lambda_3 \cos\psi/r_2)}{\lambda_1 - \lambda_2 \cos\psi + (\lambda_3 \sin\psi/r_2)} \tag{33}$$

Proof: The proof is a direct consequence of defining the degenerate and nondegenerate regions for PP (see Sec. II.B).

With this analysis, the optimal control problem for the case PP can be solved numerically. An analytical solution to the equations of optimality subject to arbitrary initial conditions is at this point elusive. Numerical results for the cases CB and PP are shown in Sec. III.D. Next, we present a suboptimal strategy for PP that is easy to implement in practice. This suboptimal strategy of the evader is based on geometric arguments and is discussed in the next subsection.

C. Suboptimal Strategy for Pure Pursuit

The optimal strategy for PP can be intuitively understood as one where the evader chooses its heading so that it does not favor any one of the two pursuers, finally resulting in simultaneous capture by both pursuers. With this motivation, a suboptimal strategy is constructed, and its performance is compared with the optimal one in the case of identical pursuers. In this regard, the time-to-capture relation given in Eq. (7) is exercised. For a nondegenerate problem, the evader's heading for which both P_1 and P_2 take equal time to reach the evader can be found from their initial positions using the expression

$$\frac{r_1(0)(u_1 + v\cos\theta)}{u_1^2 - v^2} = \frac{r_2(0)(u_2 + v\cos\theta)}{u_2^2 - v^2} \tag{34}$$

In general, Eq. (34) has two solutions resulting in simultaneous capture, assuming that the evader follows a constant heading. For the given initial conditions, the solution to Eq. (34) that provides maximum capture time is chosen as the suboptimal strategy. If the problem is degenerate (with respect to P_1 or P_2), then Eq. (34) has no solution.

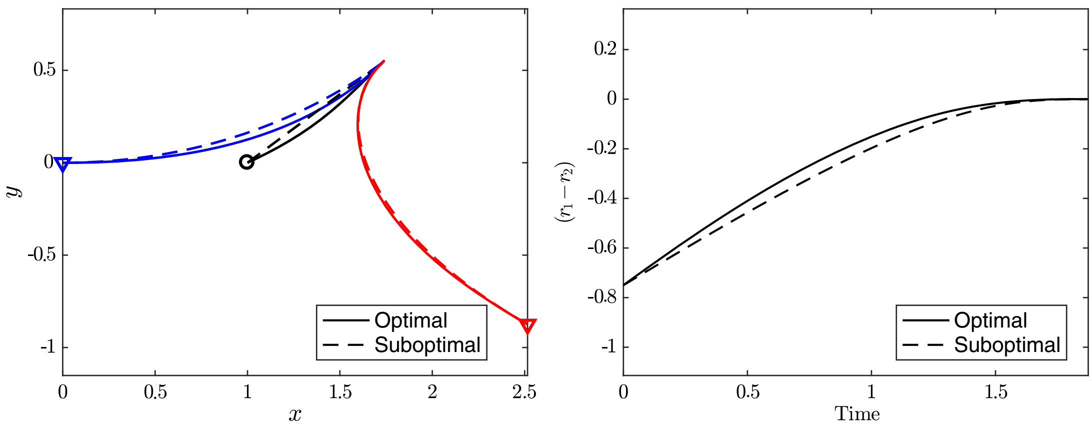
D. Numerical Simulations

This subsection demonstrates the aforementioned strategies using simulations performed for the cases CB and PP with different initial conditions. For simplicity, we assume that the speeds of the pursuers are the same and are set to $u_1 = u_2 = 1$, whereas the speed of the evader is set to $v = 0.5$, unless specified otherwise. The radius of capture is chosen as $\epsilon = 0.001$.

The optimal strategy for CB is straightforward. The software package GPOPS-II [41] was used to simulate the test cases and validate the presented theory. Figure 7a presents the trajectories of the players for the initial conditions, $p_1 = (0, 0)$, $p_2 = (3.427, -1.763)$, $p_E = (1, 0)$, which make the problem degenerate. Clearly, the optimal strategy is a pure evasion from P_1, and P_2 does not affect the evader's trajectory. An example for the nondegenerate case is presented in Fig. 7a for the initial conditions, $p_1 = (0, 0)$, $p_2 = (2.732, -1)$, $p_E = (1, 0)$. It can be observed that the optimal evading strategy involves simultaneous capture with constant heading.

The simulation results for a nondegenerate case of PP, obtained using GPOPS-II, can be seen in Fig. 8. Figure 8a presents the trajectories of the players for initial conditions $p_1 = (0, 0)$, $p_2 = (2.516, -0.875)$, $p_E = (1, 0)$. In the reduced state space, these positions correspond to $r_1(0) = 1$, $r_2(0) = 1.75$, and $\psi(0) = -\pi/6$. The optimal capture time is $t_c = 1.874$. The difference between the relative distances $(r_1 - r_2)$ is shown in Fig. 8b.

As expected, simultaneous capture is observed in these figures. Also, the difference in the relative distances, $(r_1 - r_2)$, becomes zero only at the final time. This suggests that the evader is equidistant from both the pursuers just before it gets captured. The same behavior has been observed in all the simulations that were carried out. The suboptimal strategy is also compared against the optimal strategy in Fig. 8. The (constant) heading obtained from the suboptimal strategy is $\theta = 0.6378$ (36.54 deg) with a capture time of $t_c = 1.868$. Note that the capture time and the variation in $(r_1 - r_2)$ are comparable to the corresponding results obtained using the optimal strategy; see Fig. 8b. Furthermore, a comparative study was carried out to gauge the performance of this suboptimal strategy. For this purpose, the following parameters were chosen: $r_1(0) = 1$, $u_1 = u_2 = u = 1$. The speed of the evader v was varied from 0.3 to 0.7. For each v, 140 different initial conditions ($r_2(0)$, $\psi(0)$) were considered spanning the nondegenerate area for the chosen $r_1(0)$ and u. Table 1 presents the results of this comparative study. Though the average percentage variation of the time to capture increases with the evader's speed v, the variation is less than 1% for all the evader speeds considered. The maximum percentage variation is only 2%. It can be observed that the suboptimal strategy



a) Trajectories

b) Difference between the relative distances r_1 and r_2

Fig. 8 Performance of the optimal and suboptimal strategies for a nondegenerate case in I2; black: evader, blue: P_1, red: P_2 (Please refer the online version for colors.).

is easily implementable, and its performance is similar to the optimal one. Hence, the suboptimal strategy can be considered for all practical purposes.

IV. Optimal Evading Strategies with a Stationary Pursuer

In this section, and without loss of generality, it is assumed that both pursuers are identical and that one of the pursuers remains stationary during the game. This scenario may result, for instance, from the implementation of a relay–pursuit strategy, according to which only one pursuer is assigned to go after the evader at every instant of time [26,38]. The pursuer whose Voronoi cell contains the evader is assigned to be the active pursuer to chase the evader. The other pursuer, designated as the inactive pursuer, stays at its original location and plays the role of a guard. The active pursuer switches when the evader enters the interior of the Voronoi cell of another pursuer. Because of the symmetry of the problem, when the evader resides on the Voronoi boundary, we can assign any one of the two pursuers to be the active pursuer. Therefore, throughout the pursuit process, we can fix one of the pursuers to be the active pursuer, while the other pursuer remains stationary, whose mere presence, however, imposes a state restriction, namely that the evader does not enter the interior of its corresponding Voronoi cell.

Without loss of generality, we choose P_1 to be the active pursuer with velocity $u_1 = u$ and P_2 to be the inactive pursuer with velocity $u_2 = 0$, assumed to be located at the origin, i.e., $p_2(0) = (0, 0)$. The equations of motion are given [Eqs. (1–3)] with $u_1 = u$ and $u_2 = 0$. The game evolves in the four-dimensional state space, $[x_1, y_1, x_E, y_E]^\top \in \mathbb{R}^4$. We consider strategies differing in the information structure and pursuit strategies used, subject to the state constraint

$$\|p_1 - p_E\| \le \|p_2 - p_E\| = \|p_E\| \tag{35}$$

This constraint restricts the evader from entering the Voronoi cell of the inactive pursuer. First, the region of nondegeneracy and the value of employing two pursuers in a relay pursuit mode is examined based on the set of initial conditions.

Table 1 Comparison table for optimal and suboptimal strategies of I2

v	Average percentage variation in t_c, %	Maximum percentage variation in t_c, %
0.3	0.0337	0.4451
0.4	0.0727	0.7653
0.5	0.1277	1.2182
0.6	0.1704	1.6883
0.7	0.2343	2.2487

A. Region of Nondegeneracy

The problem is nondegenerate for a given set of initial conditions if the inactive pursuer affects the outcome of the game (i.e., if the optimal evading strategy is not pure evasion from the active pursuer). Therefore, for a given set of initial conditions, and with the evader following a pure evasion strategy from the active pursuer, if the evader enters the Voronoi section of the inactive pursuer before it gets captured, then the problem is nondegenerate and vice versa. Because the position of the inactive pursuer is fixed at the origin, given the initial position of the active pursuer, the region of nondegeneracy is defined as the set of the evader's initial positions for which the problem is nondegenerate. Note that the region of nondegeneracy is the same for both R-CB and R-PP, given the active pursuer's initial position, unlike the case of two active pursuers analyzed in Sec. II. This is because of the fact that P_1 and P_2 are identical in terms of their speed capabilities, and P_2 is stationary in a relay pursuit problem.

Proposition 2 [38]: Consider the pursuit–evasion problem stated in Sec. IV. The evader will be captured before entering the Voronoi cell of the inactive pursuer while moving along the LOS and away from the active pursuer if and only if the quadratic equation

$$at^2 + bt + c = 0 \tag{36}$$

where $a = u^2 - 2uv$, $b = 2[(ux_1(0) - vx_1(0) - ux_E(0))\cos\theta_E(0) + (uy_1(0) - vy_1(0) - uy_E(0))\sin\theta(0)]$, and $c = x_1(0)^2 + y_1(0)^2 - 2(x_E(0)x_1(0) + y_E(0)y_1(0))$, does not have a solution inside the interval $[0, t_f]$, where t_f is obtained from Eq. (4), and $\theta_E(0)$ is determined by the equations

$$\cos\theta_E(0) = \frac{x_E(0) - x_1(0)}{\|p_E(0) - p_1(0)\|}, \qquad \sin\theta_E(0) = \frac{y_E(0) - y_1(0)}{\|p_E(0) - p_1(0)\|} \tag{37}$$

Notice that, when $v < u/2$, there exists no initial position for the evader such that the condition in proposition 2 is satisfied. Therefore, the optimal control for the evader is always to move along the LOS of P_1 when $v < u/2$. Henceforth, we assume that $v > u/2$.

To find the explicit expression for the region in which the condition of proposition 2 is not satisfied, and without loss of generality, let the initial position of the active pursuer be $p_1 = (x_1(0), 0)$. A schematic for the nondegenerate region with $p_1 = (-2, 0)$, $v = 0.8$, and $u = 1$ is shown in Fig. 9. The shaded region depicts the evader's initial positions for which the condition of proposition 2 is not satisfied. That is, if the evader starts from a position inside the shaded region, it will not be able to move along the LOS of P_1 throughout the pursuit without violating the state constraint. We denote the three vertices of the shaded region by A, B, and C, where A resides on the line segment between the active pursuer and the inactive pursuer, and B and C are on the Voronoi boundary.

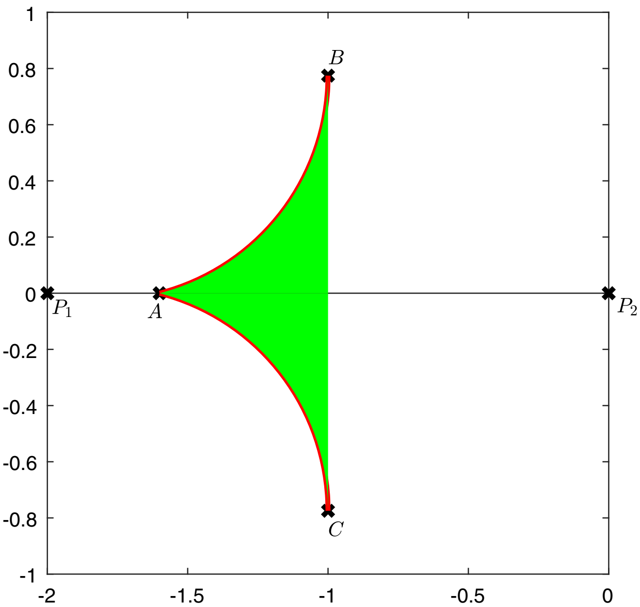


Fig. 9 Region of nondegeneracy.

If the evader starts on the line segment P_1P_2 at an initial position $(x_E(0), 0)$ that does not violate the state constraint, while moving along the LOS of P_1, it must be captured by P_1 before it reaches the boundary of the Voronoi cell. Thus, $(|x_1(0)| - |x_E(0)|)/(u - v) \le (|x_E(0)| - |x_1(0)|/2)/(v - u/2)$. After simplification, we have $|x_E(0)| \ge |x_1(0)|v/u$. Because A is at the boundary of the region and the previous inequality is linear with respect to $x_E(0)$, we thus obtain $A = (x_1(0)v/u, 0)$.

Because B is the uppermost point of the shaded region that is also on the Voronoi boundary, when the evader starts at B, its velocity for staying on the Voronoi boundary should coincide with its velocity for moving along the LOS. On the boundary, we have $x_1(x_1 - 2x_E) + y_1(y_1 - 2y_E) = 0$. Hence, by taking a time derivative, we obtain $-2(u\|p_E - p_1\| - vx_1 \cos\theta_E - vy_1 \sin\theta_E) = 0$. By plugging in $x_E(0) = x_1(0)/2$, $y_E(0) = \beta$, $y_1(0) = 0$ in the previous equations, one obtains

$$\cos\theta_E(0) = -\frac{u(x_1(0)^2/4 + \beta^2)^{1/2}}{vx_1(0)} \tag{38}$$

On the other hand, for the evader to move along the LOS of P_1, $\theta_E(0)$ satisfies

$$\cos\theta_E(0) = \frac{-x_1(0)/2}{(x_1(0)^2/4 + \beta^2)^{1/2}} \tag{39}$$

Equating Eq. (38) with Eq. (39) and solving for β, we have $B = (x_1(0)/2, \sqrt{v/(2u) - 1/4}|x_1(0)|)$. And $C = (x_1(0)/2, -\sqrt{v/(2u) - 1/4}|x_1(0)|)$, which by the nature of symmetry is the reflection of B about the x axis.

The curves AB and AC are arcs of circles and satisfy the equation

$$\begin{aligned} 0 &= (((v-u)x_1(0) + ux)(x - x_1(0)) + uy^2)^2 \\ &\quad - u(2v-u)x_1(0)(2x - x_1(0))((x - x_1(0))^2 + y^2) \end{aligned} \tag{40}$$

which is derived from $0 = b^2 - 4ac$, where a, b, and c are defined in proposition 2, by plugging in Eq. (37) and the initial condition for the active pursuer. The optimal evading strategies can now be analyzed in the nondegenerate regions for R-CB and R-PP.

B. Case of Relay Constant-Bearing Strategy

For R-CB, it is assumed that the active pursuer P_1 follows a constant-bearing strategy. The reduced state space dynamics are given by

$$\dot{r}_1 = v\cos(\theta_E - \varphi_{10}) - u\cos(\theta_1 - \varphi_{10}) \tag{41}$$

$$\dot{r}_2 = v\cos(\theta_E - \varphi_2) \tag{42}$$

$$\dot{\varphi}_2 = \frac{v}{r_2}\sin(\theta_E - \varphi_2) \tag{43}$$

where φ_{10} is the initial LOS angle of P_1. Because P_1 follows a constant-bearing strategy, θ_1 is determined from the equation

$$\dot{\varphi}_1 = \frac{1}{r_1}[v\sin(\theta_E - \varphi_{10}) - u\sin(\theta_1 - \varphi_{10})] = 0 \tag{44}$$

These equations are similar to the ones presented in Sec. III.A, but they differ in the sense that the second pursuer is stationary.

A schematic of the proposed pursuit–evasion problem can be seen in Fig. 10a. The boundary conditions are given by $r_1(0) = \|p_E(0) - p_1(0)\|$, $r_2(0) = \|p_E(0) - p_2(0)\|$, $\varphi_2(0) = \varphi_{20}$, $r_1(t_c) = \epsilon$, $r_2(t_c)$ and $\varphi_2(t_c)$ are free, where φ_{20} is the initial LOS angle of the inactive pursuer P_2. The state constraint [Eq. (35)] imposed on the relay pursuit problem can be expressed as

$$\mathcal{S} = \frac{1}{2}(r_1^2 - r_2^2) \le 0 \tag{45}$$

Note that the inequality constraint [Eq. (45)] imposed on the problem involves only the state variables. Therefore, we have to take the time derivative for the constraint $\mathcal{S}$ and substitute the equations of motion until an explicit dependence on the control variable occurs. The qth-order time derivative in which this first happens plays a role in the Hamiltonian [42]. To this end, we first use the fact

$$\begin{aligned} \dot{\mathcal{S}} &= r_1\dot{r}_1 - r_2\dot{r}_2 \\ &= r_1(v\cos(\theta_E - \varphi_{10}) - u\cos(\theta_1 - \varphi_{10})) - vr_2\cos(\theta_E - \varphi_2) \end{aligned} \tag{46}$$

It can be seen that the control variable θ appears explicitly in the first time derivative. The Hamiltonian can then be expressed as

$$\begin{aligned} H(r_1, r_2, \psi_2, \lambda_1, \lambda_2, \lambda_3, \mu) &= -1 + \lambda_1\dot{r}_1 + \lambda_2\dot{r}_2 + \lambda_3\dot{\varphi}_2 + \mu\dot{\mathcal{S}} \\ &= -1 + (\lambda_1 + \mu r_1)(v\cos(\theta_E - \varphi_{10}) - u\cos(\theta_1 - \varphi_{10})) \\ &\quad + (\lambda_2 - \mu r_2)v\cos(\theta_E - \varphi_2) + \frac{\lambda_3 v}{r_2}\sin(\theta_E - \varphi_2) \end{aligned} \tag{47}$$

where λ_1, λ_2, λ_3, and μ are the costates, and μ satisfies the Kuhn–Tucker and complementary slackness conditions (i.e., for $\mathcal{S} \ne 0$, $\mu = 0$, and $\mathcal{S} = 0, \mu \ge 0$). The transversality conditions are given by $\lambda_2(t_c) = 0$, $\lambda_3(t_c) = 0$, $H(t_c) = 0$. Furthermore, because the Hamiltonian has no explicit dependency on time, $H(t) = 0$ for all $t \in [0, t_c]$. The optimal control can be obtained from Pontryagin's minimum principle, using the expression

$$\begin{aligned} &(\lambda_1 + \mu r_1)\sin(\theta_1 - \theta_E) - (\lambda_2 - \mu r_2)\sin(\theta_E - \varphi_2)\cos(\theta_1 - \varphi_{10}) \\ &\quad + \frac{\lambda_3}{r_2}\cos(\theta_E - \varphi_2) = 0 \end{aligned} \tag{48}$$

Proposition 3: Consider the time R-CB optimal control problem. If the initial conditions are such that the problem is nondegenerate with $u > v > u/2$, then the optimal control of the evader $\theta_E^*(t)$ satisfies

$$\begin{cases} \lambda_1 \sin(\theta_1(t) - \theta_E^*(t)) - \lambda_2 \sin(\theta_E^*(t) - \varphi_2)\cos(\theta_1(t) - \varphi_{10}) + \lambda_3 \cos(\theta_E^*(t) - \varphi_2)/r_2 = 0, & t \in [0, \tau_1], \\ r_1(v\cos(\theta_E^*(t) - \varphi_{10}) - u\cos(\theta_1(t) - \varphi_{10})) - vr_2 \cos(\theta_E^*(t) - \varphi_2) = 0, & t \in [\tau_1, \tau_2], \\ \theta_E^*(t) = \varphi_{10}, & t \in [\tau_2, t_c] \end{cases}$$

where $\theta_1(t)$ is obtained using the expression

$$v \sin(\theta_E^*(t) - \varphi_{10}) = u \sin(\theta_1(t) - \varphi_{10}), \quad t \in [0, \tau_2] \tag{49}$$

Furthermore, τ_2 satisfies the switching condition:

$$(v - u)r_1(\tau_2) - vr_2(\tau_2)\cos(\varphi_{10} - \varphi_2(\tau_2)) = 0 \tag{50}$$

C. Case of Relay Pure-Pursuit Strategy

In this case, P_1 follows a pure-pursuit strategy. A schematic of the proposed relay pursuit problem is shown in Fig. 10b. This problem is similar to the one presented in Sec. III.B and can be analyzed in the three-dimensional reduced state space, but it differs from it by the fact that P_2 is now stationary. It can be solved with the use of states and the control input θ in the reduced state space presented in Sec. III.B. The equations of motion are now given by

$$\dot{r}_1 = -u + v\cos\theta \tag{51}$$

$$\dot{r}_2 = -v\cos(\psi - \theta) \tag{52}$$

$$\dot{\psi} = \frac{v}{r_2}\sin(\psi - \theta) - \frac{v}{r_1}\sin\theta \tag{53}$$

whereas the constraint given in Eq. (45) and the boundary conditions for r_1 and r_2 are the same as in the R-CB case; $\psi(0) = \psi_0$, and $\psi(t_c)$ is free. Note that a time derivative of the constraint $\mathcal{S}$ has to be taken to write the Hamiltonian for this optimal control problem and

$$\dot{\mathcal{S}} = r_1\dot{r}_1 - r_2\dot{r}_2 = r_1(-u + v\cos\theta) + vr_2\cos(\psi - \theta) \tag{54}$$

Therefore, the Hamiltonian is

$$H(r_1, r_2, \psi, \lambda_1, \lambda_2, \lambda_3, \mu) = -1 + (\lambda_1 + \mu r_1)(-u + v\cos\theta) + (-\lambda_2 + \mu r_2)v\cos(\psi - \theta) + \lambda_3\left(\frac{v}{r_2}\sin(\psi - \theta) - \frac{v}{r_1}\sin\theta\right) \tag{55}$$

where λ_1, λ_2, λ_3, and μ are the costates, and μ satisfies the Kuhn–Tucker and complementary slackness conditions (i.e., for $\mathcal{S} \neq 0$, $\mu = 0$, and $\mathcal{S} = 0, \mu \geq 0$). The transversality conditions are given by $\lambda_2(t_c) = 0$, $\lambda_3(t_c) = 0$ and $H(t_c) = 0$. Again, the Hamiltonian has no explicit dependency on time, and therefore the optimal Hamiltonian remains constant at zero. The optimal control can be obtained similarly from Pontryagin's minimum principle, using the expression

$$\tan\theta^* = -\frac{(\lambda_2 - \mu r_2)\sin\psi + (\lambda_3/r_1) + (\lambda_3\cos\psi/r_2)}{\lambda_1 - (\lambda_2 - \mu r_2)\cos\psi + (\lambda_3\sin\psi/r_2)} \tag{56}$$

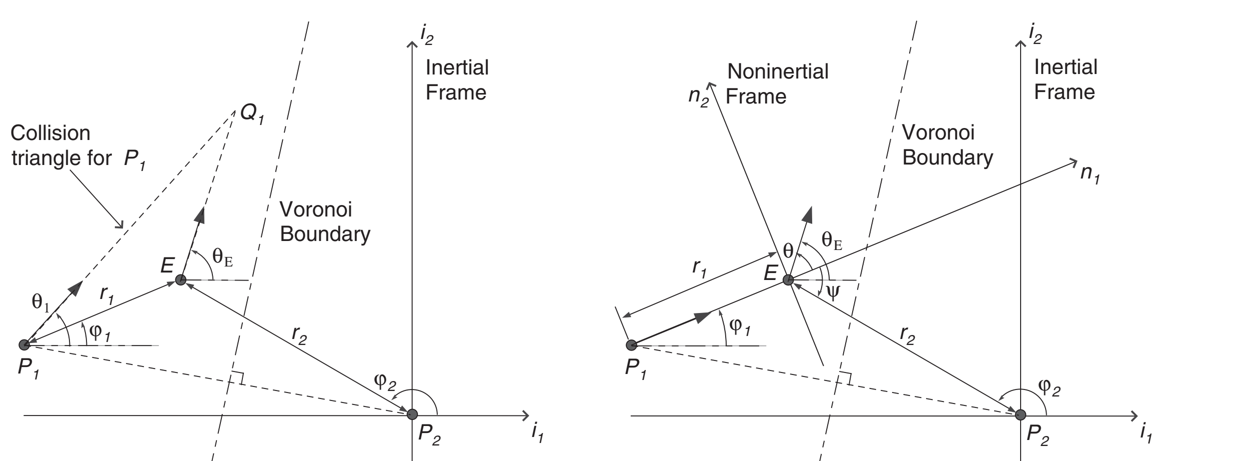
Proposition 4 [38]: Consider the R-PP time-optimal control problem. If the initial conditions are such that the problem is nondegenerate with $u > v > u/2$, then the optimal control of the evader $\theta^*(t)$ satisfies

$$\tan\theta^*(t) = \begin{cases} -\dfrac{\lambda_2\sin\psi + (\lambda_3/r_1) + (\lambda_3\cos\psi/r_2)}{\lambda_1 - \lambda_2\cos\psi + (\lambda_3\sin\psi/r_2)}, & t \in [0, \tau_1], \\ \dfrac{q - \sigma p\sqrt{p^2 + q^2 - 1}}{p + \sigma q\sqrt{p^2 + q^2 - 1}}, & t \in [\tau_1, \tau_2], \\ 0, & t \in [\tau_2, t_c] \end{cases}$$

where $p = (vr_1 + vr_2\cos\psi)/(ur_1)$, $q = (vr_2\sin\psi)/(ur_1)$, $\sigma = \mathrm{sgn}(q)$. Furthermore, τ_2 satisfies the switching condition

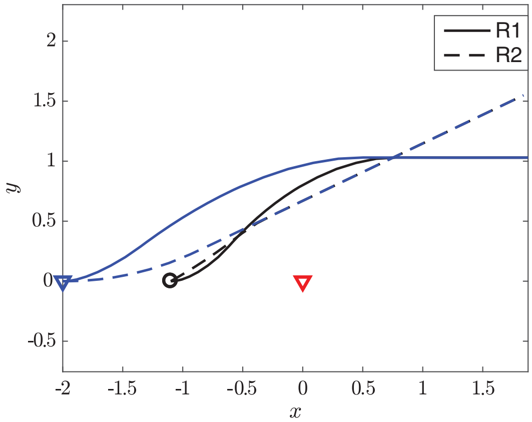
$$(v - u)r_1(\tau_2) + vr_2(\tau_2)\cos\psi(\tau_2) = 0 \tag{57}$$

Summarizing the previous analyses, we conclude that, for both cases R-CB and R-PP, the optimal trajectory of the evader involves three periods. First, the evader moves inside the Voronoi cell of the active pursuer in a way such that the optimal conditions (corresponding transversality conditions, Erdmann corner conditions, etc.) are satisfied before it hits the Voronoi boundary. The evader then moves along the boundary until the switching condition [Eq. (50) for R-CB, Eq. (57) for R-PP] is satisfied. Finally, the evader moves along the LOS of P_1 until capture occurs.



a) R-CB b) R-PP

Fig. 10 Schematics of the proposed relay pursuit problems.



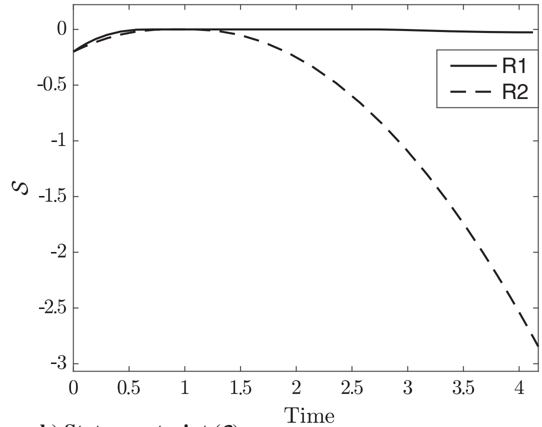


Fig. 11 **Performance of the optimal strategies for a nondegenerate cases in R1 and R2; black: evader, blue: P_1, red: P_2 (Please refer the online version for colors.).**

D. Numerical Simulations

In this subsection, the optimal strategies for R-CB and R-PP are demonstrated using an example with nondegenerate initial conditions. The speed of the active pursuer (P_1) is $u = 1$, and that of the evader is $v = 0.8$. The initial position of P_1 is $(-2, 0)$, and the evader's initial position is at $(-1.1, 0)$, which is in the nondegenerate region; see Fig. 9. Because the nondegenerate region is the same in both cases, their corresponding optimal strategies can be compared in this example. The problem is solved using GPOPS-II.

The simulated results are presented in Fig. 11. The time to capture in the case of R-PP is $t_c = 4.172$, whereas in R-CB, $t_c = 4.125$. The R-CB case is lower than R-PP because, in the latter, the pursuers have an advantage because they have information of both the evader's position and velocity. The trajectories of the players can be seen in Fig. 11a. Because P_1 follows a constant-bearing strategy, its LOS angle is constant throughout the time ($\varphi_{10} = 0$). The state constraint can be analyzed using Fig. 11b. The three periods in both optimal strategies can be observed in that plot. The interesting observation here is that, in R-CB, the evader hits the Voronoi boundary earlier compared to R-PP, but the evader stays on the boundary for longer time and eventually follows pure evasion from P_1. This is because of the difference in P_1's strategy, which also affects the dynamics of the Voronoi boundary and the switching condition.

V. Conclusions

The two-pursuer/one-evader problem is analyzed in two different scenarios, assuming both pursuers to be superior to the evader in terms of their speed capabilities, thus guaranteeing capture. The first scenario involves pursuers that have access to information of the evader's position and velocity and use this information to follow a constant-bearing strategy. In the second scenario, the pursuers have access to the evader's position only and hence, they follow a pure-pursuit strategy. The time-optimal evading strategy is identified in both scenarios. Because obtaining a closed-form solution in the second scenario is elusive at this point, a competitive suboptimal strategy that can be practically implemented is identified and is compared against the optimal strategy. The regions of nondegeneracy are used to investigate the utility of employing two pursuers to more efficiently capture the evader. If the initial positions of the players are such that the problem is degenerate, then one of the pursuers does not play any role in the game, and the optimal evading strategy is pure evasion from the other pursuer. Optimal evading strategies against relay pursuit are also investigated by keeping one pursuer stationary. The results of this paper provide a potential framework for solving larger classes of multiplayer time-optimal pursuit–evasion games under different information structures.

Appendix: Proof of Proposition 3

If the initial condition of the evader is not on the boundary of the Voronoi cell of the inactive pursuer, it follows that the control is given by Eq. (48) with $\mu = 0$, and hence

$$\lambda_1 \sin(\theta_1(t) - \theta_E^*(t)) - \lambda_2 \sin(\theta_E^*(t) - \varphi_2)\cos(\theta_1(t) - \varphi_{10}) + \frac{\lambda_3}{r_2}\cos(\theta_E^*(t) - \varphi_2) = 0, \quad \text{for } t \in [0, \tau_1] \tag{A1}$$

where $\theta_1(t)$ is obtained from Eq. (49), which is a consequence of Eq. (44). The evader will follow this strategy until some time $\tau_1 > 0$, when it hits the Voronoi boundary, and then it will stay on the boundary, defined by $\mathcal{S} = 0$, for $t \in [\tau_1, \tau_2]$. It follows that, when the evader moves along the Voronoi boundary, the control to use should satisfy

$$\dot{\mathcal{S}} = r_1(v\cos(\theta_E^*(t) - \varphi_{10}) - u\cos(\theta_1(t) - \varphi_{10})) - vr_2\cos(\theta_E^*(t) - \varphi_2) = 0, \quad \text{for } t \in [\tau_1, \tau_2] \tag{A2}$$

The optimal value of the multiplier μ^* for $t \in [\tau_1, \tau_2]$ can be immediately computed from Eq. (48) with boundary condition $\mu(\tau_2^-) = \mu(\tau_2^+) = 0$. After the evader leaves the constraint at time $t = \tau_2$, and before capture, $\mu^* = 0$, and thus the adjoint equations are linear in their respective costates whose solution subject to the transversality conditions is given by $\lambda_2(t) = \lambda_3(t) = 0$ for all $t \in [\tau_2, t_c]$. Therefore, when $t \in [\tau_2, t_c]$, from Eq. (48), $\theta_E^*(t) = \theta_1(t) = \varphi_{10}$. By imposing Erdmann's corner conditions at the entry and exit points from the state constraint [42], and after some tedious but rather straightforward calculations, one obtains $\theta_E^*(\tau_1^-) = \theta_E^*(\tau_1^+)$ and $\theta_E^*(\tau_2^-) = \theta_E^*(\tau_2^+)$. Hence, the control is continuous at τ_1 and τ_2. Because $\theta_E^*(\tau_2^+) = \varphi_{10}$, it follows that $\theta_E^*(\tau_2^-) = \varphi_{10}$, and hence the evader will leave the boundary when the evader's velocity is parallel to the current the initial LOS angle of P_1. Because after the switching at $t = \tau_2$ we have $\theta_E^*(\tau_2) = \theta_1(\tau_2) = \varphi_{10}$, the switching condition to leave the boundary follows immediately from Eq. (A2) and is given by Eq. (50). This completes the proof.

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