A COMPLEX ANALYTIC SOLUTION FOR THE ATTITUDE MOTION OF A NEAR-SYMMETRIC RIGID BODY UNDER BODY-FIXED TORQUES

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Abstract. Although analytic solutions for the attitude motion of a rigid body are available, for several special cases, a comprehensive theory does not exist in the literature for the more complicated problems found in spacecraft dynamics. In the present paper, analytic solutions in complex form are derived for the attitude motion of a near symmetric rigid body under the influence of arbitrary, but constant body-fixed torques. The solution is extremely compact, which enables efficient and rapid machine computation. Numerical simulations reveal that the solution is very accurate when applied to typical spinning spacecraft problems.

Keywords: analytic solutions, rigid body motion, kinematics

1. Introduction

Mathematicians have been working on the problem of rigid body motion for over two centuries. However, an analytic solution to the general problem of an arbitrary rigid body under the influence of arbitrary external torques, is far from complete. In fact, most existing analytic theories apply to highly idealized cases, such as torque-free or totally symmetric bodies. Solutions have been obtained for these and other several special cases by Euler, Jacobi, Poinsot and others, and are reported by Leimanis (1965) and Grammel (1954). Unfortunatelly however, these are hardly of practical importance to the more complex problems encountered in spacecraft dynamics and control. In fact, prior to the advent of jet propulsion, the problem of the self-excited rigid body, that is, a body under the influence of body-fixed torques, was mainly of academic interest, and most of the previous analytic theory was merely concerned with the case when the applied torques are dependent on the actual orientation of the body. Furthermore, even for these simple cases, such as the the case of torque-free motion of a general rigid body, where an analytic solution involving Jacobian elliptic functions has existed since the late 1800's, most modern authors of classical mechanics texts have elected not to discuss the details of the explicit analytic solution, in favor of the motion analogy to an ellipsoid rolling on an inavariable plane, first given by Poinsot (1851).

As a result of the little attention that has been paid to the development of a comprehensive analytic theory which treats the rigid body motion, scientists and engineers have come to rely on numerical methods for the solution of the problem. Even though such numerical solutions are easily found by computer simulations, analytic solutions can provide deeper insight into the problem, and can be used in obtaining quick solutions over large intervals of time, in error analyses, and in computer algorithms for onboard computations. Recently, new interest has been revived in the area of analytic solutions for the motion of spinning spacecraft. Analytical formulations have been obtained for satellite attitude computations, which significantly extend the classical torque-free and rigid body assumption of Poinsot motion (Cochran, 1972; Kraige and Junkins, 1976; Kraige and Skaar, 1977; Cochran and Shu, 1983). Other authors (Junkins *et al.*, 1973; Morton *et al.*, 1974) have also developed new formulations for Poinsot motion itself. Current interest in the area of analytic solutions for spinning spacecrafts is carried on, mainly because they have been found to be extremely useful in control problems and stability analyses associated with this class of vehicles (Likins, 1967; Larson and Likins, 1973; Junkins and Turner, 1980; Golubev and Demidov, 1984; Branets *et al.*, 1984; Winfree and Cochran, 1986) Among the recent developments in this field we can briefly mention especially the work by Larson and Likins (1974), where they obtained a closed-form solution for linearized equations in which transverse torques appear, but the spin rate is constant, and also the work by Cochran *et al.* (1982) where an exact solution was obtained for the free motion of a dual-spin spacecraft. For a symmetric rigid body subject to body-fixed torques about its principal axes a solution is given by Bodewadt (1952), however the solution for the orientation of the body in inertial space, is incorrect in these references for reasons explained in (Longuski, 1984). The case of near symmetry is dealt with by Longuski (1980), and its solution is exact for the symmetric case. The accuracy of the solutions have been tested, and the results are reported by Kia and Longuski (1984). Price (1981), using Longuski's solution as a first order approximation, has developed a semi-analytic solution in the form of power series in one of the applied torques. Although the series converge very rapidly, the method is limited to selected time intervals, so it has short term validity. Van der Ha (1984) gives a perturbation solution for the attitude motion under body-fixed torques, based on the ratio of transverse to spin rotation rate as the small parameter, but his solution is also valid only for short time intervals.

The scope of the present paper is to provide analytic solutions to the problem of the attitude evolution of a near-symmetric rigid body under arbitrary but constant body-fixed torques. The use of complex variables allows the solution to be expressed in a very compact form. The solution of Euler velocities is given in terms of a complex Fresnel integral function, and it is exact for a symmetric body. The solution for Euler angles is more involved, due to the difficulty in evaluating certain integrals in closed form, and it is limited by the assumption that the two angles defining the direction of the spin axis must be small. The accuracy of the solution is tested by numerical simulations and comparison with the solutions of the governing differential equations. Two cases are presented here, the first a spin-up maneuver from 3.15 to 10 rpm, and the second a spin-down maneuver through zero spin rate. Specific parameters were taken from the Galileo spacecraft, and the results reveal an excellent agreement between the "exact " numerical integration solution, and the analytic solution.

2. Solution for angular velocities

Euler's equations for motion of a rigid body with principal axes at the center of mass are

$$M_x = I_x\dot{\omega}_x+(I_z-I_y)\omega_y\omega_z \tag{1a}$$

$$M_y = I_y\dot{\omega}_y+(I_x-I_z)\omega_z\omega_x \tag{1b}$$

$$M_z = I_z\dot{\omega}_z+(I_y-I_x)\omega_x\omega_y \tag{1c}$$

where M_x,M_y,M_z are assumed to be constant body-fixed torques, I_x,I_y,I_z are the moments of inertia about the principal axes and $\omega_x,\omega_y,\omega_z$ are the angular velocity components along the same axes. For a nearly symmetric rigid body $I_x \approx I_y$, and one can immediately solve (1c) to obtain

$$\omega_z(t)\approx\frac{M_z}{I_z}t+\omega_{z0}\ , \qquad \omega_z(0) = \omega_{z0} \tag{2}$$

The approximation (2) is not only valid for nearly symmetric bodies, but also for the important practical case of spin-stabilized vehicles (such as rockets and spacecrafts), since then both Euler velocities ω_x, and ω_y tend to remain smaller than ω_z, so that their product $\omega_x\omega_y$ can be discarded in a first order approximation. Such a case implies that spinning is about the principal axis I_z with constant torque M_z, whereas M_x and M_y acting as disturbance torques lying in the transverse plane. If the magnitude of the disturbance torques M_x , M_y is small compared to the axial torque M_z as is often the case, the transverse angular rates ω_x, ω_y will indeed appear to be much smaller than the axial spin rate ω_z. This explains why ω_x and ω_y are often referred to, as angular velocities errors.

Substitute (2) into (1a),(1b) to get

$$M_x = I_x\dot{\omega}_x+(I_z-I_y)\omega_y(\frac{M_z}{I_z}t+\omega_{z0}) \tag{3a}$$

$$M_y = I_y\dot{\omega}_y+(I_x-I_z)\omega_x(\frac{M_z}{I_z}t+\omega_{z0}) \tag{3b}$$

Note that although we have assumed $I_x=I_y$ for the solution of ω_z, we have kept the distinction between I_x and I_y in the equations for ω_x and ω_y. This appears to be a trivial extension of the symmetric rigid body case, however as it will be shown, this assumption has significant consequences to the accuracy of the solution.

Rearranging terms, equations (3) can be written in the following convenient form

$$\dot{\Omega}_x+(At+B)\Omega_y = F_x \tag{4a}$$

$$\dot{\Omega}_y-(At+B)\Omega_x = F_y \tag{4b}$$

where

$$\Omega_x=\omega_x\sqrt{k_y}\ ,\quad \Omega_y=\omega_y\sqrt{k_x}\ ,\quad F_x=M_x/I_x\sqrt{k_y}\ ,\quad F_y=M_y/I_y\sqrt{k_x}$$
$$k_x=(I_z-I_y)/I_x,\quad k_y=(I_z-I_x)/I_y \tag{5}$$
$$A=k\alpha,\quad B=k\beta,\quad \alpha=M_z/I_z,\quad \beta=\omega_{z0},\quad k=\sqrt{k_x k_y}$$

The above definitions hold when the spinning axis - here assumed to be I_z - is the one which corresponds to the maximum or the minimum moment of inertia. For the sake of consistency, we will assume that the principal moments are ordered $I_z>I_y>I_x$ and hence, the spinning is about the axis of the major principal axis of inertia. Caution should be taken however, for the case in which I_z is the minor principal axis, because then, k_x and k_y should be taken with a negative sign so that the square roots in (5) are well-defined. The case where spinning is about the intermediate moment of inertia axis will not be considered here, since it will always result in unstable motion.

Introducing the complex variables

$$\Omega = \Omega_x + i\Omega_y\,, \qquad F = F_x + iF_y \tag{6}$$

we can combine (4a),(4b) into the following scalar equation

$$\dot{\Omega} - i(At+B)\Omega = F \tag{7}$$

This is a linear differential equation with time-varying coefficient. For the constant spin-rate case it reduces to a linear time-invariant differential equation, which can easily be solved by standard methods of operational calculus, for several special cases of forcing functions (Kurzhals, 1967). Physically Ω represents the trace of the total transverse rate velocity vector in the *skewed* body-fixed *xy* plane. The term *skewed* arises from the fact that Ω is not the actual transverse velocity vector - this would be $\omega_x + i\omega_y$ - but it is related to it by (5). That is, when the actual velocity vector traces a unit cirlce in *xy* plane, the Ω vector traces an ellipse with semiaxes k_x and k_y. This discrepancy is the result of the assumed asymmetry, and it is removed for an axially symmetric body.

It can be easily verified that the solution of (7) is given by

$$\Omega(t) = \Omega_0 \exp[i(A\frac{t^2}{2}+Bt+C)] + \exp[i(A\frac{t^2}{2}+Bt+C)]F\int_0^t \exp[-i(A\frac{t^2}{2}+Bt+C)]dt \tag{8}$$

where the first term of the right hand side of the equation represents the homogeneous part of the solution, and the second term represents the particular part due to the forcing function F. For reasons that will become obvious later, we choose the constant C to be equal to

$$C = \frac{B^2}{2A} \tag{9}$$

Then Ω_0 is related to the initial condition on $\Omega(t)$ as follows

$$\Omega(0) = \Omega_0 \exp(i\frac{B^2}{2A}) \tag{10}$$

Thus, given the initial conditions $\omega_x(0)$, $\omega_y(0)$, $\omega_z(0)$, and the mass properties of the body k_x, k_y, one can use equation (10), in order to determine Ω_0. Note from (8), that the choice of C affects only the homogeneous part of the solution, so we can always pick the value of the constant C arbitrarily, as long as we define the relation between the constant Ω_0 and the initial conditions $\Omega(0)$ in a consistent manner, as done here in (10). With this choice of C, and recalling that F is constant, equation (8) can be rewritten

$$\Omega(t) = \Omega_0 \exp[i\frac{(At+B)^2}{2A}] + \exp[i\frac{(At+B)^2}{2A}]F\int_0^t \exp[-i\frac{(A\tau+B)^2}{2A}]d\tau \tag{11}$$

In order to evaluate the integral involved in (11), consider the following transformation

$$\sigma = \frac{(A\tau+B)^2}{2|A|} \tag{12}$$

then

$$(A\tau+B)d\tau = s_3 d\sigma \quad \text{and} \quad (A\tau+B) = s_2\sqrt{2|A|\sigma} \tag{13}$$

where

$$s_3 = \mathrm{sgn}(A) \quad \text{and} \quad s_2 = \mathrm{sgn}(A\tau+B) \tag{14}$$

and sgn(.) is the signum function, defined by

$$\mathrm{sgn}(x) = \begin{cases} 1 & \text{if } x>0 \\ 0 & \text{if } x=0 \\ -1 & \text{if } x<0 \end{cases}$$

Note that $s_3 = +1$ corresponds to a spin-up maneuver, and $s_3 = -1$ corresponds to a spin-down maneuver. The case $s_3 = 0$ corresponds to constant spin-rate and it will not be considered here. Under the previous definitions, the integral involved in (11) becomes

$$I_\omega = \int_0^t \exp[-i\frac{(A\tau+B)^2}{2A}]d\tau = \frac{1}{\sqrt{2|A|}}\int_{\sigma_0}^{\sigma_1} \mathrm{sgn}(\tau+\frac{B}{A})\ \frac{\exp(-is_3\sigma)}{\sqrt{\sigma}}d\sigma \tag{15}$$

where

$$\sigma_1 = \frac{(At+B)^2}{2|A|} \quad \text{and} \quad \sigma_0 = \frac{B^2}{2|A|} \tag{16}$$

Integrals of the form

$$C_2(x) = \frac{1}{\sqrt{2\pi}}\int_0^x \frac{\cos(\eta)}{\sqrt{\eta}}d\eta \quad , \quad S_2(x) = \frac{1}{\sqrt{2\pi}}\int_0^x \frac{\sin(\eta)}{\sqrt{\eta}}d\eta \quad , \tag{17}$$

are called Fresnel Integrals, and have been extensively studied, and their values have been tabulated (Abramowitz and Stegan, 1972). If we now define the complex Fresnel integral function by

$$E(x) = C_2(x) - i\, s_3 S_2(x) = \frac{1}{\sqrt{2\pi}}\int_0^x \frac{\exp(-is_3\sigma)}{\sqrt{\sigma}}d\sigma \tag{18}$$

we can readily evaluate I_ω as

$$I_\omega = \sqrt{\frac{\pi}{|A|}}\left\{\mathrm{sgn}(t+\frac{B}{A})E[\frac{(At+B)^2}{2|A|}] - \mathrm{sgn}(\frac{B}{A})E[\frac{B^2}{2|A|}]\right\} \tag{19}$$

$$= \sqrt{\frac{\pi}{|A|}}\,[s_1 E(\sigma_1) - s_0 E(\sigma_0)]$$

where

$$s_1 = \mathrm{sgn}(t+\frac{B}{A}) \quad \text{and} \quad s_0 = \mathrm{sgn}(\frac{B}{A}) \tag{20}$$

3. Evaluation of Fresnel integrals

Fresnel integrals are notorious for their difficulty in approximating over a large range of their parameter. However, an excellent approximation based on the τ–method of Lanczos (1956), and given by Boersma (1960) is satisfactory . According to this method, two approximations are used, one valid for small values of the parameter, and the other valid for large values of the parameter.

If we define the function

$$f(x) = \int_0^x \frac{e^{-it}}{\sqrt{2\pi t}}dt = C_2(x) - i\, S_2(x) \tag{21}$$

then

$$f(x) = e^{-ix}\sum_{n=0}^{11}(a_n+ib_n)\left(\frac{x}{4}\right)^{n+\frac{1}{2}} \qquad \text{for } 0 \le x \le 4 \tag{22}$$

$$f(x) = \frac{1-i}{2} + e^{-ix}\sum_{n=0}^{11}(c_n+id_n)\left(\frac{4}{x}\right)^{n+\frac{1}{2}} \qquad \text{for } x > 4 \tag{23}$$

The numerical values of the coefficients a_n , b_n , c_n and d_n are given by Boersma (1960). The maximum error is 1.6×10^{-9} for the first approximation and 0.5×10^{-9} for the second approximation. The complex functions E(x) and f(x) are related by

$$E(x) = \begin{cases} f(x) & \text{if } s_3=+1 \\ f^*(x) & \text{if } s_3=-1 \end{cases} \tag{24}$$

where the asterisk indicates complex conjugate. The advantage of this approximation, is that it provides both Fresnel integrals in complex form, as required by equation (19). It is then an easy matter to separate the real and imaginary parts, if desired.

Other approximations for Fresnel integrals and for integrals of Fresnel integrals, using asymptotic and/or series expansions or rational functions can be found in Abramowitz and Stegan (1972).

4. Solution for the Euler angles

If we use a 3-1-2 Euler angle sequence to describe the orientation of the body-fixed reference frame, with respect to an inertially fixed reference frame, the following kinematic equations hold:

$$\dot{\phi}_x = \omega_x\cos\phi_y + \omega_z\sin\phi_y \tag{25a}$$

$$\dot{\phi}_y = \omega_y-(\omega_z\cos\phi_y-\omega_x\sin\phi_y)\tan\phi_x \tag{25b}$$

$$\dot{\phi}_z = (\omega_z\cos\phi_y-\omega_x\sin\phi_y)\sec\phi_x \tag{25c}$$

A small angle approximation for ϕ_x,ϕ_y reduces the previous system of equations to

$$\dot{\phi}_x = \omega_x + \phi_y\omega_z \tag{26a}$$

$$\dot{\phi}_y = \omega_y - \phi_x\omega_z \tag{26b}$$

$$\dot{\phi}_z = \omega_z - \phi_y\omega_x \tag{26c}$$

If we also assume that the product $\phi_y\omega_x$ is small compared to ω_z, we can immediately solve for ϕ_z to get

$$\phi_z(t) = \int_0^t \omega_z(\tau)d\tau \tag{27}$$

Note that the differential equations (26a) and (26b) are independent of ϕ_z, hence the accuracy of the solution of ϕ_x, and ϕ_y will not be affected by dropping the term $-\phi_y\omega_x$ in (26c). If however, one wishes a more precise solution for ϕ_z, it may be possible to reinstate the ignored term as a perturbation.

Using the expression for $\omega_z(t)$ from (2), one can readily perform the integration for $\phi_z(t)$ to obtain

$$\phi_z(t) = \frac{1}{2}\frac{M_z}{I_z}t^2 + \omega_{z0}t + \phi_{z0}, \qquad \phi_z(0) = \phi_{z0} \tag{28}$$

Introducing the complex variables

$$\phi = \phi_x + i\phi_y \quad \text{and} \quad \omega = \omega_x + i\omega_y \tag{29}$$

we can combine (26a) and (26b) into a simple scalar equation

$$\dot{\phi} + i\omega_z \phi = \omega \tag{30}$$

The solution of (30) has the same form as for the case of the angular velocities, where now the forcing term is $\omega(t)$. The solution is given by

$$\phi(t) = \phi_0 \exp[-i(\alpha\frac{t^2}{2}+\beta t+\gamma)] + \exp[-i(\alpha\frac{t^2}{2}+\beta t+\gamma)]\int_0^t \omega(\tau)\exp[i(\alpha\frac{\tau^2}{2}+\beta\tau+\gamma)]d\tau \tag{31}$$

Again, the choice of γ affects only the homogeneous part in (31) so we can choose the constant γ and the initial conditions such that

$$\gamma = \frac{\beta^2}{2\alpha}, \qquad \phi(0) = \phi_0 \exp(-i\frac{\beta^2}{2\alpha}) \tag{32}$$

Then we can rewrite (31) in the form

$$\phi(t) = \phi_0 \exp[-i\frac{(\alpha t+\beta)^2}{2\alpha}] + \exp[-i\frac{(\alpha t+\beta)^2}{2\alpha}]\int_0^t \omega(\tau)\exp[i\frac{(\alpha\tau+\beta)^2}{2\alpha}]d\tau \tag{33}$$

Recall that $\omega(t) = \omega_x(t)+i\omega_y(t)$ and $\Omega(t) = \Omega_x(t)+i\Omega_y(t)$ thus, we can express the angular velocity $\omega(t)$ in terms of $\Omega(t)$ as follows

$$\omega(t) = \frac{\Omega_x(t)}{\sqrt{k_y}} + i\frac{\Omega_y(t)}{\sqrt{k_x}} = \frac{\Omega(t)+\Omega^*(t)}{2\sqrt{k_y}} + \frac{\Omega(t)-\Omega^*(t)}{2\sqrt{k_x}} \tag{34}$$

Using this relation the integral in (33) can be rewritten as

$$I_\phi = \int_0^t \omega(\tau)\exp[i\frac{(\alpha\tau+\beta)^2}{2\alpha}]d\tau = \frac{\sqrt{k_x}+\sqrt{k_y}}{2\sqrt{k_x k_y}} I_{\phi_1} + \frac{\sqrt{k_x}-\sqrt{k_y}}{2\sqrt{k_x k_y}} I_{\phi_2} \tag{35}$$

where

$$I_{\phi_1} = \int_0^t \Omega(\tau)\exp[i\frac{(\alpha\tau+\beta)^2}{2\alpha}]d\tau, \quad \text{and} \quad I_{\phi_2} = \int_0^t \Omega^*(\tau)\exp[i\frac{(\alpha\tau+\beta)^2}{2\alpha}]d\tau \tag{36}$$

Let $\lambda = 1/k$, then from (5) $\alpha = \lambda A$ and $\beta = \lambda B$. Using the already known solution for $\Omega(t)$ and the new independent variable σ introduced in (12), we can rewrite the integral I_{ϕ_1} as

$$I_{\phi_1} = \left[\Omega_0 - F\sqrt{\frac{\pi}{|A|}} s_0 E(\sigma_0)\right]\int_0^t \exp[i(\lambda+1)\frac{(A\tau+B)^2}{2A}]d\tau \tag{37}$$
$$+ F\sqrt{\frac{\pi}{|A|}}\int_0^t \exp[i(\lambda+1)\frac{(A\tau+B)^2}{2A}]\mathrm{sgn}(\tau+\frac{B}{A})E(\sigma)d\tau$$

It is not difficult to show that the first integral in the above equation is easily evaluated as in (15), with an obvious change of independent variable, as follows

$$J_{\phi_1} = \int_0^t \exp[i(\lambda+1)\frac{(A\tau+B)^2}{2A}]d\tau \tag{38}$$
$$= \sqrt{\frac{\pi}{|A|(\lambda+1)}}\left\{s_1 E^*[(\lambda+1)\sigma_1] - s_0 E^*[(\lambda+1)\sigma_0]\right\}$$

The evaluation of the second integral in (37) is more involved. Use the transformation (12) to rewrite the integral in the form

$$J_{\phi_2} = \int_0^t \exp[i(\lambda+1)\frac{(A\tau+B)^2}{2A}]\mathrm{sgn}(\tau+\frac{B}{A})E(\sigma)d\tau = \frac{1}{\sqrt{2|A|}}[W_1(\lambda,\sigma_1)-W_1(\lambda,\sigma_0)] \tag{39}$$

where

$$W_1(\lambda,x)=\int_o^x \frac{\exp(is_3(\lambda+1)\eta)E(\eta)}{\sqrt{\eta}}d\eta \tag{40}$$

is a function to be evaluated later. Then, I_{ϕ_1} takes the final from

$$I_{\phi_1}=\left[\Omega_0-F\sqrt{\frac{\pi}{|A|}}s_0E(\sigma_0)\right]\sqrt{\frac{\pi}{|A|(\lambda+1)}}\left\{s_1E^*[(\lambda+1)\sigma_1]-s_0E^*[(\lambda+1)\sigma_0]\right\} \tag{41}$$
$$+\frac{F}{|A|}\sqrt{\frac{\pi}{2}}[W_1(\lambda,\sigma_1)-W_1(\lambda,\sigma_0)]$$

In a similar way, one can show that the integral I_{ϕ_2} is given by

$$I_{\phi_2}=\left[\Omega^*{}_0-F^*\sqrt{\frac{\pi}{|A|}}s_0E^*(\sigma_0)\right]\sqrt{\frac{\pi}{|A|(\lambda-1)}}\left\{s_1E^*[(\lambda-1)\sigma_1]-s_0E^*[(\lambda-1)\sigma_0]\right\} \tag{42}$$
$$+\frac{F^*}{|A|}\sqrt{\frac{\pi}{2}}[W_2(\lambda,\sigma_1)-W_2(\lambda,\sigma_0)]$$

where

$$W_2(\lambda,x)=\int_o^x \frac{\exp(is_3(\lambda-1)\eta)E^*(\eta)}{\sqrt{\eta}}d\eta \tag{43}$$

The evaluation of the integrals $W_1(\lambda,x)$ and $W_2(\lambda,x)$ will be discussed next. Without loss of generality, and for the sake of brevity, we will consider only the case when $s_3=+1$, since for the case $s_3=-1$ one can simply substitute $W_1(\lambda,x)$ and $W_2(\lambda,x)$ in (41),(42) by their complex conjugates. First recall that the expression for E(x) involves according to (24), two different approximations, one valid for $0\le x\le 4$ and the other valid for $x>4$. Thus, we can rewrite $W_i(\lambda,x)$, (i=1,2) as

$$W_i(\lambda,x)=\begin{cases} W'_i(\lambda,x) & 0\le x\le 4 \\ W'_i(\lambda,4)+W''_i(\lambda,x) & x>4 \end{cases} \tag{44}$$

where

$$W''_1(\lambda,x)=\int_4^x \frac{\exp(is_3(\lambda+1)\eta)E(\eta)}{\sqrt{\eta}}d\eta \tag{45}$$

$$W''_2(\lambda,x)=\int_4^x \frac{\exp(is_3(\lambda-1)\eta)E^*(\eta)}{\sqrt{\eta}}d\eta \tag{46}$$

After substitution of (24) into (40) and (43), and integrating term by term, we get

$$W'_1(\lambda,x)=\frac{1}{2\lambda}\sum_{n=0}^{11}\frac{(a_n+ib_n)}{(4\lambda)^n}I'_n(x) \qquad \text{for } 0\le x\le 4 \tag{47a}$$

$$W''_1(\lambda,x)=\frac{1-i}{2}\left[\frac{2\pi}{\lambda+1}\right]^{1/2}\left\{E^*[(\lambda+1)x]-E^*[(\lambda+1)4]\right\} \tag{47b}$$
$$+2\sum_{n=0}^{11}(c_n+id_n)(4\lambda)^nI''_n(x) \qquad \text{for } x>4$$

and

$$W'_2(\lambda,x)=\frac{1}{2\lambda}\sum_{n=0}^{11}\frac{(a_n-ib_n)}{(4\lambda)^n}I'_n(x) \qquad \text{for } 0\le x\le 4 \tag{48a}$$

$$W''_2(\lambda,x)=\frac{1+i}{2}\left[\frac{2\pi}{\lambda-1}\right]^{1/2}\left\{E^*[(\lambda-1)x]-E^*[(\lambda-1)4\right\} \tag{48b}$$
$$+2\sum_{n=0}^{11}(c_n-id_n)(4\lambda)^nI''_n(x) \qquad \text{for } x>4$$

where

$$I'_n(x) = \int_0^{\lambda x} \exp(i\eta)\eta^n d\eta\ , \quad \text{for } 0 \le x \le 4, \quad n=0,1,2,...,11 \tag{49}$$

$$I''_n(x) = \int_{\lambda 4}^{\lambda x} \frac{\exp(i\eta)}{\eta^{n+1}} d\eta\ , \quad \text{for } x > 4, \quad n=0,1,2,...,11 \tag{50}$$

The previous sequences of integrals can be evaluated recursively, using the relationships

$$I'_{n+1}(x) = -i(\lambda x)^{n+1}\exp(i\lambda x) + i(n+1)I'_n(x) \qquad n=0,1,2,...,11 \tag{51}$$

$$I''_{n+1}(x) = -\frac{\exp(i\lambda x)}{(n+1)(\lambda x)^{n+1}} + \frac{\exp(i\lambda 4)}{(n+1)(\lambda 4)^{n+1}} + i\frac{I''_n(x)}{n+1} \qquad n=0,1,2,...,11 \tag{52}$$

The first integrals of the above sequences are

$$I'_0(x) = \int_0^{\lambda x} \exp(i\eta)d\eta = -i(\exp(i\lambda x)-1) \tag{53a}$$

$$I''_0(x) = \int_{\lambda 4}^{\lambda x} \frac{\exp(i\eta)}{\eta} d\eta = [\mathrm{Ci}(\lambda x)-\mathrm{Ci}(\lambda 4)] + i[\mathrm{Si}(\lambda x)-\mathrm{Si}(\lambda 4)] \tag{53b}$$

Where Si(x) and Ci(x) are the well-known sine and cosine integrals defined by

$$\mathrm{Si}(x) = \int_0^x \frac{\sin(t)}{t} dt \qquad \mathrm{Ci}(x) = \bar{\gamma} + \ln(x) + \int_0^x \frac{\cos(t)-1}{t} dt \tag{54}$$

and $\bar{\gamma}$ = Euler's constant (= 0.57721 ...). The evaluation of the sine and cosine integrals in (54) can be easily performed uniformly, by rational approximations (Abramowitz and Stegan, 1972).

We should mention in passing, that one should be careful with the definition of the complex function E(x) since its argument should be always positive. It is a well-known fact however, that the following relations hold between the principal moments of inertia of an arbitrary rigid body

$$I_x+I_y>I_z\ ,\ I_x+I_z>I_y\ ,\ I_y+I_z>I_x \tag{55}$$

From the first and second equations, along with the definitions for k_x and k_y, we get that

$$-1 < k_x < 1 \quad \text{and} \quad -1 < k_y < 1 \tag{56}$$

As mentioned at the beginning, for a rigid body spinning about one of the two stable principal axes, $k = \sqrt{k_x k_y} > 0$. Hence, the parameter k satisfies the inequality $0 < k < 1$, and since $\lambda = 1/k$, we have that

$$1 < \lambda < \infty \tag{57}$$

As a consequence, both $\lambda+1 > 0$ and $\lambda-1 > 0$ and the arguments of E(x) are well-defined in equations (41),(42),(47b) and (48b).

5. Discussion of the solution

Taking advantage of the special symmetric structure of the problem, we have used a complex analytic approach to derive analytic solutions for the problem of the attitude motion of a self-excited rigid body. The use of complex variables allowed for the formulation of the solution in a very compact form, which is appealing especially for machine computations. This is very important, since it appears that for future applications, realistic compact analytical expressions modelling the attitude evolution, will become vital in on board attitude control software. Certain difficulties arise from the sign functions in the solution, but these were included, in order to give the solution in a complete form, i.e., a solution for both Eulerian rates and angles, valid for both spin-up and spin-down maneuvers, and valid also in the critical neighborhood of zero-spin rate case. This is the first time that such a complete solution for this problem is reported in the literature, as far as the authors know.

Although, even in complex form the analytic solutions are considerably shorter than previous related results, they are still quite lengthy for hand computations. Nevertheless, definite conclusions can be drawn about the asymptotic behaviour of the solution. By keeping in the solution for example, only those terms that create secular effects, one can capture the essential behavior of the motion, thus gaining an invaluable insight to the nature of the problem. In fact, it has been shown in the past, that even such simplified procedures, can be proven to be extremely successful for the study of attitude motion and control of modern spacecraft (Longuski, 1989).

6. Numerical examples

The application of the theory is illustrated by means of practical examples, such as spin-up or spin-down maneuvers of the Galileo spacecraft. Two cases are examined, the first case a spin-up maneuver from $\omega_z(0) = 3.15$ rpm to $\omega_z(t_f) = 10$ rpm. The second is spin-down maneuver from $\omega_z(0) = 3.15$ rpm to $\omega_z(t_f) = -3.15$ rpm. For both cases, the following initial conditions are assumed

$$\omega_x(0) = \omega_y(0) = 0 \tag{58}$$

$$\phi_x(0) = \phi_y(0) = \phi_z(0) = 0 \tag{59}$$

The mass properties for the Galileo spacecraft are

$$I_x = 2985 \text{ kg–m}^2, \quad I_y = 2729 \text{ kg–m}^2, \quad I_z = 4183 \text{ kg–m}^2 \tag{60}$$

In general, transverse torques arise during spin-up or spin-down maneuvers, due to error sources such as thruster misalignment or thruster mismatch. The Galileo spacecraft is a rather extreme example of a spacecraft that uses a single thruster for the spin-up and spin-down maneuvers. Moreover, the center of mass does not lie in the plane of the applied thrust force. As a result, there are significant torques about all three body-fixed axes. The torques generated about the body axes, are given by

$$M_x = -1.253 \text{ N–m}, \quad M_y = -1.494 \text{ N–m}, \quad M_z = \pm 13.5 \text{ N–m} \tag{62}$$

where the plus sign in M_z corresponds to spin-up, and the minus sign to spin-down.

6.1. CASE 1: SPIN-UP FROM 3.15 TO 10 RPM

The analytic solutions for the attitude motion are compared to the "exact" solutions which are found by numerical integration of equations (1) and (25). Fig. 1 compares the exact solution for $\omega_x(t)$ with the analytic solution. In Fig. 1a both exact and analytic solutions are displayed, but they are indistinguishable from one another. Their difference, presented in Fig. 1b, has oscillatory behaviour, with a linearly increasing envelope. The same plot indicates that the analytic solution for $\omega_x(t)$ deviates from the exact solution by only about 0.1%. Similar results were found also for the solution for $\omega_y(t)$. Fig.2 demonstrates that the linearity assumption (2) for $\omega_z(t)$ is reasonable, since the error indicates a discrepancy of only about 0.01% from the exact solution.

In Fig. 3, the exact solution for $\phi_x(t)$ is compared to the analytic solution. The discrepancy between the exact and analytic solutions is not apparent in Fig. 4a, but Fig. 4b, which displays their difference reveals that the error is of the order 0.5%. In Fig. 5 the analytic solution for $\phi_z(t)$ is shown to be within about 0.01% of the exact solution.

6.2. CASE 2: SPIN-DOWN FROM 3.15 TO -3.15 RPM

The second example examines the very important case of despinning, possibly through the region of zero spin rate. Low spin-rate, in conjuction with the nonlinear rate coupling effect inherent in the Euler equations, can have catastrophic consequences, as was vividly demonstrated by the GEOS-1 satellite experiment (Van der Ha, 1984).

Plots reveal that the assumption $\omega_z(t) = M_z/I_z + \omega_{z0}$ still remains valid, although the error with the exact solution has increased to 0.3%. The solutions for $\omega_x(t)$ and $\omega_y(t)$ are still very accurate, and up to the point when $\omega_z(t)$ crosses the critical zero spin rate value, the error is of the same order of magnitude as in the previous case. At very low spin rates, however, the transverse torques create large angular displacements, and the small angle approximation for $\phi_x(t)$, $\phi_y(t)$ is no longer valid. The kinematics equations have entered the region of nonlinearity, which is clearly illustrated by the phase shift in Fig. 8 and Fig. 9. From this point of view, a nonlinear method, such as the Poincaré's or Lindstedt's method of small parameters (Blaquiere, 1966), could be proven to be useful, in a second order approximation of the solution. Regardless of this fact, the solutions are still qualitatively correct, and the analytic solution predicts the time history of the attitude orientation very closely. The degradation of the accuracy of the solution at the low spin rate region should be expected; further simulations however, not presented herein, have shown that low spin-rate by itself is not a matter of concern. Rather the relative magnitude of the axial to the transverse torques i.e., $M_z/(M_x^2+M_y^2)^{1/2}$, acting during the time that the body is in the neighborhood of zero spin rate, has proven to be the major factor for the inaccuracy of the analytic solution.

7. Conclusions

Analytic solutions have been derived for the attitude motion of spinning, self-excited near-symmetric rigid body. The complex representation enabled the solution to take a compact form, especially suitable for implementation in maneuver or attitude control software. The solution assumes exact axial symmetry in order to write the solution for the angular velocity about the spinning axis in a linear form, but keeps the distinction of the moments of inertia in the other two equations for the angular velocities. A small angle approximation allowed the Euler angles to be given as the solution of a linear, time-varying system with the expression for the angular velocities acting as a forcing function. Numerical simulations reveal that the solutions are very accurate in describing the rotational motion of a typical spacecraft. Current and previous research indicates, that such analytic solutions, although cumbersome for hand computations, can be extremely helpful in capturing the fundamental behavior of the motion, by neglecting the non-secular terms, and provide insight into the mechanics of the motion, which cannot be derived from numerical solutions.

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