Laplacian Cooperative Attitude Control of Multiple Rigid Bodies

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***Abstract*— Motivated by the fact that linear controllers can stabilize the rotational motion of a rigid body, we propose in this paper a control strategy that exploits graph theoretic tools for cooperative control of multiple rigid bodies. The control objective is to stabilize the system to a configuration where the rigid bodies will have a common orientation and common angular velocity. The control law respects the limited information each rigid body has with respect to the rest of the team. Specifically, each rigid body is equipped with a control law that is based on the Laplacian matrix of the communication graph, which encodes the limited communication capabilities between the team members. Similarly to the linear case, the convergence of the multi-agent system relies on the connectivity of the communication graph.**

I. INTRODUCTION

Cooperative distributed control of multiple vehicles has gained increased attention in recent years in the control community, due to the fact that it provides feasible solutions to large-scale multi-agent problems, in terms both of complexity and computational load.

Among the various specifications the control design aims to impose on the multi-agent team is the state-agreement or consensus problem, i.e. convergence of the multi-agent system to a common configuration. This design objective has been extensively pursued in the last few years. In most cases, vehicle motion is modelled by a single integrator [5],[1], while double integrator models have also been considered [9]. A recent review of the various approaches for solving the consensus problem when the underlying dynamics are linear is found in [7]. A common analysis tool that is frequently used to model these distributed systems is algebraic graph theory [2].

Motivated by the fact that linear controllers can stabilize a rigid body [11], in this paper we propose a control strategy that exploits graph theoretic tools for cooperative control of multiple rigid bodies. The control objective is to stabilize the system to a configuration where all the rigid bodies have a common orientation and common angular velocity. When the desired angular velocity is set to zero for all agents, all the satellites end up at relative orientations which can be defined a priori (and which can also be zero for the case of the same desired orientation for all the satellites). The proposed control law for each agent respects the limited information each rigid body has with respect to the rest of the team.

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Cooperative control of multiple rigid bodies has been addressed recently by many authors [3], [12], [4]. While these papers use distributed consensus algorithms to achieve the desired objective, they are not directly related to the algebraic graph theoretic framework encountered in this work. Specifically, here we equip each rigid body with a control law that is based on the Laplacian matrix of the communication graph, which encodes the limited communication capabilities between the team members. Similarly to the linear case, the convergence of the multi-agent system relies on the connectivity of the communication graph. The results are also extended to the case when each rigid body converges to a desired–not necessarily zero–orientation with respect to each of the agents with which it can communicate. We should note that similar results to the ones presented in this paper, were derived at almost the same time in the recent paper [6].

The rest of the paper is organized as follows: Section II describes the system and the three problems treated in this paper. Assumptions regarding the communication topology between the agents are also presented, and modelled in terms of an undirected graph. Section III begins with some background on algebraic graph theory that is used in the sequel, and proceeds with the development of the proposed distributed feedback control strategy. This strategy drives the multi-agent team to a common configuration. In the same section the stability analysis for each of the problems introduced in Section II is given. Computer simulations are included in Section IV to illustrate the success of the proposed approach. Section V summarizes the results of this paper and indicates some current research efforts.

II. SYSTEM AND PROBLEM DEFINITION

Consider a team of N rigid bodies (henceforth called agents) indexed by $\mathcal{N} = \{1, \ldots, N\}$. The dynamics of agent i are given by [11]:

$$J_i \dot{\omega}_i = S(\omega_i) J_i \omega_i + u_i, \quad i \in \mathcal{N}, \tag{1}$$

where $\omega_i \in \mathbb{R}^3$ is the angular velocity vector in each satellite's body fixed frame, $u_i \in \mathbb{R}^3$ is the acting torque vector, and J_i is the symmetric inertia matrix of agent i. The matrix $S(\cdot)$ denotes a skew-symmetric matrix representing the cross product between two vectors, i.e. $S(v_1)v_2 = -v_1 \times v_2$.

In this paper, the orientation of the rigid bodies with respect to the inertial frame are described in terms of the Modified Rodriguez Parameters (MRPs)[8], [10]. The kinematics of agent i in terms of the MRPs, are given by:

$$\dot{\sigma}_i = G_i(\sigma_i) \omega_i, \quad i \in \mathcal{N}, \tag{2}$$

where the matrix G_i is given by

$$G_i(\sigma_i) = \frac{1}{2}\left(\frac{1-\sigma_i^\intercal \sigma_i}{2} I_3 - S_i(\sigma_i) + \sigma_i \sigma_i^\intercal\right),$$

and has the following properties [11]

$$\sigma_i^\intercal G_i(\sigma_i)\,\omega_i = \left(\frac{1+\sigma_i^\intercal \sigma_i}{4}\right)\sigma_i^\intercal \omega_i, \qquad (3)$$

$$G_i(\sigma_i)\,G_i^\intercal(\sigma_i) = \left(\frac{1+\sigma_i^\intercal \sigma_i}{4}\right)^2 I_3. \qquad (4)$$

Each agent is assigned a subset $\mathcal{N}_i \subset \mathcal{N}$ from the rest of the team, called agent i's *communication set*, that includes the agents with which it can communicate in order to achieve the desired objective. The limited inter-agent communication is encoded in terms of a *communication graph*:

Definition 1: The *communication graph* $\mathcal{G} = \{V, E\}$ is an undirected graph that consists of a set of vertices $V = \{1, \ldots, N\}$ indexed by the team members, and a set of edges, $E = \{(i,j) \in V \times V : j \in \mathcal{N}_i\}$ containing pairs of nodes that represent inter-agent communication specifications.

We assume that the formation graph is undirected, in the sense that $i \in \mathcal{N}_j \Leftrightarrow j \in \mathcal{N}_i, \quad \forall i,j \in \mathcal{N},\ i \neq j$.. It is obvious that $(i,j) \in E$ if and only if $i \in \mathcal{N}_j \Leftrightarrow j \in \mathcal{N}_i$.

The control law is of the form

$$u_i = u_i(\omega_i, \sigma_i, \omega_j, \sigma_j),\ j \in \mathcal{N}_i \qquad (5)$$

representing the limited communication capabilities of each agent. The three problems treated in this paper can now be stated as follows (denoted by P1, P2, P3):

- **P1** Derive distributed control laws of the form (5) that drive the team of N rigid bodies to a common configuration with respect to both orientation and angular velocities.
- **P2** Derive distributed control laws of the form (5) that drive the team of N rigid bodies to a common zero angular velocity and common orientation.
- **P3** Derive distributed control laws of the form (5) that drive the team of N rigid bodies to a configuration where all rigid bodies have the same angular velocity, while their final relative orientations are prescribed a priori.

III. Control Design and Stability Analysis

A. Tools from Algebraic Graph Theory

In this subsection we review some tools from algebraic graph theory [2] that we use in the sequel.

For an undirected graph $\mathcal{G}$ with n vertices, the *adjacency matrix* $A = A(\mathcal{G}) = (a_{ij})$ is the $n \times n$ symmetric matrix given by $a_{ij} = 1$, if $(i,j) \in E$ and $a_{ij} = 0$, otherwise. If there is an edge connecting two vertices i, j, i.e. $(i,j) \in E$, then i, j are called *adjacent*. A *path* of length r from a vertex i to a vertex j is a sequence of $r+1$ distinct vertices starting with i and ending with j such that consecutive vertices are adjacent. If there is a path between any two vertices of $\mathcal{G}$, then $\mathcal{G}$ is called *connected* (otherwise it is called *disconnected*). The *degree* d_i of vertex i is the number of its neighboring vertices, i.e. $d_i = \{\#j : (i,j) \in E\} = |\mathcal{N}_i|$. Let Δ be the $n \times n$ diagonal matrix of d_i's. The (combinatorial) *Laplacian* of $\mathcal{G}$ is the symmetric positive semidefinite matrix $L = \Delta - A$. The Laplacian matrix L captures many topological properties of the graph. Of particular interest is the fact that for a connected graph, the Laplacian has a single zero eigenvalue and the corresponding eigenvector is the vector of ones, denoted by $\overrightarrow{\mathbf{1}}$.

B. Proposed Control Strategy-Problem 1

For Problem 1 we propose the feedback control strategy for agent i as follows:

$$u_i = -G_i(\sigma_i)\sum_{j \in \mathcal{N}_i}(\sigma_i - \sigma_j) - \sum_{j \in \mathcal{N}_i}(\omega_i - \omega_j). \qquad (6)$$

This control strategy respects the limited communication ruling between the members of the multi-agent team. Under this control strategy the following theorem holds:

Theorem 1: Assume that the communication graph is connected. Then the control strategy (6) is a solution to Problem 1.

Proof: Let $u, \omega, \sigma \in \mathbb{R}^{3N}$ be the stack vectors of all the control inputs, the angular velocities and the orientations of the multi-agent team, respectively. Then it is easily derived from (6) that

$$u = -G^\intercal(\sigma)(L \otimes I_3)\,\sigma - (L \otimes I_3)\,\omega,$$

where

$$G(\sigma) = \mathrm{blockdiag}(G_1(\sigma_1), \ldots, G_N(\sigma_N)),$$

where L denotes the Laplacian of the associated communication graph, and $\otimes$ denotes the standard Kronecker product between two matrices. Let us now choose

$$V(\sigma, \omega) = \sum_{i=1}^{N}\left(\frac{1}{2}\omega_i^\intercal J_i \omega_i\right) + \frac{1}{2}\sigma^\intercal (L \otimes I_3)\,\sigma$$

as a candidate Lyapunov function. Function V is positive semidefinite.

The level sets of V define compact sets in the product space of agents' angular velocities and *relative* orientations. Specifically, the set $\Omega_c = \{(\omega, \sigma) : V(\sigma, \omega) \leq c\}$ for $c > 0$ is closed by continuity of V. For all $(\omega, \sigma) \in \Omega_c$ we have

$$\omega_i^\intercal J_i \omega_i \leq 2c \Rightarrow \|\omega_i\| \leq \sqrt{\frac{2c}{\lambda_{\min}(J_i)}}.$$

Furthermore, we also have

$$\sigma^\intercal (L \otimes I_3)\,\sigma \leq 2c \Rightarrow \frac{1}{2}\sum_{i=1}^{N}\sum_{j \in \mathcal{N}_i}\|\sigma_i - \sigma_j\|^2 \leq 2c \Rightarrow$$
$$\Rightarrow \|\sigma_i - \sigma_j\|^2 \leq 4c, \qquad \forall (i,j) \in E.$$

Connectivity of $\mathcal{G}$ ensures that the maximum length of a path connecting two vertices of the graph is at most $N-1$. Hence $\|\sigma_i - \sigma_j\| \leq 2\sqrt{c(N-1)}$, for all $i, j \in \mathcal{N}$.

Differentiating now V with respect to time, we get

$$\dot{V}(\sigma, \omega) = \sum_{i=1}^{N}(\omega_i^\intercal J_i \dot{\omega}_i) + \sigma^\intercal (L \otimes I_3)\,\dot{\sigma}$$
$$= u^\intercal \omega + \sigma^\intercal (L \otimes I_3)\,G(\sigma)\,\omega.$$

With the choice of the control law in (6) we get

$$\dot{V}(\sigma,\omega) = -\omega^{\intercal}\left(L\otimes I_3\right)\omega \leq 0.$$

By LaSalle's invariance principle, the system converges to the largest invariant set inside the set

$$M = \left\{(\sigma,\omega) : \omega^{\intercal}\left(L\otimes I_3\right)\omega = 0\right\}.$$

Since $L \otimes I_3$ is positive semidefinite, if follows that $(L\otimes I_3)\omega = 0$ which implies that

$$L\omega^1 = L\omega^2 = L\omega^3 = 0, \tag{7}$$

where $\omega^1,\omega^2,\omega^3 \in \mathbb{R}^N$ are the stack vectors of the three coefficients of the agents' angular velocities, respectively. Connectivity of the communication graph implies that L has a simple zero eigenvalue with corresponding eigenvector $\overrightarrow{\mathbf{1}}$. Equation (7) now implies that $\omega^1,\omega^2,\omega^3$ are eigenvectors of L corresponding to the zero eigenvalue, thus they belong to span$\{\overrightarrow{\mathbf{1}}\}$. Hence $\omega_i = \omega_j$ for all $i,j\in\mathcal{N}$, implying that all ω_i's converge to a common value at steady state.

We next proceed to show that this common value ω^* is constant. Inside the set M, we have $\left(L\otimes I_3\right)\omega = 0$ hence also $\left(L\otimes I_3\right)\dot{\omega} = 0$, which yields $L\dot{\omega}^1 = L\dot{\omega}^2 = L\dot{\omega}^3 = 0$ and following the same argument as before, that $\dot{\omega}^1,\dot{\omega}^2,\dot{\omega}^3 \in \mathrm{span}\{\overrightarrow{1}\}$. As a result, we have shown that $\dot{\omega}_i = \dot{\omega}^*$ for all $i\in\mathcal{N}$, that is, the angular accelerations converge to a common value as well. Inside the set M, we also have

$$J_i\dot{\omega}_i = S\left(\omega_i\right)J_i\omega_i + u_i \Rightarrow \omega_i^{\intercal}J_i\dot{\omega}_i = \omega_i^{\intercal}\,u_i \Rightarrow$$
$$\Rightarrow \omega_i^{\intercal}\left(J_i\dot{\omega}_i + G_i\left(\sigma_i\right)\sum_{j\in\mathcal{N}_i}\left(\sigma_i-\sigma_j\right)\right) = 0,$$

or in stack vector form,

$$\omega^{\intercal}\left(J\dot{\omega} + G^{\intercal}(\sigma)\left(L\otimes I_3\right)\sigma\right) = 0,$$

where $J \triangleq \mathrm{blockdiag}\left(J_1,\ldots,J_N\right)$. Now since $\omega^1,\omega^2,\omega^3 \in \mathrm{span}\{\overrightarrow{\mathbf{1}}\}$ the last equation implies that

$$\sum_{i=1}^{N}\left(J_i\dot{\omega}_i + G_i\left(\sigma_i\right)\sum_{j\in\mathcal{N}_i}\left(\sigma_i-\sigma_j\right)\right) = 0,$$

or

$$\dot{\omega}_i = \dot{\omega}^* = -\left(\sum_{i=1}^{N}J_i\right)^{-1}\sum_{i=1}^{N}G_i\left(\sigma_i\right)\sum_{j\in\mathcal{N}_i}\left(\sigma_i-\sigma_j\right),$$

for all $i\in\mathcal{N}$. Using $\tilde{J} \triangleq \left(\sum_{i=1}^{N}J_i\right)^{-1}$ we now have

$$\sum_{i=1}^{N}\dot{\omega}_i = \sum_{i=1}^{N}\dot{\omega}^* = -N\tilde{J}\sum_{i=1}^{N}\sum_{i=1}^{N}G_i\left(\sigma_i\right)\sum_{j\in\mathcal{N}_i}\left(\sigma_i-\sigma_j\right)$$
$$\Rightarrow \sum_{i=1}^{N}\dot{\omega}_i = -N\tilde{J}\left(\sum_{i=1}^{N}G_i\left(\sigma_i\right)\right)\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\left(\sigma_i-\sigma_j\right).$$

The fact that the communication graph is undirected implies

$$\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\left(\sigma_i-\sigma_j\right) = 0,$$

and hence

$$\sum_{i=1}^{N}\dot{\omega}_i = \sum_{i=1}^{N}\dot{\omega}^* = 0 \Rightarrow \dot{\omega}^* = 0.$$

It follows that the common angular acceleration of the rigid bodies is zero, and therefore the common angular velocity for all agents, ω^*, is in fact, constant. This, in turn, implies that $\dot{\omega}_i = 0$ for all $i\in\mathcal{N}$ and from equation (1) the control inputs of each rigid body must also be zero. Hence we have $u = 0$ for all trajectories inside the set M, which implies

$$G^{\intercal}\left(\sigma\right)\left(L\otimes I_3\right)\sigma = 0 \Rightarrow G\left(\sigma\right)G^{\intercal}\left(\sigma\right)\left(L\otimes I_3\right)\sigma = 0$$

or

$$\left(\Sigma\otimes I_3\right)\left(L\otimes I_3\right)\sigma = \left(\Sigma L\otimes I_3\right)\sigma = 0, \tag{8}$$

where

$$\Sigma = \mathrm{diag}\left(\left(\frac{1+\sigma_1^{\intercal}\sigma_1}{4}\right)^2,\ldots,\left(\frac{1+\sigma_N^{\intercal}\sigma_N}{4}\right)^2\right).$$

It follows from (8) that

$$\Sigma L\sigma^1 = \Sigma L\sigma^2 = \Sigma L\sigma^3 = 0,$$

where $\sigma^1,\sigma^2,\sigma^3 \in \mathbb{R}^N$ are the stack vectors of the three coefficients of the agents' orientations, respectively. The spectral properties of L are retained under the multiplication with the positive definite diagonal matrix Σ. Hence the σ_i's converge to a common value as well as $t\to\infty$. ■

Remark 1: It should be noted at this point that while the control law (6) guarantees that the agents will converge to a configuration where $\omega_1(t) = \ldots = \omega_N(t) = \omega^*$, with ω^* constant, and $\sigma_1(t) = \ldots = \sigma_N(t) = \sigma^*(t)$, as $t\to\infty$, it is nonetheless not guaranteed that ω^* will be equal to zero. Consecutively, it is not guaranteed that $\sigma^*(t)$ will reach a constant value. The latter is guaranteed if we add the additional constraint that $\omega^* = 0$. This is achieved with the treatment of Problem 2, which is discussed in the sequel.

C. Proposed Control Strategy-Problem 2

Theorem 1 guarantees that the team of rigid bodies will converge to a common constant angular velocity, while their orientations will eventually have a common value, which may not remain constant. In fact, it will not be constant unless the common, final angular velocity all satellites converge to is zero (see Remark 1). In order to ensure that all agents converge to the same constant orientation, in this section we show that it is sufficient that one agent has a damping element on the angular velocity. Without loss of generality, we assume that this is agent 1. The following theorem is the main result of this section:

Theorem 2: Assume that the communication graph is connected. Then the control strategy

$$u_i = -G_i^{\intercal}\left(\sigma_i\right)\sum_{j\in\mathcal{N}_i}\left(\sigma_i-\sigma_j\right) - \sum_{j\in\mathcal{N}_i}\left(\omega_i-\omega_j\right) - a_i\omega_i, \tag{9}$$

where $i = 1, \dots, N$ and $a_i = 1$, if $i = 1$, and $a_i = 0$, otherwise, is a suitable solution to Problem 2.

Proof: We choose again

$$V(\sigma,\omega) = \sum_{i=1}^{N} \left(\frac{1}{2}\omega_i^{\intercal} J_i \omega_i\right) + \frac{1}{2}\sigma^{\intercal} (L \otimes I_3)\, \sigma$$

as a candidate Lyapunov function. Differentiating with respect to time and after some manipulation we get

$$\dot{V}(\sigma,\omega) = -\omega^{\intercal} (L \otimes I_3)\, \omega - \|\omega_1\|^2 \leq 0.$$

It follows that ω remains bounded. By LaSalle's invariance principle, the system converges to the largest invariant set inside the set

$$M = \left\{(\sigma,\omega) : (\omega^{\intercal} (L \otimes I_3)\, \omega = 0) \wedge (\omega_1 = 0)\right\}.$$

Similarly to the proof of Theorem 1, the condition $\omega^{\intercal} (L \otimes I_3)\, \omega = 0$ guarantees that all ω_i's converge to a common value. Since $\omega_1 = 0$, this common value is zero. Furthermore, the orientations of the agents converge to a common value, which is constant, due to the fact that $\omega_i = 0$ for all $i \in \mathcal{N}$. ∎

D. Proposed Control Strategy-Problem 3

In the discussion thus far, it has been assumed that it is desirable that all rigid bodies converge to the same orientation. For some applications (i.e., Earth monitoring or stellar observation using a satellite cluster with a large baseline) it may be necessary for the rigid bodies to acquire and maintain a certain (perhaps nonzero) relative orientation among themselves. The relative orientation for each pair of rigid bodies may be different, and can be dictated by the mission requirements. In this section, we thus impose the specification that for each pair $(i,j) \in E$, there exists a desired relative orientation $\sigma_{ij}^d \in \mathbb{R}^3$ to which we wish the pair of rigid bodies to converge. Next, we show how to modify the control law (6) in order to achieve this objective.

Throughout this section, we assume that there are no conflicting inter-agent objectives. Hence, we assume that $\sigma_{ij}^d = -\sigma_{ji}^d, \quad \forall\, i,j \in \mathcal{N},\ i \neq j$. The next theorem proposes a control law to achieve the objective stated previously.

Theorem 3: Assume that the communication graph is connected. Then the control strategy

$$u_i = -G_i^{\intercal}(\sigma_i) \sum_{j \in \mathcal{N}_i} \left(\sigma_i - \sigma_j - \sigma_{ij}^d\right) - \sum_{j \in \mathcal{N}_i} (\omega_i - \omega_j), \quad (10)$$

where $i = 1, \dots, N$ is a suitable solution for Problem 3.

Proof: For each agent i, we define the "cost function"

$$\gamma_i(\sigma) = \frac{1}{2} \sum_{j \in \mathcal{N}_i} \left\|\sigma_i - \sigma_j - \sigma_{ij}^d\right\|^2,$$

and we introduce

$$V(\sigma,\omega) = \sum_{i=1}^{N} \left(\frac{1}{2}\omega_i^{\intercal} J_i \omega_i\right) + \frac{1}{2}\sum_{i=1}^{N} \gamma_i(\sigma)$$

as a candidate Lyapunov function. We then have

$$\dot{V}(\sigma,\omega) = \sum_{i=1}^{N} \omega_i^{\intercal} J_i \dot{\omega}_i + \frac{1}{2}\sum_{i=1}^{N} \nabla\gamma_i \dot{\sigma}.$$

With a slight abuse of notation we rewrite the last term in the previous equation as

$$\nabla\gamma_i = \begin{bmatrix} \frac{\partial^{\intercal}\gamma_i}{\partial\sigma_1} & \dots & \frac{\partial^{\intercal}\gamma_i}{\partial\sigma_N} \end{bmatrix},$$

where,

$$\frac{\partial\gamma_i}{\partial\sigma_j} = \begin{cases} \sum_{j\in\mathcal{N}_i} (\sigma_i - \sigma_j) + \sigma_{ii}^d, & i = j, \\ -\sigma_i + \sigma_j + \sigma_{ij}^d, & j \in \mathcal{N}_i,\ j \neq i, \\ 0, & j \notin \mathcal{N}_i. \end{cases}$$

where we have defined $\sigma_{ii}^d = -\sum_{j\in\mathcal{N}_i} \sigma_{ij}^d$. Hence,

$$\begin{aligned} \sum_{i=1}^{N} \frac{\partial\gamma_i}{\partial\sigma_j} &= \frac{\partial\gamma_j}{\partial\sigma_j} + \sum_{i\in\mathcal{N}_j} \frac{\partial\gamma_i}{\partial\sigma_j} \\ &= \sum_{i\in\mathcal{N}_j} (\sigma_j - \sigma_i) + \sigma_{jj}^d + \sum_{i\in\mathcal{N}_j} (-\sigma_i + \sigma_j + \sigma_{ij}^d) \\ &= 2\sum_{i\in\mathcal{N}_j} \sigma_j - 2\sum_{i\in\mathcal{N}_j} \sigma_i + 2\sigma_{jj}^d \\ &= 2 d_j \sigma_j - 2\sum_{i\in\mathcal{N}_j} \sigma_i + 2\sigma_{jj}^d. \end{aligned}$$

It follows that

$$\begin{aligned} &\sum_{i=1}^{N} \nabla\gamma_i = \sum_{i=1}^{N} \begin{bmatrix} \frac{\partial\gamma_i}{\partial\sigma_1} & \cdots & \frac{\partial\gamma_i}{\partial\sigma_N} \end{bmatrix} \\ &= 2\begin{bmatrix} d_1\sigma_1 & \cdots & d_N\sigma_N \end{bmatrix} - 2\begin{bmatrix} \sum_{j\in\mathcal{N}_1} \sigma_j & \cdots & \sum_{j\in\mathcal{N}_N} \sigma_j \end{bmatrix} \\ &\quad + 2\begin{bmatrix} \sigma_{11}^d & \cdots & \sigma_{NN}^d \end{bmatrix} \end{aligned}$$

and finally,

$$\sum_{i=1}^{N} \nabla\gamma_i = 2\left((L \otimes I_3)\,\sigma + c_\ell\right)^{\intercal}, \quad (11)$$

where $c_\ell = \left[\sigma_{11}^d \cdots \sigma_{NN}^d\right]^{\intercal}$. Using (11), $\dot{V}$ can be written as

$$\dot{V}(\sigma,\omega) = u^{\intercal}\omega + \left((L \otimes I_3)\,\sigma + c_\ell\right)^{\intercal} G(\sigma)\,\omega.$$

From (10) it can be easily derived that

$$u = -G^{\intercal}(\sigma)\left((L \otimes I_3)\,\sigma + c_\ell\right) - (L \otimes I_3)\,\omega.$$

With the previous choice of the control law we get

$$\dot{V}(\sigma,\omega) = -\omega^{\intercal} (L \otimes I_3)\,\omega \leq 0.$$

By LaSalle's invariance principle, the system converges to the largest invariant set inside the set

$$M = \left\{(\sigma,\omega) : \omega^{\intercal} (L \otimes I_3)\,\omega = 0\right\}.$$

The condition $\omega^{\intercal} (L \otimes I_3)\,\omega = 0$ guarantees again that all ω_i's converge to a common value. Following the proof of Theorem 1, we deduce that the invariance of M, along with $u = 0$ implies that $G^{\intercal}(\sigma)\left((L \otimes I_3)\,\sigma + c_\ell\right) = 0$ or $G(\sigma)\,G^{\intercal}(\sigma)\left((L \otimes I_3)\,\sigma + c_\ell\right) = 0$, which yields

$(\Sigma \otimes I_3)\left((L \otimes I_3)\,\sigma + c_\ell\right) = 0$. Finally, it follows that $(L \otimes I_3)\,\sigma + c_\ell = 0$ due to the positive definiteness of Σ.

For all $i \in \mathcal{N}$, let σ_i^d denote the desired orientation of agent i with respect to the global coordinate frame. It is then obvious that $\sigma_{ij}^d = \sigma_i^d - \sigma_j^d$ for all $(i,j) \in E$ for all possible desired final orientations. Define $\sigma_i - \sigma_j - \sigma_{ij}^d = \sigma_i - \sigma_j - (\sigma_i^d - \sigma_j^d) = \tilde{\sigma}_i - \tilde{\sigma}_j$. Then we have

$$(L \otimes I_3)\sigma + c_\ell = 0 \Rightarrow (L \otimes I_3)\,\tilde{\sigma} = 0 \Rightarrow$$
$$L\tilde{\sigma}^1 = L\tilde{\sigma}^2 = L\tilde{\sigma}^3 = 0,$$

where $\tilde{\sigma}^1, \tilde{\sigma}^2, \tilde{\sigma}^3$ are the stack vectors of each of the three coefficients of $\tilde{\sigma}$ of the agents' orientations, respectively. The fact that the communication graph is connected implies that L has a simple zero eigenvalue with corresponding eigenvector the vector of ones, $\overrightarrow{\mathbf{1}}$. This guarantees that each one of the vectors $\tilde{\sigma}^1, \tilde{\sigma}^2, \tilde{\sigma}^3$ are eigenvectors of L belonging to span$\{\overrightarrow{\mathbf{1}}\}$. Therefore all $\tilde{\sigma}_i$ are equal to a common vector value, say c. Hence $\tilde{\sigma}_i = c$ for all $i \in \mathcal{N}$ which implies that $\sigma_i - \sigma_j = \sigma_{ij}^d \quad j \in \mathcal{N}_i, \quad \forall i, j$. We conclude that the agents converge to the desired, specified configuration of relative orientations. ■

Remark 2: Similarly to Remark 1, it should be noted that while the control law (10) guarantees that the agents will converge to a configuration where $\omega_1(t) = \cdots = \omega_N(t) = \omega^*$, where ω^* is a constant value, and $\sigma_i(t) - \sigma_j(t) = \sigma_{ij}^d$ for all $(i,j) \in E$, it is not guaranteed that ω^* will be equal to zero. Consecutively, it is not guaranteed that each $\sigma_i(t)$ will reach a constant value. The latter is guaranteed if we add the additional constraint that ω^* tends to zero. This is accomplished with the extension of Theorem 2 in this case. The treatment is straightforward and is omitted here.

IV. Simulations

To verify the results of the previous section we provide next computer simulations of the proposed control designs. The first simulation involves four rigid bodies evolving under the control strategy (6). The Laplacian of the communication graph encoding the static communication ruling has been selected to be of the form

$$L = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

The initial conditions on the angular velocities and orientations were chosen as: $\sigma_1(0) = [0.01, -0.01, 0]^\intercal$, $\sigma_2(0) = [-0.01, 0.03, 0]^\intercal$, $\sigma_3(0) = [0.01, 0.01, 0]^\intercal$, $\sigma_4(0) = [0, 0, 0]^\intercal$, $\omega_1(0) = [0.02, 0, 0]^\intercal$, $\omega_2(0) = [0, 0.01, 0]^\intercal$, $\omega_3(0) = [0, 0, 0.01]^\intercal$, $\omega_4(0) = [0, 0, -0.01]^\intercal$. The inertia matrix of the four rigid bodies was chosen as $J = \mathrm{diag}\,(20, 15, 10)$.

Figure 1 shows the plots of the angular velocities and orientations of the four rigid bodies with respect to time under the feedback law (6). We observe that the system behaves as expected. The angular velocities converge to a common non-zero value. Consequently, the orientations converge to a common value which varies with time.

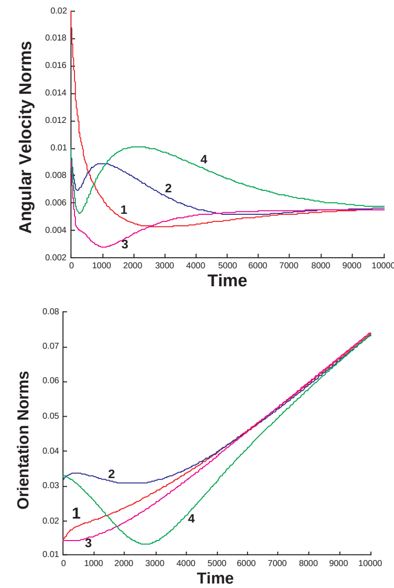

Fig. 1. Plots of the angular velocities and orientations the four rigid bodies with respect to time under the feedback law (6).

In the second simulation we add a damping element to the controller of agent 1, as in the feedback strategy (9). Figure 2 shows the plots of the angular velocities and orientations of the four rigid bodies with respect to time under the feedback law (9). We observe that all angular velocities converge to zero, while the corresponding body orientations converge to a common, and constant this time, zero value.

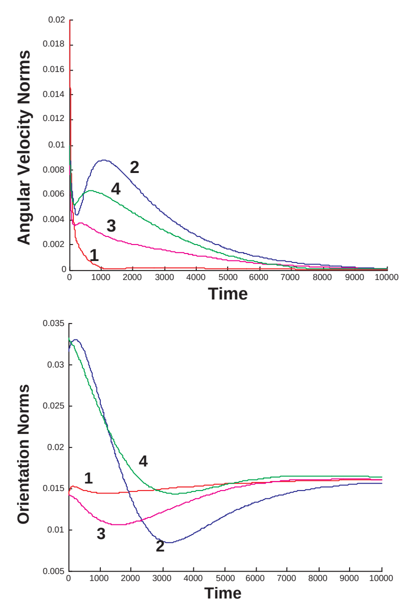

Fig. 2. Plots of the angular velocities and orientations the four rigid bodies with respect to time under the feedback law (9).

The third simulation involves four rigid bodies evolving under the control strategy (10). The same Laplacian matrix and the same initial conditions for the angular velocities as in the previous simulations were used, while the initial conditions for the orientations were chosen as $\sigma_1(0) = [0.046, -0.1, 0.018]^\intercal$, $\sigma_2(0) = [0, 0.21, 0]^\intercal$, $\sigma_3(0) = [0, 0, -0.1]^\intercal$, $\sigma_4(0) = [0, -0.026, 0.1]^\intercal$. The desired relative orientations between the members of the team were chosen as $\sigma_{12}^d = [0, 0, 0]^\intercal$, $\sigma_{13}^d = [0, 0, 0.02]^\intercal$, $\sigma_{34}^d = [0, -0.03, 0]^\intercal$. Note that these specifications correspond to the following desired configuration: $\sigma_1^1 = \sigma_2^1 = \sigma_3^1 = \sigma_4^1$, $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma_4^2 - 0.03$, $\sigma_1^3 = \sigma_2^3$, $\sigma_4^3 = \sigma_3^3$ and $\sigma_1^3 = \sigma_3^3 + 0.02$.

Figure 3 shows the plots of the angular velocities along with the and the "cost functions" γ_i for each rigid body evolving under the control law (10). We observe that the angular velocities converge to a common value, while the γ_i of each rigid body tend to zero. Therefore, the desired relative orientations between each agent with the other agents belonging to its communication set are achieved. This is also verified in Figure 4 where the relative orientation coefficients converge to the desired values.

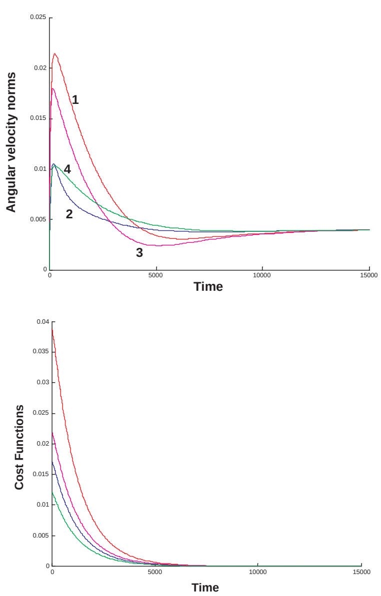

Fig. 3. Plots of the angular velocities and the cost function γ_i of the four rigid bodies with respect to time under the feedback law (10).

V. Conclusions

We proposed a distributed control strategy that exploits graph theoretic tools for cooperative control of multiple rigid bodies. The control objective was the stabilization of the overall system to a configuration where all the rigid bodies have a common orientation and common angular velocity. Similarly to the linear case, the convergence of the multi-agent system was shown to rely on the connectivity of the communication graph. We also extended our results to the case where each rigid body aims to converge to a desired (not necessarily zero) orientation with respect to each of the agents with which it can communicate. Further research efforts involve the cases of switching interconnection topology, as well as the case of unidirectional communication ruling.

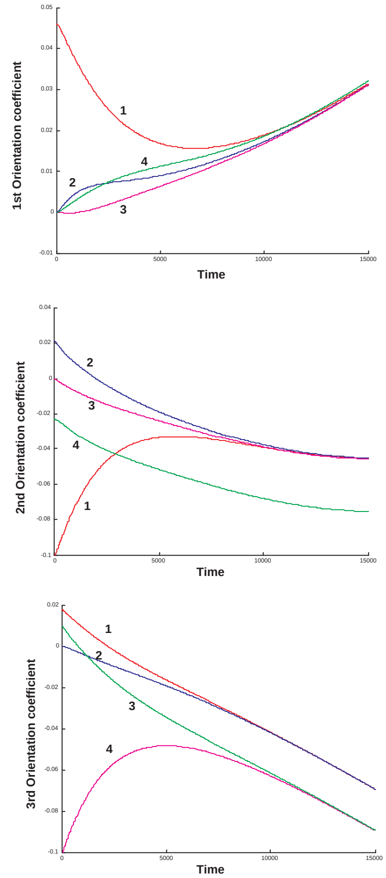

Fig. 4. Plots of the orientations of the four rigid bodies with respect to time for each of the three coefficients under the feedback law (10).

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