Dynamic Games and Applications
https://doi.org/10.1007/s13235-019-00304-4

Multiplayer Pursuit-Evasion Games in Three-Dimensional Flow Fields

Wei Sun[1] · **Panagiotis Tsiotras**[2] · **Anthony J. Yezzi**[3]

Abstract

In this paper, we deal with a pursuit-evasion differential game between multiple pursuers and multiple evaders in the three-dimensional space under dynamic environmental disturbances (e.g., winds, underwater currents). We first recast the problem in terms of partitioning the pursuer set and assign each pursuer to an evader. We present two algorithms to partition the pursuer set from either the pursuer's perspective or the evader's perspective. Within each partition, the problem is reduced into a multi-pursuer/single-evader game. This problem is then addressed through a reachability-based approach. We give conditions for the game to terminate in terms of reachable set inclusions. The reachable sets of the pursuers and the evader are obtained by solving their corresponding level set equations through the narrow band level set method. We further demonstrate why fast marching or fast sweeping schemes are not applicable to this problem for a general class of disturbances. The time-optimal trajectories and the corresponding optimal strategies can be retrieved afterward by traversing these level sets. The proposed scheme is implemented on problems with both simple and realistic flow fields.

Keywords Multiplayer pursuit-evasion · Flow field · Differential game · Reachable set · Level set method

---

This work has been supported by NSF award CMMI-1662542.

✉ Wei Sun
wsun@ou.edu

Panagiotis Tsiotras
tsiotras@gatech.edu

Anthony J. Yezzi
ayezzi@ece.gatech.edu

[1] School of Aerospace and Mechanical Engineering, University of Oklahoma, Norman, OK 73019, USA

[2] School of Aerospace Engineering and Institute for Robotics and Intelligent Machines, Georgia Institute of Technology, Atlanta, GA 30332, USA

[3] School of Electrical and Computer Engineering, Georgia Institute of Technology, Atlanta, GA 30332, USA

Published online: 15 February 2019

1 Introduction

There is a plethora of work in the literature that deal with multiplayer pursuit-evasion problems. In general, these are difficult problems to solve analytically, owing to the delicate nature of the problem itself [15]. Different techniques have been utilized to simplify and subsequently deal with this problem. Single integrator kinematics are commonly assumed in most multiplayer pursuit problems. Under such an assumption, the pursuit problem can be further divided into group-pursuit and relay-pursuit problems. A group-pursuit problem refers to the case where several (or all) pursuers act simultaneously to capture the target(s), whereas in a relay-pursuit problem only one pursuer actively chases each target. In the category of group-pursuit problems, conditions for target interception between multiple pursuers and a single evader in the simplest form is studied by Pshenichnyi [27]. The result is extended to simultaneous k-capture in [6] and ϵ-capture in [16], where capture occurs when a pursuer is within an ϵ distance from the evader. Conditions for guaranteed evasion and the corresponding evading strategies are studied in [8] and [14]. Pursuit-evasion between one evader and countably many pursuers is investigated in [13]. In terms of relay-pursuit problems, generalized Voronoi diagrams have been utilized to assign active pursuers dynamically to capture the evader. Capture is achieved through a relay of the pursuers in a multi-pursuer/one-evader problem [2,3]. This idea has also been applied in [30] to deal with known environmental disturbances and in [10] for cooperative relay tracking of targets. Some results exist for cases with more general dynamics for the agents (pursuers/evaders), but extra assumptions are made for the problem to be tractable. Pursuit-evasion between a group of pursuers and a single evader with linear time-varying dynamics is studied in [26] and was generalized later on to the case of multiple evaders [4]. Group pursuit of a target under the so-called soft capture, where capture occurs only when at least one pursuer and the target have identical orientation, velocity, and acceleration, is investigated in [25] under the assumption of linear time-invariant dynamics. A multiplayer extension of the classical Homicidal Chauffeur game [15] is discussed in [7], where a chain formation of faster, yet less maneuverable, pursuers is utilized to ensure capture of a single slower, but agile evader. However, no optimality analysis of such strategy is given.

In this paper, we consider a multi-pursuer/multi-evader pursuit-evasion problem of agents having single integrator kinematics under a known external dynamic flow field in the 3D Euclidean space. The reason we consider dynamic environmental conditions is that the presence of time-varying or spatially varying flows may significantly affect the vehicles' motion and the corresponding strategy. This is the case, for example, when either the pursuers or the evaders (or both) are small autonomous underwater vehicles (AUV) or small unmanned aerial vehicles (UAV). We solve this problem through a two-step approach. We first recast the problem as a partition of the pursuer set and assign each pursuer to an evader. Subsequently, the original problem is separated into a collection of sub-games involving multiple pursuers and one evader. We find the optimal trajectories of the agents in each sub-game through a reachable set method. Specifically, we utilize the level set method [24,28] to generate the reachable sets of both the evader and the pursuers and retrieve the corresponding optimal control actions at the current location of the agents by backward propagation of their respective reachable sets.

The major contributions of the paper are summarized below:

(a) We provide a two-step approach to deal with the multi-pursuer/multi-evader pursuit-evasion problem under a dynamic flow field in the 3D Euclidean space. The approach leads to a decomposition of the original problem into a pursuer assignment problem in which the subsequent multi-pursuer/one-evader games are much easier to solve.

(b) We propose two algorithms for the pursuer assignment problem. One is from the pursuer's perspective and is valid for pursuers with different capacity and general flow fields. The other algorithm relies on the generation of the Zermelo–Voronoi diagram (ZVD), which is more efficient in cases when the number of pursuers is higher than the number of evaders, but assumes identical pursuers and a time-invariant flow field. A method to generate the ZVD is also presented.

(c) We extend the result in [31] to solve the multi-pursuer/one-evader game under a dynamic flow field in the 3D Euclidean space.

(d) We utilize the narrow band level method to solve the level set equations arising from the propagation of the reachable sets of the pursuers and the evaders, as well as the backward reachable set of the pursuers. We also demonstrate why fast marching or fast sweeping schemes are not applicable to our problem under general disturbances.

2 Problem Formulation

We consider a pursuit-evasion game under an external flow field between n pursuers and m evaders. Henceforth, we will refer to the pursuers and the evaders collectively as "agents". The dynamics of the pursuers P_i, $i \in \mathcal{I}$, and the evaders E_j, $j \in \mathcal{J}$, where $\mathcal{I} = \{1, 2, \dots, n\}$ and $\mathcal{J} = \{1, 2, \dots, m\}$, are given by

$$\dot{X}_P^i = u_P^i + w(X_P^i, t), \quad X_P^i(t_0) = X_{P_0}^i, \tag{1}$$

$$\dot{X}_E^j = u_E^j + w(X_E^j, t), \quad X_E^j(t_0) = X_{E_0}^j, \tag{2}$$

where $X_P^i \in \mathcal{D} \subset \mathbb{R}^3$ denotes the position of the ith pursuer and $X_E^j \in \mathcal{D} \subset \mathbb{R}^3$ denotes the position of the jth evader. Here $\mathcal{D}$ is a compact subset of $\mathbb{R}^3$ that denotes the user-specified region of interest to the problem and u_P^i is the control input (i.e., velocity) of the ith pursuer such that $u_P^i \in \mathcal{U}_P^i$, for all $i \in \mathcal{I}$. The set $\mathcal{U}_P^i$ consists of all measurable functions whose range is included in the set $U_P^i = \{u \in \mathbb{R}^3, |u| \leqslant \bar{u}_P^i\}$, where $|\cdot|$ represents the Euclidean norm and $\bar{u}_P^i$, $i \in \mathcal{I}$, are constants. If $u_P^i \in \mathcal{U}_P^i$, we say that u_P^i is an *admissible* control for the ith pursuer. Similarly, u_E^j is the control input of the jth evader such that $u_E^j \in \mathcal{U}_E^j$, where $\mathcal{U}_E^j$ consists of all measurable functions whose range is included in the set $U_E^j = \{u \in \mathbb{R}^3, |u| \leqslant \bar{u}_E^j\}$. We say that u_E^j is an *admissible* control of the jth evader if $u_E^j \in \mathcal{U}_E^j$. In (1), $w(X, t) \in \mathbb{R}^3$ represents an exogenous dynamic flow, but it could also represent an endogenous drift owing to the nonlinear dynamics of the agent. It is reasonable to assume that the magnitude of this flow is bounded from above by some constant, that is, there exists a constant $\bar{w}$ such that $|w(X, t)| \leq \bar{w}$, for all $(X, t) \in \mathcal{D} \times [t_0, \infty)$. In order to ensure existence and uniqueness of solutions of the differential equations (1) and (2), it will be assumed that $w(X, t)$ is Lipschitz continuous in X and piecewise continuous in t.

The pursuers aim to intercept the evaders and minimize the overall time-to-capture, if possible, whereas the evaders try to avoid capture as long as possible. It is assumed that all the players have perfect knowledge of the dynamics of the system given by (1) and (2), the constraint sets U_P^i and U_E^j, as well as the current state $\bar{X} = [X_P^{1\mathsf{T}}, X_P^{2\mathsf{T}}, \dots, X_P^{n\mathsf{T}}, X_E^{1\mathsf{T}}, , X_E^{2\mathsf{T}}, \dots, X_E^{m\mathsf{T}}]^\mathsf{T} \in \mathbb{R}^{3(n+m)}$. The game begins at time $t = t_0$ with initial positions $X_{P_0}^i$, $i \in \mathcal{I}$, and $X_{E_0}^j$, $j \in \mathcal{J}$, for the pursuers and the evaders, respectively, and terminates when each evader coincides with at least one of the pursuers, in which case capture occurs. That is, capture implies that for each $j \in \mathcal{J}$, there exists $i^j \in \mathcal{I}$ and a terminal time $T^j \geq t_0$ such that $X_P^{i^j}(T^j) = X_E^j(T^j)$. The overall time-to-capture is then denoted as $T^o = \max_{j\in\mathcal{J}} T^j$.

We assume that each pursuer can capture at most one evader, that is, once a pursuer captures an evader, the pursuer does not re-enter the game to chase another evader. Such an assumption is valid in cases where the pursuers are disposable (such as missiles) or the number of pursuers is large. Subject to this assumption, and in accordance with the previous notion of capture, it can be seen that a necessary condition for the game to terminate is $n \geq m$. In other words, there should be more pursuers than evaders for all the evaders to be captured. Henceforth, we assume that this condition holds, that is, there are more pursuers than evaders.

The original problem is intractable due to the existence of the external flow field and the large number of possible pairings between the evaders and the pursuers. Therefore, instead of attempting to solve this problem head on, we propose to decompose the problem into two steps. In the first step, we apply a partition to the set of pursuers by assigning each pursuer to an evader at the beginning of the game. In the second step, the pursuit-evasion game between each evader and its assigned (paired) pursuer(s) is solved to obtain the optimal trajectories and controls of the pursuer(s) and the evader. The problems in these two steps are formally stated as follows.

Problem 1 Given the system described by (1) and (2), find a partition $\{\mathcal{I}_j\}_{j\in\mathcal{J}}$ of the pursuer set $\mathcal{I}$ such that, for each $i \in \mathcal{I}_j$, it follows that $T^*(X_{P_0}^i, X_{E_0}^j) \leq T^*(X_{P_0}^i, X_{E_0}^k)$ for all $k \in \mathcal{J}$, where $T^*(X_{P_0}^i, X)$ denotes the minimum time-to-reach from $X_{P_0}^i$ to $X \in \mathbb{R}^3$ under the dynamics of the ith pursuer.

Problem 2 For some $j \in \mathcal{J}$, consider a pursuit-evasion game in the presence of an external flow field between multiple pursuers P_i, $i \in \mathcal{I}_j$ and evader E_j. The dynamics of the pursuers are given by (1), where $i \in \mathcal{I}_j$. The pursuers aim to intercept the evader whose dynamics is given by (2). Without loss of generality, let $\mathcal{I}_j = \{1, \ldots, k\}$. The game begins at initial time $t_0 = 0$ with initial positions $\bar{X}_0 = [X_{E_0}^{\mathsf{T}}, X_{P_0}^{1\mathsf{T}}, X_{P_0}^{2\mathsf{T}}, \ldots, X_{P_0}^{k\mathsf{T}}]^{\mathsf{T}}$, and terminates when at least one of the pursers reaches the location of the evader. The objective is to find the open-loop representation [5] of the optimal strategies of the pursuers and the evader.

3 Problem Analysis

3.1 Pursuer Assignment

As mentioned in Sect. 2, we decompose the problem into two steps. The first step assigns each pursuer to an evader, and the second step solves the pursuit-evasion game between each evader and its assigned pursuers in the presence of the external flow field. In this subsection, we focus on the first step, that is, obtaining a pursuer assignment such that each pursuer is paired with an evader.

Before we present the algorithms to solve Problem 1, we start our analysis by introducing some basic definitions and facts about reachable sets.

Definition 1 [29] The *reachable set* $\mathcal{R}(X_0, t)$ at time $t \geq t_0$ of a system of the form (1) or (2) starting at initial condition $X(t_0) = X_0$ is the set of all points that can be reached by the agent at time t.

In particular, the reachable set of the ith pursuer at time $\tau \geq t_0$, denoted by $\mathcal{R}^i_P(X^i_{P_0}, \tau)$, is the set of all points $X \in \mathbb{R}^3$ such that there exists a trajectory satisfying (1) for all $t \in [t_0, \tau]$ with $X_P(t_0) = X^i_{P_0}$ and $X_P(\tau) = X$.

Definition 2 [20] The boundary of the reachable set is the *reachability front*.

The reachability front of the ith pursuer at time $t \geq t_0$ will be denoted by $\partial\mathcal{R}^i_P(X^i_{P_0}, t)$.

The following theorem, due to Filippov, gives the conditions to ensure the compactness of the reachable sets of the players.

Theorem 1 [17] *Given a control system*

$$\dot{X} = f(X, t, u), \quad X(t_0) = X_0,$$

with u taking values in U, assume that its solutions exist on a time interval $[t_0, t_f]$ for all controls u and that for every pair (X, t) the set $\{f(X, t, u) : u \in U\}$ is compact and convex. Then the reachable set $\mathcal{R}(X_0, t)$ is compact for each $t \in [t_0, t_f]$.

Filippov's theorem can be applied to the systems affine in controls given by (1) and (2) and a given time interval $[t_0, t_f]$. For such systems, the sets $\{u^i_P + w(X^i_P, t) : u^i_P \in U^i_P\}$ and $\{u^j_E + w(X^j_E, t) : u^i_E \in U^j_E\}$ are compact and convex for each (X^i_P, t) and (X^j_E, t), respectively. Under the previous assumptions on $w(X, t)$, the solutions exist for the differential equations (1) and (2) on the time interval $[t_0, t_f]$. Hence, it can be concluded that the reachable sets of the pursuers and the evaders, denoted by $\mathcal{R}^i_P(X^i_{P_0}, t)$ and $\mathcal{R}^j_E(X^j_{E_0}, t)$, respectively, are compact, thus closed and bounded in $\mathbb{R}^3$, for each $t \in [t_0, t_f]$. The compactness of the reachable sets ensures the existence of a minimum-time optimal control for each player to reach the boundary of its reachable set.

Given the previous definitions, we present the following proposition.

Proposition 1 *Given the system described by (1) and (2), and the partition $\{\mathcal{I}_j, j \in \mathcal{J}\}$ of the pursuer set $\mathcal{I}$ introduced in Problem 1, then $i \in \mathcal{I}$ is part of the partition $\mathcal{I}_j$ if $t^i_j = \min_{k\in\mathcal{J}} t^i_k$, where $t^i_k := \inf\{t \in [t_0, \infty) : X^k_{E_0} \in \mathcal{R}^i_P(X^i_{P_0}, t)\}$, that is, t^i_k is the first time such that $X^k_{E_0}$ is inside $\mathcal{R}^i_P(X^i_{P_0}, t)$.*

Proof It can be easily seen that the proposition is true as long as $t^i_j = T^*(X^i_{P_0}, X^j_{E_0})$. Indeed, let $\{\mathcal{I}'_j, j \in \mathcal{J}\}$ be a partition of the pursuer set $\mathcal{I}$ such that $i \in \mathcal{I}'_j$ if $t^i_j = \min_{k\in\mathcal{J}} t^i_k$. Then $\{\mathcal{I}'_j\}$ coincides with $\{\mathcal{I}_j\}$ when $t^i_j = T^*(X^i_{P_0}, X^j_{E_0})$. Therefore, it suffices to show that $t^i_j = T^*(X^i_{P_0}, X^j_{E_0})$.

Consider now the definition of $\mathcal{R}^i_P(X^i_{P_0}, t)$. Since $\mathcal{R}^i_P(X^i_{P_0}, t)$ contains all the points the ith pursuer can reach at time t starting from $X^i_{P_0}$, and since t^i_j is defined as the first time such that $X^j_{E_0} \in \mathcal{R}^i_P(X^i_{P_0}, t^i_j)$, it can be seen that t^i_j is the minimum time-to-reach from $X^i_{P_0}$ to $X^j_{E_0}$. In other words, $t^i_j = T^*(X^i_{P_0}, X^j_{E_0})$. Assume now, on the contrary, that $t^i_j \neq T^*(X^i_{P_0}, X^j_{E_0})$, then we have two cases to consider. First, $t^i_j < T^*(X^i_{P_0}, X^j_{E_0})$ and second, $t^i_j > T^*(X^i_{P_0}, X^j_{E_0})$. Suppose that $t^i_j < T^*(X^i_{P_0}, X^j_{E_0})$. This clearly contradicts the fact that $T^*(X^i_{P_0}, X^j_{E_0})$ is the minimum time-to-reach from $X^i_{P_0}$ to $X^j_{E_0}$. If, on the other hand, $t^i_j > T^*(X^i_{P_0}, X^j_{E_0})$, it follows that $X^j_{E_0} \in \mathcal{R}^i_P(X^i_{P_0}, T^*(X^i_{P_0}, X^j_{E_0}))$ which means that capture occurs at a time prior to t^i_j. This contradicts the definition $t^i_j = \inf\{t \in [t_0, \infty) : X^j_{E_0} \in \mathcal{R}^i_P(X^i_{P_0}, t)\}$. Finally, the assumption $t^i_j \neq T^*(X^i_{P_0}, X^j_{E_0})$ leads to a contradiction and thus $t^i_j = T^*(X^i_{P_0}, X^j_{E_0})$. □

The previous proposition provides us with an approach to obtain the partition of the pursuer set through propagation of their reachable sets. Specifically, we propose the following Algorithm 1 to find the partition $\{\mathcal{I}_j,\ j \in \mathcal{J}\}$ of the pursuer set $\mathcal{I}$.

Algorithm 1 Initial pursuer assignment through reachable sets

For each pursuer $P_i, i \in \mathcal{I}$:

(a) Propagate its reachable set from the initial position and time.
(b) Stop the propagation when the initial position of an evader (say E_k) enters the reachable set of this pursuer.
(c) Place i in the cell $\mathcal{I}_k$.

Notice that in the aforementioned pursuer assignment algorithm, the reachable set of each pursuer needs to be propagated forward in time in order to find the assignment for all pursuers. This may require a lot of computation when the number of pursuers is large. On the other hand, when the problem has identical pursuers, the assignment can be determined from the perspective of the evaders. Specifically, Problem 1 can be solved by generating the so-called Zermelo–Voronoi Diagram (ZVD) from the initial positions of the evaders and by assigning pursuers to the evader whose Zermelo–Voronoi cell contains those pursers. The definition of ZVD is given as follows.

Definition 3 [2] A *Zermelo–Voronoi Diagram (ZVD)* is a partition of the space for a given (finite) set of generators, such that each element in this partition is associated with a particular generator in the following sense: an agent that resides in a particular set of the partition at a given time instant can arrive at the generator associated with this set faster than any other agent that may be located anywhere outside this set at the same time instant. Each set of the partition associated with one generator is also called a *Zermelo–Voronoi cell* of that generator.

From the definition of the ZVD, it is clear that if a pursuer resides in the Zermelo-Voronoi cell of an evader at time $t = t_0$, then the pursuer can reach the initial position of this evader faster than it can reach the initial position of any other evader. Therefore, the partition of the pursuer set defined in Problem 1 can be achieved by determining which Zermelo-Voronoi cell each pursuer resides in. The corresponding algorithm is summarized in Algorithm 2. It is clear that the key step in Algorithm 2 is to generate the ZVD. This is discussed in greater detail in Sect. 4.2.

Algorithm 2 Initial pursuer assignment through ZVD

(a) Generate the ZVD from the initial positions of the evaders.
(c) For each pursuer $P_i, i \in \mathcal{I}$, place i in the cell $\mathcal{I}_k$ if pursuer P_i resides in the Zermelo–Voronoi cell of evader E_k.

Remark 1 We hereby assume that there is always a sufficient number of pursuers such that $\mathcal{I}_j \neq \emptyset$, for all $j \in \mathcal{J}$. This assumption ensures that each evader can be actively chased by at least one pursuer. For cases where this assumption is not satisfied, it can be seen later on that there may exists pursuers who are assigned to an evader but are not involved in the active pursuit of that evader. We can then assign some of these inactive pursuers to the evaders who do not yet have any pursuers assigned to them. This scheme will not be discussed further, so as not to distract from the key results of this paper.

3.2 Multi-Pursuers/One-Evader Game

In this subsection, we utilize the reachable set analysis to find conditions for capture of the evader and also derive the corresponding optimal controls and the trajectories of the agents in the sub-game between multiple pursuers and one evader. Reachable sets provide a quick snapshot of all possible future trajectories of the agent and thus succinctly encode all possible future positions of the agent under any possible control action. Knowledge of the reachable set of each pursuer and the evader can then be used to draw conclusions about the potential meeting of the two at some future time (or not). In this paper, we use this intuition behind the information conveyed by the reachable set of each player to solve the pursuit-evasion problem under minimal assumptions about the maximum number of agents and the environment they operate in. Since the computation of the reachable sets for each pursuer can be done independently from the other pursuers, the proposed method is decentralized and scales well with the number of pursuers, something that is not the case with more traditional approaches that require directly the solution of a HJI partial differential equation [5].

The evader plays a game against the pursuers and we would like to find the set of terminal points of all "safe" evader trajectories, that is, the set of all terminal points of the evader at time t, whose trajectories do not pass through any reachable sets of the pursuers at *any* time in the interval $[t_0, t]$. The formal definition of such set is given as follows.

Definition 4 Given the reachable sets of the pursuers, we define the *usable reachable set* of the evader at time $t \geq t_0$ as

$$\mathcal{R}_E^\star(X_{E0}, t) = \left\{ X \in \mathcal{D} : X = X_E(t) \text{ and } X_E(\tau) \notin \bigcup_{i=1}^{n} \mathcal{R}_P^i(X_{P0}^i, \tau), \forall \tau \in [t_0, t] \right\}. \tag{3}$$

From this definition, it is clear that $\mathcal{R}_E^\star(X_{E0}, t) \subseteq \mathcal{R}_E(X_{E0}, t)$. The definition implies that $\mathcal{R}_E^\star(X_{E0}, t)$ is the set of all terminal points of the evader at time t, whose trajectories do not pass through any reachable sets of the pursuers at *any* time in the interval $[t_0, t]$.

Suppose now that at some time $t_c > t_0$, and for some $i \in \mathcal{I}$, we have that $\mathcal{R}_E(X_{E0}, t_c) \subseteq \mathcal{R}_P^i(X_{P0}^i, t_c)$. It follows that, for each $u_E \in \mathcal{U}_E$, there exists $u_P^i \in \mathcal{U}_P^i$ such that $X_P^i(t_c) = X_E(t_c)$. In other words, capture of the evader is guaranteed at time t_c by the ith pursuer. Note that $\mathcal{R}_E^\star(X_{E0}, t_c) = \emptyset$ in this case.

If, on the other hand, for some $t_e > t_0$, we have that $\mathcal{R}_E^\star(X_{E0}, t_e) \neq \emptyset$, it follows that there exists $u_E \in \mathcal{U}_E$ such that capture can be avoided in the time interval $[t_0, t_e]$, no matter how the pursuers choose their (admissible) controls. In other words, if $\mathcal{R}_E^\star(X_{E0}, t_e) \neq \emptyset$, the game will not terminate in the time interval $[t_0, t_e]$.

The previous observations lead to the following theorem, which is the main theoretical result of this paper. It is used later on in order to develop an efficient numerical algorithm for solving the pursuit-evasion game with multiple pursuers in the presence of a known dynamic flow field.

Theorem 2 *Let $T = \inf\{t \in [t_0, +\infty] : \mathcal{R}_E^\star(X_{E0}, t) = \emptyset\}$. If $T < \infty$, then capture of the evader is guaranteed for any time greater than T, while the evader can always escape within a time smaller than T. Hence, T is the time-to-capture if both players play optimally. Furthermore, let X_f denote the location where the evader is captured by at least one of the pursuers. Then we have that*

$$X_f \in \mathcal{X} = \left\{ X \in \mathcal{D} : X = X_E(T) \textit{ and } X_E(\tau) \notin \bigcup_{i=1}^{n} \mathcal{R}_P^i(X_{P0}^i, \tau), \forall \tau \in [t_0, T) \right\}. \tag{4}$$

Proof The proof is almost identical to the proof of the main theorem in [31], and is thus omitted. □

From the previous theorem, it can be seen that capture is guaranteed when $\mathcal{R}_E^\star(X_{E0}, t) = \emptyset$ for some $t \in [t_0, \infty)$. Furthermore, $\mathcal{R}_E^\star(X_{E0}, t)$ can be computed by keeping track of the reachable sets of the pursuers and the evader at all times prior to the capture time. However, under some constraints on the speeds of the players, this criterion can be replaced with an instantaneous condition which is easier to check and implement. We present the simplified condition on the capture of the evader as follows.

Proposition 2 [31] *If $\bar{v} \le \min_{i\in\mathcal{I}} \bar{u}_i$, the set $\mathcal{R}_E^\star(X_{E0}, t)$ satisfies*

$$\mathcal{R}_E^\star(X_{E0}, t) = \mathcal{R}_E(X_{E0}, t) \setminus \bigcup_{i=1}^{n} \mathcal{R}_P^i(X_{P0}^i, t), \tag{5}$$

for all $t \ge t_0$. In such cases, the condition $\mathcal{R}_E^\star(X_{E0}, t) = \emptyset$ is equivalent to the condition

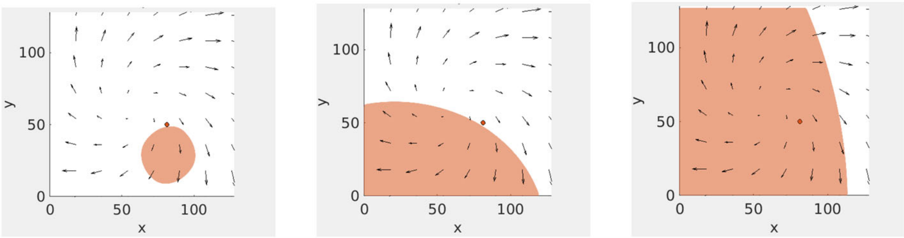
$$\mathcal{R}_E(X_{E0}, t) \subseteq \bigcup_{i=1}^{n} \mathcal{R}_P^i(X_{P0}^i, t). \tag{6}$$

The previous expression states that in the case when $\bar{v} \le \min_{i\in\mathcal{I}} \bar{u}_i$ the optimal time-to-capture is the first time instant when the union of the reachable sets of the pursuers $\bigcup_{i=1}^{n} \mathcal{R}_P^i(X_{P0}^i, t)$ completely covers the reachable set of the evader $\mathcal{R}_E(X_{E0}, t)$. If $\bar{v} > \bar{u}_i$, for some $i \in \mathcal{I}$ (the relative maximum speed of the evader is larger than that of the ith pursuer), the relation (5) may not always hold. Some admissible evader trajectories may temporarily enter the reachable set of the ith pursuer and exit later on. This is not allowable. To eliminate this possibility, in such cases $\mathcal{R}_E^\star(X_{E0}, t)$ can be determined by treating the reachable set of the ith pursuer as a dynamic "forbidden" region for the evader [19,21]. That is, whenever the reachable set of the evader intersects the reachable set of any of the pursuers, we can either stop the evolution of the intersected part of the evader's reachable set or let it evolve at the same speed as the reachable set of the pursuer. This way, we can ensure that the terminal points of the admissible trajectories of the evader that temporarily enter the reachable set of the pursuer and exit later on are not included in the usable reachable set of the evader.

4 Numerical Construction

4.1 Narrow Band Level Set Method

Upon constructing the reachable sets of the pursuers and the evaders by numerically solving their corresponding Hamilton–Jacobi (HJ) equations, one may be tempted to utilize the fast marching method [28], which is a well-known and efficient approach to track the evolution of a closed surface as a function of time with speed in the normal direction along the propagating surface. However, one prerequisite to using the fast marching method is that each grid point should not be revisited after the level set propagation. This condition is not satisfied in our problem due to the existence of the external flow field. Such phenomenon can be demonstrated in the following two-dimensional example [1]. The reachable sets of the pursuer at time $t = 1, 3, 5$ are depicted in Fig. 1. As shown in Fig. 1a, the pursuer (shown by the red dot) starts at $P_0 = [81, 50]^\top$ at time $t = 0$. Therefore, $P_0 \in \mathcal{R}_P(P_0, 0)$. However, $P_0 \notin \mathcal{R}_P(P_0, 1)$, as is depicted in Fig. 1a, where the orange region represents the reachable set of the pursuer



(a) **(b)** **(c)**

Fig. 1 **a** Reachable set of pursuer at time $t = 1$. **b** Reachable set of pursuer at time $t = 3$. **c** Reachable set of pursuer at time $t = 5$

at time $t = 1$. The initial position of the pursuer is outside the reachable set at $t = 1$ and is still outside the reachable set at $t = 3$, as shown in Fig. 1b. Later on, at $t = 5$, P_0 reenters the reachable set $\mathcal{R}_P(P_0, 5)$, as illustrated in Fig. 1c. This indicates that P_0 is revisited by the reachable set.

The previous observation forces us to utilize a more sophisticated method to construct the reachable sets, while retaining some level of efficiency. In particular, we apply the narrow band level set method [24,28] to construct the reachable sets of all players.

The level set method evolves the reachability front by embedding it as a hyper-surface in a higher dimension, where time is the additional dimension. The level set formulation provides an implicit representation of the front, which offers several advantages over an explicit representation [24,28]. The standard level set method would initialize with a mesh grid and update the value on every grid point to correspond to the motion of the surface. Instead of this full matrix approach, we adopt a more efficient method known as the narrow band approach that performs an update of the value in a neighborhood of the zero-level set [12,22,28]. This approach can save a lot of computation effort with regard to the operation on the entire computational domain. In particular, the narrow band method saves computation by limiting the set of the grid over which the Hamilton–Jacobi equation is solved, and it also reduces the cost of computing the signed distance function at fewer grid points.

The choice of the implicit function for this embedding to represent the reachability front. is not unique. The signed distance function is one of the most common choices and will be utilized in this paper. Its definition is given as follows.

Definition 5 The *signed distance function* $\varphi(X)$ with respect to a set $\mathcal{R}$ is defined as

$$\varphi(X) = \begin{cases} \min\limits_{Y \in \partial \mathcal{R}} |X - Y|, & \text{if } X \notin \mathcal{R}, \\ -\min\limits_{Y \in \partial \mathcal{R}} |X - Y|, & \text{if } X \in \mathcal{R}. \end{cases} \tag{7}$$

Recall that, for any $c \in \mathbb{R}$, the *c-level set* of a φ is the set $\{X : \varphi(X) = c\}$. We hereby utilize the signed distance function from the reachable set to track the evolution of the fronts of the reachable sets of all agents. This is achieved by expressing the reachable front at time t as the zero-level set of the corresponding signed distance function. Assuming that the signed distance function with respect to the ith pursuer reachable set $\mathcal{R}^i_P(X^i_{P_0}, t)$ at time t is $\phi^i_P(X, t)$, then the evolution of the reachable front $\partial \mathcal{R}^i_P(X^i_{P_0}, t)$ is governed by the viscosity solution of the Hamilton–Jacobi equation [18,20]

$$\frac{\partial \phi^i_P(X, t)}{\partial t} + \bar{u}|\nabla \phi^i_P(X, t)| + \nabla \phi^i_P(X, t) w(X, t) = 0, \tag{8}$$

with initial condition $\phi_P^i(X, t_0) = |X - X_{P_0}^i|$. Note that $\mathcal{R}_P^i(X_{P_0}^i, t) = \{X \in \mathcal{D} : \phi_P^i(X, t) \leq 0\}$ and $\partial\mathcal{R}_P^i(X_{P_0}^i, t) = \{X \in \mathcal{D} : \phi_P^i(X, t) = 0\}$.

Similarly, the reachable front $\partial\mathcal{R}_E(X_{E_0}, t)$ of the evader is computed by solving the Hamilton–Jacobi equation

$$\frac{\partial \phi_E(X, t)}{\partial t} + \bar{v}|\nabla\phi_E(X, t)| + \nabla\phi_E(X, t)w(X, t) = 0, \tag{9}$$

with initial condition $\phi_E(X, t_0) = |X - X_{E_0}|$, where $\phi_E(X, t)$ is the signed distance function with respect to the reachable set $\mathcal{R}_E(X_{E_0}, t)$ of the evader at time t.

In the case where the condition in Proposition 2 is not satisfied, we need to track $\partial\mathcal{R}_E^\star(X_{E_0}, t)$ in order to determine the optimal time-to-capture. Instead of propagating $\partial\mathcal{R}_E^\star(X_{E_0}, t)$ directly, we propagate an intermediate reachable front $\partial\tilde{\mathcal{R}}_E(X_{E_0}, t)$ which can be computed by solving the following modified version of the Hamilton–Jacobi equation

$$\frac{\partial \tilde{\phi}_E(X, t)}{\partial t} + \tilde{v}(t)\,|\nabla\tilde{\phi}_E(X, t)| + \nabla\tilde{\phi}_E(X, t)w(X, t) = 0, \tag{10}$$

where

$$\tilde{v}(t) = \begin{cases} \min\limits_{i\in\mathcal{I}} \bar{u}_i, & \text{if } \min\limits_{i\in\mathcal{I}} \phi_P^i(X, t) < 0, \\ \bar{v}, & \text{otherwise,} \end{cases} \tag{11}$$

and initial condition $\tilde{\phi}_E(X, t_0) = |X - X_{E_0}|$.

The main idea here is to treat the reachable sets of the pursuers as moving obstacles and propagate $\tilde{\mathcal{R}}_E(X_{E_0}, t)$ with the maximum speed of the evader $\bar{v}$ for the parts that fall outside the union of the reachable sets of the pursuers, and to keep pace with the propagation of the reachable set of the slowest pursuer when the front of the evader enters any reachable set of the pursuers. By doing this, we make sure that the front of the evader does not grow out of the union of the reachable sets of the pursuers. The parts of the reachable front of the evader that do not encounter the reachable sets of the pursuers remain unaffected by the change of speed from $\bar{v}$ to $\tilde{v}$, since these changes are only performed for points inside the reachable sets of the pursuers.

Let $\tilde{\mathcal{R}}_E(X_{E_0}, t) = \{X \in \mathcal{D} : \tilde{\phi}_E(X, t) \leq 0\}$. At every time instant t, by construction, $\tilde{\mathcal{R}}_E(X_{E_0}, t)$ excludes all the points X such that $X = X_E(t)$ and $X \notin \bigcup_{i=1}^n \mathcal{R}_P^i(X_{P_0}^i, t)$, while $X \in \bigcup_{i=1}^n \mathcal{R}_P^i(X_{P_0}^i, \tau)$, for some $\tau \in [t_0, t)$. It follows that

$$\mathcal{R}_E^\star(X_{E_0}, t) = \tilde{\mathcal{R}}_E(X_{E_0}, t)\Big\backslash \bigcup_{i=1}^{n} \mathcal{R}_P^i(X_{P_0}^i, t).$$

Moreover, since $\mathcal{R}_P^i(X_{P_0}^i, t) = \{X \in \mathcal{D} : \phi_P^i(X, t) \leq 0\}$, the usable reachable set of the evader can also be represented in a form that is more suitable for numerical calculations, that is,

$$\mathcal{R}_E^\star(X_{E_0}, t) = \{X \in \mathcal{D} : \tilde{\phi}_E(X, t) \leq 0 \quad \text{and} \quad \min_{i\in\mathcal{I}} \phi_P^i(X, t) \geq 0\}.$$

4.2 Generation of ZVD

In this subsection, we discuss how to numerically generate the ZVD at the initial time step with the evaders as generators. Once the ZVD is generated, it can, in turn, be utilized in Algorithm 2 to determine the initial partition of the pursuers.

To this end, consider the dynamics of the pursuer given by

$$\dot{X}_P = u_P + w(X_P), \quad X_P(t_0) = X_{P_0}, \tag{12}$$

where $u_P \in \mathcal{U}_P$, for all $i \in \mathcal{I}$. It is assumed here that all the pursuers are identical (which results in dropping the superscript i), and the dynamic flow is time invariant, for reasons that will become clear soon after. We take Eq. (12) and reverse the time direction by replacing t with $\tau = T^*(X_{P_0}, X_{\text{end}}) - t$, where X_{end} denotes some terminal position of the pursuer. We thus obtain the retrograde evolution of the states as follows

$$\mathring{X}_P = -u_P - w(X_P), \tag{13}$$

where $(\,\mathring{}\,)$ denotes the derivative with respect to the retrograde time τ. Notice that (13) is valid owing to the fact that $T^*(X_{P_0}, X_{\text{end}})$ does not show up explicitly in (13), which results from the previous time-invariant flow assumption.

Given (13), our next step is to find a way to generate the *backward* reachable set $\mathscr{R}_P^j(X_{E_0}^j, t)$ of the pursuer at time t, starting from the initial position $X_{E_0}^j$ of the evader E^j, $j \in \mathcal{J}$. The set $\mathscr{R}_P^j(X_{E_0}^j, t)$ contains all the initial positions of the pursuer $X \in \mathcal{D}$ that can reach $X_{E_0}^j$ with minimum time $T^*(X, X_{E_0}^j) \leq t$. Similar to the approach utilized in Sect. 4.1, it is assumed that the signed distance function with respect to $\mathscr{R}_P^j(X_{E_0}^j, t)$ is denoted by $\psi_P^j(X, t)$, then the evolution of its reachable front is governed by the viscosity solution of the Hamilton–Jacobi equation

$$\frac{\partial \psi_P^j(X, t)}{\partial t} + \bar{u}|\nabla \psi_P^j(X, t)| - \nabla \psi_P^j(X, t) w(X) = 0, \tag{14}$$

with initial condition $\psi_P^j(X, t_0) = |X - X_{E_0}^j|$.

Note that the backward reachable set can be described as $\mathscr{R}_P^j(X_{E_0}^j, t) = \{X \in \mathcal{D} : \psi_P^j(X, t) \leq 0\}$. Furthermore, for each $X \in \mathcal{D}$, the minimum time-to-reach can be calculated as

$$T^*(X, X_{E_0}^j) = \inf\{t \in [t_0, +\infty] : X \in \mathscr{R}_P^j(X_{E_0}^j, t)\}. \tag{15}$$

Let the augmented representation of the ZVD with respect to the generators $\{X_{E_0}^j, j \in \mathcal{J}\}$ be denoted as $\{(X, k)\}$, where $X \in \mathcal{D}$ and $k \in \mathcal{J}$ is the indicator of which Zermelo–Voronoi cell X resides in. Then the ZVD can also be written as $\{(X, \operatorname{argmin}_{j \in \mathcal{J}} T^*(X, X_{E_0}^j))\}$, per Definition 3. Therefore, the ZVD can be generated through the propagation of $\mathscr{R}_P^j(X_{E_0}^j, t)$, $j \in \mathcal{J}$ and comparison of $T^*(X, X_{E_0}^j)$ for each $j \in \mathcal{J}$ to decide which Zermelo–Voronoi cell X belongs to, for all $X \in \mathcal{D}$.

4.3 Pursuer Classification

For problems with a large number of pursuers, it is very likely that not all pursuers are involved in the optimal capture. In certain applications, such as when the pursuers are subject to energy or fuel limitations, or when they play a dual role as pursuers and guards of a certain region of responsibility, it may be beneficial that some of the pursuers remain inactive. In group-pursuit problems involving several pursuers, we may therefore classify the pursuers according to their level of involvement as either active or inactive.

In particular, we can divide the pursuer set into two distinct subsets. One subset consists of all *active* pursuers, while the second subset contains the *inactive* pursuers that do not chase

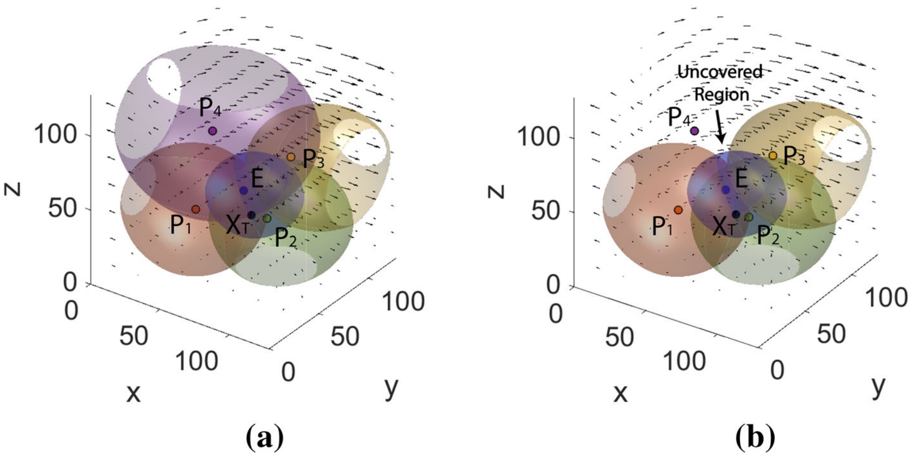


Fig. 2 **a** Level sets of four pursuers and one evader at time T. **b** Level sets of three pursuers and one evader at time T. Level set of pursuer P_4 is removed to show the reachable set of the evader is not fully covered by the union of reachable sets of P_1, P_2 and P_3

the evader. We also refer to the pursuers in the latter subset as *guards*, since they remain inactive during the optimal pursuit when the evader stays along its optimal trajectory or join the chase if the evader plays suboptimally and deviates from its original trajectory. Once the capture point X_f is found, the active pursuers can be identified as the pursuers whose boundary of the reachable sets at time T intersects X_f, while the rest of the pursuers are guards.

The classification of the pursuer set into active pursuers and guards can be demonstrated by the situation depicted in Fig. 2a. As shown in this figure, the reachability fronts of pursuers P_1, P_2 and P_3 at the capture time T coincide at the terminal position X_f. These three pursuers need to reach X_f at time T to ensure capture of the evader. Hence, these are the active pursuers. On the other hand, pursuer P_4 cannot reach X_f within time T, but its reachable set covers a portion of the reachable set of the evader. This can been seen in Fig. 2b where the reachable set of pursuer P_4 is removed to show that the reachable set of the evader is not fully covered by the union of reachable sets of P_1, P_2 and P_3 at time T. Therefore, P_4 acts as a guard. It is also worth noting that if one would like to account for the possibility that the evader may not maneuver optimally, then the process of classifying active pursuers and guards should be repeated at each time step. Otherwise, a pursuer that has been initially classified as a guard might remain inactive even if the evader moves toward it and away from the active pursuers.

4.4 Time-Optimal Paths

In this section, we show how the optimal controls of the evaders and the active pursuers, as well as their corresponding optimal trajectories, can be retrieved from the computed level sets.

We first consider the active pursuers. Since they can reach X_f at time T, it is clear that X_f resides on the boundary of their reachable sets, otherwise capture would have occurred earlier. Therefore, when ϕ_P^i's are differentiable, the optimal trajectory for each active pursuer satisfies [20]

$$\frac{\mathrm{d}X_P^{i*}}{\mathrm{d}t} = \bar{u}^i \frac{\nabla \phi_P^i}{|\nabla \phi_P^i|} + w(X_P^{i*}, t), \quad X_P^{i*}(0) = X_{P_0}^i, \quad i \in \mathcal{I}_A, \tag{16}$$

where $\mathcal{I}_A \subseteq \mathcal{I}$ denotes the index set of the active pursuers. The corresponding optimal controls of the active pursuers are thus

$$u_P^{i*} = \bar{u}^i \frac{\nabla \phi_P^i}{|\nabla \phi_P^i|}, \quad i \in \mathcal{I}_A. \tag{17}$$

In order to find the optimal trajectories of the active pursuers, we propagate backward the dynamics (16) starting from X_f until it reaches $X_{P_0}^i$, $i \in \mathcal{I}_A$.

As for the evader, there are two possible outcomes after the termination of the evolution of the reachable sets of the pursuers and the evader. One possibility is that at the terminal time T, X_f resides on $\partial\tilde{\mathcal{R}}_E(X_{E_0}, T)$ (or $\partial\mathcal{R}_E(X_{E_0}, T)$ when $\bar{v} \le \min_{i\in\mathcal{I}} \bar{u}^i$). In this case, it follows that the boundary of the reachable set of the evader is not fully covered for all $t < T$. When differentiable, the optimal trajectory of the evader is then unique and it satisfies the differential equation

$$\frac{\mathrm{d}X_E^*}{\mathrm{d}t} = \bar{v}\frac{\nabla \phi_E}{|\nabla \phi_E|} + w(X_E^*, t), \quad X_E^*(0) = X_{E_0}. \tag{18}$$

The corresponding optimal control for the evader is given by

$$u_E^* = \bar{v}\frac{\nabla \phi_E}{|\nabla \phi_E|}. \tag{19}$$

Similar to the case of the pursuers, we propagate backward the dynamics (18) starting from X_f until the trajectory reaches X_{E_0} to find the optimal trajectory of the evader.

It may also happen that X_f lies in the interior of $\tilde{\mathcal{R}}_E(X_{E_0}, T)$ (or the interior of $\mathcal{R}_E(X_{E_0}, T)$ when $\bar{v} \le \min_{i\in\mathcal{I}} \bar{u}_i$). This situation occurs when there exists $t_c \in (t_0, T)$ such that $\partial\mathcal{R}_E(X_{E_0}, t) \subset \bigcup_{i=1}^n \mathcal{R}_P^i(X_{P_0}^i, t)$, for all $t \in [t_c, T]$. However, some part of the interior of $\mathcal{R}_E(X_{E_0}, t)$ may not be covered until time T. In this case, the optimal control of the evader is not necessarily unique. In particular, the control of the evader can be chosen from the set

$$\mathcal{U}_E^* = \left\{ u_E \in \mathcal{U}_E : X \text{ satisfies (2) and } X(\tau) \notin \bigcup_{i=1}^n \mathcal{R}_P^i(X_{P_0}^i, \tau), \forall \tau \in [t_0, T] \right\}. \tag{20}$$

It follows that an optimal evading strategy for the evader is valid, as long as it can bring the evader to X_f at time T, without getting captured by any of the pursuers prior to time T.

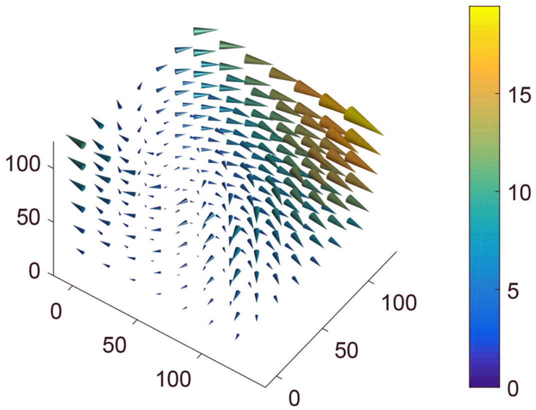
5 Simulation Results

In this section, we present simulation results for the multiplayer pursuit-evasion problem under distinct flow fields.

We first consider a state-dependent wind field that resembles the shape of a hurricane whose wind field snapshot can be found in [11]. Letting $X = [x, y, z]^\intercal \in \mathbb{R}^3$, this much simplified approximation model is generated by letting

$$w(X) = \sqrt{\frac{z}{h}} A(X - X_s), \quad X \in \mathcal{D}, \tag{21}$$

Fig. 3 An analytical flow field that approximates a hurricane



where

$$A = \begin{bmatrix} 0.2 & 0.3 & 0 \\ -0.15 & 0.1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{22}$$

In (21), $X_s = [55, 40, 0]$ denotes the location of the flow singularity and $\mathcal{D} = [0, 128]^3$ represents the 3D space. Also in (21), $h = 128$ denotes the height of the 3D space, and A is a 3×3 matrix, whose structure captures the local characteristics of the flow singularity. Notice that due to the chosen value of A, the flow field along the z axis is zero. Also, for each fixed z, the flow field in the xy-plane approximates the velocity field of a vortex with a linear vector field from the Rankine model [9]. The multiplier $\sqrt{z/h}$ scales the magnitude of the flow field along the z axis so that the flow intensifies as the height increases. The vector field of the external flow field is depicted in Fig. 3.

Here we address a five-pursuers/two-evaders problem. At time $t_0 = 0$, the five pursuers are located at $X^1_{P_0} = [60, 20, 30]^\intercal$, $X^2_{P_0} = [70, 60, 50]^\intercal$, $X^3_{P_0} = [90, 80, 70]^\intercal$, $X^4_{P_0} = [50, 80, 60]^\intercal$ and $X^5_{P_0} = [25, 70, 45]^\intercal$, respectively. Their corresponding maximum speeds are given by $\bar{u}_1 = 30$, $\bar{u}_2 = 30$, $\bar{u}_3 = 40$, $\bar{u}_4 = 50$ and $\bar{u}_5 = 40$. The initial locations of the evaders are given by $X^1_{E_0} = [50, 50, 40]^\intercal$, $X^2_{E_0} = [70, 90, 75]^\intercal$ and their maximum speeds are $\bar{v}_1 = 20$, $\bar{v}_2 = 15$, respectively. Note that, in this example, the maximum speed of the evaders is smaller than the speed of the pursuers. Therefore, we only need to propagate the reachability front of the evaders and pursuers in order to recover $\mathcal{R}^\star_E(X_{E_0}, t)$.

We start the simulation by propagating the reachability front of each pursuer until it coincides with one of the evaders to decide which evader the pursuer is assigned to. In this example, P_1, P_2 and P_5 are assigned to evader E_1 and P_3, P_4 are assigned to evader E_2.

For each sub-game, we utilize the reachable set algorithm to find the optimal terminal time for the evader to be captured by the pursuers. The optimal time-to-capture for the first sub-game between P_1, P_2, P_5 and E_1 is $T_1 = 1.20$ and the optimal time-to-capture of the second sub-game is $T_2 = 0.76$. So the overall time-to-capture is $T_o = 1.20$. The reachable fronts of the pursuers and the evaders at their corresponding capture time, as well as the optimal trajectories of the active pursuers and the evaders, are shown in Fig. 4. The red, yellow and purple color surfaces represent the reachable fronts of the pursuers P_1, P_2 and P_5 at the terminal time T_1, respectively. The union of the reachable sets of these pursuers fully cover the reachable set of the evader E_1 at time T_1 denoted by the blue surface and the evader is captured at X_{T_1}. The red dashed and blue lines represent the optimal trajectories of the

Fig. 4 Pursuit-evasion between 5 pursuers and 2 evader in an analytical flow field

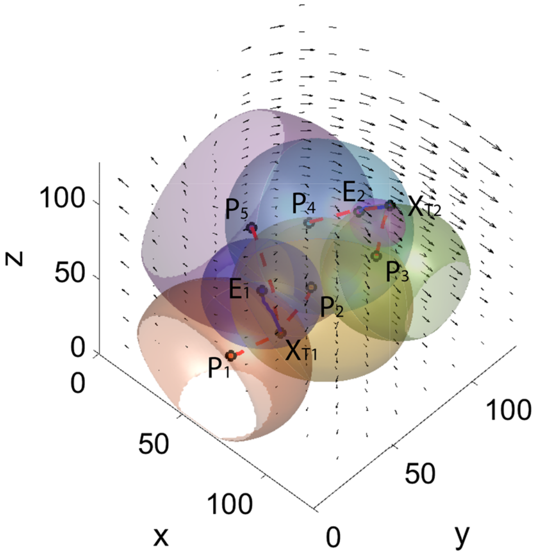
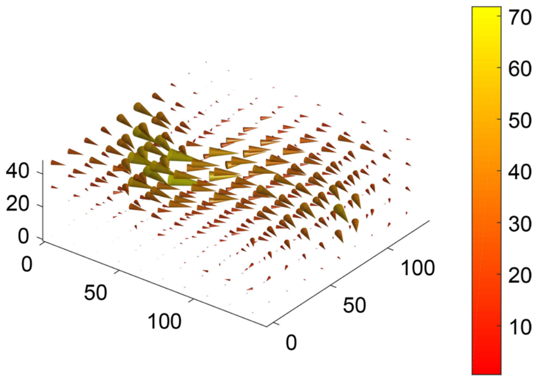


Fig. 5 A realistic flow field from MATLAB



pursuers and evaders, respectively. Black arrows on the background represent the external flow field.

Next, we apply our algorithm to a pursuit-evasion problem using a MATLAB default wind dataset [23]. The external flow field generated from this dataset is depicted in Fig. 5. Through this example, we demonstrate that the proposed algorithm can handle scenarios with complex spatial flow fields.

The initial positions of the pursuers and the evaders are set as $X^1_{P0} = [45, 20, 30]^\intercal$, $X^2_{P0} = [55, 45, 45]^\intercal$, $X^3_{P0} = [50, 70, 35]^\intercal$, $X^4_{P0} = [45, 95, 40]^\intercal$ and $X^1_{E0} = [70, 40, 40]^\intercal$, $X^2_{E0} = [60, 90, 30]^\intercal$, respectively. Their corresponding maximum speeds are given by $\bar{u}_1 = 40$, $\bar{u}_2 = 30$, $\bar{u}_3 = 30$, $\bar{u}_4 = 35$ and $\bar{v}_1 = 15$, $\bar{v}_2 = 20$. After propagation of the reachable sets of the pursuers, it is determined that P_3, P_4 will go after E_1 while E_2 will be actively chased by P_1 and P_2. Capture of E_1 occurs at time $T_1 = 1.00$ and E_2 is captured at time $T_2 = 0.98$. The optimal paths of the pursuers and the evaders are depicted in Fig. 6. The reachability fronts of the pursuers P_1 through P_4 are illustrated in purple, green, red and yellow colors. Similarly, the reachability fronts of the evaders E_1, E_2 are depicted by blue

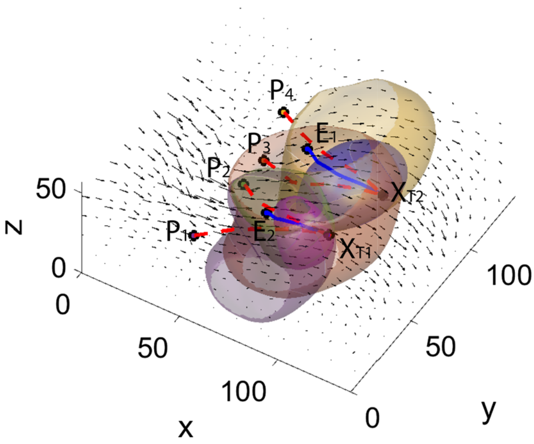


Fig. 6 Pursuit-evasion between 4 pursuers and 2 evaders in a realistic wind field

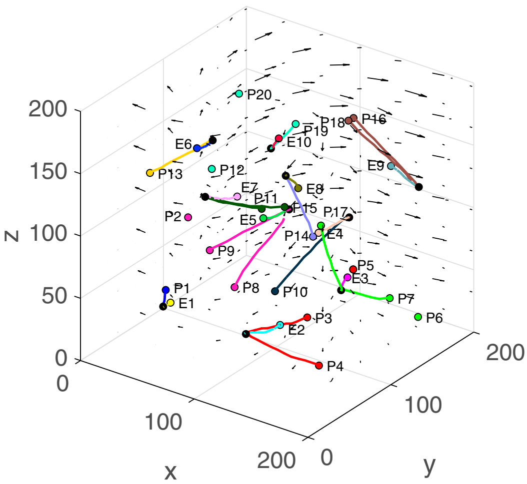


Fig. 7 Pursuer assignment through Algorithm 1 and the optimal trajectories of pursuit-evasion between 20 pursuers and 10 evaders

and magenta surfaces. The optimal trajectories of the pursuers and evaders are shown as red dashed and blue lines, respectively.

Finally, consider an example between 20 identical pursuers and 10 evaders under an state-dependent flow field. The external flow field in this case is given by

$$w(X) = \sqrt{\frac{z}{h_2}} B(X - X_{s2}), \quad X \in \mathcal{D}_2 \tag{23}$$

where

$$B = \begin{bmatrix} 0.1 & 0.1 & 0 \\ -0.1 & 0.1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{24}$$

In (21), $X_{s2} = [90, 80, 0]$, $\mathcal{D}_2 = [0, 200]^3$ and $h_2 = 200$.

We apply both Algorithms 1 and 2 to assign each pursuer to an evader. The results can be found in Figs. 7 and 8, respectively. The paired pursuers and evaders are illustrated in opposite colors. The ZVD is also depicted in Fig. 8 to highlight the assignment of pursuers.

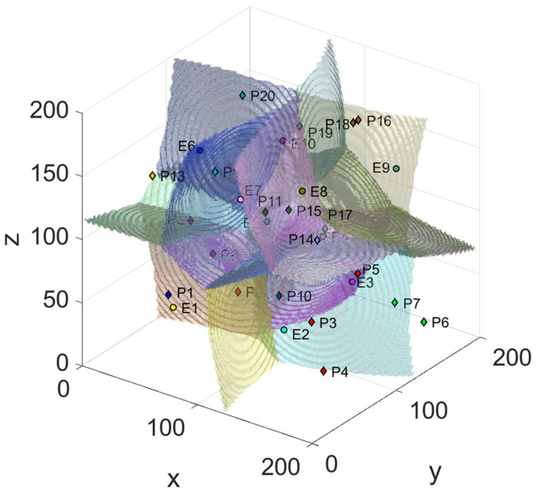


Fig. 8 Pursuer assignment through Algorithm 2 and the corresponding ZVD of pursuit-evasion between 20 pursuers and 10 evaders

In both cases, P_1 is assigned to evader E_1. P_3, P_4 and P_5 are assigned to evader E_2. P_6, P_7 and P_{17} are assigned to evader E_3. Also, E_4 is paired with P_{10}. E_5 is paired with P_2, P_8 and P_9. E_6 is paired with P_{13}. Furthermore, P_{11} and P_{15} are assigned to evader E_7. P_{14} is assigned to evader E_8. P_{16} and P_{18} are assigned to evader E_9. At last, P_{12}, P_{19} and P_{20} are assigned to evader E_{10}. The resulting multi-pursuers/one-evader sub-games are then solved to find optimal trajectories of the active pursuers and evaders, which are also depicted in Fig. 7. Among all the pursuers, P_2 P_5, P_6, P_{12} and P_{20} are identified as guards.

6 Conclusion

In this paper, we have considered differential games between multiple pursuers and multiple evaders in a three-dimensional space subject to an external dynamic flow field. The problem is divided into a pursuer assignment problem where a partition of the pursuer set is formed in order to pair the pursuers with the evaders and solve the subsequent sub-game between each evader and its assigned pursuers. Reachability set analysis is utilized for both problems and the narrow band level set method is adopted to propagate the reachable sets of all the players. It is also demonstrated why the narrow band level set method is chosen over the fast marching or fast sweeping schemes, which are not applicable to the differential game problem under general disturbances.

The pursuer assignment problem is first achieved by pairing each pursuer with the evader that first enters the reachable set of this pursuer among all the evaders. An algorithm is presented to partition the pursuer set through the generation of the ZVD from the initial positions of the evaders under the identical pursuer and time-invariant flow field assumption. The second algorithm is preferred in cases where the number of pursuers are much larger than that of the evaders. A numerical method to generate the ZVD from the propagation of backward reachable sets of the pursuers is also presented. Furthermore, when the maximum speed of the evader is less than the maximum speed of each pursuer, a simplified criterion to capture the evader is derived. Depending on whether a pursuer guards some part of the reachable set of the evader or actively chases the evader, the pursuers can be classified as guards or active pursuers. The optimal trajectories and controls of the pursuers and the evaders are retrieved by backward propagation along the corresponding levels of the reachable sets. The proposed solution scheme is demonstrated by applying it to multiplayer pursuit-evasion games taking place both in artificial flow fields and realistic wind fields.

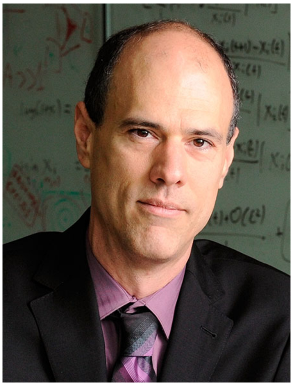
References

1. Bakolas E (2014) Optimal guidance of the isotropic rocket in the presence of wind. J Optim Theory Appl 162(3):954–974
2. Bakolas E, Tsiotras P (2010) The Zermelo–Voronoi diagram: a dynamic partition problem. Automatica 46(12):2059–2067
3. Bakolas E, Tsiotras P (2012) Relay pursuit of a maneuvering target using dynamic Voronoi diagrams. Automatica 48:2213–2220
4. Bannikov AS (2009) A non-stationary problem of group pursuit. J Comput Syst Sci Int 48(4):527–532
5. Başar T, Olsder GJ (1999) Dynamic noncooperative game theory, vol 23. SIAM, Philadelphia
6. Blagodatskikh AI (2009) Simultaneous multiple capture in a simple pursuit problem. J Appl Math Mech 73(1):36–40
7. Bopardikar SD, Bullo F, Hespanha JP (2009) A cooperative homicidal chauffeur game. Automatica 45(7):1771–1777
8. Chernous'ko FL (1976) A problem of evasion from many pursuers. J Appl Math Mech 40(1):11–20
9. Corpetti T, Memin E, Pérez P (2003) Extraction of singular points from dense motion fields: an analytic approach. J Math Imaging Vis 19(3):175–198
10. Dong L, Chai S, Zhang B, Nguang SK, Li X (2016) Cooperative relay tracking strategy for multi-agent systems with assistance of voronoi diagrams. J Frankl Inst 353(17):4422–4441
11. Earth Observatory (2012) Comparing the winds of Sandy and Katrina. https://earthobservatory.nasa.gov/IOTD/view.php?id=79626&src=ve
12. Gómez P, Hernandez J, López J (2005) On the reinitialization procedure in a narrow-band locally refined level set method for interfacial flows. Int J Numer Methods Eng 63(10):1478–1512
13. Ibragimov GI (2005) Optimal pursuit with countably many pursuers and one evader. Differ Equ 41(5):627–635
14. Ibragimov GI, Salimi M, Amini M (2012) Evasion from many pursuers in simple motion differential game with integral constraints. Eur J Oper Res 218(2):505–511
15. Isaacs R (1999) Differential games: a mathematical theory with applications to warfare and pursuit, control and optimization. Courier Dover Publications, Mineola
16. Khaidarov BK (1984) Positional I-capture in the game of a single evader and several pursuers. J Appl Math Mech 48(4):406–409
17. Liberzon D (2011) Calculus of variations and optimal control theory: a concise introduction. Princeton University Press, Princeton, NJ, chap. 4.5
18. Lolla T (2016) Path planning and adaptive sampling in the coastal ocean. PhD thesis, Department of Mechanical Engineering, Massachusetts Institute of Technology
19. Lolla T, Ueckermann MP, Yiğit K, Haley PJ Jr, Lermusiaux PFJ (2012) Path planning in time dependent flow fields using level set methods. In: IEEE international conference on robotics and automation. St. Paul, MN, pp 166–173
20. Lolla T, Lermusiaux PFJ, Ueckermann MP, Haley PJ Jr (2014) Time-optimal path planning in dynamic flows using level set equations: theory and schemes. Ocean Dyn 64(10):1373–1397
21. Lolla T, Haley PJ Jr, Lermusiaux PFJ (2015) Path planning in multi-scale ocean flows: coordination and dynamic obstacles. Ocean Model 94:46–66. https://doi.org/10.1016/j.ocemod.2015.07.013
22. Malladi R, Sethian JA, Vemuri BC (1995) Shape modeling with front propagation: a level set approach. IEEE Trans Pattern Anal Mach Intell 17(2):158–175
23. MATLAB (2015) version 8.6.0 (R2015b). The MathWorks Inc., Natick, Massachusetts
24. Osher S, Fedkiw R (2006) Level set methods and dynamic implicit surfaces, vol 153. Springer, New York
25. Petrov NN, Shuravina IN (2009) On the "soft" capture in one group pursuit problem. J Comput Syst Sci Int 48(4):521–526
26. Pittsyk M, Chikrii AA (1982) On a group pursuit problem. J Appl Math Mech 46(5):584–589
27. Pshenichnyi BN (1976) Simple pursuit by several objects. Cybern Syst Anal 12(3):484–485
28. Sethian JA (1999) Level set methods and fast marching methods: evolving interfaces in computational geometry, fluid mechanics, computer vision, and materials science, vol 3. Cambridge University Press, Cambridge

29. Sontag ED (2013) Mathematical control theory: deterministic finite dimensional systems, vol 6. Springer, New York
30. Sun W, Tsiotras P (2017) Sequential pursuit of multiple targets under external disturbances via Zermelo–Voronoi diagrams. Automatica 81:253–260. https://doi.org/10.1016/j.automatica.2017.03.015
31. Sun W, Tsiotras P, Lolla T, Subramani DN, Lermusiaux PFJ (2017) Multiple-pursuer/one-evader pursuit-evasion game in dynamic flowfields. J Guid Control Dyn 40(7):1627–1637

Wei Sun is an Assistant Professor in the School of Aerospace and Mechanical Engineering at the University of Oklahoma. He previously held postdoc appointments at the University of Washington and the Georgia Institute of Technology. He received his B.S. degree in Mathematics from the Peking University, China, in 2010. He got his two M.S. degrees in Aerospace Engineering and Mathematics, as well as a Ph.D. degree in Aerospace Engineering in 2017 from the Georgia Institute of Technology. His research interests include optimal control, differential games, reinforcement learning and trajectory optimization with applications in motion planning, and pursuit evasion games.

Panagiotis Tsiotras is the David and Andrew Lewis Chair Professor in the School of Aerospace Engineering at the Georgia Institute of Technology (Georgia Tech), where he is the Director of the Dynamics and Controls Systems Laboratory (DCSL) in the same school and an Associate Director of the Institute for Robotics and Intelligent Machines at Georgia Tech. He received his Ph.D. degree in Aeronautics and Astronautics from Purdue University and also holds degrees in Mathematics and Mechanical Engineering. He has held visiting research appointments at MIT, JPL, INRIA Rocquencourt, and Mines ParisTech. His research interests include optimal control of nonlinear systems with applications to ground, aerial and space vehicle autonomy. He has served in the Editorial Boards of the Transactions on Automatic Control, the IEEE Control Systems Magazine, the AIAA Journal of Guidance, Control and Dynamics and the journal Dynamics and Control. He is the recipient of the NSF CAREER award, the Outstanding Aerospace Engineer award from Purdue and the IEEE Technical Excellence Award in Aerospace Control. He is a Fellow of AIAA, and IEEE, and a member of the Phi Kappa Phi, Tau Beta Pi, and Sigma Gamma Tau Honor Societies.



Anthony J. Yezzi holds the position of Julian Hightower Chair Professor within the School of Electrical and Computer Engineering at Georgia Institute of Technology where he directs the Laboratory for Computational Computer Vision. He has over twenty years of research experience in shape optimization via geometric partial differential equations. He obtained his Ph.D. in Electrical Engineering in December 1997 from the University of Minnesota with a minor in mathematics. After completing a postdoctoral research appointment at Massachusetts Institute of Technology, he joined the faculty at Georgia Tech in August 1999. Dr. Yezzi's research lies primarily within the fields of image processing and computer vision with a particular emphasis on medical imaging and 3D surface reconstruction. He has consulted for a number of companies including GE, 3M, MZA, Philips, Picker, and VTI. His work spans a wide range of image processing and vision problems including image denoising, edge-detection, segmentation, shape analysis, multi-frame stereo reconstruction, visual tracking, and registration. Some central themes of his research include curve and surface evolution, differential geometry, partial differential equations, and shape optimization.