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A Reduced-Effort Control Law for Underactuated Rigid Bodies

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Abstract

Recent results show that a nonsmooth, time-invariant feedback control law can be used to stabilize an axi-symmetric rigid body using only two control torques to the zero equilibrium. This method, however, may require a significant amount of control effort, especially for initial conditions close to an equilibrium manifold. In this paper we propose a modification of the previous control law which reduces the control effort required. The new control law renders the equilibrium manifold unstable and drives the trajectories of the closed-loop system into a "safe" region where the original control law can be subsequently used.

1 Introduction

The problem of stabilization of a rigid body using less than three control inputs has received a lot of attention in the recent literature. Both the problems of the stabilization of the dynamics, and the stabilization of the kinematics have been treated in the literature [1, 2, 3, 4, 5, 6]. The stabilization problem of the complete system, i.e., the dynamics and the kinematics, has been addressed in [7, 8, 9, 10, 11, 12, 13]. For example, the attitude stabilization of an axially symmetric rigid body using two independent control torques was studied by Krishnan, *et al.* in [8, 9]. If the uncontrolled principal axis is not the axis of symmetry the system is strongly accessible and small time locally controllable [9]. When the uncontrolled axis coincides with the axis of symmetry, the complete system fails to be controllable or even accessible. However, the system equations are strongly accessible and small time locally controllable in the case of zero spin rate. A nonlinear control approach was developed in [8], which achieves arbitrary reorientation for this restricted case. In [14, 15] the authors presented a new formulation of the attitude kinematics and used it in [10, 11] in order to solve the same problem avoiding the successive switchings of [8]. The previous results address the axi-symmetric body case. The non-symmetric case is much more difficult and it is treated in [12, 13, 16].

In this paper, we provide a modification of the control law presented in [11] for the attitude stabilization of an axi-symmetric rigid body using two independent control torques. Because the system has an equilibrium manifold which includes the origin, Brockett's necessary condition for smooth stabilizability is not satisfied and thus, any stabilizing control law is necessarily nonsmooth. (Stabilizing *time-varying* smooth control laws may still exist, however.) This nonsmoothness is evident in [11] in the form of the non-differentiability of the control law at the origin. Because of this singularity at the origin, this control law may take large values, especially for initial conditions close to the equilibrium manifold. Compared to the control law in [11] the control law proposed in this paper remedies this high control authority problem by driving the trajectories of the closed-loop system away from the singular equilibrium manifold and to a region in the state space where the high authority part of the control input remains small and bounded. The procedure is simple and can be easily validated from phase portrait considerations. A numerical example illustrates the control effort improvement using the new control law.

2 Kinematics of the Attitude Motion

The orientation of a rigid spacecraft can be specified using various parameterizations, for example, Eulerian Angles, Euler Parameters, Cayley-Rodrigues Parameters, Cayley-Klein parameters, etc. Recently, a new parameterization using a pair of a complex and a real coordinate was introduced [14, 15] which was shown to have some significant advantages for attitude analysis and control problems [11, 17, 18]. According to these results, the relative orientation between two given reference frames can be represented by *two rotations*, one corresponding to the real coordinate (z) and the other corresponding to the complex coordinate (w). Let $R_1(z)$ and $R_2(w)$ denote the rotation matrices corresponding to the two elementary rotations associated with z and w, respectively. The total rotation matrix is then the product of these two rotation matrices

$$R(w, z) = R_2(w)R_1(z) \tag{1}$$

An explicit formula for $R(w, z)$ can be found in [15].

The kinematic equations provide the geometric constraints of the motion which relate the rates of the kinematic parameters z and w to the angular velocity vector expressed in body coordinates. As was shown in [11, 15], these equations can be written compactly in terms of w and z as follows

$$\begin{aligned} \dot{w} &= -i\,\omega_3 w + \frac{\omega}{2} + \frac{\bar{\omega}}{2} w^2 && \text{(2a)} \\ \dot{z} &= \omega_3 + Im(\omega \bar{w}) && \text{(2b)} \end{aligned}$$

where $\omega = \omega_1 + i\,\omega_2$, $w = w_1 + i\,w_2$, $i = \sqrt{-1}$, bar denotes complex conjugate, and $Re(\cdot)$ and $Im(\cdot)$ denote real and imaginary parts of a complex number respectively. Moreover, notice that these equations can take the convenient form

$$\begin{aligned} \frac{d}{dt}|w|^2 &= (1 + |w|^2)Re(\omega \bar{w}) && \text{(3a)} \\ \dot{z} &= \omega_3 + Im(\omega \bar{w}) && \text{(3b)} \end{aligned}$$

where $|\cdot|$ denotes absolute value. In Eq. (3b) only the imaginary part of the product $\omega \bar{w}$ appears, while in Eq. (3a) only the real part appears. This duality (or anti-symmetry) of Eqs. (3a) and (3b) is desirable and can be used to derive stabilizing control laws for the system of Eqs. (2).

Note in passing that since $w = 0$ if and only if $|w| = 0$ stabilization of the system in Eqs. (2) is equivalent to stabilization of the system Eqs. (3).

3 Problem Statement

Consider the system of Eqs. (2), equivalently Eqs. (3), where the angular velocity vector is the assumed control input. In realistic situations three (or even two) independent control *torques* could be used to shape the velocity vector instead. For an axi-symmetric body (say, about the 3-axis), there is no direct control over ω_3 if the applied torque vector lies in the plane which is perpendicular to the symmetry axis. In such a case ω_3 remains constant, and if initially it is $\omega_3(0) \neq 0$, no control input can bring the system (2) to the equilibrium [8]. The system is not controllable to the equilibrium but it is controllable to the submanifold $w = 0$ in the (z, w)-space. For a more detailed discussion on this issue, one may peruse [8, 9, 11, 15].

According to the previous discussion assuming an axi-symmetric body, the stabilization to the equilibrium of the system (2) really makes sense only if $\omega_3 \equiv 0$. In this case, the system simplifies to

$$\begin{aligned} \dot{w} &= \frac{\omega}{2} + \frac{\bar{\omega}}{2} w^2 && \text{(4a)} \\ \dot{z} &= Im(\omega \bar{w}) && \text{(4b)} \end{aligned}$$

This system can be stabilized to the origin, but any time-invariant stabilizing control law has to be necessarily *nonsmooth* since Eqs. (4) fail Brockett's necessary condition for smooth stabilizability [19]. One is therefore compelled to use nonsmooth (time-invariant) stabilizers for this system.

4 Previous Results

In Ref. [11] a nonsmooth control law was proposed for the system described by Eqs. (4). The proposed control law in [11] was motivated by the decoupling of these equations with respect to the product $\omega \bar{w}$, as it is evident from the discussion following Eqs. (3). This control law is given by

$$\omega = -\kappa w - i\mu \frac{z}{\bar{w}} \tag{5}$$

where $\mu > \kappa/2 > 0$. The closed loop system in terms of $|w|$ and z is given by

$$\begin{aligned} \frac{d|w|^2}{dt} &= -\kappa |w|^2 (1 + |w|^2) && \text{(6a)} \\ \dot{z} &= -\mu z && \text{(6b)} \end{aligned}$$

which is clearly exponentially stable. As can be easily inferred by observing Eqs. (5) and (3) the first term in the control (5) has an effect only on the differential equation for w, whereas the second term in Eq. (5) has an effect only on the differential equation for z.

The main disadvantage of the control law in Eq. (5) is that the last term, which involves the ratio $z/\bar{w}$, may become unbounded without careful choice of the gains. The previously imposed gain condition $\mu > \kappa/2$ ensures that the rate of decay of z is at least as large as the rate of decay of w, such that their ratio remains bounded. Actually, one can easily establish from Eqs. (6) that, for $\mu > \kappa/2$, along the solutions of the system one has $z/\bar{w} \to 0$ as $t \to \infty$.

Introducing the variable $v = |w|^2$ the system (6) takes the form

$$\begin{aligned} \dot{v} &= -\kappa(1 + v)v && \text{(7a)} \\ \dot{z} &= -\mu z && \text{(7b)} \end{aligned}$$

Notice that this is a system evolving in $\mathbb{R}_+ \times \mathbb{R}$. Typical trajectories and the vector field of the closed-loop system (7) for $\kappa = 1$ and $\mu = 2$ are shown in Fig. 1.

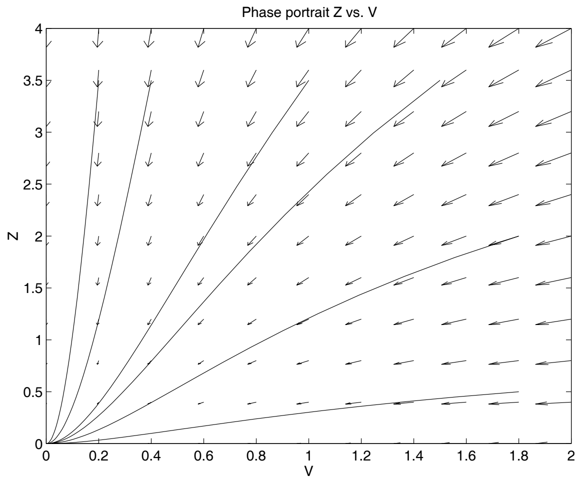

Figure 1: Phase portrait of system in Eqs. (8).

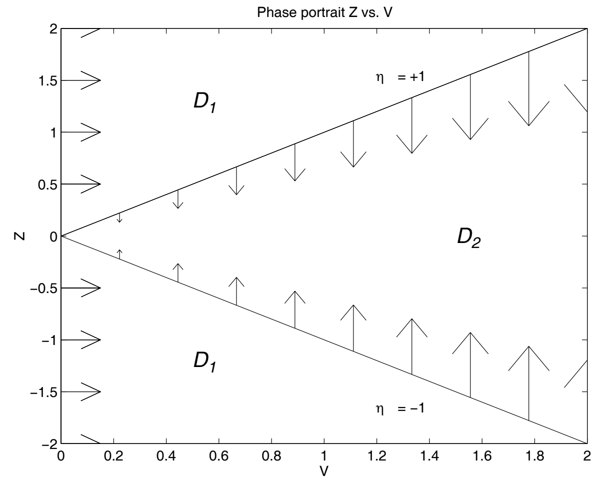

Figure 2: Regions $\mathcal{D}_1$ and $\mathcal{D}_2$ in (z, v) phase space.

(Since z does not change sign it suffices to plot only the $z > 0$ case.)

Although the ratio $z/\bar{w}$, and hence the control effort ω, remains bounded by proper choice of control gains, the control input ω may take large values in the region where w is small. From Eq. (6a) $|w(t)| \leq |w(0)|$ for all $t \geq 0$ and for small initial conditions $w(0)$ the control law may use a substantial amount of energy, especially in regions where $|z|$ is large. In Fig. 1, for example, the region which is close to the z axis is clearly undesirable. We wish to modify the control law in Eq. (5) such that the vector field close to the z axis points away from this axis. In short, the idea is to divide the (z, v) phase space into two regions according to the value of the ratio

$$\eta = \frac{z}{v} \tag{8}$$

This ratio is a direct indication of the relative magnitude between z and w. Owing to the non-differentiability of the term $z/\bar{w}$ this ratio should be kept small in order to avoid high control effort. Hence, if initially the states are in an undesirable region where η attains large values, the feedback control strategy should drive the trajectories to a "safe" region in the state space where η remains relatively small. Without loss of generality, we choose as undesirable the region where $|\eta| > 1$ and as desirable the region where $|\eta| \leq 1$. These two regions, denoted by $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively, are therefore defined by

$$\begin{aligned} \mathcal{D}_1 &= \{(z, v) \in \mathbb{R} \times \mathbb{R}_+ : \infty > |\eta| > 1\} && \text{(9a)} \\ \mathcal{D}_2 &= \{(z, v) \in \mathbb{R} \times \mathbb{R}_+ : |\eta| \leq 1\} && \text{(9b)} \end{aligned}$$

These two regions are shown in Fig. 2.

5 Main Results

The proposed modification of the control law in Eq. (5) is simple. We use positive feedback for v when the trajectory is in region $\mathcal{D}_1$, while keeping z approximately constant (or decreasing). This will make the manifold $v = 0$ (equivalently, $w = 0$) unstable and the trajectories will move towards the region $\mathcal{D}_2$ and subsequently stay there. The control law in region $\mathcal{D}_2$ is essentially the same as in Eq. (5). Notice that, by definition, inside the region $\mathcal{D}_2$ we have $|\eta| \leq 1$, and since $|z|/|\bar{w}| = |\eta||w|$ we can ensure that $\omega(\cdot)$ will not take excessive values as long as the trajectories remain in $\mathcal{D}_2$. These statements will be made more precise in the sequel.

5.1 Proposed Control Law

The proposed control law for the system in Eqs. (4) is defined by

$$\omega = -\kappa(\eta) w - i\mu(\eta) \frac{z}{\bar{w}} \tag{10}$$

where

$$\begin{aligned} \kappa(\eta) &= \frac{2\kappa_c}{\pi} \arctan\left(\rho(1 - |\eta|)\right) && \text{(11a)} \\ \mu(\eta) &= \frac{\mu_c}{\pi} \arctan\left(\rho(1 - |\eta|)\right) + \frac{\mu_c}{2} && \text{(11b)} \end{aligned}$$

and $0 < \kappa_c < \mu_c$. From Eqs. (11) we have immediately that

$$-\kappa_c \leq \kappa(\eta) \leq \kappa_c, \qquad 0 \leq \mu(\eta) \leq \mu_c \tag{12}$$

for all $\eta \in \mathbb{R}$. In fact, we have that $-\kappa_c \leq \kappa(\eta) < 0$ and $0 \leq \mu(\eta) < \frac{\mu_c}{2}$ for all $(z, v) \in \mathcal{D}_1$ and $0 \leq \kappa(\eta) \leq \kappa_c$ and $\frac{\mu_c}{2} \leq \mu(\eta) \leq \mu_c$ for all $(z, v) \in \mathcal{D}_2$.

The next theorem gives the main result of the paper.

Theorem 1 *Consider the system (4) and let the control law as in equations (10)-(11) with* $0 < \kappa_c < \mu_c$.

Then for initial conditions $(z(0), w(0)) \in \mathbb{R} \times (\mathbb{C}\backslash\{0\})$ *the following properties hold*

(i) $w(t) \neq 0, \forall\, t \geq 0$.

(ii) the trajectory $(z(\cdot), w(\cdot))$ *is bounded and* $\lim_{t\to\infty}(z(t), w(t)) = 0$.

(iii) the control law $\omega(\cdot)$ *is bounded and it has bounded derivative.*

With the control law (10) the closed loop system takes the form

$$\dot{v} = -\kappa(\eta)(1+v)v \tag{13a}$$

$$\dot{z} = -\mu(\eta)z \tag{13b}$$

where $v = |w|^2$ and η as in equation (8). From Eq. (12) we have that z decays monotonically for all initial conditions, whereas v increases in the region $\mathcal{D}_1$ and decreases in $\mathcal{D}_2$. The result is that the trajectories of (13) tend to $\mathcal{D}_2$ and then to the origin, as required.

Before we prove Theorem 1 we need to establish the following two lemmas.

Lemma 5.1 *The region* $\mathcal{D}_2$ *is invariant for the system (13).*

Proof: The boundary of the set $\mathcal{D}_2$ is given by the two lines $\eta = \pm 1$ (cf. Fig. 2). On the boundary of $\mathcal{D}_2$ we have that $\kappa(\eta) = 0$ and $\mu(\eta) = \mu_c/2$. The vector field on the boundary of $\mathcal{D}_2$ is therefore

$$\dot{v} = 0 \tag{14a}$$

$$\dot{z} = -\frac{\mu_c}{2}z \tag{14b}$$

which points into the interior of $\mathcal{D}_2$. Therefore trajectories in $\mathcal{D}_2$ cannot escape this region and thus it is invariant for the closed-loop system (13). ∎

This lemma establishes that for initial conditions in $\mathcal{D}_2$ the trajectories of the closed-loop system remain in $\mathcal{D}_2$ for all times. Equivalently, if at some time $t' \geq 0$ the trajectory enters $\mathcal{D}_2$, it stays in $\mathcal{D}_2$ for all $t \geq t'$. Figure 2 shows the vector field on the boundary of $\mathcal{D}_2$.

Lemma 5.2 *Consider the system (13). For all initial conditions* $(z, v) \in \mathcal{D}_1$ *the trajectories enter the region* $\mathcal{D}_2$ *in finite time.*

Proof: As long as $(z, v) \in \mathcal{D}_1$ from Eq. (??) we have that $0 \leq \mu(\eta) < \mu_c/2$. This implies that z is bounded. Actually, $|z(t)| \leq |z(0)|$ for all $t \geq 0$. Note that z does not change sign for all $t \geq 0$. Without loss of generality we can assume that $z(0) \geq 0$ (the case $z(0) \leq 0$ being similar). If $(z(0), v(0)) \in \mathcal{D}_1$ then, by definition $\eta(0) > 1$. The derivative of η in $\mathcal{D}_1$ is then

$$\begin{aligned} \dot{\eta} &= \frac{\dot{z}}{v} - \frac{z}{v^2}\dot{v} \\ &= -\mu(\eta)\eta + \kappa(\eta)(1+v)\eta \\ &\leq -\mu(\eta)\eta \leq 0 \end{aligned} \tag{15}$$

since $\kappa(\eta) < 0$ and $v > 0$; hence η is bounded in $\mathcal{D}_1$. Let $cl\,\mathcal{D}_1$ denote the closure of $\mathcal{D}_1$ in $\mathbb{R}^2$, that is,

$$\begin{aligned} cl\,\mathcal{D}_1 = \mathcal{D}_1 \quad &\cup \quad \{(z, v) \in \mathbb{R} \times \mathbb{R}_+ : |\eta| = 1\} \\ &\cup \quad \{(z, v) \in \mathbb{R} \times \mathbb{R}_+ : v = 0\} \end{aligned} \tag{16}$$

Then it is an easy exercise to show that $\dot{\eta} \neq 0$ for all $(z, v) \in cl\,\mathcal{D}_1\backslash\{(0,0)\}$. Hence there exists $\epsilon > 0$ such that $\dot{\eta} < -\epsilon$ in $\mathcal{D}_1$ and consequently, η monotonically decreases. Thus, every trajectory starting in $\mathcal{D}_1$ will leave this set and enter $\mathcal{D}_2$ in finite time. ∎

Notice that the set $\{(z, v) \in \mathbb{R}\times\mathbb{R}_+ : v = 0 \text{ and } z \neq 0\}$ is an unstable manifold for the closed-loop system. Figure 2 shows the vector field on the boundary of $\mathcal{D}_1$. The following corollary follows directly from Lemmas 5.1 and 5.2.

Corollary 5.1 *Consider system (13). For all initial conditions* $(z(0), v(0)) \in \mathbb{R} \times (\mathbb{R}_+\backslash\{0\})$ η *is bounded for all* $t \geq 0$.

We are now ready to give the proof of Theorem 1.

Proof: [Theorem 1] From Eqs. (13a) and (12) we have that

$$\dot{v} \geq -\kappa_c(1+v)v \tag{17}$$

where $\kappa_c > 0$. The solution of the differential equation

$$\dot{x} = -\kappa_c(1+x)x, \qquad x(0) = x_0 > 0 \tag{18}$$

is given by

$$x(t) = \frac{1}{c_0 e^{\kappa_c t} - 1} \tag{19}$$

where $c_0 = (x_0 + 1)/x_0$. Clearly, $x(t) \neq 0$ for all $t \geq 0$ and $\lim_{t\to\infty} x(t) = 0$. Therefore $v(\cdot)$ is bounded below by the solutions of the differential equation (18) subject to initial condition $x_0 = v(0)$. Hence, $|w(t)| \neq 0$ for all $t \geq 0$ and $w(\cdot)$ approaches the origin asymptotically.

We now show that $\lim_{t\to\infty}(z(t), v(t)) = 0$. If $(z(0), v(0)) \in \mathcal{D}_2$ then according to Lemma 5.1 $(z(t), v(t)) \in \mathcal{D}_2$ for all $t \geq 0$ and $\mathcal{D}_2$ is an invariant set for the closed-loop system. Consider now the positive definite, radially unbounded function $V : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$V(z, v) = \frac{1}{2}v^2 + \frac{1}{2}z^2, \qquad \forall\,(z, v) \in \mathcal{D}_2 \tag{20}$$

The derivative of V along the trajectories of (13) is

$$\dot{V} = -\kappa(\eta)(1+v)v^2 - \mu(\eta)z^2 \leq 0, \qquad \forall (z,v) \in \mathcal{D}_2 \tag{21}$$

therefore, the trajectories are bounded in $\mathcal{D}_2$. Moreover, $\dot{V} = 0$ if and only if $\kappa(\eta)(1+v) + \mu(\eta)\eta^2 = 0$. Using the definitions of $\kappa(\eta)$ and $\mu(\eta)$ in $\mathcal{D}_2$ and recalling that $v > 0$, one establishes that the last equality is not satisfied in $\mathcal{D}_2$ unless $z = v = 0$. By LaSalle's theorem, the system is asymptotically stable for all initial conditions in $\mathcal{D}_2$. To finish the proof, recall from Lemma 5.2 that if $(z(0), v(0)) \in \mathcal{D}_1$ then $|z|$ is bounded by $|z(0)|$ and there exist a time $t' > 0$ such that $(z(t'), v(t')) \in \mathcal{D}_2$. This implies that for all $t' \geq t \geq 0$ the trajectories in $\mathcal{D}_1$ are bounded, and are confined inside the strip $|z(t)| \leq |z(0)|$. However, according to the previous discussion, the trajectory with initial condition $(z(t'), v(t'))$ has the property that $\lim_{t\to\infty}(z(t), v(t)) = 0$. Therefore, we have shown that for all $(z(0), v(0)) \in \mathbb{R} \times (\mathbb{R}_+ \backslash \{0\})$ the trajectories remain bounded and $\lim_{t\to\infty}(z(t), v(t)) = 0$ By the definition of v this implies that

$$\lim_{t\to\infty}(z(t), w(t)) = 0 \tag{22}$$

In order to show that ω is bounded, write $z/\bar{w} = \eta w$. From Eq. (10) one obtains that

$$|\omega| \leq \kappa_c |w| + \mu_c |\eta||w| \tag{23}$$

From Corollary 5.1 we have that for all initial conditions $(z(0), w(0)) \in \mathbb{R} \times (\mathbb{C}\backslash\{0\})$ η is bounded. Since w is also bounded, from Eq. (23) it follows that ω is bounded.

From Eq. (4a) it follows immediately that $\dot{w}$ is also bounded. Moreover, since

$$\dot{\eta} = -\mu(\eta)\eta + \kappa(\eta)(1+v)\eta \tag{24}$$

and $\mu(\eta), \kappa(\eta), v$ and η are all bounded, we have that $\dot{\eta}$ is bounded.

The derivative of ω is given by

$$\dot{\omega} = -\dot{\kappa}(\eta)w - \kappa(\eta)\dot{w} - i\,\dot{\mu}(\eta)\eta\, w - i\,\mu(\eta)\dot{\eta}w - i\,\mu(\eta)\eta\dot{w} \tag{25}$$

Let us now assume that $\eta(0) > 0$. Then $\eta(t) > 0$ for all $t \geq 0$. Thus in the subsequent discussion, without loss of generality, we assume that $|\eta| = \eta$. The case when $\eta(0) < 0$ is treated similarly. Using Eqs. (11) one has

$$\dot{\kappa}(\eta) = -\frac{2\kappa_c}{\pi}\frac{\rho}{1+\rho^2(1-\eta)^2}\,\dot{\eta} \tag{26a}$$

$$\dot{\mu}(\eta) = \frac{\mu_c}{\pi}\frac{\rho}{1+\rho^2(1-\eta)^2}\,\dot{\eta} \tag{26b}$$

Since $\dot{\eta}$ is bounded, $\dot{\kappa}(\eta)$ and $\dot{\mu}(\eta)$ are both bounded. Finally, the boundedness of $\dot{\omega}$ follows directly from Eq. (25) and the fact all the terms in the right hand side of this equation are bounded. ■

The vector field and the corresponding trajectories of the closed-loop system with the control law in Eq. (10) is shown in Fig. 3 (compare with Fig. 1).

Remark 5.1 It should be clear that the choice of the *arctan* function in the definition of $\kappa_c(\eta)$ and $\mu_c(\eta)$ is not restrictive. One could have used a "saturation-type" function in order to switch between the appropriate values of the gains in regions $\mathcal{D}_1$ and $\mathcal{D}_2$. We have used a differentiable function instead, because in applications the desired angular velocity history $\omega(\cdot)$ has to be generated by some control torques through the dynamics. This requires differentiability (with respect to time) of $\omega(\cdot)$. This is also the reason we require that, in addition to ω, the derivative of ω is also bounded.

Remark 5.2 Theorem 1 shows that for all initial conditions $w(0) \neq 0$ the control law (10) drives the system trajectories to the origin. This control law cannot be used if $w(0) = 0$ (and $z \neq 0$). Linearization of system (4) about $w = 0$, however, results in

$$\dot{w} = \frac{\omega}{2} \tag{27a}$$

$$\dot{z} = 0 \tag{27b}$$

and choosing, for example, a constant control $\omega = \omega_c \in \mathbb{C}$, one can move away from the z axis into the $\mathcal{D}_1$ region; once in $\mathcal{D}_1$, use of the control (10) drives the system to the origin.

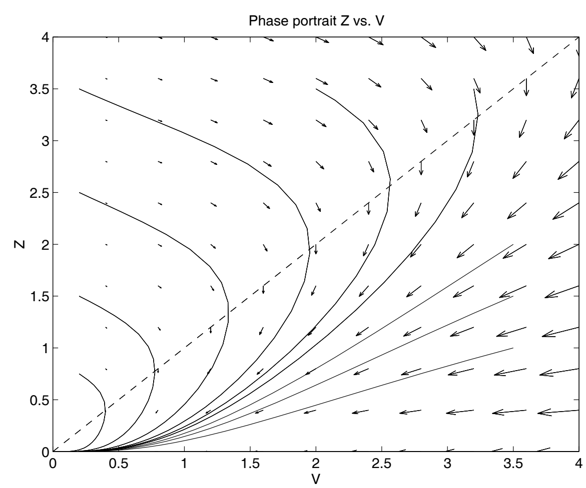

Figure 3: Phase portrait of system in Eqs. (11).

6 Numerical Example

To illustrate the previous theoretical analysis, we have simulated the differential equations (4) with the two control laws in Eqs. (5) and (10). The gains are chosen as $\kappa_c = 0.5$ and $\mu_c = 2$. The value of the parameter $\rho = 2$. The initial conditions were taken as $w(0) = 0.3 - i\,0.25$ and $z(0) = 2.5$. The results are shown in Figures 4 and 5. Figure 4 shows the corresponding closed-loop trajectories, and Fig. 5 shows the magnitude of $|\omega|^2$. Solid lines correspond to the new control law in Eq. (10) and the dashed lines correspond to the previous control law given in Eq. (5).

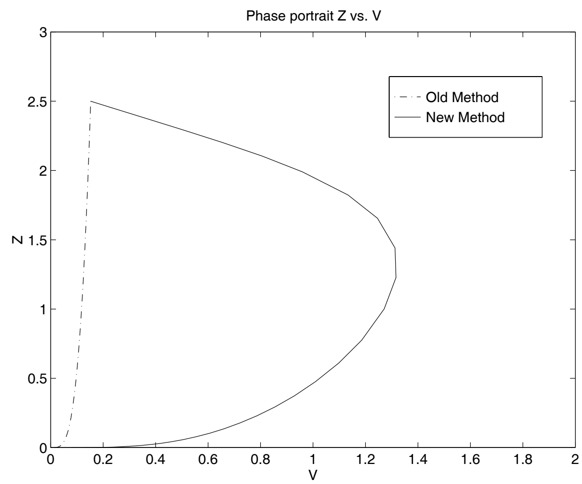

Figure 4: Closed-loop trajectories for the two methods.

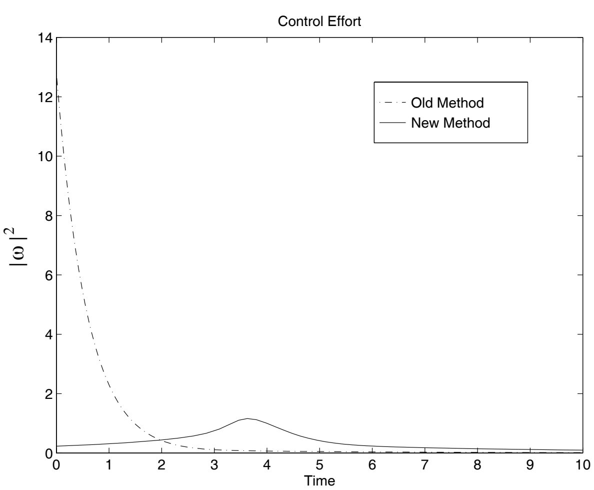

Figure 5: Control effort for the two methods.

7 Conclusions

We have constructed a nonsmooth control law which stabilizes the kinematics of an underactuated rigid body. We have shown that the control law is well defined and it uses considerably less control effort than a previously derived control law. A numerical example is provided for comparison of the two control laws. Future research will be directed towards implementing the proposed control law through the dynamical equations, as well as extending these results to general nonholonomic systems. Actually, the rigid body problem subject to two control inputs is only but one example of an underactuated mechanical system. Several other examples include systems subject to nonholonomic, i.e., non-integrable constraints. The application of the proposed control law for general nonholonomic systems in chain/power form will be addressed in a future article.

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