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Feedback Navigation in an Uncertain Flowfield and Connections with Pursuit Strategies

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This paper presents several classes of control laws for steering an agent, that is, an aerial or marine vehicle, in the presence of a both temporally and spatially varying drift field induced by local winds/currents. The navigation problem is addressed assuming various information patterns about the drift field in the vicinity of the agent. In particular, three cases are considered: namely, when the agent has complete information about the local drift, when the drift field is partially known, and when the drift field is completely unknown. By first establishing a duality between the navigation problem and a special class of problems of pursuit of a maneuvering target, several navigation schemes are presented, which are appropriately tailored to the fidelity of the information about the local drift available to the agent. The proposed navigation laws are dual to well-known pursuit strategies, such as pure pursuit, parallel guidance/navigation, line-of-sight guidance, motion camouflage, and pursuit with neutralization. Simulation results are presented to illustrate the theoretical developments.

Nomenclature

C^1	=	set of continuously differentiable functions
$\mathsf{e}_{\mathsf{x}}^1$	=	unit vector parallel to the line of sight
$\mathsf{e}_{\mathsf{x}}^2$	=	unit vector perpendicular to the line of sight
LC	=	set of Lipschitz continuous functions
ℓ_{LS}	=	ray defined by the line of sight
$\mathbb{R}^2$	=	set of two-dimensional real vectors
T_f	=	arrival time, s
u	=	forward velocity vector of the agent, m/s
$\bar{u}$	=	maximum forward speed of the agent, m/s
$u_{\mathcal{P}}$	=	velocity vector of the pursuer, m/s
w	=	known component of the drift field, m/s
$\bar{w}$	=	upper bound on the norm of w, m/s
x	=	position vector of the agent, m
$\mathsf{x}_{\mathcal{P}}$	=	position vector of the pursuer, m
$\mathsf{x}_{\mathcal{T}}$	=	position vector of the maneuvering target, m
Δw	=	unknown component of the drift field, m/s
$\Delta\bar{w}$	=	upper bound on the norm of Δw, m/s
ε	=	tolerance of miss-target error, m
λ	=	angle of the line of sight measured with respect to a fixed reference axis, rad

I. Introduction

This paper deals with the problem of characterizing navigation laws for steering an agent in the presence of a both spatially and temporally varying drift field induced by local winds or currents. The problem is a variation of the classical Zermelo navigation problem (ZNP) [1] , which seeks a navigation law to steer an agent with single integrator kinematics to a prescribed destination in the presence of drift in minimum time. In contrast to the solution of the classical ZNP, which yields noncausal/anticipative controllers that require, in general, global and perfect knowledge of the drift field, the objective of this work is to characterize instead causal/nonanticipative steering laws that require only partial and local knowledge of the drift field; consequently, these navigation laws are robust to uncertainties arising from incomplete information about the local drift field dynamics.

Semi-analytical and numerical solutions to the ZNP have been recently reported in [2–8]. In all these references, it is assumed that the agent has a priori, perfect and global information about the drift field. In this work, the navigation problem is addressed in the more realistic case when the information about the drift available to the agent is limited, and possibly uncertain. In particular, three cases are considered: 1) the agent has perfect and reliable knowledge of the local drift, 2) the knowledge of the local drift field is imperfect, and 3) the local drift field is completely unknown. With the proposed navigation schemes, useful insights can be gleaned for a large spectrum of applications, ranging from path planning, vehicle routing, to motion coordination for, say, environmental monitoring or surveillance and reconnaissance missions in the presence of drift, thus extending the available results in the literature, which typically deal with cases when the drift is either a priori known or completely ignored [3,5,8–13].

The main contribution of this work is the characterization of feedback navigation laws that are tailored to the fidelity of the information about the local drift available to the agent. The design of these feedback navigation laws is based on the duality between the navigation problem and a special class of pursuit problems of a maneuvering target. This duality was originally demonstrated for special cases of the drift vector field, namely, when the drift is constant, when it varies uniformly with time, and when it is a time-varying affine field [6,14–17], and it is established for general, both temporally and spatially varying, drift fields in this work. After having elucidated the connection between the navigation and the pursuit problems, several navigation laws that are dual to some well-known pursuit strategies are presented. First, two classes of navigation laws that require perfect, but only local, information about the drift field are introduced. The navigation laws of the first class constraint the agent to move along the line of sight, that is, the direction defined from the agent's position to its destination (line-of-sight navigation), whereas the second navigation law is the dual to a well-known pursuit strategy, namely, *line of sight* or *three-point guidance* [18]. Feedback navigation laws that are robust to model uncertainties induced by the incomplete information about the drift field in the vicinity of the agent are subsequently presented. Finally, the navigation problem in the presence of a completely unknown drift field is addressed by employing a feedback navigation law that steers the agent's forward velocity so that it always points toward its destination. This navigation law is the dual to the well-known

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pure-pursuit strategy, also known as *hound-hare pursuit* [18] or *direct-bearing pursuit* [19]. For each of the proposed navigation laws, a set of sufficient conditions for the convergence of the agent to its destination in finite time are presented. These conditions highlight an important distinction between the problems of pursuit and navigation, namely, that the navigation problem may be feasible for cases when its equivalent pursuit problem is not feasible. This situation occurs as a result of the different underlying assumptions in the formulation of the two problems. In particular, in the navigation problem, the notional opponent of the agent, whose evading strategy is induced by the local drift field, does not necessarily act as an adversarial, noncooperative opponent, as is the case in the classical pursuit problem. In addition, the optimality (or near optimality) of the proposed navigation schemes is highlighted by elucidating their interpretation as gradient descent control laws derived from heuristics of the time-to-come function taking into account the information about the local drift available to the agent. Besides the novel contribution of this work regarding the characterization of causal/nonanticipative feedback navigation laws in the presence of an uncertain drift field, this paper also has a pedagogical value stemming from useful insights gained by reinterpreting and reevaluating known pursuit strategies from an *information-centric* perspective.

The rest of the paper is organized as follows. Section II discusses the formulation of the navigation problem in the presence of a both time and spatially varying drift field. Section III introduces navigation laws that require perfect knowledge of the local drift. The problem of navigation with imperfect or complete lack of information about the local drift is discussed and analyzed in Sections IV and V, respectively. Section VI highlights the interpretation of the presented navigation laws as gradient descent laws in terms of appropriate performance indices. Simulation results are presented in Sec. VII. Finally, Sec. VIII concludes the paper with a summary of remarks.

II. Formulation of the Navigation Problem

Consider an agent whose kinematics are described by

$$\dot{\mathsf{x}} = u + w(\mathsf{x}) + \Delta w(t, \mathsf{x}), \qquad \mathsf{x}(0) = \mathsf{x}_0 \tag{1}$$

where $\mathsf{x} := [x, \quad y]^\mathsf{T} \in \mathbb{R}^2$ and $\mathsf{x}_0 := [x_0, \quad y_0]^\mathsf{T} \in \mathbb{R}^2$ denote the position vector of the agent at time t and $t = 0$, respectively, and u is the control input (velocity vector) of the agent. It is assumed that $u \in \mathcal{U}$, where $\mathcal{U}$ consists of all piecewise continuous functions taking values in the set $U = \{u \in \mathbb{R}^2\colon |u| \leq \bar{u}\}$, where $\bar{u}$ is a positive constant (maximum allowable forward speed), and $|\cdot|$ denotes the standard Euclidean vector norm. Furthermore, $w(\mathsf{x}) + \Delta w(t, \mathsf{x})$ is the drift induced by the winds/currents in the vicinity of the agent. In particular, $w(\mathsf{x})$ denotes the component of the local drift that is perfectly known to the agent, and which is assumed to be at least C^1 in the domain of interest. The term $\Delta w(t, \mathsf{x})$ denotes the unknown component of the drift and is assumed to be a piecewise continuous function of time t, and C^1 with respect to the agent's position x. Furthermore, it is assumed that there exist $\bar{w} > 0$ and $\Delta\bar{w} > 0$, such that

$$|w(\mathsf{x})| \leq \bar{w}, \qquad |\Delta w(t, \mathsf{x})| \leq \Delta\bar{w}, \quad \text{for all } t \geq 0 \quad \text{and} \quad \mathsf{x} \in \mathbb{R}^2 \tag{2}$$

A. Formulation of the Minimum-Time Navigation Problem

First, the classical ZNP [1] is revisited. The ZNP deals with the characterization of a navigation law to steer an agent, whose kinematics are described by Eq. (1), to a prescribed destination in minimum time, in the special case when the drift is perfectly known, that is, when $\Delta w(t, \mathsf{x}) \equiv 0$ (deterministic minimum-time problem).

Problem 1 (ZNP) Let the system described by Eq. (1) with $\Delta w(t, \mathsf{x}) \equiv 0$. Determine a control input $u_* \in \mathcal{U}$, such that

1) The trajectory $\mathsf{x}_*\colon [0, T_f] \mapsto \mathbb{R}^2$ generated by the control u_* satisfies the boundary conditions

$$\mathsf{x}_*(0) = \mathsf{x}_0, \qquad \mathsf{x}_*(T_f) = 0 \tag{3}$$

2) The control u_* minimizes, along the trajectory x_*, the cost functional $J(u) := T_f$, where $0 \leq T_f < \infty$ is the free final time.

It can be shown that the control law that solves Problem 1 has necessarily the following structure:

$$u_* = \bar{u}[\cos\theta_*, \quad \sin\theta_*]^\mathsf{T} \tag{4}$$

where θ_* satisfies the following differential equation, known as the *navigation formula* (for more details, see, for example, [20,21])

$$\begin{aligned}\dot{\theta}_* = \nu_x(\mathsf{x}_*)\sin^2\theta_* - \mu_y(\mathsf{x}_*)\cos^2\theta_* + (\mu_x(\mathsf{x}_*) \\ - \nu_y(\mathsf{x}_*))\cos\theta_*\sin\theta_*\end{aligned} \tag{5}$$

where $w := [\mu, \quad \nu]^\mathsf{T}$ and $\mu_x := \partial\mu/\partial x$, $\mu_y := \partial\mu/\partial y$, $\nu_x := \partial\nu/\partial x$, and $\nu_y := \partial\nu/\partial y$. It follows that the candidate optimal control u_* is determined up to a single parameter, namely, $\bar{\theta} = \theta_*(0) \in [0, 2\pi)$, from Eqs. (4) and (5); the optimal control is consequently written as $u_*(t; \bar{\theta})$. One immediately observes that a candidate optimal control of the ZNP depends explicitly on the current position vector x_*, as well as both the drift w and its Jacobian matrix $\partial w/\partial\mathsf{x}$, through the navigation formula (5). Therefore, the ZNP cannot be solved in practice, unless the agent has a priori perfect and global knowledge of the drift vector field $w(\mathsf{x})$, in which case the ZNP can be addressed as a standard, deterministic two-point boundary-value minimum-time problem. The objective of this work is to derive feedback navigation laws that require information about the drift field only in the vicinity of the agent, and which are completely independent of the Jacobian of the drift field (navigation with local information).

B. Formulation of the Navigation Problem with Local Information

Next, the problem of characterizing feedback navigation laws for different information patterns regarding the drift in the vicinity of the agent is considered. To this end, let the kinematics of the agent be described by Eq. (1) as before, but with the distinctive difference that $u(\mathsf{x})$ is a state feedback control law. In particular, it is assumed that $u \in \mathcal{U}_f$, where

$$\mathcal{U}_f := \{f \in \mathrm{LC}(\mathbb{R}^2\backslash\{0\})\colon f(\mathsf{x}) \in U, \quad \forall\ \mathsf{x} \neq 0\}$$

and where $\mathrm{LC}(\mathbb{R}^2\backslash\{0\})$ denotes the set of all locally Lipschitz continuous functions on $\mathbb{R}^2\backslash\{0\}$. Different information patterns regarding the drift in the vicinity of the agent are considered:

1) The drift is perfectly known only in the vicinity of the agent, that is, $w(\mathsf{x}) \not\equiv 0$, and $\Delta w(t, \mathsf{x}) \equiv 0$.

2) The drift is not known perfectly, that is, $w(\mathsf{x}) \not\equiv 0$, and $\Delta w(t, \mathsf{x}) \not\equiv 0$.

3) The drift is completely unknown, that is, $w(\mathsf{x}) \equiv 0$, and $\Delta w(t, \mathsf{x}) \not\equiv 0$.

Next, the navigation problem, when the drift field is only locally known, is formulated.

Problem 2. Let the system described by Eq. (1), where, at every instant of time t, only the local drift field $w(\mathsf{x})$ is known. Given $\varepsilon > 0$, determine a control input $u \in \mathcal{U}_f$, such that the trajectory $\mathsf{x}\colon [0, T_f] \mapsto \mathbb{R}^2$ generated by the control u satisfies, for every $|\mathsf{x}_0| > \varepsilon$, the boundary conditions

$$\mathsf{x}(0) = \mathsf{x}_0, \qquad |\mathsf{x}(T_f)| \leq \varepsilon \tag{6}$$

for some $0 \leq T_f < \infty$.

One of the differences between Problem 1 and Problem 2 is that in the formulation of the latter, the requirement that the agent should exactly reach its destination in minimum time has been relaxed. Instead, in Problem 2, and in order to account for the possibility of imperfect knowledge of the local drift field, it is only required that the agent reaches a ball of radius ε centered at $\mathsf{x} = 0$ in finite time. Furthermore, the control law that solves Problem 2 has been restricted to the class of (time-invariant) state feedback control laws, which satisfy standard regularity properties guaranteeing that the mathematical model of the closed-loop system is well posed.

C. Navigation Problem as a Problem of Pursuit of a Maneuvering Target

Next, the interpretation of the navigation Problem 2 as a problem of pursuit of a maneuvering target is discussed. To this end, consider a pursuer and a moving target whose kinematics are described by the following set of equations:

$$\dot{\mathsf{x}}_{\mathcal{P}} = u_{\mathcal{P}}(\mathsf{x}_{\mathcal{P}}, \mathsf{x}_{\mathcal{T}}), \qquad \mathsf{x}_{\mathcal{P}}(0) = \mathsf{x}_0 \tag{7}$$

$$\dot{\mathsf{x}}_{\mathcal{T}} = -w(\mathsf{x}_{\mathcal{P}}, \mathsf{x}_{\mathcal{T}}) - \Delta w(t, \mathsf{x}_{\mathcal{P}}, \mathsf{x}_{\mathcal{T}}), \qquad \mathsf{x}_{\mathcal{T}}(0) = 0 \tag{8}$$

where

$$\mathsf{x}_{\mathcal{P}} := [x_{\mathcal{P}}, \quad y_{\mathcal{P}}]^{\mathsf{T}} \in \mathbb{R}^2, \qquad \mathsf{x}_{\mathcal{T}} := [x_{\mathcal{T}}, \quad y_{\mathcal{T}}]^{\mathsf{T}} \in \mathbb{R}^2$$

are the position vectors of the pursuer and the moving target at time t, respectively. In addition, $u_{\mathcal{P}} \in \mathcal{U}_{\mathcal{P},f}$, where

$$\mathcal{U}_{\mathcal{P},f} := \{f \in \mathrm{LC}(\mathbb{R}^4 \backslash \mathcal{M}) \colon f(\mathsf{x}) \in U, \ \forall \, \mathsf{x} \notin \mathcal{M}\}$$

and $\mathcal{M} = \{(\mathsf{x}_{\mathcal{P}}, \mathsf{x}_{\mathcal{T}}) \in \mathbb{R}^4 \colon \mathsf{x}_{\mathcal{P}} = \mathsf{x}_{\mathcal{T}}\}$. Furthermore, $-w(\mathsf{x}_{\mathcal{P}}, \mathsf{x}_{\mathcal{T}}) - \Delta w(t, \mathsf{x}_{\mathcal{P}}, \mathsf{x}_{\mathcal{T}})$ is the target's velocity, where $w(\mathsf{x}_{\mathcal{P}}, \mathsf{x}_{\mathcal{T}})$ (the known component of the instantaneous target's velocity) and $\Delta w(t, \mathsf{x}_{\mathcal{P}}, \mathsf{x}_{\mathcal{T}})$ (the unknown component of the instantaneous target's velocity) satisfy the same regularity conditions as in the formulation of the ZNP. Next, a problem of pursuit of a maneuvering target, which, as shown later, turns out to be equivalent to Problem 2, is presented.

Problem 3. Let the kinematics of a pursuer and a moving target be described by Eqs. (7) and (8), respectively, and assume that, at each instant of time, the pursuer has only knowledge of $-w(\mathsf{x})$. Given $\varepsilon > 0$, find a control law $u_{\mathcal{P}} \in \mathcal{U}_{\mathcal{P},f}$, such that the trajectories $\mathsf{x}_{\mathcal{P}}(\cdot; u_{\mathcal{P}})$ and $\mathsf{x}_{\mathcal{T}}(\cdot; -w - \Delta w)$ generated by $u_{\mathcal{P}}$ and $-w - \Delta w$, respectively, satisfy, for all $|\mathsf{x}_0| > \varepsilon$, the boundary conditions

$$\mathsf{x}_{\mathcal{P}}(0) = \mathsf{x}_0, \qquad \mathsf{x}_{\mathcal{T}}(0) = 0$$
$$|\mathsf{x}_{\mathcal{P}}(T_f; u_{\mathcal{P}}) - \mathsf{x}_{\mathcal{T}}(T_f; -w - \Delta w)| \le \varepsilon \tag{9}$$

for some $0 \le T_f < \infty$.

Let one consider the special case when

$$u_{\mathcal{P}}(\mathsf{x}_{\mathcal{P}}, \mathsf{x}_{\mathcal{T}}) = u_{\mathcal{P}}(\mathsf{x}_{\mathcal{P}} - \mathsf{x}_{\mathcal{T}}), \qquad w(\mathsf{x}_{\mathcal{P}}, \mathsf{x}_{\mathcal{T}}) = w(\mathsf{x}_{\mathcal{P}} - \mathsf{x}_{\mathcal{T}})$$
$$\Delta w(t, \mathsf{x}_{\mathcal{P}}, \mathsf{x}_{\mathcal{T}}) = \Delta w(t, \mathsf{x}_{\mathcal{P}} - \mathsf{x}_{\mathcal{T}})$$

By taking $\mathsf{x} = \mathsf{x}_{\mathcal{P}} - \mathsf{x}_{\mathcal{T}}$ and $u = u_{\mathcal{P}}$, it is easy to see that

$$\begin{aligned} \dot{\mathsf{x}} &= u_{\mathcal{P}}(\mathsf{x}_{\mathcal{P}} - \mathsf{x}_{\mathcal{T}}) + w(\mathsf{x}_{\mathcal{P}} - \mathsf{x}_{\mathcal{T}}) + \Delta w(t, \mathsf{x}_{\mathcal{P}} - \mathsf{x}_{\mathcal{T}}) \\ &= u(\mathsf{x}) + w(\mathsf{x}) + \Delta w(t, \mathsf{x}) \end{aligned} \tag{10}$$

Furthermore,

$$\mathsf{x}(0) = \mathsf{x}_{\mathcal{P}}(0) - \mathsf{x}_{\mathcal{T}}(0) = \mathsf{x}_0$$
$$|\mathsf{x}(T_f)| = |\mathsf{x}_{\mathcal{P}}(T_f; u_{\mathcal{P}}) - \mathsf{x}_{\mathcal{T}}(T_f; -w - \Delta w)| \le \varepsilon$$

Therefore, a navigation law u that solves Problem 2 is also a pursuit law $u_{\mathcal{P}}$ that solves Problem 3, and vice versa. This correspondence between Problem 2 and Problem 3 is an illustration of the duality between the navigation problem and the problem of pursuit of a maneuvering target, in the special case when both the motions of the pursuer and the target are described by single integrator kinematics, and, in addition, their strategies are functions of their relative positions with respect to each other. By making use of this duality between the navigation and the pursuit problems, navigation laws that are dual to well-known pursuit strategies are proposed in the next section. Furthermore, the equivalence of some intuitive solutions to the navigation problem with standard pursuit strategies is established.

III. Navigation with Perfect Local Drift Information

First, a class of feedback laws solving Problem 2, when the agent has perfect knowledge of the local drift, that is, when $\Delta w(t, \mathsf{x}) \equiv 0$, is considered. Before proceeding with the presentation of this class of navigation laws, a few geometric concepts that shall be extensively used throughout this paper are introduced. In particular, it is assumed that a moving frame $(\mathsf{e}_{\mathsf{x}}^1, \mathsf{e}_{\mathsf{x}}^2)$ is attached to the current position of the agent x, where $\mathsf{e}_{\mathsf{x}}^1 := -\mathsf{x}/|\mathsf{x}|$, and $\mathsf{e}_{\mathsf{x}}^2 := \mathbf{S}\mathsf{e}_{\mathsf{x}}^1$, for all $\mathsf{x} \in \mathbb{R}^2 \backslash \{0\}$, and where

$$\mathbf{S} := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Note that $\mathsf{e}_{\mathsf{x}}^1$ is the unit vector parallel to the direction toward its destination (origin) as observed by the agent, whereas $\mathsf{e}_{\mathsf{x}}^2$ is the unit vector perpendicular to $\mathsf{e}_{\mathsf{x}}^1$. The ray emanating from the agent's current position parallel to $\mathsf{e}_{\mathsf{x}}^1$ is henceforth referred to as the line of sight (LOS), and it will be denoted by $\ell_{\mathrm{LS}}(\mathsf{x}) := \{\mathsf{z} \in \mathbb{R}^2 \colon \mathsf{z} = \rho\mathsf{x}, \rho \in [0, 1]\}$. After some algebraic manipulations, one can show that

$$\dot{\mathsf{e}}_{\mathsf{x}}^1 = -\frac{\langle \dot{\mathsf{x}}, \mathsf{e}_{\mathsf{x}}^2 \rangle}{|\mathsf{x}|} \mathsf{e}_{\mathsf{x}}^2, \qquad \dot{\mathsf{e}}_{\mathsf{x}}^2 = \frac{\langle \dot{\mathsf{x}}, \mathsf{e}_{\mathsf{x}}^2 \rangle}{|\mathsf{x}|} \mathsf{e}_{\mathsf{x}}^1 \tag{11}$$

Furthermore, let λ denote the angle of the LOS measured with respect to some fixed reference direction, as illustrated in Fig. 1. It follows readily from Eq. (11) that the rate of change of λ is given by

$$\dot{\lambda}(\mathsf{x}) = -\frac{\langle \dot{\mathsf{x}}, \mathsf{e}_{\mathsf{x}}^2 \rangle}{|\mathsf{x}|} \tag{12}$$

The following identity will be useful in the subsequent discussion,

$$2|\mathsf{x}| \frac{\mathrm{d}}{\mathrm{d}t} |\mathsf{x}| = \frac{\mathrm{d}}{\mathrm{d}t} |\mathsf{x}|^2 = \frac{\mathrm{d}}{\mathrm{d}t} \langle \mathsf{x}, \mathsf{x} \rangle = 2 \langle \dot{\mathsf{x}}, \mathsf{x} \rangle \tag{13}$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t} |\mathsf{x}| = \frac{\langle \dot{\mathsf{x}}, \mathsf{x} \rangle}{|\mathsf{x}|} = -\langle \dot{\mathsf{x}}, \mathsf{e}_{\mathsf{x}}^1 \rangle, \quad \text{for all } \mathsf{x} \in \mathbb{R}^2 \backslash \{0\} \tag{14}$$

A. Line-of-Sight Feedback Navigation Laws

In this section, a class of feedback navigation laws that steer the agent to its destination, such that the agent remains at all times on $\ell_{\mathrm{LS}}(\mathsf{x}_0)$ is presented. In particular, two different navigation laws, which constraint the agent to travel along the LOS by canceling the component of the drift perpendicular to $\mathsf{e}_{\mathsf{x}_0}^1$, are considered.

The first navigation law steers the agent toward its destination while the latter maintains, at all times, maximum forward speed $\bar{u}$ as it travels along $\ell_{\mathrm{LS}}(\mathsf{x}_0)$. The situation is illustrated in Fig. 2a. This navigation law will be henceforth referred to as the optimal line-of-sight (OLOS) navigation, since among all navigation laws that steer the agent along the original LOS, it is the one that pointwise maximizes the speed along the ensuing path. The analytic expression of this feedback law is given by

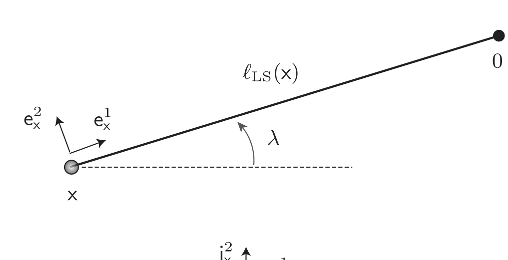
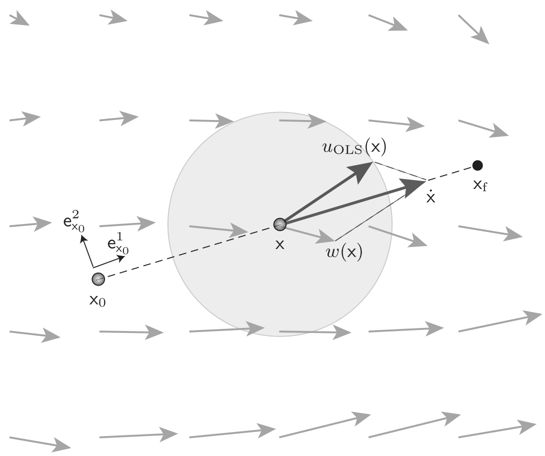
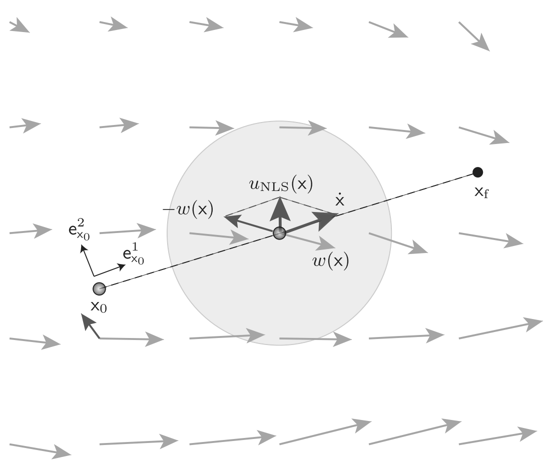


Fig. 1 Global and local frames of reference.



a) Optimal LOS navigation



b) LOS navigation with drift neutralization

Fig. 2 Motion along the LOS direction is achieved when the agent's forward velocity can cancel the drift component perpendicular to the LOS direction.

$$u_{\mathrm{OLS}}(\mathsf{x}) = u_{\mathrm{OLS},1}(\mathsf{x})\mathsf{e}^1_{\mathsf{x}_0} + u_{\mathrm{OLS},2}(\mathsf{x})\mathsf{e}^2_{\mathsf{x}_0}$$
$$u_{\mathrm{OLS},1}(\mathsf{x}) := \sqrt{\bar{u}^2 - \langle w(\mathsf{x}), \mathsf{e}^2_{\mathsf{x}_0}\rangle^2}$$
$$u_{\mathrm{OLS},2}(\mathsf{x}) := -\langle w(\mathsf{x}), \mathsf{e}^2_{\mathsf{x}_0}\rangle \tag{15}$$

The following proposition provides sufficient conditions for the feasibility of the navigation law (15).

Proposition 1.Let $\varepsilon > 0$ and $\Delta w(t, \mathsf{x}) \equiv 0$. Then, for all $|\mathsf{x}_0| > \varepsilon$, the navigation law (15) will drive the system (1) to the set $\{\mathsf{x}\colon |\mathsf{x}| \leq \varepsilon\}$ in finite time, provided there exist $\bar{w}_1 > 0$ and $\bar{w}_2 > 0$, such that

$$|\langle w(\mathsf{x}), \mathsf{e}^1_{\mathsf{x}_0}\rangle| \leq \bar{w}_1 < \sqrt{\bar{u}^2 - \bar{w}_2^2} \tag{16}$$

$$|\langle w(\mathsf{x}), \mathsf{e}^2_{\mathsf{x}_0}\rangle| \leq \bar{w}_2 < \bar{u} \tag{17}$$

for all $\mathsf{x} \in \ell_{\mathrm{LS}}(\mathsf{x}_0)$. Finally, the time of travel satisfies the upper bound

$$T_f \leq \frac{|\mathsf{x}_0| - \varepsilon}{\sqrt{\bar{u}^2 - \bar{w}_2^2} - \bar{w}_1} < \infty \tag{18}$$

Proof. Note that Eq. (17) guarantees that $u_{\mathrm{OLS},1}(\mathsf{x})$ is well defined along $\ell_{\mathrm{LS}}(\mathsf{x}_0)$. Furthermore, in view of Eq. (17), it follows that

$$u_{\mathrm{OLS},1}(\mathsf{x}) = \sqrt{\bar{u}^2 - \langle w(\mathsf{x}), \mathsf{e}^2_{\mathsf{x}_0}\rangle} \geq \sqrt{\bar{u}^2 - \bar{w}_2^2} \tag{19}$$

In addition, it follows readily, after plugging Eq. (15) into Eq. (1), and in light of Eqs. (16) and (19), that

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathsf{x}| = -\langle \dot{\mathsf{x}}, \mathsf{e}^1_{\mathsf{x}_0}\rangle = -\langle u_{\mathrm{OLS}}(\mathsf{x}) + w(\mathsf{x}), \mathsf{e}^1_{\mathsf{x}_0}\rangle$$
$$= -u_{\mathrm{OLS},1}(\mathsf{x}) - \langle w(\mathsf{x}), \mathsf{e}^1_{\mathsf{x}_0}\rangle \leq -\sqrt{\bar{u}^2 - \bar{w}_2^2} + \bar{w}_1 \tag{20}$$

Note that Eq. (16) implies that the right-hand-side of Eq. (20) is strictly negative, and thus, the navigation law (15) will drive the system (1) to the set $\{\mathsf{x}\colon |\mathsf{x}| \leq \varepsilon\}$ in finite time, for all $|\mathsf{x}_0| > \varepsilon$. Furthermore, Eq. (18) follows after integrating both sides of Eq. (20). □

Note that Proposition 1 implies that the navigation law (15) solves Problem 2, provided the drift component perpendicular to $\mathsf{e}^1_{\mathsf{x}_0}$ can be canceled by the agent's control actions, and furthermore, the projection of the drift on $-\mathsf{e}^1_{\mathsf{x}_0}$ (opposite of the LOS direction) never dominates the forward speed of the agent. The reader should notice here that conditions (16) and (17) may hold even if $|w(\mathsf{x})| > \bar{u}$, for some $\mathsf{x} \in \ell_{\mathrm{LS}}(\mathsf{x}_0)$. Thus, the standard assumption, which is typically made in problems of pursuit of a maneuvering target, where the pursuer is assumed to have a speed advantage over the target, has been relaxed. Note that if the target is faster than the pursuer, then the former can always escape capture by simply traveling along the original LOS direction with its maximum speed. In the problem of navigation, the assumptions for the feasibility of the navigation law (15) can be relaxed given that the notional maneuvering target, whose velocity is $-w(\mathsf{x})$, may not necessarily act as an adversarial, non-cooperative opponent, in contrast to the classical pursuit problem.

Next, a second navigation law that will enforce motion of the agent along $\ell_{\mathrm{LS}}(\mathsf{x}_0)$ is introduced.

The expression of this control law is given by

$$u_{\mathrm{NLS}}(\mathsf{x}) := u_{\mathrm{NLS},1}(\mathsf{x})\mathsf{e}^1_{\mathsf{x}_0} + u_{\mathrm{NLS},2}(\mathsf{x})\mathsf{e}^2_{\mathsf{x}_0}$$
$$u_{\mathrm{NLS},1}(\mathsf{x}) := \bar{u} - |w(\mathsf{x})| - \langle w(\mathsf{x}), \mathsf{e}^1_{\mathsf{x}_0}\rangle$$
$$u_{\mathrm{NLS},2}(\mathsf{x}) := -\langle w(\mathsf{x}), \mathsf{e}^2_{\mathsf{x}_0}\rangle \tag{21}$$

The interpretation of navigation law (21) is as follows: The agent first completely "cancels" the effect of the drift, and subsequently allocates the remaining control authority along the original LOS. The navigation law (21) may be particularly useful during the last phase of the navigation process and, in particular, as the agent approaches its final destination. Note that the navigation law (21) can also be written as follows

$$u_{\mathrm{NLS}}(\mathsf{x}) = -w(\mathsf{x}) + (\bar{u} - |w(\mathsf{x})|)\mathsf{e}^1_{\mathsf{x}_0} \tag{22}$$

The situation is illustrated in Fig. 2b. One important observation here is that $|u_{\mathrm{NLS}}(\mathsf{x})| \not\equiv \bar{u}$, for all $\mathsf{x} \in \ell_{\mathrm{LS}}(\mathsf{x}_0)$, that is, the agent may not necessarily maintain maximum forward speed along its ensuing path. This may be useful when the agent is approaching a landing/docking point (rendezvous problem), where a "smooth" final approach is more important than a fast one. Note, furthermore, that $|u_{\mathrm{NLS}}(\mathsf{x})| = \bar{u}$ only if $w(\mathsf{x}) = -|w(\mathsf{x})|\mathsf{e}^1_{\mathsf{x}_0}$, in which case, the navigation laws (15) and (21) turn out to be exactly the same.

The following proposition provides a sufficient condition for the feasibility of the navigation law (21).

Proposition 2. Let $\varepsilon > 0$ and $\Delta w(t, \mathsf{x}) \equiv 0$. Then, for all $|\mathsf{x}_0| > \varepsilon$, the navigation law (21) will drive the system (1) to the set $\{\mathsf{x}\colon |\mathsf{x}| \leq \varepsilon\}$ in finite time, provided there exists $\bar{w} > 0$, such that

$$|w(\mathsf{x})| \leq \bar{w} < \bar{u}, \qquad \text{for all } \mathsf{x} \in \ell_{\mathrm{LS}}(\mathsf{x}_0) \tag{23}$$

Finally, the time of travel satisfies the upper bound

$$T_f \leq \frac{|\mathsf{x}_0| - \varepsilon}{\bar{u} - \bar{w}} < \infty \tag{24}$$

Proof. Note that Eq. (23) implies that the component of the drift $w(\mathsf{x})$ can be canceled by the agent's forward velocity. In addition, by plugging Eq. (21) into Eq. (1), and by virtue of Eq. (23), it follows readily that

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathsf{x}| = -\langle \dot{\mathsf{x}}, \mathsf{e}^{1}_{\mathsf{x}_0}\rangle = -\langle u_{\mathrm{NLS}}(\mathsf{x}) + w(\mathsf{x}), \mathsf{e}^{1}_{\mathsf{x}_0}\rangle$$
$$= -\langle(\bar{u} - |w(\mathsf{x})|)\mathsf{e}^{1}_{\mathsf{x}_0}, \mathsf{e}^{1}_{\mathsf{x}_0}\rangle \le -(\bar{u} - \bar{w}) \tag{25}$$

The rest of the proof is similar to the proof of Proposition 1, and it is thus omitted. □

One of the main drawbacks of the feedback law (21), compared with Eq. (15), is that for its application it is necessary that the control authority of the agent always dominates the drift as the agent moves along the original LOS. Note that Eq. (23) is more restrictive than conditions (16) and (17). Another restriction of the navigation law (21) has to do with the fact that, as it has already been mentioned, when the agent is driven by this law, it may not maintain constant forward speed along its ensuing path. This may be an undesirable situation for several applications, say, fixed-wing unmanned aerial vehicles, where the forward speed of the aircraft must remain, at all times, above stall speed. On the other hand, as it has already been mentioned, the navigation law (21) may be more practical than Eq. (15), when, for example, a smooth final approach is more preferable than a quick one.

It is interesting to note that the control law (21) corresponds to a pursuit strategy known as “pursuit with neutralization” [15]. With this strategy, the pursuer first neutralizes the action of its opponent (maneuvering target) and, subsequently, uses the remaining control authority (provided the pursuer has a speed advantage over its opponent) to diminish their relative distance.

B. Three-Point Navigation and LOS Guidance

Next, a navigation scheme that, in contrast to the navigation laws (15) and (21), does not require the forward velocity of the agent to dominate the component of the drift perpendicular to the LOS direction is presented. The proposed navigation is derived from a well-known pursuit strategy, namely, the LOS or three-point guidance law [18]. It turns out that this pursuit strategy enforces the geometric constraint of motion camouflage with respect to a fixed point [22,23], which stipulates, in turn, that the position vector of the pursuer with respect to the reference point x_0 is, at all times, parallel to the position vector of the target with respect to the pursuer. Equivalently, the pursuer always lies on the line segment defined by the target’s current position and the reference point x_0. It is worth-mentioning that the term “motion camouflage” was first coined by Srinivasan and Davey to describe an effective deception strategy adopted by various animal and insect species, where a pursuer (the shadower) conceals its apparent motion from an evader (the shadowee) by emulating the optical flow produced by a stationary point [22]. By eliminating any motion parallax, the pursuer’s motion reduces the ability of the evader to accurately obtain depth information regarding its actual relative distance from the pursuer [22]. Depending on whether the distance of the fixed reference point from the pursuer is finite or infinite, one refers to “motion camouflage with respect to a fixed point” and to “motion camouflage with respect to a point at infinity,”, respectively. While in the former case the pursuer’s strategy is to match the angular velocity of its motion with that of the target, in the latter, the pursuer’s line of sight has a fixed direction in space.

Note that the LOS guidance law is a pursuit strategy that entails two LOS directions, namely, the direction from x_0 to $\mathsf{x}_{\mathcal{P}}$, and the direction from $\mathsf{x}_{\mathcal{P}}$ to $\mathsf{x}_{\mathcal{T}}$. Alternatively, the same pursuit strategy involves three points of interest, namely, x_0, $\mathsf{x}_{\mathcal{P}}$ and $\mathsf{x}_{\mathcal{T}}$, which must remain collinear at all times. The situation is illustrated in Fig. 3.

In this section, the applicability of the LOS guidance law to the navigation problem, when the drift field is only partially known, is examined. To this end, let $\lambda_{\mathcal{P}}$ and $\lambda_{\mathcal{T}}$ denote, respectively, the angular positions of the pursuer and the target from x_0 with respect to some fixed reference direction, at time t. With the aid of Fig. 3, one can observe that the motion camouflage condition implies that $\dot{\lambda}_{\mathcal{P}} = \dot{\lambda}_{\mathcal{T}}$.

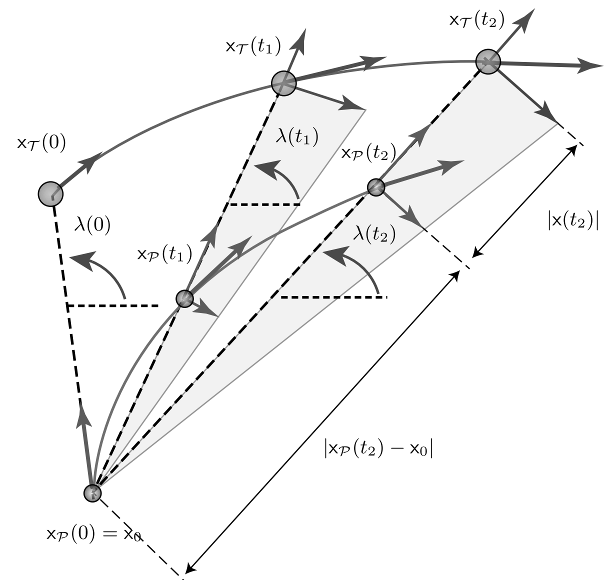


Fig. 3 LOS or three-point guidance is synonymous to motion camouflage with respect to a fixed point.

Thus, the components of the velocity of both the target and the pursuer perpendicular to $\mathsf{e}^{1}_{\mathsf{x}}$ (or $\mathsf{e}^{1}_{\mathsf{x}_{\mathcal{P}}}$) satisfy

$$\frac{\langle u_{\mathcal{P}}, \mathsf{e}^{2}_{\mathsf{x}}\rangle}{|\mathsf{x}_{\mathcal{P}} - \mathsf{x}_0|} = -\frac{\langle w(\mathsf{x}_{\mathcal{P}} - \mathsf{x}_{\mathcal{T}}), \mathsf{e}^{2}_{\mathsf{x}}\rangle}{|\mathsf{x}_{\mathcal{T}} - \mathsf{x}_0|} = -\frac{\langle w(\mathsf{x}), \mathsf{e}^{2}_{\mathsf{x}}\rangle}{|\mathsf{x}_{\mathcal{T}} - \mathsf{x}_0|}$$
$$= -\frac{\langle w(\mathsf{x}), \mathsf{e}^{2}_{\mathsf{x}}\rangle}{|\mathsf{x}_{\mathcal{P}} - \mathsf{x}_0| + |\mathsf{x}|} \tag{26}$$

in light of the identity

$$|\mathsf{x}_{\mathcal{T}} - \mathsf{x}_0| = |\mathsf{x}_{\mathcal{T}} - \mathsf{x}_{\mathcal{P}}| + |\mathsf{x}_{\mathcal{P}} - \mathsf{x}_0| = |\mathsf{x}| + |\mathsf{x}_{\mathcal{P}} - \mathsf{x}_0| \tag{27}$$

which follows, in turn, from the collinearity of x_0, $\mathsf{x}_{\mathcal{P}}$, and $\mathsf{x}_{\mathcal{T}}$. Therefore,

$$\langle u_{\mathcal{P}}, \mathsf{e}^{2}_{\mathsf{x}}\rangle = -\frac{|\mathsf{x}_{\mathcal{P}} - \mathsf{x}_0|}{|\mathsf{x}_{\mathcal{P}} - \mathsf{x}_0| + |\mathsf{x}|}\langle w(\mathsf{x}), \mathsf{e}^{2}_{\mathsf{x}}\rangle \tag{28}$$

and the expression of the pursuit strategy $u_{\mathcal{P}}$ for LOS guidance is given by

$$u_{\mathcal{P}}(\mathsf{x}, \mathsf{x}_{\mathcal{P}}) := u_{\mathcal{P},1}(\mathsf{x}, \mathsf{x}_{\mathcal{P}})\mathsf{e}^{1}_{\mathsf{x}} + u_{\mathcal{P},2}(\mathsf{x}, \mathsf{x}_{\mathcal{P}})\mathsf{e}^{2}_{\mathsf{x}}$$
$$u_{\mathcal{P},1}(\mathsf{x}, \mathsf{x}_{\mathcal{P}}) := \sqrt{\bar{u}^2 - u^{2}_{\mathcal{P},2}(\mathsf{x}, \mathsf{x}_{\mathcal{P}})}$$
$$u_{\mathcal{P},2}(\mathsf{x}, \mathsf{x}_{\mathcal{P}}) := -\frac{|\mathsf{x}_{\mathcal{P}} - \mathsf{x}_0|}{|\mathsf{x}_{\mathcal{P}} - \mathsf{x}_0| + |\mathsf{x}|}\langle w(\mathsf{x}), \mathsf{e}^{2}_{\mathsf{x}}\rangle \tag{29}$$

Note that the pursuit strategy (29) depends explicitly on both x and $\mathsf{x}_{\mathcal{P}}$. Therefore, the control law (29) cannot be used directly as a navigation law for the system (1), since it depends on $\mathsf{x}_{\mathcal{P}}$, in addition to the current location of the agent x. Before applying the control law (29) to the navigation problem, the kinematic model described by Eq. (1) needs to be dynamically extended to the following kinematic model:

$$\dot{\mathsf{x}} = u_{\mathrm{TPN}}(\mathsf{x}, \mathsf{x}_{\mathcal{P}}) + w(\mathsf{x}), \qquad \mathsf{x}(0) = \mathsf{x}_0 \tag{30}$$

$$\dot{\mathsf{x}}_{\mathcal{P}} = u_{\mathrm{TPN}}(\mathsf{x}, \mathsf{x}_{\mathcal{P}}), \qquad \mathsf{x}_{\mathcal{P}}(0) = \mathsf{x}_0 \tag{31}$$

where $u_{\mathrm{TPN}}(\mathsf{x}, \mathsf{x}_{\mathcal{P}}) := u_{\mathcal{P}}(\mathsf{x}, \mathsf{x}_{\mathcal{P}})$. The control law u_{TPN} is henceforth referred to as the three-point navigation law.

One noteworthy observation for the three-point navigation law u_{TPN} is that the component of $u_{\mathcal{P}}$ perpendicular to the LOS direction never dominates the component of the drift along the same direction, as it follows readily from Eq. (29). Consequently, the agent driven by

Eq. (29) does not travel along the LOS. This fact may incur some loss of performance, in terms of minimizing the arrival time, when compared with Eq. (15) (see also the discussion in Sec. VI regarding the local optimality of Eq. (15)). On the other hand, the applicability of the control law (29) may not be limited to navigation problems where the control authority of the agent can cancel the term $\langle w(\mathsf{x}), \mathsf{e}_{\mathsf{x}}^2\rangle$. This is in contrast to the navigation laws (15) and (21), which cannot guarantee convergence of the agent to its destination in the case when the component of the drift along $\mathsf{e}_{\mathsf{x}_0}^2$ is stronger than the control authority of the agent. Another advantage of the navigation law (29), compared with Eqs. (15) and (21), is its robustness in the presence of unknown drift. This is demonstrated in Sec. IV.B.

Proposition 3. Let $\varepsilon > 0$ and $\Delta w(t, \mathsf{x}) \equiv 0$. Then, for all $|\mathsf{x}_0| > \varepsilon$, the navigation law (29) will drive the system (1) to the set $\{\mathsf{x}\colon |\mathsf{x}| \leq \varepsilon\}$ in finite time, provided there exist $\bar{w}_1 > 0$ and $\bar{w}_2 > 0$, such that

$$|\langle w(\mathsf{x}), \mathsf{e}_{\mathsf{x}}^1\rangle| \leq \bar{w}_1 < \sqrt{\bar{u}^2 - \bar{w}_2^2} \tag{32}$$

$$|\langle w(\mathsf{x}), \mathsf{e}_{\mathsf{x}}^2\rangle| \leq \bar{w}_2 \tag{33}$$

for all $\mathsf{x} \in \mathbb{R}^2\backslash\{0\}$. Furthermore, the time of travel satisfies the upper bound

$$T_f \leq \frac{|\mathsf{x}_0| - \varepsilon}{\sqrt{\bar{u}^2 - \bar{w}_2^2} - \bar{w}_1} < \infty \tag{34}$$

It follows from Eq. (33) that

$$|u_{\mathcal{P},2}(\mathsf{x}, \mathsf{x}_{\mathcal{P}})| = \frac{|\mathsf{x}_{\mathcal{P}} - \mathsf{x}_0|}{|\mathsf{x}_{\mathcal{P}} - \mathsf{x}_0| + |\mathsf{x}|} |\langle w(\mathsf{x}), \mathsf{e}_{\mathsf{x}}^2\rangle| \leq |\langle w_{\mathcal{T}}(\mathsf{x}), \mathsf{e}_{\mathsf{x}}^2\rangle| \leq \bar{w}_2 \tag{35}$$

which implies, in turn, that

$$u_{\mathcal{P},1}(\mathsf{x}, \mathsf{x}_{\mathcal{P}}) = \sqrt{\bar{u}^2 - u_{\mathcal{P},2}^2(\mathsf{x}, \mathsf{x}_{\mathcal{P}})} \geq \sqrt{\bar{u}^2 - \bar{w}_2^2}$$

for all $(\mathsf{x}, \mathsf{x}_{\mathcal{P}}) \in \mathbb{R}^4\backslash\{0, \mathsf{x}_0\}$. Furthermore, it follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\mathsf{x}| &= -\langle \dot{\mathsf{x}}, \mathsf{e}_{\mathsf{x}}^1\rangle = -\langle u_{\mathrm{TPN}}(\mathsf{x}) + w(\mathsf{x}), \mathsf{e}_{\mathsf{x}}^1\rangle \\ &= -u_{\mathcal{P},1}(\mathsf{x}_{\mathcal{P}}, \mathsf{x}) - \langle w(\mathsf{x}), \mathsf{e}_{\mathsf{x}}^1\rangle \leq -\sqrt{\bar{u}^2 - \bar{w}_2^2} + \bar{w}_1\end{aligned} \tag{36}$$

The rest of the proof follows similarly to the proof of Proposition 1, and thus it is omitted. □

C. Navigation with Local Drift Information and Pursuit with Motion Camouflage

A common theme in both the navigation laws (15) and (21) is that when the agent is driven by either of these two control laws, its direction of motion is constant and parallel to $\mathsf{e}_{\mathsf{x}_0}^1$ (the original LOS direction). The interpretation of the previous observation, within the context of the problem of pursuit of a maneuvering target, is that the relative position vector of the pursuer from the target remains, at all times, parallel to a constant vector, namely, $\mathsf{e}_{\mathsf{x}_0}^1$. Equivalently, the relative angular position of the target from the pursuer, and vice versa, is constant. Therefore, both the pursuit strategies $u_{\mathcal{P}}(\mathsf{x}) = u_{\mathrm{OLS}}(\mathsf{x})$ and $u_{\mathcal{P}}(\mathsf{x}) = u_{\mathrm{NLS}}(\mathsf{x})$ satisfy the so-called requirement for *motion camouflage* with respect to a point *at infinity* [22], also known in the field of missile guidance as the condition for *parallel guidance/navigation* [18]. Note that motion camouflage with respect to a point at infinity results in a navigation strategy where the original LOS direction remains always fixed, as illustrated in Fig. 4.

Another way to reach the same conclusion is by showing that when the pursuer is driven by either the control law (15) or Eq. (21), the LOS angle λ remains constant during the course of the pursuit. In particular, in light of Eq. (12),

$$\dot{\lambda} = -\frac{\langle \dot{\mathsf{x}}_{\mathcal{P}} - \dot{\mathsf{x}}_{\mathcal{T}}, \mathsf{e}_{\mathsf{x}}^2\rangle}{|\mathsf{x}_{\mathcal{P}} - \mathsf{x}_{\mathcal{T}}|} = -\frac{\langle \dot{\mathsf{x}}, \mathbf{S}\mathsf{e}_{\mathsf{x}}^1\rangle}{|\mathsf{x}|} \tag{37}$$

It is easy to show that when the pursuer is driven by either the control law (15) or Eq. (21), the vector $\dot{\mathsf{x}} = \dot{\mathsf{x}}_{\mathcal{P}} - \dot{\mathsf{x}}_{\mathcal{T}}$ remains parallel to $\mathsf{e}_{\mathsf{x}}^1 \equiv \mathsf{e}_{\mathsf{x}_0}^1$. Consequently, the inner product in the numerator of Eq. (37) is zero, given that $\mathbf{S}$ is a skew symmetric matrix, and thus λ is constant at all times. Note that when the agent is steered by either the LOS navigation law (15) or Eq. (21), it will remain on the original LOS during its course to its destination, and thus the points x, x_0, and the origin $\mathsf{x} = 0$ will always be collinear. Thus, both of the navigation laws (15) and (21) satisfy the condition for motion camouflage with respect to a *fixed point*, namely, x_0, rather than the condition for motion camouflage with respect to a *point at infinity*, which is satisfied, when Eq. (15) or Eq. (21) are used as pursuit strategies.

The three-point navigation law is derived directly from the pursuit strategy (29), which satisfies, by construction, the geometric condition for motion camouflage with respect to a fixed point, namely, x_0. Note that the geometric condition for motion camouflage with respect to neither a fixed point (that is, collinearity of x_0, x and the origin) nor a point at infinity (that is, $\dot{\lambda} = 0$) are necessarily satisfied when the control (29) is used as a navigation law.

IV. Navigation with Imperfect Information

In this section, feedback navigation laws for the case when the information about the local drift field available to the agent is imperfect are presented. The proposed navigation laws are derived from the control laws presented in Sec. III, after the necessary modifications reflecting the lack of complete knowledge of the drift field have been carried out. Specifically, note that the control laws (15) and (21) depend on the initial LOS direction $\mathsf{e}_{\mathsf{x}0}^1$ and its normal direction $\mathsf{e}_{\mathsf{x}0}^2$, and both of them remain constant throughout. By updating the initial LOS direction with the most current LOS direction $\mathsf{e}_{\mathsf{x}}^1$ and its corresponding normal direction by $\mathsf{e}_{\mathsf{x}}^2$, the control law can use the most up-to-date information of its relative position to its destination. In other words, the drift components along the current LOS direction and its perpendicular entail enough information about the prevailing wind/current field so that the controller can compensate its effect on the ensuing path of the agent.

A. Robust LOS Navigation Laws with Imperfect Local Information of the Drift

One important remark from the discussion in Sec. III is that for the implementation of both the navigation laws (15) and (21), the agent must have perfect knowledge of the local drift at every instant of time. If the local drift is not known perfectly, however, that is, $\Delta w(t, \mathsf{x}) \not\equiv 0$, then the navigation laws (15) and (21) will not successfully cancel the component of the drift perpendicular to the LOS direction. Consequently, the agent may fail to reach its destination. To alleviate this deficiency, two variations of the navigation laws (15) and (21), which are robust to model uncertainties induced by the incomplete knowledge of the local drift field, are introduced.

The adopted approach is based on the observation that, in contrast to the pursuit problem, where motion camouflage is often used to introduce the element of deception, the enforcement of the geometric condition for motion camouflage in the navigation problem has no apparent practical value. Therefore, one can relax the motion camouflage requirement and consider instead the following modification of the navigation law (15):

$$\begin{aligned} u_{\mathrm{OLS}}^{\star}(\mathsf{x}) &= u_{\mathrm{OLS},1}^{\star}(\mathsf{x})\mathsf{e}_{\mathsf{x}}^1 + u_{\mathrm{OLS},2}^{\star}(\mathsf{x})\mathsf{e}_{\mathsf{x}}^2 \\ u_{\mathrm{OLS},1}^{\star}(\mathsf{x}) &= \sqrt{\bar{u}^2 - \langle w(\mathsf{x}), \mathsf{e}_{\mathsf{x}}^2\rangle^2} \\ u_{\mathrm{OLS},2}^{\star}(\mathsf{x}) &= -\langle w(\mathsf{x}), \mathsf{e}_{\mathsf{x}}^2\rangle \end{aligned} \tag{38}$$

Note that the navigation laws (15) and (38) are almost identical modulo the replacement of $\mathsf{e}_{\mathsf{x}_0}^1$ and $\mathsf{e}_{\mathsf{x}_0}^2$ by $\mathsf{e}_{\mathsf{x}}^1$ and $\mathsf{e}_{\mathsf{x}}^2$, respectively, which is induced, in turn, by the relaxation of the geometric constraint of motion camouflage. As shown below, the navigation

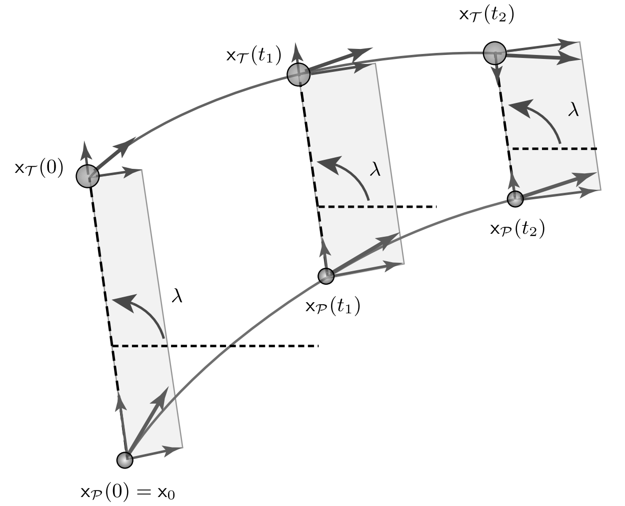


Fig. 4 Motion camouflage with respect to a point at infinity is synonymous to parallel guidance/navigation, where the LOS angle between the pursuer and the target, when measured with respect to some fixed reference direction, remains constant at all times.

law (38) is more robust than the original navigation law (15) in the presence of unknown drift. The performance of Eq. (38) in terms of minimizing the arrival time, however, is still compromised by the existence of an unknown drift. In particular, in the presence of a nonzero component of the unknown drift along e^2_{x}, the inertial velocity of the agent will not point toward the agent's destination, as is the case with the navigation law (15), when the local drift is perfectly known. Therefore, there exists an offset error between the direction of the inertial velocity of the agent and the current LOS, which incurs some loss of performance (see also the discussion in Sec. VI regarding the interpretation of the LOS as the direction that maximizes the rate of decrease of the distance of the agent from its destination). The situation is illustrated in Fig. 5.

The following proposition furnishes sufficient conditions for the feasibility of the navigation law (38).

Proposition 4. Let $\varepsilon > 0$. Then, for all $|\mathsf{x}_0| > \varepsilon$, the navigation law (38) will drive the system (1) to the set $\{\mathsf{x} \in \mathbb{R}^2\colon |\mathsf{x}| \le \varepsilon\}$ in finite time, provided there exist $\bar{w}_1 > 0$ and $\bar{w}_2 > 0$, such that

$$|\langle w(\mathsf{x}) + \Delta w(t, \mathsf{x}), e^1_{\mathsf{x}} \rangle| \le \bar{w}_1 < \sqrt{\bar{u}^2 - \bar{w}_2^2} \tag{39}$$

$$|\langle w(\mathsf{x}), e^2_{\mathsf{x}} \rangle| \le \bar{w}_2 < \bar{u} \tag{40}$$

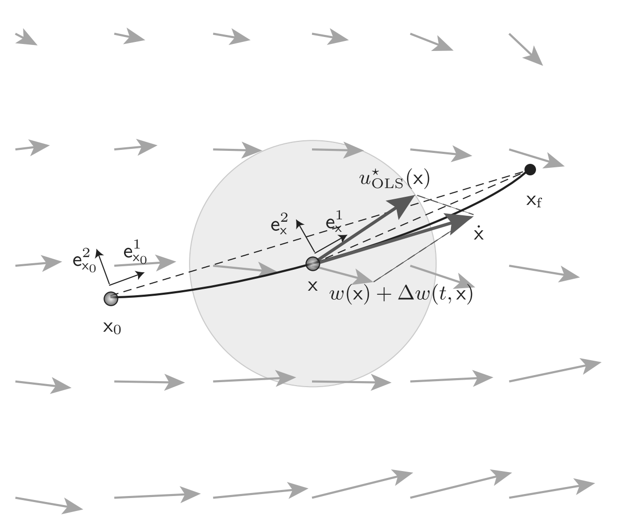


Fig. 5 Robust LOS navigation. The direction of motion of the agent does not always align with the current LOS owing to the presence of the unknown drift component Δw.

for all $\mathsf{x} \in \mathbb{R}^2 \backslash \{0\}$. Furthermore, the arrival time satisfies the upper bound

$$T_f \le \frac{|\mathsf{x}_0| - \varepsilon}{\sqrt{\bar{u}^2 - \bar{w}_2^2} - \bar{w}_1} < \infty \tag{41}$$

Proof. The proof follows similarly to the proof of Proposition 1, and thus it is omitted. □

Similarly, one can consider a variation of the navigation law (21), whose expression is given by

$$\begin{aligned} u^{\star}_{\mathrm{NLS}}(\mathsf{x}) &:= u^{\star}_{\mathrm{NLS},1}(\mathsf{x}) e^1_{\mathsf{x}} + u^{\star}_{\mathrm{NLS},2}(\mathsf{x}) e^2_{\mathsf{x}} \\ u^{\star}_{\mathrm{NLS},1}(\mathsf{x}) &:= \bar{u} - |w(\mathsf{x})| - \langle w(\mathsf{x}), e^1_{\mathsf{x}} \rangle \\ u^{\star}_{\mathrm{NLS},2}(\mathsf{x}) &:= -\langle w(\mathsf{x}), e^2_{\mathsf{x}} \rangle \end{aligned} \tag{42}$$

The following proposition presents sufficient conditions for the feasibility of the navigation law (42).

Proposition 5. Let $\varepsilon > 0$. Then, for all $|\mathsf{x}_0| > \varepsilon$, the navigation law (42) will drive the system (1) to the set $\{\mathsf{x} \in \mathbb{R}^2\colon |\mathsf{x}| \le \varepsilon\}$ in finite time provided there exist $\bar{w} > 0$ and $\Delta\bar{w}_1 > 0$, such that

$$|w(\mathsf{x})| \le \bar{w} < \bar{u} \tag{43}$$

$$|\langle \Delta w(t, \mathsf{x}), e^1_{\mathsf{x}} \rangle| \le \Delta\bar{w}_1 < \bar{u} - \bar{w} \tag{44}$$

for all $t \ge 0$ and $\mathsf{x} \in \mathbb{R}^2 \backslash \{0\}$. Furthermore, the time T_f satisfies the upper bound

$$T_f \le \frac{|\mathsf{x}_0| - \varepsilon}{\bar{u} - \bar{w} - \Delta\bar{w}_1} < \infty \tag{45}$$

Proof. The proof follows similarly to the proof of Proposition 2, and thus it is omitted. □

If one uses the navigation laws (38) or (42) as pursuit strategies for Problem 3, then the condition for motion camouflage with respect to a point at infinity will not be satisfied. This comes as a consequence of the fact that any discrepancies between the actual and the known drift would result in a nonzero $\dot{\lambda}$, in general. In particular, it can easily be shown that

$$\dot{\lambda} = -\frac{\langle \Delta w(t, \mathsf{x}), e^2_{\mathsf{x}} \rangle}{|\mathsf{x}|} \tag{46}$$

Since $\dot{\lambda}$ is not zero for $\Delta w(t, \mathsf{x}) \ne 0$, the constant LOS angle requirement (the condition for motion camouflage with respect to a point at infinity) is not satisfied. Another important observation from Eq. (46) is that as $|\mathsf{x}| \to 0$, $\dot{\lambda}$ grows unbounded, which implies, in turn, that the normal acceleration of the agent along its ensuing path grows unbounded as well; this is an undesirable, from the application point of view, situation. The following proposition furnishes a sufficient condition for $\dot{\lambda}$ to remain bounded at all times.

Proposition 6. Let $\varepsilon > 0$, and let all assumptions of Propositions 4 and 5 hold. Furthermore, assume that there exists $\Delta\bar{w} > 0$, such that

$$|\Delta w(t, \mathsf{x})| \le \Delta\bar{w}, \qquad \text{for all } t \ge 0 \quad \text{and} \quad \mathsf{x} \in \mathbb{R}^2 \tag{47}$$

If $\Delta w(t, \mathsf{x}) = \mathcal{O}(|\mathsf{x}|)$, as $|\mathsf{x}| \to 0$ uniformly for all $t \ge 0$, then $\dot{\lambda}$ remains bounded for all $t \in [0, T_f]$ and for all $|\mathsf{x}_0| > \varepsilon$.

Proof. By hypothesis, there exists $k(\varepsilon) > 0$, such that $|\Delta w(t, \mathsf{x})| \le k(\varepsilon)|\mathsf{x}|$, for all $t \ge 0$ and $|\mathsf{x}| \le \varepsilon$. Furthermore, by virtue of the Cauchy–Schwartz inequality, it follows that

$$\frac{|\langle \Delta w(t, \mathsf{x}), e^2_{\mathsf{x}} \rangle|}{|\mathsf{x}|} \le \frac{|\Delta w(t, \mathsf{x})|}{|\mathsf{x}|} \le \frac{\Delta\bar{w}}{\varepsilon} \quad \text{for all } t \ge 0 \quad \text{and} \quad \mathsf{x} \in \{\mathsf{y} \in \mathbb{R}^2\colon |\mathsf{y}| > \varepsilon\} \tag{48}$$

In light of Eqs. (46) and (48), it follows that

$$|\dot{\lambda}| = \frac{|\langle \Delta w(t,\mathsf{x}), \mathsf{e}_{\mathsf{x}}^2\rangle|}{|\mathsf{x}|} \leq \max\{k(\varepsilon), \Delta\bar{w}/\varepsilon\} < \infty$$
$$\text{for all } t \geq 0 \quad \text{and} \quad \mathsf{x} \in \mathbb{R}^2\backslash\{0\} \tag{49}$$

and thus completing the proof. □

B. Robust Three-Point Navigation

In case the local drift is not perfectly known, that is, when $\Delta w(t,\mathsf{x})\not\equiv 0$, the pursuit strategy (29) will not satisfy the condition for motion camouflage with respect to a fixed point, that is, the points $\mathsf{x}_\mathcal{P}$, $\mathsf{x}_\mathcal{T}$ and x_0 may not be collinear at all times. Since the enforcement of the motion camouflage condition has no apparent practical value for the navigation problem, one can proceed with the design of a navigation law, at the geometric level, by relaxing the motion camouflage constraint. In particular, it is assumed that the condition for motion camouflage is satisfied with respect to a moving point, denoted henceforth by $\mathsf{x}_0^\star(t)$, rather than with respect to the fixed point x_0. This variation of the navigation law (29) is denoted by $u_{\mathrm{TPN}}^\star$.

Let the moving reference point $\mathsf{x}_0^\star(t)$ be defined, for all $t \geq 0$, by the following set of equations:

$$|\mathsf{x}_\mathcal{P}(t) - \mathsf{x}_0^\star(t)| = |\mathsf{x}_\mathcal{P}(t) - \mathsf{x}_0| \tag{50}$$

$$\langle \mathsf{x}_\mathcal{P}(t) - \mathsf{x}_0^\star(t), \mathsf{e}_\mathsf{x}^1\rangle = |\mathsf{x}_\mathcal{P}(t) - \mathsf{x}_0^\star(t)| \tag{51}$$

It follows readily from Eq. (51) that $\mathsf{x}_\mathcal{P}(t)$, $\mathsf{x}_0^\star(t)$, and $\mathsf{x}_\mathcal{T}(t)$ are collinear for all $t \geq 0$, and, furthermore,

$$|\mathsf{x}_\mathcal{T}(t) - \mathsf{x}_0^\star(t)| = |\mathsf{x}_\mathcal{P}(t) - \mathsf{x}_0^\star(t)| + |\mathsf{x}_\mathcal{T}(t) - \mathsf{x}_\mathcal{P}(t)| \tag{52}$$

The situation is illustrated in Fig. 6. It follows readily that, at each time $t \geq 0$, the moving reference point $\mathsf{x}_0^\star(t)$ belongs to the intersection of a circle centered at x_0 with radius $|\mathsf{x}_\mathcal{P}(t) - \mathsf{x}_0|$ with the line defined by $\mathsf{x}_\mathcal{T}(t)$ and $\mathsf{x}_\mathcal{P}(t)$. As it shall be explained later, the exact location of $\mathsf{x}_0^\star(t)$ will not affect the analytic expression of the navigation law. Indeed, in light of Eq. (52), it follows that

$$\frac{\langle u_{\mathrm{TPN}}^\star(\mathsf{x},\mathsf{x}_\mathcal{P}), \mathsf{e}_\mathsf{x}^2\rangle}{|\mathsf{x}_\mathcal{P} - \mathsf{x}_0^\star|} = -\frac{\langle w(\mathsf{x}), \mathsf{e}_\mathsf{x}^2\rangle}{|\mathsf{x}_\mathcal{T} - \mathsf{x}_0^\star|} = -\frac{\langle w(\mathsf{x}), \mathsf{e}_\mathsf{x}^2\rangle}{|\mathsf{x}_\mathcal{P} - \mathsf{x}_0^\star| + |\mathsf{x}|} \tag{53}$$

Finally, since by construction $|\mathsf{x}_\mathcal{P}(t) - \mathsf{x}_0^\star(t)| \equiv |\mathsf{x}_\mathcal{P}(t) - \mathsf{x}_0|$, for all $t \geq 0$, it follows that

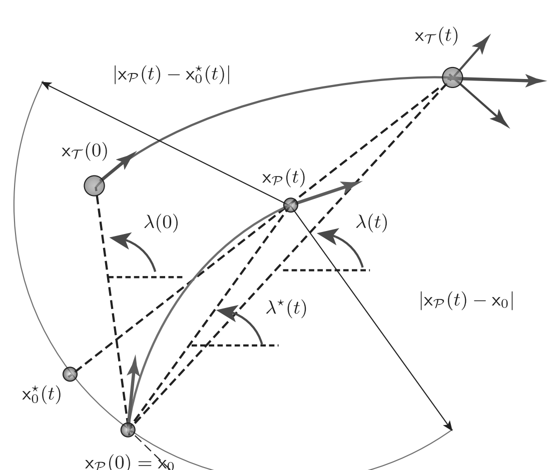


Fig. 6 Three-point guidance or motion camouflage with respect to a moving point $\mathsf{x}_0^\star$ rather than x_0. The condition for motion camouflage with respect to the fixed point x_0 is violated when $\lambda(t) \neq \lambda^\star(t)$.

$$\frac{\langle u_{\mathrm{TPN}}^\star(\mathsf{x},\mathsf{x}_\mathcal{P}), \mathsf{e}_\mathsf{x}^2\rangle}{|\mathsf{x}_\mathcal{P} - \mathsf{x}_0^\star|} = \frac{\langle u_{\mathrm{TPN}}^\star(\mathsf{x},\mathsf{x}_\mathcal{P}), \mathsf{e}_\mathsf{x}^2\rangle}{|\mathsf{x}_\mathcal{P} - \mathsf{x}_0|} = -\frac{\langle w(\mathsf{x}), \mathsf{e}_\mathsf{x}^2\rangle}{|\mathsf{x}_\mathcal{P} - \mathsf{x}_0| + |\mathsf{x}|} \tag{54}$$

Therefore, one can easily conclude from Eq. (54) that $u_\mathcal{P}^\star = u_\mathcal{P}$, or, equivalently,

$$u_{\mathrm{TPN}}^\star(\mathsf{x},\mathsf{x}_\mathcal{P}) = u_{\mathrm{TPN}}(\mathsf{x},\mathsf{x}_\mathcal{P}) \tag{55}$$

Thus, the analytic expressions of the three-point-navigation law derived after relaxing the motion camouflage constraint and the original three-point-navigation law (29) are exactly the same. On the grounds of the previous observation, one concludes that the navigation law (29) is robust to model uncertainties of the local drift. The following proposition follows readily from the previous discussion.

Proposition 7. Let $\varepsilon > 0$. Then, for all $|\mathsf{x}_0| > \varepsilon$, the navigation law (29) will drive the system (1) to the set $\{\mathsf{x}\colon |\mathsf{x}| \leq \varepsilon\}$ in finite time, provided there exist $\bar{w}_1 > 0$ and $\bar{w}_2 > 0$, such that

$$|\langle w(\mathsf{x}) + \Delta w(t,\mathsf{x}), \mathsf{e}_{\mathsf{x}_0}^1\rangle| \leq \bar{w}_1 < \sqrt{\bar{u}^2 - \bar{w}_2^2} \tag{56}$$

$$|\langle w(\mathsf{x}), \mathsf{e}_\mathsf{x}^2\rangle| \leq \bar{w}_2 \tag{57}$$

for all $t \geq 0$ and $\mathsf{x} \in \mathbb{R}^2\backslash\{0\}$. Finally, the time of travel satisfies the upper bound

$$T_f \leq \frac{|\mathsf{x}_0| - \varepsilon}{\sqrt{\bar{u}^2 - \bar{w}_2^2} - \bar{w}_1} < \infty \tag{58}$$

V. Navigation in Unknown Drift

In this section, the problem of steering the agent in the presence of a completely unknown drift field, that is, when $w(\mathsf{x}) \equiv 0$ and $\Delta w(t,\mathsf{x}) \equiv 0$, is considered.

The feedback navigation law

$$u_{\mathrm{PP}}(\mathsf{x}) = \bar{u}\mathsf{e}_\mathsf{x}^1 \tag{59}$$

steers the agent's forward velocity to always point toward its destination. It is worth-mentioning that due to the absence of any knowledge about the local drift at x, the navigation law (59) steers the inertial velocity of the agent so that it points toward a direction different than the LOS. This fact may incur some loss of performance, in terms of minimizing the arrival time, when compared with, for example, the navigation law (15) (see also the discussion in Sec. VI). The situation is illustrated in Fig. 7. On the other hand, one of the main advantages of the navigation law (59) is that it is completely independent of the drift $\Delta w(t,\mathsf{x})$, and thus, it is robust to model uncertainties induced by the local drift. The navigation law (59) is the dual to the well-known pure-pursuit or hound-hare pursuit strategy [18], where the pursuer's velocity vector always points toward the current position of the target. The following proposition provides a sufficient condition for the feasibility of the navigation law (59).

Proposition 8. Let $\varepsilon > 0$ and $w(\mathsf{x}) \equiv 0$. Then, for all $|\mathsf{x}_0| > \varepsilon$, the navigation law (59) will drive the system (1) to the set $\{\mathsf{x} \in \mathbb{R}^2\colon |\mathsf{x}| \leq \varepsilon\}$ in finite time, provided there exists $\Delta\bar{w}_1 > 0$, such that

$$|\langle \Delta w(t,\mathsf{x}), \mathsf{e}_\mathsf{x}^1\rangle| \leq \Delta\bar{w}_1 < \bar{u}, \qquad \text{for all } t \geq 0 \quad \text{and} \quad \mathsf{x} \in \mathbb{R}^2 \tag{60}$$

Furthermore, the arrival time satisfies the upper bound

$$T_f \leq \frac{|\mathsf{x}_0| - \varepsilon}{\bar{u} - \Delta\bar{w}_1} < \infty \tag{61}$$

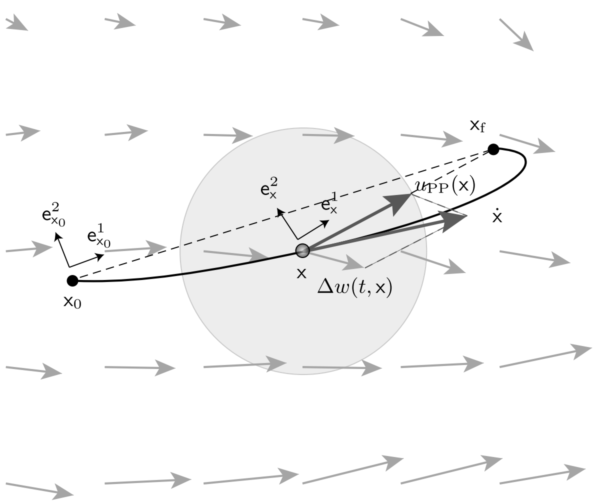


Fig. 7 Direct-bearing navigation. The forward velocity of the agent is aligned with the LOS direction at every instant in time.

Proof. It follows readily that

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathsf{x}| = -\langle \dot{\mathsf{x}}, \mathsf{e}_{\mathsf{x}}^1\rangle = -\langle u_{\mathrm{PP}}(\mathsf{x}) + \Delta w(t,\mathsf{x}), \mathsf{e}_{\mathsf{x}}^1\rangle$$
$$= -\bar{u} - \langle \Delta w(t,\mathsf{x}), \mathsf{e}_{\mathsf{x}}^1\rangle \leq -\bar{u} + \Delta\bar{w}_1 \tag{62}$$

The rest of the proof follows similar to the proof of Proposition 1, and thus it is omitted. □

Table 1 summarizes the navigation control laws developed in terms of the corresponding information pattern of the local drift field. It is important to remind the reader the duality between these navigation control laws and the corresponding pursuit strategies, as indicated in the last column of Table 1.

VI. Gradient Descent Laws for Navigation in a Flowfield with Limited Information

In this section, the proposed navigation laws are reinterpreted as gradient descent laws in terms of different performance indices. In particular, it is shown that when the agent is driven by the presented navigation laws, the direction of either the agent's forward or inertial velocity is parallel to the opposite of the gradient of various performance indices; consequently, the velocity vectors point toward the direction of the maximum rate of decrease of the performance indices. To simplify the presentation, it is henceforth assumed that

$$|w(\mathsf{x}) + \Delta w(t,\mathsf{x})| < \bar{u}, \qquad \text{for all } t \geq 0 \quad \text{and} \quad \mathsf{x} \in \mathbb{R}^2 \tag{63}$$

First, it is shown that the navigation law (38) is a (pseudo-)gradient descent law in terms of an estimate of the time required for the agent to reach its destination (time-to-come). In particular, a simple estimate of the time-to-come, which is henceforth denoted by $\tilde{T}(\mathsf{x})$, is given by

$$\tilde{T}(\mathsf{x}) := \frac{\sqrt{\langle \mathsf{x}, w(\mathsf{x})\rangle^2 + (\bar{u}^2 - |w(\mathsf{x})|^2)|\mathsf{x}|^2}}{\bar{u}^2 - |w(\mathsf{x})|^2} - \frac{\langle \mathsf{x}, w(\mathsf{x})\rangle}{\bar{u}^2 - |w(\mathsf{x})|^2} \tag{64}$$

Note that $\tilde{T}(\mathsf{x})$ is the minimum time required for the agent located at x, at time $t = \tau$, to reach the origin, assuming that the drift will remain constant and equal to $w(\mathsf{x})$, for all $t \geq \tau$.

Let $\nabla\tilde{T}(\mathsf{x})$ denote the gradient of $\tilde{T}(\mathsf{x})$, which is, in general, a function of x, $w(\mathsf{x})$ and $\partial w/\partial \mathsf{x}$. Because, by hypothesis, the Jacobian $\partial w/\partial \mathsf{x}$ is unknown to the agent, a pseudogradient operator acting on $\tilde{T}(\mathsf{x})$, denoted by $\tilde{\nabla}\tilde{T}(\mathsf{x})$, where

$$\tilde{\nabla}\tilde{T}(\mathsf{x}) := \nabla\tilde{T}(\mathsf{x})|_{\frac{\partial w}{\partial \mathsf{x}}=0}$$

is introduced instead. It is straightforward to show that the (pseudo-) gradient descent control law

$$u_{\mathrm{PGDN}}(\mathsf{x}) := -\bar{u}\frac{\tilde{\nabla}\tilde{T}(\mathsf{x})}{|\tilde{\nabla}\tilde{T}(\mathsf{x})|} \tag{65}$$

satisfies $u_{\mathrm{PGDN}}(\mathsf{x}) = u^\star_{\mathrm{OLS}}(\mathsf{x})$, for all $\mathsf{x} \in \mathbb{R}^2\backslash\{0\}$. This facts highlights the local optimality of the navigation law (38), in terms of minimizing the arrival time.

Next, it is shown that the LOS navigation law (38) can also be interpreted as a quickest descent control law [24] in terms of the Euclidean distance of the agent from its destination. In other words, when the agent is driven by the law (38), then the rate of decrease of the agent's distance from its destination is locally maximized. In particular, the time derivative of $V(\mathsf{x}) = |\mathsf{x}|$ evaluated along the trajectories of the system (1), after closing the loop with $u(\mathsf{x}) = u^\star_{\mathrm{OLS}}(\mathsf{x})$, is pointwise minimized.

Proposition 9. The navigation law $u^\star_{\mathrm{OLS}}(\mathsf{x})$ is the quickest descent law for the system (1) with respect to the descent function $V(\mathsf{x}) = |\mathsf{x}|$.

Proof. Let $u(\mathsf{x}) \in \mathcal{U}_f$. The time derivative of $V(\mathsf{x})$ evaluated along the trajectories of system (1), after closing the loop, is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}V(\mathsf{x}) = \nabla V(\mathsf{x})\dot{\mathsf{x}} = -\langle u(\mathsf{x}) + w(\mathsf{x}) + \Delta w(t,\mathsf{x}), \mathsf{e}_{\mathsf{x}}^1\rangle \tag{66}$$

where the identity $\nabla V(\mathsf{x}) = \mathsf{x}/|\mathsf{x}| = -\mathsf{e}_{\mathsf{x}}^1$ has been used. It follows readily that the quickest descent control $u(\mathsf{x})$ satisfies

$$u(\mathsf{x}) + w(\mathsf{x}) = \max_{|u|\leq\bar{u}}\langle u(\mathsf{x}) + w(\mathsf{x}), \mathsf{e}_{\mathsf{x}}^1\rangle\mathsf{e}_{\mathsf{x}}^1 \tag{67}$$

Equation (67) implies that

$$\langle u(\mathsf{x}) + w(\mathsf{x}), \mathsf{e}_{\mathsf{x}}^2\rangle = 0$$

and $|u(\mathsf{x})| = \bar{u}$. Therefore, $\langle u(\mathsf{x}), \mathsf{e}_{\mathsf{x}}^2\rangle = -\langle w(\mathsf{x}), \mathsf{e}_{\mathsf{x}}^2\rangle$, which implies, in turn, that

$$\langle u(\mathsf{x}), \mathsf{e}_{\mathsf{x}}^1\rangle = \sqrt{\bar{u}^2 - \langle w(\mathsf{x}), \mathsf{e}_{\mathsf{x}}^2\rangle^2}$$

Therefore, it follows that $u(\mathsf{x}) = u^\star_{\mathrm{OLS}}(\mathsf{x})$, for all $\mathsf{x} \in \mathbb{R}^2\backslash\{0\}$, thus completing the proof. □

Finally, the direct-bearing navigation law (59) can also be viewed as a gradient descent control law. In particular, it is easy to show that

Table 1 Proposed navigation laws for different information patterns

Navigation law	Expression	Information pattern	Pursuit strategy
$u_{\mathrm{OLS}}(\mathsf{x})$	$\sqrt{\bar{u}^2 - \langle w(\mathsf{x}), \mathsf{e}_{\mathsf{x}_0}^2\rangle^2}\mathsf{e}_{\mathsf{x}_0}^1 - \langle w(\mathsf{x}), \mathsf{e}_{\mathsf{x}_0}^2\rangle\mathsf{e}_{\mathsf{x}_0}^2$	$w(\mathsf{x})\not\equiv 0$, $\Delta w(t,\mathsf{x}) \equiv 0$	Parallel guidance
$u_{\mathrm{NLS}}(\mathsf{x})$	$(\bar{u} - \lvert w(\mathsf{x})\rvert - \langle w(\mathsf{x}), \mathsf{e}_{\mathsf{x}_0}^1\rangle)\mathsf{e}_{\mathsf{x}_0}^1 - \langle w(\mathsf{x}), \mathsf{e}_{\mathsf{x}_0}^2\rangle\mathsf{e}_{\mathsf{x}_0}^2$	$w(\mathsf{x})\not\equiv 0$, $\Delta w(t,\mathsf{x}) \equiv 0$	Pursuit with neutralization
$u_{\mathrm{TPN}}(\mathsf{x}, \mathsf{x}_{\mathcal{P}})$	$\sqrt{\bar{u}^2 - u_{\mathrm{TPN},2}^2(\mathsf{x}, \mathsf{x}_{\mathcal{P}})}\mathsf{e}_{\mathsf{x}}^1 + u_{\mathrm{TPN},2}(\mathsf{x}, \mathsf{x}_{\mathcal{P}})\mathsf{e}_{\mathsf{x}}^2$, $u_{\mathrm{TPN},2}(\mathsf{x}, \mathsf{x}_{\mathcal{P}}) = -\frac{\lvert\mathsf{x}_{\mathcal{P}} - \mathsf{x}_0\rvert}{\lvert\mathsf{x}_{\mathcal{P}} - \mathsf{x}_0\rvert + \lvert\mathsf{x}\rvert}\langle w(\mathsf{x}), \mathsf{e}_{\mathsf{x}}^2\rangle$	$w(\mathsf{x})\not\equiv 0$, $\Delta w(t,\mathsf{x}) \equiv 0$	LOS guidance
$u^\star_{\mathrm{OLS}}(\mathsf{x})$	$\sqrt{\bar{u}^2 - \langle w(\mathsf{x}), \mathsf{e}_{\mathsf{x}}^2\rangle^2}\mathsf{e}_{\mathsf{x}}^1 - \langle w(\mathsf{x}), \mathsf{e}_{\mathsf{x}}^2\rangle\mathsf{e}_{\mathsf{x}}^2$	$w(\mathsf{x})\not\equiv 0$, $\Delta w(t,\mathsf{x})\not\equiv 0$	Parallel guidance
$u^\star_{\mathrm{NLS}}(\mathsf{x})$	$(\bar{u} - \lvert w(\mathsf{x})\rvert - \langle w(\mathsf{x}), \mathsf{e}_{\mathsf{x}}^1\rangle)\mathsf{e}_{\mathsf{x}}^1 - \langle w(\mathsf{x}), \mathsf{e}_{\mathsf{x}}^2\rangle\mathsf{e}_{\mathsf{x}}^2$	$w(\mathsf{x})\not\equiv 0$, $\Delta w(t,\mathsf{x})\not\equiv 0$	Pursuit with neutralization
$u^\star_{\mathrm{TPN}}(\mathsf{x}, \mathsf{x}_{\mathcal{P}})$	$\sqrt{\bar{u}^2 - u_{\mathrm{TPN},2}^2(\mathsf{x}, \mathsf{x}_{\mathcal{P}})}\mathsf{e}_{\mathsf{x}}^1 + u_{\mathrm{TPN},2}(\mathsf{x}, \mathsf{x}_{\mathcal{P}})\mathsf{e}_{\mathsf{x}}^2$, $u_{\mathrm{TPN},2}(\mathsf{x}, \mathsf{x}_{\mathcal{P}}) = -\frac{\lvert\mathsf{x}_{\mathcal{P}} - \mathsf{x}_0\rvert}{\lvert\mathsf{x}_{\mathcal{P}} - \mathsf{x}_0\rvert + \lvert\mathsf{x}\rvert}\langle w(\mathsf{x}), \mathsf{e}_{\mathsf{x}}^2\rangle$	$w(\mathsf{x})\not\equiv 0$, $\Delta w(t,\mathsf{x})\not\equiv 0$	LOS guidance
$u_{\mathrm{PP}}(\mathsf{x})$	$\bar{u}\mathsf{e}_{\mathsf{x}}^1$	$w(\mathsf{x}) \equiv 0$, $\Delta w(t,\mathsf{x})\not\equiv 0$	Pure pursuit

$$u_{\mathrm{PP}}(\mathsf{x}) \equiv -\bar{u}\frac{\nabla V(\mathsf{x})}{|\nabla V(\mathsf{x})|} \tag{68}$$

where $V(\mathsf{x}) := |\mathsf{x}|$. Note that $V(\mathsf{x})$ is a reasonable heuristic function in terms of the time-to-come for the navigation problem in a completely unknown drift field (this follows readily by setting $w(\mathsf{x}) \equiv 0$ in Eq. (64)). An interesting question is when, and under which conditions, the direct-bearing navigation law (59) is a minimum-time control law for Problem 2. The following proposition addresses the previous questions.

Proposition 10. Let $\varepsilon > 0$. The navigation law (59) is a minimum-time control law of the ZNP provided there exists a Lipschitz continuous function f: $[\varepsilon, \infty) \mapsto \mathbb{R}$, such that $\langle w(\mathsf{x}), \mathsf{e}_{\mathsf{x}}^{1} \rangle = f(|\mathsf{x}|)$. Furthermore, the system (1) will converge to the set $\{\mathsf{x}\colon |\mathsf{x}| \leq \varepsilon\}$ in finite time, for all $|\mathsf{x}_0| > \varepsilon$, if and only if $f(z) < \bar{u}z$, for all $\varepsilon \leq z \leq |\mathsf{x}_0|$. In addition, the final arrival time is given by

$$T_f = \int_{\varepsilon}^{|\mathsf{x}_0|} \frac{z\mathrm{d}z}{\bar{u}z - f(z)} \tag{69}$$

Proof. The reader can refer to [17]. □

Proposition 10 highlights a rather surprising result, namely, that although the measurement of the local drift $w(\mathsf{x})$ does not appear at all in the expression of the navigation law (59), in contrast to all the other navigation laws presented in this paper, which explicitly account for the local drift, the direct-bearing navigation law (59) can be the minimum-time navigation law for some drift fields.

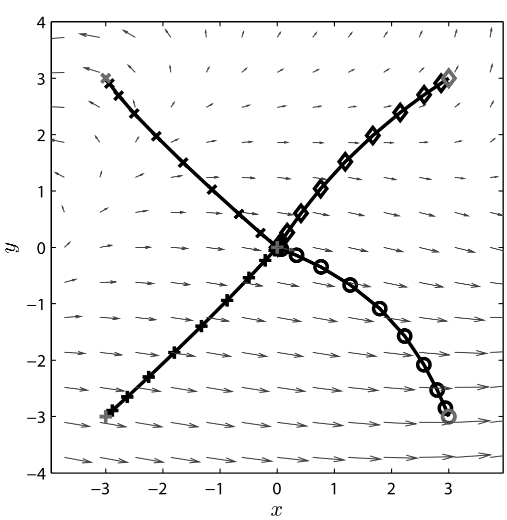
VII. Simulation Results

In this section, simulation results that illustrate the previous developments are presented. The drift field is assumed to be expressed as the vector sum of a uniform flow component and the local flow induced by a finite number of distinct, nonlinear flow singularities [25]. In particular, it is assumed that the known part of the drift $w(\mathsf{x})$ can be modeled as follows

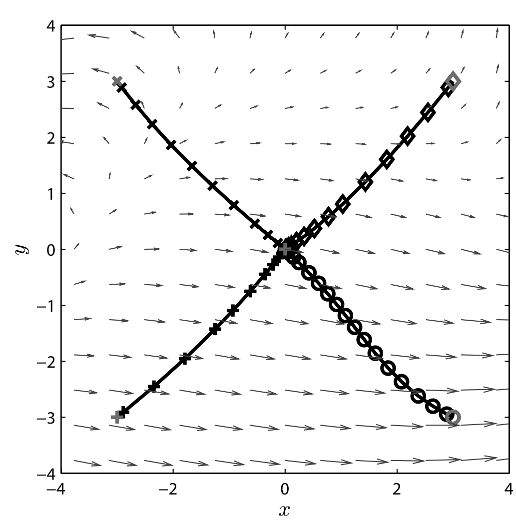
$$w(\mathsf{x}) = w^0 + \sum_{i=1}^{n_s} \alpha_i^{-1}(|\mathsf{x} - \mathsf{x}_{s_i}|)\mathbf{A}_i(\mathsf{x} - \mathsf{x}_{s_i}) \tag{70}$$

where n_s is the number of flow singularities, x_{s_i} is the location of the ith flow singularity, α_i: $[0, \infty) \mapsto \mathbb{R}$ is a continuous function, which may vanish only at $\mathsf{x} = \mathsf{x}_{s_i}$, and $\mathbf{A}_i$ is a 2×2 matrix, whose structure captures the local characteristics of the ith flow singularity [26]. Note that the flow model given in Eq. (70) extends the model adopted in [26] to account for multiple flow singularities located at distinct positions.

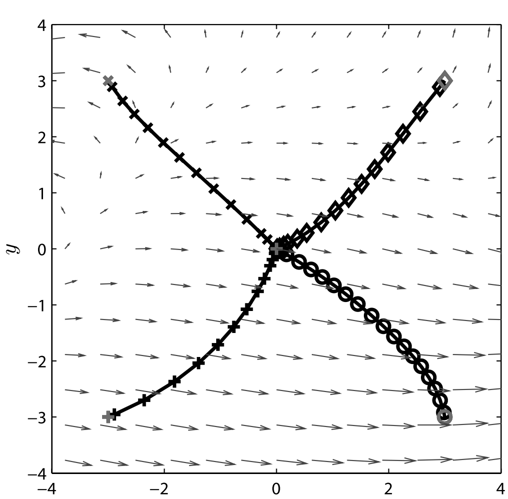
The following problem data were used in the numerical simulations:



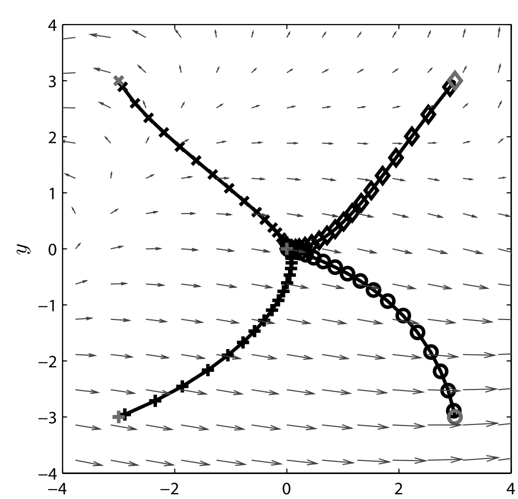
a) Minimum-time navigation in a perfectly known drift field



b) Robust, optimal LOS navigation with imperfect knowledge of the local drift



c) Three-point navigation with imperfect knowledge of the local drift



d) Direct-bearing navigation in an unknown drift field

Fig. 8 Trajectories toward the origin of an agent driven by the robust optimal LOS, the three-point and the direct-bearing navigation laws.

$$\bar{u} = 1, \qquad n_s = 2, \qquad \mathbf{A}_1 = \mathbf{S}, \qquad \mathbf{A}_2 = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\mathsf{x}_{s_1} = [5, \quad 0]^\mathsf{T}, \qquad \mathsf{x}_{s_2} = [-6, \quad -4]^\mathsf{T}, \qquad w^0 = [0, \quad 0]^\mathsf{T}$$
$$\alpha_1(|\mathsf{x} - \mathsf{x}_{s_1}|) = |\mathbf{A}_1(\mathsf{x} - \mathsf{x}_{s_1})|/0.3$$
$$\alpha_2(|\mathsf{x} - \mathsf{x}_{s_2}|) = |\mathbf{A}_2(\mathsf{x} - \mathsf{x}_{s_2})|/0.4$$

Furthermore, it is assumed that the unknown part of the drift field is given by

$$\Delta w(t, \mathsf{x}) = \sqrt{3}|\mathsf{x}|/6[0.3(1 - \cos(t/\pi)), \quad -0.25]^\mathsf{T}$$

Figure 8 illustrates the trajectories of the agent, when the latter is steered by the minimum-time control law of the ZNP (Fig. 8a), the robust optimal LOS navigation law (38) (Fig. 8b), the three-point navigation law (29) (Fig. 8c), and the direct-bearing navigation law (59) (Fig. 8d). For the computation of the minimum-time paths, GPOPS [27], which is an open-source software for numerical optimal control, has been used. One can observe from Figs. 8b–8d that, despite the presence of the unknown part $\Delta w(t, \mathsf{x})$ of the local drift field, the agent driven by the robust optimal LOS, the three-point and the direct-bearing navigation laws successfully reaches its destination. Furthermore, it is observed that the geometry of the ensuing paths of the agent, when the agent is far away from its destination and it is driven by the navigation laws (29) and (59), exhibit notable similarities, as is illustrated in Figs. 8c and 8d. The ensuing paths of the agent are also similar when the agent is close to its destination and is driven by the navigation laws (38) and (29), as is illustrated in Figs. 8b and 8c. The last two observations are justified by the fact that the navigation law (29) becomes approximately equal to Eq. (59), for large $|\mathsf{x}|$ [in light of Eq. (28), the component of Eq. (29) along e_x^2 becomes approximately equal to zero as $|\mathsf{x}| \to \infty$], whereas it approximates Eq. (38), for $|\mathsf{x}|$ sufficiently small [in light of Eq. (28), the component of Eq. (29) along e_x^2 becomes approximately equal to $-\langle w(\mathsf{x}), \mathsf{e}_\mathsf{x}^2 \rangle$ as $|\mathsf{x}| \to 0$].

VIII. Conclusions

This paper presents several classes of navigation laws for steering an agent in the presence of a both temporally and spatially varying drift field, by investigating the navigation problem for different information patterns about the drift field. The analysis, which is based on the duality between the navigation problem and a special class of problems of pursuit of a maneuvering target, brings to light some interesting findings related to the effectiveness of the proposed navigation laws in terms of coping with model uncertainties of the drift field dynamics, as well as in terms of minimizing the arrival time. In particular, it was shown that the effectiveness of the line-of-sight navigation law, which is the dual to the parallel guidance law, in terms of steering the agent to its destination, is impaired by the incomplete knowledge of the local drift field. A robust modification of the line-of-sight navigation law, which was derived by employing simple geometric arguments, was subsequently proposed. In contrast to the line-of-sight navigation law, the three-point navigation law, may successfully steer the agent to its destination in the presence of unknown drift. Furthermore, it was shown that the direct-bearing navigation law, which is the dual to the pure-pursuit strategy, furnishes a straightforward solution to the navigation problem in a completely unknown drift field. One important observation is that all of the proposed navigation laws that account for the unknown component of the drift reduce to the direct-bearing navigation law in the limiting case when the known component of the drift vanishes. The analysis of the planar navigation problem presented in this work can be easily extended to the three-dimensional navigation problem, given that the adopted approach was based on tools from vector analysis. Future work includes the use of the proposed navigation laws for the design of novel protocols for motion coordination, and dynamic routing problems for groups of agents traveling in the presence of an uncertain drift field.

References

[1] Zermelo, E., "Über das Navigationproble bei Ruhender oder Veränderlicher Windverteilung," *Zeitschrift für Angewandte Mathematik und Mechanik*, Vol. 11, 1931, pp. 114–124.
doi:10.1002/zamm.19310110205

[2] Nishida, T., Sugihara, K., and Kimura, M., "Stable Marker-Particle Method for the Voronoi Diagram in a Flow Field," *Journal of Computational and Applied Mathematics*, Vol. 202, No. 2, 2007, pp. 377–391.
doi:10.1016/j.cam.2006.01.035

[3] Petres, C., Pailhas, Y., Patron, P., Petillot, Y., Evans, J., and Lane, D., "Path Planning for Autonomous Underwater Vehicles," *IEEE Transactions on Robotics and Automation*, Vol. 23, No. 2, 2007, pp. 331–341.
doi:10.1109/TRO.2007.895057

[4] Davis, R. E., Leonard, N. E., and Fratantoni, D. M., "Routing Strategies for Underwater Gliders," *Deep Sea Research Part II: Topical Studies in Oceanography*, Vol. 56, Nos. 3–5, 2009, pp. 173–187.
doi:10.1016/j.dsr2.2008.08.005

[5] Bakolas, E., and Tsiotras, P., "Minimum-Time Paths for a Light Aircraft in the Presence of Regionally-Varying Strong Winds," AIAA Infotech at Aerospace, AIAA Paper 2010–3380, Atlanta, April 2010.

[6] Bakolas, E., and Tsiotras, P., "Optimal Pursuit of Moving Targets using Dynamic Voronoi Diagrams," *Proceedings of IEEE International Conference on Decision and Control*, IEEE Press, Piscataway, NJ, Dec. 2010, pp. 7431–7436.

[7] Rhoads, B., Mezić, I., and Poje, A., "Minimum Time Feedback Control of Autonomous Underwater Vehicles," *Proceedings of 49th IEEE Conference on Decision and Control*, IEEE Press, Piscataway, NJ, Dec. 2010, pp. 5828–5834.

[8] Soulignac, M., "Feasible and Optimal Path Planning in Strong Current Fields," *IEEE Transactions on Robotics and Automation*, Vol. 27, No. 1, 2011, pp. 89–98.
doi:10.1109/TRO.2010.2085790

[9] Ceccarelli, N., Enright, J. J., Frazzoli, E., Rasmussen, S. J., and Schumacher, C. J., "Micro UAV Path Planning for Reconnaissance in Wind," *Proceedings of the 2007 American Control Conference*, IEEE Press, Piscataway, NJ, July 2007, pp. 5310–5315.

[10] Paley, D. A., and Peterson, C., "Stabilization of Collective Motion in a Time-Invariant Flowfield," *Journal of Guidance, Control, and Dynamics*, Vol. 32, No. 3, 2009, pp. 771–779.
doi:10.2514/1.40636

[11] Bullo, F., Cortés, J., and Martinez, S., *Distributed Control of Robotic Networks*, Applied Mathematics Series, Princeton Univ. Press, Princeton, NJ, 2009, pp. 139–246, Chaps. 3–5.

[12] Hernadez, S., and Paley, D. A., "Three-Dimensional Motion Coordination in a Spatiotemporal Flowfield," *IEEE Transactions on Automatic Control*, Vol. 55, No. 12, 2010, pp. 2805–2810.
doi:10.1109/TAC.2010.2070710

[13] Mesbahi, M., and Egerstedt, M., *Graph Theoretic Methods in Multiagent Networks*, Applied Mathematics Series, Princeton Univ. Press, Princeton, NJ, 2010, pp. 117–190, Chaps. 6–7.

[14] Kelley, H. J., "Guidance Theory and Extremal Fields," *IRE Transactions on Automatic Control*, Vol. 7, No. 5, Oct. 1962, pp. 75–82.
doi:10.1109/TAC.1962.1105503

[15] Hajek, O., *Pursuit Games: An Introduction to the Theory and Applications of Differential Games of Pursuit and Evasion*, 2nd ed., Dover, Mineola, NY, 2008, pp. 1–7, Chap. 1.

[16] Bakolas, E., and Tsiotras, P., "The Zermelo-Voronoi Diagram: a Dynamic Partition Problem," *Automatica*, Vol. 46, No. 12, 2010, pp. 2059–2067.
doi:10.1016/j.automatica.2010.09.003

[17] Bakolas, E., and Tsiotras, P., "Optimal Pursuer and Moving Target Assignment using Dynamic Voronoi Diagrams," *Proceedings of the American Control Conference*, IEEE Press, Piscataway, NJ, 2011, pp. 5444–5449.

[18] Shneydor, N. A., *Missile Guidance and Pursuit: Kinematics, Dynamics and Control*, Horwood, West Sussex, England, U.K., 1998, pp. 11–100, Chaps. 2–4.

[19] Justh, E. W., and Krishnaprasad, P. S., "Steering Laws for Motion Camouflage," *Proceedings of the Royal Society of London, Series A: Mathematical and Physical Sciences*, Vol. 462, 2006, pp. 3629–3643.
doi:10.1098/rspa.2006.1742

[20] Carathéodory, C., *Calculus of Variations and Partial Differential Equations of First Order*, 3rd ed., American Mathematical Society, Washington, D.C., 1999, pp. 239–247, 370–373, Chaps. 14, 18.

[21] Bryson, A. E., and Ho, Y. C., *Applied Optimal Control*, Taylor &

Francis, Levittown, PA, 1975, pp. 76–77, Chap. 2.
[22] Srinivasan, M. V., and Davey, M., "Strategies for Active Camouflage of Motion," *Proceedings of the Royal Society of London, Series B: Biological Sciences*, Vol. 259, No. 1354, 1995, pp. 19–25. doi:10.1098/rspb.1995.0004
[23] Glendinning, P., "The Mathematics of Motion Camouflage," *Proceedings of the Royal Society of London, Series B: Biological Sciences*, Vol. 271, No. 1538, 2004, pp. 477–481. doi:10.1098/rspb.2003.2622
[24] Vincent, L. T., and Grantham, W. J., *Nonlinear and Optimal Control Systems*, Wiley–Interscience, New York, 1997, pp. 273–313, Chap. 5.
[25] Corpetti, T., Mémin, E., and Pérez, P., "Extraction of Singular Points from Dense Motion Fields: An Analytic Approach," *Journal of Mathematical Imaging and Vision*Vol. 19, No. 3, 2003, pp. 175–198. doi:10.1023/A:1026352203836
[26] Petrich, J., Woolsey, C. A., and Stilwell, D. J., "Planar Flow Identification for Improved Navigation of Small AUVs," *Ocean Engineering*, Vol. 36, No. 1, 2009, pp. 119–131. doi:10.1016/j.oceaneng.2008.10.002
[27] Rao, A. V., Benson, D. A., Darby, C., Patterson, M. A., Francolin, C., Sanders, I., and Huntington, G. T., "Algorithm 902: GPOPS, A MATLAB Software for Solving Multiple-Phase Optimal Control Problems Using the Gauss Pseudospectral Method," *ACM Transactions on Mathematical Software*, Vol. 37, No. 2, April 2010, pp. 22:1–22:39. doi:10.1145/1731022.1731032