Analytical Solutions for a Spinning Rigid Body Subject to Time-Varying Body-Fixed Torques, Part I: Constant Axial Torque

J. M. Longuski
Assoc. Professor.

P. Tsiotras
Ph.D. Candidate.

School of Aeronautics and Astronautics,
Purdue University,
West Lafayette, IN 47907

Analytic solutions are derived for the general attitude motion of a near-symmetric rigid body subject to time-varying torques in terms of certain integrals. A methodology is presented for evaluating these integrals in closed form. We consider the case of constant torque about the spin axis and of transverse torques expressed in terms of polynomial functions of time. For an axisymmetric body with constant axial torque, the resulting solutions of Euler's equations of motion are exact. The analytic solutions for the Eulerian angles are approximate owing to a small angle assumption, but these apply to a wide variety of practical problems. The case when all three components of the external torque vector vary simultaneously with time is much more difficult and is treated in Part II.

1 Introduction

Many numerical simulations are often necessary in order to understand the attitude response and control characteristics of a rigid-body vehicle. In fact, sensitivity and/or error analyses, using numerical integration, may be prohibitively expensive and time consuming, especially when a large number of problem parameters are involved. Analytical models can be of great help in obtaining a *qualitative* understanding of the complex dynamical behavior; even simple heuristic analytical results may provide a fast and relatively accurate model for maneuver analysis. This approach has been proved to be very useful in the past (Hintz and Longuski, 1985; Kia and Longuski, 1984; Longuski, Kia and Breckenridge, 1989). Moreover, for future applications it appears that the trend for autonomous on-board guidance control schemes dictates the use of compact, simple, analytical expressions modeling the attitude evolution. The development of such analytic solutions for the attitude history of rotating bodies thus provides a cornerstone for the advent of autonomous, closed-loop navigation. Even for the case of optimal open-loop control laws, analytic solutions can be very helpful in the associated two-point boundary value problem, providing a first guess for shooting-method algorithms. Analytic solutions are also important in the area of attitude reconstruction, where modern satellite telescopes require a precision of a few milliseconds of arc for the success of the mission (Bois, 1986). In such cases attitude reconstruction by numerical integration is not suitable; it is necessary to have analytical representations, using as few parameters as possible. Thus, the revival of interest in analytic solutions for the rigid-body dynamics problem comes as no surprise. In fact, attitude evolution of a rigid body has been extensively studied over the past few decades.

Analytic solutions for the attitude motion of a rigid body have been obtained recently for the *constant body-fixed torque problem* by Longuski (1980, 1991) and Tsiotras and Longuski (1991a). By this term, we mean the problem of a rotating rigid body under the influence of an external torque vector, which has *constant* components along the axes of a coordinate frame fixed in the body. Practical applications of this problem include the case of spinning satellites in space, where the torques are created internally in the body, e.g., as a result of a thruster firing. This problem is often referred to in the literature as the *self-excited rigid-body problem* and includes all the cases when the acting torques do not depend on the actual orientation of the body in inertial space. It does include, however, the case when the acting torques have a prescribed time-varying behavior in a reference frame fixed in the body. This is exactly the case that we are interested in. In this paper we assume that the transverse torques vary continuously with time, and are modeled as polynomials, or more generally, as truncated Taylor series expansions of analytic functions. The axial torque is assumed to be constant. This restriction can be relaxed, although a different procedure is necessary in order to handle the more complex integrals arising in this case (see Part II). The solution for the angular velocities is *exact* for symmetric rigid bodies, and very accurate for near-symmetric rigid bodies. In the case of the Eulerian angles, the solutions are always approximate because of a small angle assumption. We use the

Contributed by the Applied Mechanics Division of The American Society of Mechanical Engineers for publication in the Journal of Applied Mechanics.

Discussion on this paper should be addressed to the Technical Editor, Professor Lewis T. Wheeler, Department of Mechanical Engineering, University of Houston, Houston, TX 77204-4792, and will be accepted until four months after final publication of the paper itself in the Journal of Applied Mechanics.

Manuscript received by the ASME Applied Mechanics Division, Apr. 10, 1992; final revision, Oct. 22, 1992. Associate Technical Editor: R. L. Huston.

term *symmetric* (or equivalently, *axisymmetric*) for bodies with two equal principal moments of inertia. The term *near-symmetric* will refer to bodies with two principal moments of inertia that do not differ "too much." When the body is symmetric or nearly symmetric, we will also assume that the spinning axis is the third principal axis, being perpendicular to the plane defined by the two principal axes with equal (or nearly equal) moments of inertia.

Since every torque that varies continuously with time can be approximated to any desired degree of accuracy by polynomials, the analysis permits one to develop more realistic models of spacecraft torques and forces; these then may be applied to a variety of current and future problems involving chemical, ion, and solar sail propulsion. Another work by the authors (Tsiotras and Longuski, 1991b) examines the closely related problem when the acting torques are periodic functions of time, or alternatively, torques that can be expressed in terms of Fourier expansions. In such cases, solutions have been obtained for both the angular velocities and Eulerian angles; however, the analysis is more involved because the frequency of the acting transverse torques generates a two-parameter family of integrals, the explicit evaluation of which is very tedious. Since Fourier and Taylor expansions are the most common techniques of functional approximations, the results can be used in many applications. One, of course, would expect that the results of Tsiotras and Longuski (1991b) will be more helpful for discontinuous torques, or when the torques exhibit some kind of periodicity. For most of the other cases, however, the results of the current work will be more relevant and easier to apply.

2 Analytic Solution of Euler's Equations of Motion

Euler's equations of motion for principal axes at the center of mass are

$$M_x = I_x\dot{\omega}_x + (I_z - I_y)\,\omega_y\omega_z \tag{1}$$

$$M_y = I_y\dot{\omega}_y + (I_x - I_z)\,\omega_z\omega_x \tag{2}$$

$$M_z = I_z\dot{\omega}_z + (I_y - I_x)\,\omega_x\omega_y. \tag{3}$$

In these equations M_x, M_y, and M_z are the torque vector components, ω_x, ω_y, and ω_z are the angular velocity vector components, and I_x, I_y, and I_z are the principal moments of inertia. As usual, a dot represents differentiation with respect to time. No explicit analytic solutions of this system of nonlinear differential equations are known to exist for arbitrary functions of the external torques M_x, M_y, and M_z. In fact, no exact solutions are known, without the need of some simplifying assumptions, even for the case of M_x, M_y, and M_z being constant. Assuming that only M_z is constant and that the last term of Eq. (3) is small (either because of near symmetry or because of the product $\omega_x\omega_y$ being small) we obtain

$$\omega_z(t) \approx (M_z/I_z)\,t + \omega_{z0}, \quad \omega_{z0} \triangleq \omega_z(0). \tag{4}$$

This approximation has been proved very useful in previous developments (Longuski, 1980; Longuski, 1991; Tsiotras and Longuski, 1991a) and is *exact* for the case of a *symmetric* rigid body (defined here by $I_x = I_y$). Surely this approximation is very accurate also for the case of a spin-stabilized spacecraft, when both ω_x and ω_y tend to remain small, even when no symmetry assumption can be made. This approximation in the solution of Eq. (3) allows one to decouple the third-order system of nonlinear differential Eqs. (1)–(3). Therefore, assuming the validity of Eq. (4), one can merely concentrate on Eqs. (1) and (2), which now become a set of two coupled, but linear time-varying differential equations. The use of the change to the new independent variable

$$\tau(t) \triangleq \omega_z(t) \tag{5}$$

and the transformation of the dependent variables

$$\Omega_x \triangleq \omega_x\sqrt{k_y}, \quad \Omega_y \triangleq \omega_y\sqrt{k_x} \tag{6}$$

where $k_x \triangleq (I_z - I_y)/I_x$, $k_y \triangleq (I_z - I_x)/I_y$, $k \triangleq \sqrt{k_x k_y}$ allows one to combine both Eqs. (1) and (2) into the following scalar, linear but *complex* differential equation with a time-varying coefficient

$$\frac{d\Omega}{d\tau} - i\rho\tau\Omega = F. \tag{7}$$

In Eq. (7) we have

$$\Omega \triangleq \Omega_x + i\Omega_y, \quad \rho \triangleq k(I_z/M_z), \quad F \triangleq F_x + iF_y \tag{8}$$

$$F_x \triangleq (M_x/I_x)\,(I_z/M_z)\sqrt{k_y}, \quad F_y \triangleq (M_y/I_y)\,(I_z/M_z)\sqrt{k_x} \tag{9}$$

where $F = F(\tau)$ is now considered as a function of τ. It is assumed, without loss of generality, that the spin axis is the axis of the largest moment of inertia I_z, and for the sake of consistency we will assume that $I_z > I_x > I_y$. This choice will imply stable motion about the spin axis. The following procedure can be applied with a few modifications to the case when the z-axis is the minor principal axis, as well. The case when I_z represents the intermediate axis will not be considered, since it always results in unstable motion. The solution for the transverse angular velocities can be written immediately as follows:

$$\Omega(\tau) = \Omega_0 \exp\left(i\frac{\rho}{2}\tau^2\right) + \exp\left(i\frac{\rho}{2}\tau^2\right)\int_{\tau_0}^{\tau}\exp\left(-i\frac{\rho}{2}u^2\right)F(u)\,du \tag{10}$$

where $\Omega_0 \triangleq \Omega(\tau_0)\exp[-(i\rho/2)\tau_0^2]$ and $\Omega(\tau_0)$ is the initial condition in the new independent variable τ. The only difficulty that arises in the computation of the solution for the transverse angular velocities ω_x and ω_y comes from the integral of the nonhomogeneous part of the solution, due to the forcing function $F(\tau)$. We are, therefore, merely interested in computing the integral appearing in (10), that is,

$$I_\omega(\tau_0, \tau; \rho) \triangleq \int_{\tau_0}^{\tau}\exp\left(-i\frac{\rho}{2}u^2\right)F(u)\,du \tag{11}$$

for the special case when the transverse torques, M_x and M_y, are assumed to take the form of polynomials. Using the affinity of the transformation (4) and (5), one can assume, without loss of generality, the following expression for M_x and M_y in the new independent variable τ

$$M_x(\tau) = \sum_{n=0}^{m}\overline{M}_{x,n}\tau^n, \quad M_y(\tau) = \sum_{n=0}^{m}\overline{M}_{y,n}\tau^n \tag{12}$$

where $\overline{M}_{x,n}$ and $\overline{M}_{y,n}$ are constants, and some of them can be zero. (Therefore, there is no loss of generality in assuming that both M_x and M_y are polynomials of the same degree.) Equation (12) implies that because of the simple character of the transformation (5), polynomial functions in the original independent variable t, correspond to polynomials in the new independent variable τ. This is not the case, however, when M_z is not constant, and the relationship between the two expressions is not that obvious. Part II of the paper addresses this problem. Using Eqs. (8), (9), and (12) one can now write the following expression for the forcing function $F(\tau)$:

$$F(\tau) = \sum_{n=0}^{m} F_n\tau^n \tag{13}$$

where the coefficients F_n in the above expression are complex constants, given by $F_n \triangleq F_{x,n} + iF_{y,n}$ and $F_{x,n} \triangleq (\overline{M}_{x,n}/I_x)\,(I_z/M_z)\sqrt{k_y}$, $F_{y,n} \triangleq (\overline{M}_{y,n}/I_y)\,(I_z/M_z)\sqrt{k_x}$. Substitution of this representation into Eq. (11) allows one to rewrite the integral as follows:

$$\int_{\tau_0}^{\tau} \exp\left(-i\frac{\rho}{2}u^2\right)F(u)du = \sum_{n=0}^{m} F_n \int_{\tau_0}^{\tau} \exp\left(-i\frac{\rho}{2}u^2\right)u^n du = \sum_{n=0}^{m} F_n \bar{I}_n(\tau_0, \tau;\rho) \quad (14)$$

where for convenience the following integrals have been introduced:

$$\bar{I}_n(\tau_0, \tau;\rho) \triangleq \int_{\tau_0}^{\tau} \exp\left(-i\frac{\rho}{2}u^2\right)u^n du, \ n=0,1,2,\ldots, m. \quad (15)$$

Integrals of the form

$$I_n(x;\rho) \triangleq \int_0^x \exp\left(-i\frac{\rho}{2}u^2\right)u^n du \quad (16)$$

can be evaluated easily by means of the recurrence formula

$$\int_0^x \exp\left(-i\frac{\rho}{2}u^2\right)u^n du = i\frac{x^{n-1}}{\rho}\exp\left(-i\frac{\rho}{2}x^2\right) - i\frac{n-1}{\rho}\int_0^x \exp\left(-i\frac{\rho}{2}u^2\right)u^{n-2}du, \quad n=2,3,4,\ldots. \quad (17)$$

To obtain all the integrals from (17) one needs to find the first two terms of the sequence. This can be done using the following expressions:

$$I_0(x;\rho) \triangleq \int_0^x \exp\left(-i\frac{\rho}{2}u^2\right)du = \sqrt{\pi/\rho}\,\mathrm{sgn}(x)E\left(\sqrt{\rho/\pi}x\right) \quad (18)$$

$$I_1(x;\rho) \triangleq \int_0^x \exp\left(-i\frac{\rho}{2}u^2\right)u\,du = \frac{i}{\rho}\left[\exp\left(-i\frac{\rho}{2}x^2\right) - 1\right] \quad (19)$$

where $E(x)$ represents the complex Fresnel integral function of the *first kind* defined by

$$E(x) \triangleq \int_0^x \exp\left(-i\frac{\pi}{2}u^2\right)du. \quad (20)$$

There exists also the complex Fresnel integral function of the *second kind* defined by

$$E_2(x) \triangleq \frac{1}{\sqrt{2\pi}}\int_0^x \frac{\exp(-iu)}{\sqrt{u}}du. \quad (21)$$

The foregoing two functions are related by $E(x) = E_2[(\pi/2)x^2]$. Series and asymptotic representations of these functions can be found, for instance, in Abramowitz and Stegun (1972). We have assumed, without loss of generality, that in Eq. (18) ρ is positive (which corresponds to a spin-up), since from (16) one sees that for ρ negative (spin-down) one merely takes the complex conjugate of Eq. (18), that is, $I_0^*(x;\rho) = I_0(x;-\rho)$ where the asterisk denotes complex conjugation. Both cases can be handled at the same time by defining the function

$$\bar{E}(x) \triangleq \begin{cases} E(x) & \text{if } \rho>0 \\ E^*(x) & \text{if } \rho<0. \end{cases} \quad (22)$$

Since we have assumed that $I_z > I_x > I_y$ the case of $\rho=0$ is of no concern to us, since it corresponds to either $I_z = I_y$ or $I_z = I_x$, and in either case the problem reduces to the axisymmetric case. Therefore, one can solve exactly for one of the components of the angular velocities (ω_x if $I_z = I_y$ or ω_y if $I_z = I_x$) and substitute in the other two. It is clear that by relabeling the axes, the problem can be cast into the current framework of analysis. On the other hand, if $I_z = I_x = I_y$, then the solution of the system of the differential Eqs. (1)-(3) becomes trivial. Hence, in any case we can assume, without loss of generality, that $\rho \neq 0$. Using (22), we have that for *both* spin-up and spin-down, the integral $I_0(x;\rho)$ can be evaluated by

$$I_0(x;\rho) = \sqrt{\pi/|\rho|}\,\mathrm{sgn}(x)\bar{E}\left(\sqrt{|\rho|/\pi}x\right). \quad (23)$$

In this expression sgn denotes the signum function such that $\mathrm{sgn}(x)=1$, for $x>0$, $\mathrm{sgn}(x)=-1$, for $x<0$, and $\mathrm{sgn}(x)=0$, for $x=0$. Therefore, Eqs. (10) and (14) give the complete solution for the transverse components of the angular velocity vector ω_x and ω_y in terms of complex Fresnel integrals (20). The solution for ω_z was found in (4).

3 Analytic Solution of Attitude Motion

We use Eulerian angles to describe the attitude orientation of a rotating body. Although other descriptions of the attitude are also available (Euler parameters, direction cosines) the Eulerian angles have the advantage of directly representing physical quantities, making them amenable to engineering insight. We have chosen to use Eulerian angles in the present analysis mainly for this reason. Using a 3-1-2 Eulerian angle sequence (Wertz, 1980; Kane et al., 1983), the kinematics obey the following set of differential equations:

$$\dot{\phi}_x = \omega_x\cos\phi_y + \omega_z\sin\phi_y \quad (24)$$

$$\dot{\phi}_y = \omega_y - (\omega_z\cos\phi_y - \omega_x\sin\phi_y)\tan\phi_x \quad (25)$$

$$\dot{\phi}_z = (\omega_z\cos\phi_y - \omega_x\sin\phi_y)\sec\phi_x. \quad (26)$$

Under this parameterization of the kinematics, the orientation of the local body-fixed reference frame with respect to the inertial reference frame is found by first rotating the body about its z-axis through an angle ϕ_z, then rotating about its x-axis with an angle ϕ_x and finally by rotating about its y-axis by an angle ϕ_y. Note that a singularity exists in Eqs. (25) and (26) for $\phi_x = \pm\pi/2$. For a 3-1-2 sequence, ϕ_x and ϕ_y describe the attitude deviation of the spin axis from its initial orientation (assumed to be the inertial Z-axis), that is caused from the application of disturbances. For a spin-stabilized body these represent unwanted deviation of the spin axis and are often referred to as the *attitude error components*.

A small angle approximation for ϕ_x, ϕ_y is therefore quite reasonable for a spin-stabilized body, and together with the assumption that the product $\phi_y\omega_x$ in Eq. (26) is usually small compared to ω_z, reduces the previous system of equations, (24)-(26), to the following reduced system of three linear time-varying differential equations:

$$\dot{\phi}_x = \omega_x + \phi_y\omega_z \quad (27)$$

$$\dot{\phi}_y = \omega_y - \phi_x\omega_z \quad (28)$$

$$\dot{\phi}_z = \omega_z. \quad (29)$$

The solution for ϕ_z is given immediately by

$$\phi_z(t) = \int_0^t \omega_z(u)du + \phi_z(0) = \frac{1}{2}(M_z/I_z)t^2 + \omega_{z0}t + \phi_{z0}, \quad \phi_{z0} \triangleq \phi_z(0). \quad (30)$$

Since the solution of ω_z is known from (4), one can combine the two equations (27) and (28) into the following single complex scalar equation:

$$\dot{\phi} + i\omega_z\phi = \omega \quad (31)$$

where the complex variables $\phi(t) \triangleq \phi_x(t) + i\phi_y(t)$ and $\omega(t) \triangleq \omega_x(t) + i\omega_y(t)$ have been introduced. Using the new independent variable τ as was done in Eq. (5), one gets that

$$\frac{d\phi}{d\tau}+i\lambda\tau\phi=\lambda\omega, \tag{32}$$

where λ is defined by $\lambda\triangleq\rho/k=I_z/M_z$. Equation (32) is a linear time-varying differential equation in the complex plane. The complex angle ϕ represents a measure of the total deviation of the spin axis. The differential equation for ϕ has the solution

$$\phi(\tau)=\phi_0\exp\left(-i\frac{\lambda}{2}\tau^2\right)$$
$$+\lambda\exp\left(-i\frac{\lambda}{2}\tau^2\right)\int_{\tau_0}^{\tau}\exp\left(i\frac{\lambda}{2}u^2\right)\omega(u)du \tag{33}$$

where $\phi_0\triangleq\phi(\tau_0)\exp[i(\lambda/2)\tau_0^2]$ and $\phi(\tau_0)$ represents the initial condition for the transverse angles ϕ_x and ϕ_y in the new independent variable. The solution involves an expression for $\omega(\tau)$ instead of $\Omega(\tau)$, which has been already found in Eq. (10). However, it is easy to see that $\omega(\tau)$ can be expressed in terms of the already known solution of $\Omega(\tau)$ to obtain

$$\omega(\tau)=\Omega(\tau)\frac{\sqrt{k_x}+\sqrt{k_y}}{2k}+\Omega^*(\tau)\frac{\sqrt{k_x}-\sqrt{k_y}}{2k}. \tag{34}$$

Therefore, in order to solve for the Eulerian angles, one needs to evaluate the two integrals

$$I_{\phi_1}(\tau_0,\tau;\lambda,\rho)\triangleq\int_{\tau_0}^{\tau}\exp\left(i\frac{\lambda}{2}u^2\right)\Omega(u)du \tag{35}$$

$$I_{\phi_2}(\tau_0,\tau;\lambda,\rho)\triangleq\int_{\tau_0}^{\tau}\exp\left(i\frac{\lambda}{2}u^2\right)\Omega^*(u)du. \tag{36}$$

Recall that from Eqs. (10) and (11) the solution for $\Omega(\tau)$ is given by

$$\Omega(\tau)=\Omega_0\exp\left(i\frac{\rho}{2}\tau^2\right)+\exp\left(i\frac{\rho}{2}\tau^2\right)I_\omega(\tau_0,\tau;\rho) \tag{37}$$

where, from Eq. (14), $I_\omega(\tau_0,\tau;\rho)=\sum_{n=0}^{m}F_n\bar{I}_n(\tau_0,\tau;\rho)$ and $\bar{I}_n(\tau_0,\tau;\rho)$ was given in Eq. (15). If one substitutes (37) into (35) and (36), after carrying out the algebra, one obtains

$$I_{\phi_1}(\tau_0,\tau;\lambda,\rho)=\left[\Omega_0-\sum_{n=0}^{m}F_nI_n(\tau_0;\rho)\right]\int_{\tau_0}^{\tau}\exp\left(i\frac{\mu}{2}u^2\right)du$$
$$+\sum_{n=0}^{m}F_n\int_{\tau_0}^{\tau}\exp\left(i\frac{\mu}{2}u^2\right)I_n(u;\rho)\,du \tag{38}$$

$$I_{\phi_2}(\tau_0,\tau;\lambda,\rho)=\left[\Omega_0^*-\sum_{n=0}^{m}F_n^*I_n^*(\tau_0;\rho)\right]\int_{\tau_0}^{\tau}\exp\left(i\frac{\kappa}{2}u^2\right)du$$
$$+\sum_{n=0}^{m}F_n^*\int_{\tau_0}^{\tau}\exp\left(i\frac{\kappa}{2}u^2\right)I_n^*(u;\rho)\,du \tag{39}$$

where $\mu\triangleq\lambda+\rho=\lambda(1+k)$ and $\kappa\triangleq\lambda-\rho=\lambda(1-k)$. The first integral in (38) is computed by

$$\int_{\tau_0}^{\tau}\exp\left(i\frac{\mu}{2}u^2\right)du=\bar{I}_0(\tau_0,\tau;-\mu). \tag{40}$$

In similar way one handles the first integral in (39). The second integrals appearing in Eqs. (38) and (39) are of the same form and can be handled simultaneously. Therefore, we present next how one computes the second integral in (38), with the provision that the computation of the corresponding integral in (39) is found by taking the complex conjugate. Let the integral in Eq. (38) be denoted by

$$J_n(\tau_0,\tau;\mu,\rho)\triangleq\int_{\tau_0}^{\tau}\exp\left(i\frac{\mu}{2}u^2\right)I_n(u;\rho)du,\ n=0,1,2,\ldots,m. \tag{41}$$

From the recurrence formula for $I_n(u;\rho)$ given in Eq. (17), one can easily verify the following recurrence formula for $J_n(\tau_0,\tau;\mu,\rho)$:

$$J_n(\tau_0,\tau;\mu,\rho)=\frac{i}{\rho}\int_{\tau_0}^{\tau}\exp\left(i\frac{\lambda}{2}u^2\right)u^{n-1}du$$
$$-i\frac{n-1}{\rho}J_{n-2}(\tau_0,\tau;\mu,\rho),\quad n=2,3,4,\ldots. \tag{42}$$

Since one readily computes

$$\int_{\tau_0}^{\tau}\exp\left(i\frac{\lambda}{2}u^2\right)u^{n-1}du=\bar{I}_{n-1}(\tau_0,\tau;-\lambda), \tag{43}$$

the use of the above recurrence formula allows one to reduce the evaluation of the integral $J_n(\tau_0,\tau;\mu,\rho)$ to the evaluation of the first two unknown terms of the sequence, given by

$$J_0(\tau_0,\tau;\mu,\rho)\triangleq\int_{\tau_0}^{\tau}\exp\left(i\frac{\mu}{2}u^2\right)I_0(u;\rho)\,du \tag{44}$$

$$J_1(\tau_0,\tau;\mu,\rho)\triangleq\int_{\tau_0}^{\tau}\exp\left(i\frac{\mu}{2}u^2\right)I_1(u;\rho)\,du. \tag{45}$$

The integral in (45) can easily be computed using Eq. (19) as

$$J_1(\tau_0,\tau;\mu,\rho)=\frac{i}{\rho}\left[\int_{\tau_0}^{\tau}\exp\left(i\frac{\lambda}{2}u^2\right)du-\int_{\tau_0}^{\tau}\exp\left(i\frac{\mu}{2}u^2\right)du\right]$$
$$=\frac{i}{\rho}\left[\bar{I}_0(\tau_0,\tau;-\lambda)-\bar{I}_0(\tau_0,\tau;-\mu)\right]. \tag{46}$$

Using (23), the integral $J_0(\tau_0,\tau;\mu,\rho)$ takes the form

$$J_0(\tau_0,\tau;\mu,\rho)$$
$$=\sqrt{\pi/|\rho|}\int_{\tau_0}^{\tau}\sin(u)\exp\left(i\frac{\mu}{2}u^2\right)\bar{E}\left(\sqrt{|\rho|/\pi}u\right)du. \tag{47}$$

By the change of variable $\bar{u}\triangleq\sqrt{|\rho|/\pi}u$, the above integral simplifies to

$$J_0(\tau_0,\tau;\mu,\rho)=(\pi/|\rho|)\int_{\tau_0}^{\tau}\sin(\bar{u})\exp\left(i\frac{\bar{\mu}}{2}\bar{u}^2\right)\bar{E}(\bar{u})d\bar{u}$$
$$\triangleq(\pi/|\rho|)\bar{J}_0(\bar{\tau}_0,\bar{\tau};\bar{\mu}) \tag{48}$$

where $\bar{\tau}_0\triangleq\sqrt{|\rho|/\pi}\tau_0$, $\bar{\tau}\triangleq\sqrt{|\rho|/\pi}\tau$, and $\bar{\mu}\triangleq\mu\pi/|\rho|$. If one now also drops for notational convenience the bars from the arguments of $\bar{J}_0$, one has merely to examine the integral

$$\bar{J}_0(\tau_0,\tau;\mu)\triangleq\int_{\tau_0}^{\tau}\operatorname{sgn}(u)\exp\left(i\frac{\mu}{2}u^2\right)\bar{E}(u)\,du. \tag{49}$$

The above is an integral that appears in the problem when the forcing transverse torques are constants, and has been thoroughly evaluated by Tsiotras and Longuski (1991a), where series expansions of the Fresnel integral $\bar{E}(u)$ were used, in order to integrate (49) termwise. A very accurate representation of the Fresnel integral was used (Boersma, 1960) based on the τ-method of Lanczos (1956). It is interesting to note that $\bar{J}_0(\tau_0,\tau;\mu)$ is closely related to integrals appearing in the mathematical theory of rocket flight (Rosser et al., 1947). Summing up, we have that $I_{\phi_1}(\tau_0,\tau;\lambda,\rho)$ and $I_{\phi_2}(\tau_0,\tau;\lambda,\rho)$ are computed by

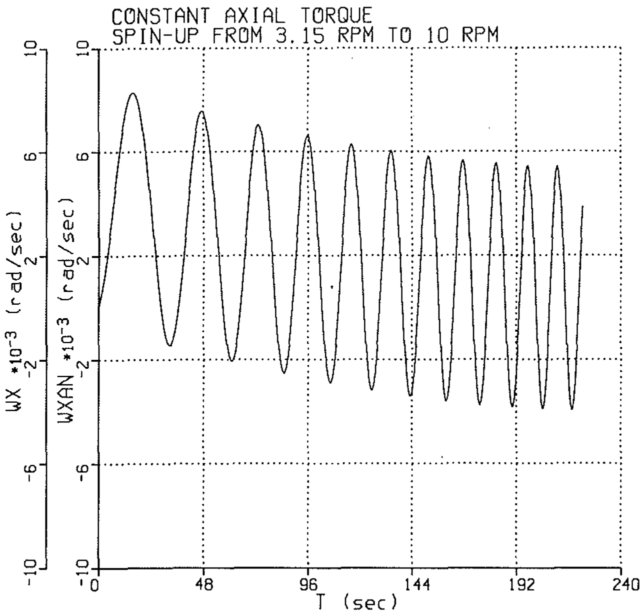

Fig. 1 Exact and analytic solutions for $\omega_x(t)$

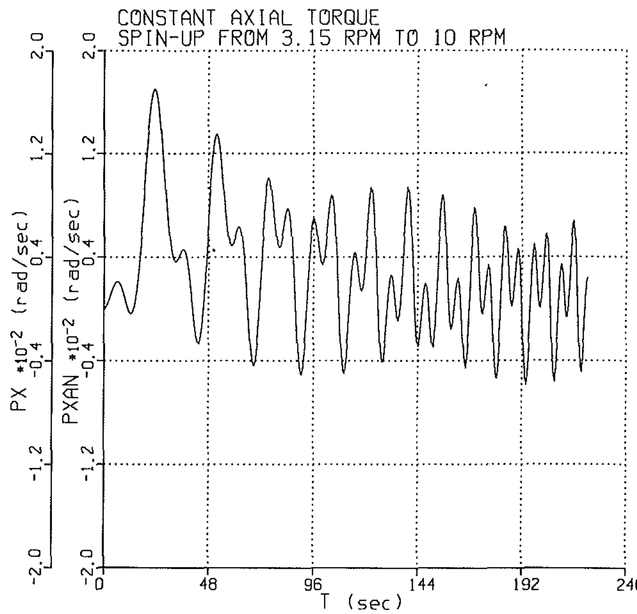

Fig. 2 Exact and analytic solutions for $\phi_x(t)$

$$I_{\phi_1}(\tau_0,\tau;\lambda,\rho)=\left[\Omega_0-\sum_{n=0}^{m}F_nI_n(\tau_0;\rho)\right]\bar{I}_0(\tau_0,\tau;-\mu)+\sum_{n=0}^{m}F_nJ_n(\tau_0,\tau;\mu,\rho) \quad (50)$$

$$I_{\phi_2}(\tau_0,\tau;\lambda,\rho)=\left[\Omega_0^*-\sum_{n=0}^{m}F_n^*I_n^*(\tau_0;\rho)\right]\bar{I}_0(\tau_0,\tau;-\kappa)+\sum_{n=0}^{m}F_n^*J_n^*(\tau_0,\tau;-\kappa,\rho). \quad (51)$$

The above relations along with (33)–(36) and (30), provide the *complete* analytical solution to the differential Eqs. (29) and (31), for the orientation of a near-symmetric spinning rigid body, when the forcing transverse torques are polynomial functions of time.

4 Numerical Example

A numerical example is used in order to demonstrate the accuracy of the analytic solutions. The inertia parameters are based on the Galileo spacecraft (Longuski, 1991) and are given by $I_x = 2985$ kg-m^2, $I_y = 2729$ kg-m^2, and $I_z = 4183$ kg-m^2. The Galileo spacecraft is a dual spin vehicle. It can operate in an all-spin mode in which the rotor and the stator revolve together, or a dual-spin mode with the stator fixed in inertial frame. The above values correspond to its all-spin mode, i.e., when the stator and the rotor are locked and are rotating with the same spin velocity. We consider first a *spin-up maneuver* from $\omega_z(0) = 3.15$ rpm to $\omega_z(t_f) = 10$ rpm. For the purpose of illustration, the transverse torques M_x and M_y are assumed to be given by

$$M_x(t) = 1.0 + 2.7\times10^{-2}t - 2.4\times10^{-4}t^2 + 5.7\times10^{-7}t^3\,\text{N-m} \quad (52)$$

$$M_y(t) = -1.5 - 9.0\times10^{-3}t + 1.2\times10^{-4}t^2 - 3.0\times10^{-7}t^3\,\text{N-m}. \quad (53)$$

The coefficients of the corresponding polynomials of M_x and M_y in terms of τ in Eq. (12) can then be computed, using (4) and (5).

The initial conditions for ω_x, ω_y, ϕ_x, ϕ_y, and ϕ_z are all assumed to be zero. The spin-up maneuver is simulated with a constant M_z torque of 13.5 N-m. The results are shown in Figs. 1 and 2. In these figures the captions refer to the *exact* solution which means a precise numerical integration of the original differential equations (Eqs. (1)–(3) and Eqs. (24)–(26)). As shown in the figures, the analytic solutions are very close to the exact solutions for the case of a spin-up maneuver. In fact, the analytic solutions are indistinguishable from the exact solutions when presented on the same plot. Only the ω_x and ϕ_x solutions are plotted since ω_y and ϕ_y exhibited similar behavior.

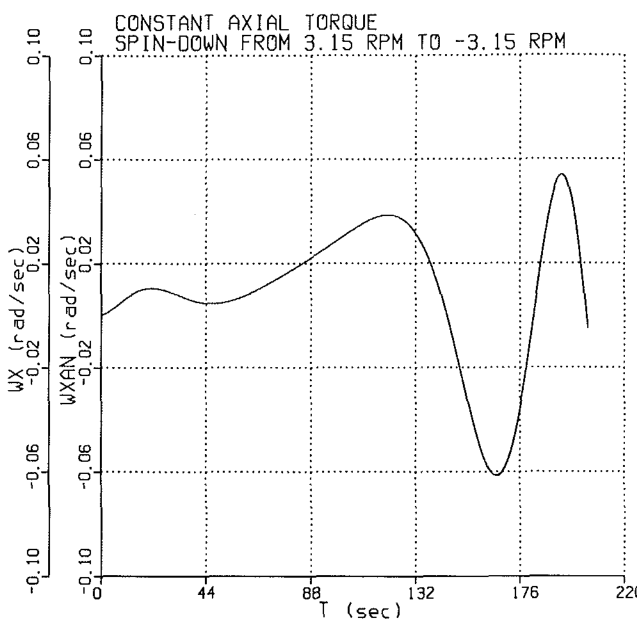

Fig. 3 Exact and analytic solutions for $\omega_x(t)$

A spin-down maneuver through zero spin rate, is considered next, from $\omega_z(0) = 3.15$ rpm to $\omega_z(t_f) = -3.15$ rpm, with all other initial conditions set to zero, as before. The torque M_z is considered to be constant at $M_z = -13.5$ N-m. The transverse torques are given again by Eqs. (52) and (53). The results are shown in Figs. 3 through 5. Once again, we see that the analytic solutions for the angular velocities cannot be differentiated from the exact solutions when plotted in the same figure. The

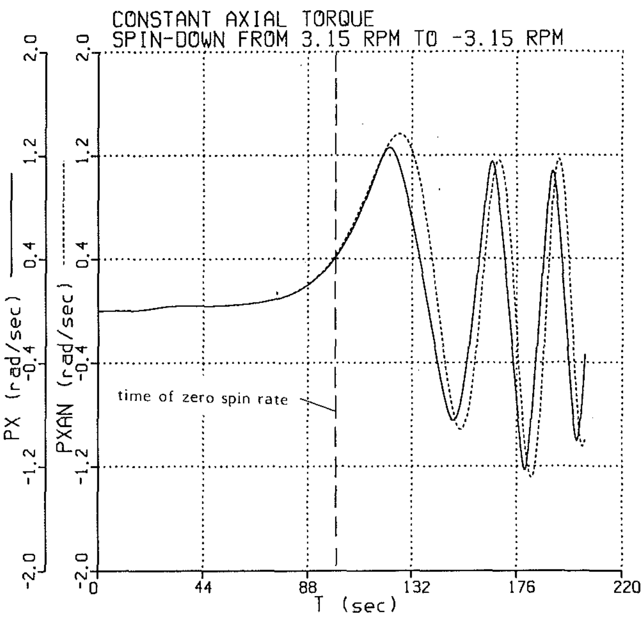

Fig. 4 Exact and analytic solutions for $\phi_x(t)$

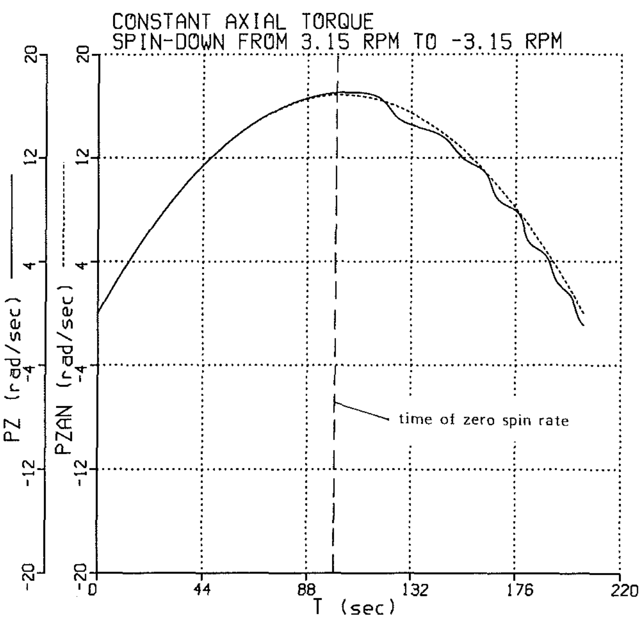

Fig. 5 Exact and analytic solutions for $\phi_z(t)$

spin-down maneuver represents a much more difficult case for the analytic solutions of the Eulerian angles when zero spin rate is reached, because the stabilizing effect of spinning about the maximum principal axis of inertia vanishes, and the transverse torques cause the angles to grow suddenly, as can be seen in Fig. 4. At this point the small angle assumption begins to break down and nonlinear behavior is exhibited. This is clear from Figs. 4 and 5 where the solid line indicates the exact solution, while the dotted line represents the analytic solution. It is interesting to note, however, that even though the value of ϕ_x exceeds 1 radian in Fig. 4, the general behavior is still qualitatively predicted by the analytic solution. For a further discussion of the large angle problem see Tsiotras and Longuski (1992).

5 Conclusions

Approximate analytic solutions have been derived for the attitude motion of a near-symmetric spinning rigid body, under the influence of transverse time-varying torques, expressed as polynomial functions. The torque about the spinning axis is assumed to remain constant. Analytic solutions were derived for both the angular velocity vector and the Eulerian angles. These solutions were shown to be very accurate when compared with the exact solutions for typical spacecraft maneuvers in a numerical example. In fact, the solution of the angular velocities are exact when the rigid body is axisymmetric. The solutions for the Eulerian angles are always approximate due to a small angle assumption, but they remain valid for a large class of practical applications. Part II of the paper extends the preceding results to include cases when all *three* components of the external torque vector are time-varying.

Acknowledgment

This research has been supported by the National Science Foundation under Grant No. MSS-9114388 (NSF program official Elbert L. Marsh).

References

Abramowitz, M., and Stegun, I. A., 1972, *A Handbook of Mathematical Functions*, Dover Publications Inc., New York.

Boersma J., 1960, "Computation of Fresnel Integrals," *Math. Comp.*, Vol. 14, p. 380.

Bois, E., 1986, "First-Order Theory of Satellite Attitude Motion Application to Hipparcos," *Celest. Mech.*, Vol. 39, pp. 309–327.

Hintz, G. R., and Longuski, J. M., 1985, "Error Analysis for the Delivery of a Spinning Probe to Jupiter," *Journal of Guidance, Control, and Dynamics*, Vol. 8, No. 3, pp. 384–390.

Kane, T. R., Likins, P. W., and Levinson, P. A., 1983, *Spacecraft Dynamics*, McGraw-Hill, New York.

Kia, T., and Longuski, J. M., 1984, "Error Analysis of Analytic Solutions for Self-Excited Near Symmetric Rigid Bodies: A Numerical Study," AIAA Paper 84-2018, AIAA/AAS Astrodynamics Conference, Seattle, WA, Aug. 20–22.

Lanczos C., 1956, *Applied Analysis*, Prentice-Hall, Englewood Cliffs, NJ.

Longuski, J. M., 1980, "Solution of Euler's Equations of Motion and Eulerian Angles for Near Symmetric Rigid Bodies Subject to Constant Moments," AIAA Paper 80-1642, AIAA/AAS Astrodynamics Conference, Danvers, MA, Aug. 11–13.

Longuski, J. M., 1991, "Real Solutions for the Attitude Motion of a Self-Excited Rigid Body," *Acta Astron.*, Vol. 25, No. 3, pp. 131–140.

Longuski, J. M., Kia, T., and Breckenridge, W. G., 1989, "Annihilation of Angular Momentum Bias During Thrusting and Spinning-up Maneuvers," *J. Astron. Sci.*, Vol. 37, No. 4, pp. 433–450.

Rosser, J. B., Newton, R. R., and Gross, G. L., 1947, *Mathematical Theory of Rocket Flight*, McGraw-Hill, New York.

Tsiotras, P., and Longuski, J. M., 1991a, "Complex Analytic Solution for the Attitude Motion of a Near-Symmetric Rigid Body Under Body-Fixed Torques," *Celest. Mech.*, Vol. 51, pp. 281–301.

Tsiotras, P., and Longuski, J. M., 1991b, "Analytic Solutions for the Attitude Motion of Spinning Rigid Bodies Subject to Periodic Torques," AAS Paper 91-404, AAS/AIAA Astrodynamics Conference, Durango, CO, Aug. 19–22.

Tsiotras, P., and Longuski, J. M., 1992, "On the Large Angle Problem in Rigid Body Attitude Dynamics," IAF Paper 92-0034, 43rd Congress of the International Astronautical Federation, Washington, D.C., Aug. 28–Sept. 5.

Wertz, J. R., 1980, *Spacecraft Attitude Determination and Control*, D. Reidel Publishing Company, Dordrecht, Holland.