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Global Asymptotic Stabilization of a Spinning Top With Torque Actuators Using Stereographic Projection

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Abstract

The dynamical equations for a spinning top are derived in which the orientation is specified by a complex variable using stereographic projection of Poisson's equations. Necessary and sufficient conditions for Lyapunov stability of the uncontrolled motion of a spinning top are given using the Energy-Casimir method. Control laws that globally asymptotically stabilize the spinning top to the sleeping motion using two torque actuators are then synthesized by employing techniques from the theory of systems in cascade form and generalized using Hamilton-Jacobi-Bellman theory with zero dynamics.

1. Introduction

In this paper we examine the problem of the global asymptotic stabilization of a spinning top with fixed vertex to a uniform, steady rotation about its axis of symmetry. This motion of the top is often referred to in the literature as the *sleeping top.* This terminology arises because a smooth, axially symmetric top with its symmetric axis vertical, might appear at first glance to be not moving at all, and hence "sleeping" [5]. Stability analysis of the sleeping motion of a spinning top is well-developed. In [14], the authors summarized the previous results and gave necessary and sufficient conditions for Lyapunov stability of the sleeping motion which simplified the earlier results given by Ge and Wu [4]. The controlled top problem, *i.e.*, applying control inputs to drive the spinning top to the sleeping motion, was also studied in [14]. In [14], the control inputs are inertially-fixed horizontal forces and the kinematic formulation was based on the 2-1-3 Euler angles. Asymptotically stabilizing control laws were derived using the feedback linearization and the Hamilton-Jacobi-Bellman theory with zero dynamics for the case of two control forces. In the case of only one control force, if the top is spinning sufficiently fast, asymptotically stabilizing control laws were developed by the Jurdjevic-Quinn technique.

Here we consider the controlled top problem using alternative control inputs, namely, body-fixed torques. It is well-known that the sleeping motion of a spinning top is Lyapunov stable if its spin rate is sufficiently high [2, 4, 5, 7, 8, 14]. We rederive the necessary and sufficient condition for stability about the vertical of an uncontrolled spinning top using the Energy-Casimir method [8, 13]. This condition coincides with the previous results of [4, 14] and implies that the top motion can be Lyapunov stable, but a minimum amount of spin rate is necessary in order to achieve stability. Later we remove this restriction and consider stabilization without any requirement on the magnitude of the spin rate.

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The results hold also for the extreme case when the spin rate remains zero. Two control torques about the top's transverse principal axes are used in order to achieve this. The formulation of the problem departs from the traditional treatment — based on Eulerian angles — and takes advantage of the formulation for the kinematics of the rotational motion developed in [9]. This kinematic formulation uses the stereographic projection of the Riemann sphere on the complex plane in order to derive a very elegant and compact equation for a complex quantity related to the direction cosines of the local vertical (the inertial Z-axis) with respect to the local body-fixed system of the top. It is based on an idea by Darboux [3], where an equation of the same form was derived in connection with some problems in classical differential geometry. However its derivation using the stereographic projection and its use in attitude kinematics was established in [9] and was first applied to attitude control problems of spinning rigid bodies in [10, 11].

2. Equations for a Spinning Top Using Stereographic Projection

In this section we derive the dynamical equations of a spinning top using stereographic projection of Poisson's equations [9, 10]. Traditionally, the motion of an (uncontrolled) spinning top is described by the Euler-Poisson system of equations [7, 14] given by

$$J_1\dot{\omega}_1 = (J_2 - J_3)\omega_2\omega_3 + mg\ell\gamma_2 \quad (1a)$$
$$J_2\dot{\omega}_2 = (J_3 - J_1)\omega_3\omega_1 - mg\ell\gamma_1 \quad (1b)$$
$$J_3\dot{\omega}_3 = (J_1 - J_2)\omega_1\omega_2 \quad (1c)$$

$$\dot{\gamma}_1 = \gamma_2\omega_3 - \gamma_3\omega_2 \quad (2a)$$
$$\dot{\gamma}_2 = \gamma_3\omega_1 - \gamma_1\omega_3 \quad (2b)$$
$$\dot{\gamma}_3 = \gamma_1\omega_2 - \gamma_2\omega_1 \quad (2c)$$

Equations (1) describe the dynamics of the motion with respect to a body-fixed reference frame located at the vertex of the top and equations (2) describe the kinematics. In (1) and (2) $\omega_1, \omega_2, \omega_3$ are the angular velocity vector components in body coordinates, m is the mass of the top, g is the gravitational constant and ℓ is the distance from the vertex to the center of mass. The parameters J_1, J_2, J_3 represent the principal moments of inertia with respect to the chosen body-fixed reference frame. The variables $\gamma_1, \gamma_2, \gamma_3$ represent the components of the unit vector in the negative gravity direction when expressed in body coordinates. Notice that for this reason they satisfy the constraint $\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$.

We assume that the top is symmetric, *i.e.*, $J_1 = J_2 \triangleq J$, and therefore from (1c) we find that ω_3 is constant. Define

$\omega_3 \triangleq \Omega$, $b \triangleq J_3\Omega/J$ and $c \triangleq 2mg\ell/J$, and the complex variables

$$\omega = \omega_1 + i\,\omega_2 \quad (3a)$$

$$\eta = \eta_1 + i\,\eta_2 \triangleq \frac{\gamma_2 - i\,\gamma_1}{1+\gamma_3} = \frac{1-\gamma_3}{\gamma_2 + i\,\gamma_1} \quad (3b)$$

where $i = \sqrt{-1}$. Using the complex variables ω and η, the equations of motion can be expressed compactly as

$$\dot{\omega} = i\,(b-\Omega)\omega + \frac{c\eta}{1+|\eta|^2} \quad (4)$$

$$\dot{\eta} = -i\,\Omega\eta + \frac{\omega}{2} + \frac{\bar{\omega}}{2}\eta^2 \quad (5)$$

where the bar denotes complex conjugate. Equation (3b) represents a stereographic projection of the unit sphere $S^2 = \{(\gamma_1,\gamma_2,\gamma_3) \in \mathbb{R}^3 : \gamma_1^2+\gamma_2^2+\gamma_3^2 = 1\}$ onto the complex plane [9, 10, 11].

The tilt angle Θ between the top symmetry axis and the inertial Z-axis is

$$\Theta = \cos^{-1}\left(\frac{1-|\eta|^2}{1+|\eta|^2}\right) \quad (6)$$

An easy calculation shows that, if $\Omega = b$, *i.e.*, $J_3 = J$, then the only equilibrium state of equations (4) and (5) is $\omega = \eta = 0$. If $\Omega \neq b$, then (apart from the trivial case $\omega = \eta = 0$) the equilibrium states of the uncontrolled motion of equations (4) and (5) satisfy

$$|\eta|^2 = \frac{c - 2\Omega(b-\Omega)}{c + 2\Omega(b-\Omega)} \quad (7)$$

and

$$\omega = i\frac{c}{(b-\Omega)}\frac{\eta}{1+|\eta|^2}. \quad (8)$$

The expression (7) can be written equivalently as

$$\gamma_3 = \frac{2\Omega(b-\Omega)}{c}. \quad (9)$$

Since $c > 0$, it can be shown that if $b^2 < 2c$ then $\omega = \eta = 0$ is the only equilibrium state. If, on the other hand, $b^2 \geq 2c$, then two cases need to be considered, namely, $\Omega > 0$ and $\Omega < 0$. If $\Omega > 0$ and $\frac{1}{2}(b + \sqrt{b^2-2c}) \geq \Omega > \frac{b}{2}$ or $\Omega \geq \frac{1}{2}(b+\sqrt{b^2+2c})$, then $\omega = \eta = 0$ is the only equilibrium state. If $\Omega > 0$ and $\frac{1}{2}(b+\sqrt{b^2+2c}) > \Omega > \frac{1}{2}(b+\sqrt{b^2-2c})$ and $\Omega \neq b$ then there are nonzero equilibrium states corresponding to the solutions of (7) or (9) and (8). Similarly, if $\Omega < 0$ and $\frac{1}{2}(b-\sqrt{b^2-2c}) \leq \Omega < \frac{b}{2}$ or $\Omega \leq \frac{1}{2}(b-\sqrt{b^2+2c})$, then $\omega = \eta = 0$ is the only equilibrium state. If $\Omega < 0$ and $\frac{1}{2}(b-\sqrt{b^2+2c}) < \Omega < \frac{1}{2}(b-\sqrt{b^2-2c})$ and $\Omega \neq b$ then there are nonzero equilibrium states corresponding to the solutions of (7) or (9) and (8). It is interesting to note that these nonzero equilibrium states correspond to a steady precession of the top. In the steady precession, $|\eta| = const.$ which implies from (6) a constant tilt angle Θ. Note that if $\Omega = 0$, then from (4) and (5) one sees that the top degenerates to an inverted spherical pendulum and has only one equilibrium state $\omega = \eta = 0$. Note also that the equations for the top reduce to those for a symmetric spacecraft in the case $g = 0$ (*i.e.*, $c = 0$), which has only one equilibrium state, namely, $\omega = \eta = 0$. Finally, it should be noted that, because of the well-known properties among the principal moments of inertia, for any physically realizable rigid body, one must have that $\Omega > \frac{b}{2}$ for the case when $\Omega > 0$, while if $\Omega < 0$, then $\Omega < \frac{b}{2}$.

3. Stability of the Free Motion of the Spinning Top

In this section we analyze the (nonlinear) Lyapunov stability of the sleeping motion of the spinning top using Lyapunov's direct method and the Energy-Casimir method [8, 13, 14]. The use of the Energy-Casimir method allows us to draw stability conclusions about conservative mechanical systems when certain independent integrals of the motion (Casimirs) are known [8, 13], by checking the definiteness of the second variation at the critical points of a "energy-like" quantity. The procedure follows [14] closely.

The linearization of the nonlinear top equations (4) and (5) about the equilibrium $\omega = \eta = 0$, corresponding to the sleeping motion, is given by

$$\begin{bmatrix} \dot{\omega} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} i\,(b-\Omega) & c \\ 1/2 & -i\,\Omega \end{bmatrix} \begin{bmatrix} \omega \\ \eta \end{bmatrix} \quad (10)$$

As can be easily calculated, the eigenvalues of the system (10) are

$$\lambda_{1,2} = i\,\frac{b-2\Omega}{2} \pm \frac{1}{2}\sqrt{-b^2+2c} \quad (11)$$

Obviously, if $b^2 < 2c$, then system (10) has eigenvalues in the open right half plane, which corresponds to instability of the linearized equation (10), and thus instability of the original nonlinear system (4) and (5). When $b^2 \geq 2c$, (10) has eigenvalues on the imaginary axis and no conclusion can be drawn about the stability of the original nonlinear system from its linearization in this case.

Let us define the quantity

$$H(\omega,\eta) \triangleq h_{c1}(\omega,\eta) - b\,h_{c2}(\omega,\eta) \quad (12)$$

where

$$h_{c1}(\omega,\eta) \triangleq |\omega|^2 + c\,\frac{1-|\eta|^2}{1+|\eta|^2} \quad (13)$$

$$h_{c2}(\omega,\eta) \triangleq \frac{2}{1+|\eta|^2}\,\mathrm{Im}(\omega\bar{\eta}) + b\,\frac{1-|\eta|^2}{1+|\eta|^2} \quad (14)$$

and where $\mathrm{Im}(\cdot)$ denotes the imaginary part of a complex number. It can be easily checked that this expression remains constant under the flow of the nonlinear system (4) and (5). The first term in (12) represents the total energy of the system, while the second term represents the angular momentum along the inertial Z-axis. It is easy to show that $\omega = \eta = 0$ is a critical point of (12), that is,

$$\frac{\partial H}{\partial \omega}(0,0) = \frac{\partial H}{\partial \eta}(0,0) = 0 \quad (15)$$

and the second variation of H at the critical point $\omega = \eta = 0$ can be computed to be

$$\delta^2 H(0,0) = \begin{bmatrix} \delta\omega & \delta\eta \end{bmatrix} \begin{bmatrix} 2 & -i\,2b \\ i\,2b & -4c+4b^2 \end{bmatrix} \begin{bmatrix} \delta\bar{\omega} \\ \delta\bar{\eta} \end{bmatrix} \quad (16)$$

The matrix in (16) is positive definite for $b^2 > 2c$ and therefore the system is (Lyapunov) stable if $b^2 > 2c$. In fact, a Lyapunov function for the system (4) and (5) is given by

$$\begin{aligned} V(\omega,\eta) &\triangleq H(\omega,\eta) - H(0,0) \\ &= |\omega|^2 + c\,\frac{1-|\eta|^2}{1+|\eta|^2} - b\Big[\frac{2}{1+|\eta|^2}\,\mathrm{Im}(\omega\bar{\eta}) \\ &+ b\,\frac{1-|\eta|^2}{1+|\eta|^2}\Big] - c + b^2 \end{aligned} \quad (17)$$

It can be seen that if $b^2 \geq 2c$, then $V(\omega,\eta) > 0$ for all $\omega, \eta \in \mathbb{C}$ although its Hessian is only positive semi-definite; see (16). Next, recall that if $b^2 < 2c$, then the sleeping motion is unstable. Therefore we conclude that the sleeping motion of the (uncontrolled) top is Lyapunov stable if and only if $b^2 \geq 2c$ [4, 14].

4. Feedback Stabilization with Two Torque Inputs

4.1 Complex Formulation

We consider the controlled top problem in which two torque actuators u_1, u_2 along two transverse principal axes perpendicular to the symmetry axis are applied to the top. Defining the complex control variable $u_c \triangleq u_1 + i u_2$, the equations of motion now yield

$$\dot{\omega} = i(b-\Omega)\omega + \frac{c\eta}{1+|\eta|^2} + u_c \tag{18}$$

$$\dot{\eta} = -i\Omega\eta + \frac{\omega}{2} + \frac{\bar{\omega}}{2}\eta^2. \tag{19}$$

The control strategy employed in this subsection is based on the results of [10] and [11]. In [10, 11], globally asymptotically stabilizing control laws were derived for the motion of a symmetric spinning rigid body in space, using two control actuators about principal axes.

Redefining the new control

$$v \triangleq \frac{c\eta}{1+|\eta|^2} + u_c \tag{20}$$

equations (18), (19) yield a system in the form

$$\dot{\omega} = i(b-\Omega)\omega + v \tag{21a}$$

$$\dot{\eta} = -i\Omega\eta + \frac{\omega}{2} + \frac{\bar{\omega}}{2}\eta^2. \tag{21b}$$

Control laws for the system of equations (21) have been obtained in [10, 11]. Using the results of [10, 11] and equation (20) we have the following globally asymptotically stabilizing control laws for the motion of a spinning top about its symmetry axis.

Theorem 4.1 *The choice of the feedback control law*

$$u_c = -i(b-\Omega)\omega - \frac{c\eta}{1+|\eta|^2} + \kappa(i\Omega\eta - \frac{\omega}{2} - \frac{\bar{\omega}}{2}\eta^2) - \alpha(\omega + \kappa\eta) \tag{22}$$

with $\kappa > 0$ and $\alpha > 0$, globally asymptotically stabilizes system (18)-(19).

Theorem 4.2 *The choice of the feedback control law*

$$\begin{aligned} u_c &= -i(b-\Omega)\omega - \frac{c\eta}{1+|\eta|^2} + \kappa(i\Omega\eta - \frac{\omega}{2} - \frac{\bar{\omega}}{2}\eta^2) \\ &- \alpha(\omega+\kappa\eta) - \eta(1+|\eta|^2) \end{aligned} \tag{23}$$

with $\kappa > 0$ and $\alpha > 0$, globally exponentially stabilizes system the (18)-(19) with rate of decay $\beta/2$, where $\beta = \min\{2\alpha, \kappa\}$.

Theorem 4.3 *The choice of the feedback control law*

$$u_c = -\kappa_1\omega - \kappa_2\eta - \frac{c\eta}{1+|\eta|^2} \tag{24}$$

with $\kappa_1 > 0$ and $\kappa_2 > 0$, globally asymptotically stabilizes the system (18)-(19).

The proofs of these theorems are shown by construction of appropriate Lyapunov functions for the corresponding closed loop systems and can be found in [10, 11] (for $v = u_c$).

Next we present a general theory of stabilization of the equations (18) and (19) of the controlled spinning top based on Hamilton-Jacobi-Bellman (HJB) theory with zero dynamics [1, 12].

4.2 Hamilton-Jacobi-Bellman Theory with Zero Dynamics

Consider a nonlinear controlled system which is affine in the control, of the form

$$\dot{x} = f(x) + g(x)u = f(x) + g_1(x)u_1 + \ldots + g_m(x)u_m \tag{25}$$

where $x \in \mathbb{R}^n$, $u = \mathrm{col}(u_1, \ldots, u_m) \in \mathbb{R}^m$ and $g_1(x), \ldots, g_m(x)$ are the column vectors of $g(x)$. We assume f and $g_1, \ldots, g_m$ are sufficiently smooth and, without loss of generality, we assume that the origin is an equilibrium state of the uncontrolled system, namely, $f(0) = 0$. In order to apply the HJB theory with zero dynamics, we *define* an artificial output function

$$y = h(x) \tag{26}$$

where $y \in \mathbb{R}^m$ and $h(x) = \mathrm{col}(h_1(x), h_2(x), \ldots, h_m(x))$. For the system (25), (26), consider the performance functional

$$J(x_0, u(\cdot)) \triangleq \int_0^\infty L(x(t), u(t))\, dt \tag{27}$$

where

$$L(x,u) \triangleq L_1(x) + L_2(x)u + u^{\mathrm{T}}Ru \tag{28}$$

and $L_1: \mathbb{R}^n \to \mathbb{R}$, $L_2: \mathbb{R}^n \to \mathbb{R}^{1\times m}$ with $L_2(0) = 0$, and $R \in \mathbb{R}^{m\times m}$ is a positive-definite matrix. The superscript T denotes, as usual, the transpose.

The following lemma is essential in characterizing the canonical form of (25).

Lemma 4.1 *Assume that the nonlinear system (25), (26) is minimum phase with relative degree $\{1, 1, \ldots, 1\}$. If the vector field $g\,(\mathrm{L}_g h)^{-1}$ is complete, then there exists a global diffeomorphism $\theta: \mathbb{R}^n \to \mathbb{R}^n$, a C^∞ function $f_0: \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$, and a C^∞ function $r: \mathbb{R}^{n-m} \times \mathbb{R}^m \to \mathbb{R}^{(n-m)\times m}$ such that, in the coordinates*

$$\begin{bmatrix} z \\ y \end{bmatrix} \triangleq \theta(x) \tag{29}$$

the differential equation (25) is equivalent to the normal form

$$\begin{bmatrix} \dot{z} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} f_0(z) + r(z,y)y \\ \mathrm{L}_f h(x) \end{bmatrix} + \begin{bmatrix} 0 \\ \mathrm{L}_g h(x) \end{bmatrix} u. \tag{30}$$

For a definition of the notion of relative degree and zero dynamics, one may consult [6]. The next theorem gives the main result for optimal nonlinear feedback of minimum phase systems with relative degree $\{1, 1, \ldots, 1\}$. The optimality of the feedback control law is guaranteed through the Hamilton-Jacobi-Bellman equation. The performance functional is assumed to include a nonquadratic state weighting and a quadratic control weighting.

Theorem 4.4 ([12]) *Consider the nonlinear system defined by equations (25), (26). Assume that the system is minimum phase with relative degree $\{1, 1, \ldots, 1\}$ and the vector field $g\,(\mathrm{L}_g h)^{-1}$ is complete. Furthermore, let $P_0 \in \mathbb{R}^{m\times m}$ and $R \in \mathbb{R}^{m\times m}$ be positive definite and let*

$V_0 : \mathbb{R}^{n-m} \to \mathbb{R}$ *be a* C^1 *positive definite function such that* $DV_0(z)f_0(z) < 0$, *for* $z \in \mathbb{R}^{n-m}, z \neq 0$. *Then define*

$$L_2^{\mathrm{T}}(x) = R(\mathrm{L}_g h)^{-1}[P_0^{-1}r^{\mathrm{T}}(z,y)DV_0(z)^{\mathrm{T}} + 2\mathrm{L}_f h] \quad (31)$$

$$V(x) = V_0(z) + y^{\mathrm{T}}P_0 y \quad (32)$$

where z, y *and* $r(z,y)$ *are defined in Lemma 4.1. Then* V *is a Lyapunov function for the closed-loop system with the control law*

$$\phi(x) = -\frac{1}{2}[\mathrm{L}_g h(x)]^{-1}[P_0^{-1}r^{\mathrm{T}}(z,y)DV_0(z)^{\mathrm{T}} + 2\mathrm{L}_f h(x)] - R^{-1}[\mathrm{L}_g h(x)]^{\mathrm{T}}P_0 h(x) \quad (33)$$

which globally asymptotically stabilizes (25) and minimizes $J(x_0, u(\cdot))$ *in the sense that*

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot)\in S(x_0)} J(x_0, u(\cdot)) = V(x_0), \quad \forall x_0 \in \mathbb{R}^n \quad (34)$$

where $J(x_0, u(\cdot))$ *is defined in (27)-(28),* $S(x_0)$ *is the set of asymptotically stabilizing control laws, and*

$$L_1(x) = \phi^{\mathrm{T}}(x)R\phi(x) - \mathrm{L}_f V(x), \quad x \in \mathbb{R}^n \quad (35)$$

The performance integrand corresponding to the optimal control law (33) is [12]

$$\begin{aligned} L(x,u) &= \{u + \frac{1}{2}(\mathrm{L}_g h)^{-1}[P_0^{-1}r^{\mathrm{T}}(z,y)DV_0(z)^{\mathrm{T}} \\ &+ 2\mathrm{L}_f h]\}^{\mathrm{T}}R\{u + \frac{1}{2}(\mathrm{L}_g h)^{-1}[P_0^{-1}r^{\mathrm{T}}(z,y) \\ &\quad DV_0(z)^{\mathrm{T}} + 2\mathrm{L}_f h]\} - DV_0(z)f_0(z) \\ &+ h^{\mathrm{T}}(x)P_0(\mathrm{L}_g h)R^{-1}(\mathrm{L}_g h)^{\mathrm{T}}P_0 h(x) \end{aligned} \quad (36)$$

which is nonnegative for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. In the above expressions $DV_0(z)$ denotes the Jacobian of V_0 with respect to z.

Remark: In the statement of Theorem 4.4 is implicit the fact that there exists an output for the nonlinear system with respect to with the overall system has relative degree $\{1, 1, \ldots, 1\}$. Therefore the existence of such an output is crucial for the Theorem 4.4 to be applicable. Note that as in the case of feedback linearizable systems the desired output may not be necessarily the given output of the system. Instead it is often up to the control designer to *choose* such an output in order to achieve relative degree one with respect to all output channels. This is not necessarily a trivial task, and a judicious choice may facilitate the analysis and the control design.

4.3 The Spinning Top

We use again the stereographic coordinates, but for convenience (and to be consistent with the standard notation in the literature of nonlinear control theory), we expand equations (18) and (19) into their real and imaginary parts. Letting $x_1 = \omega_1, x_2 = \omega_2, x_3 = \eta_1$ and $x_4 = \eta_2$, and decomposing (18) and (19) into their real and imaginary components, we can write these equations in the familiar form

$$\dot{x} = f(x) + g(x)u = f(x) + g_1(x)u_1 + g_2(x)u_2 \quad (37)$$

where

$$f(x) \triangleq \begin{bmatrix} -(b-\Omega)x_2 + cx_3/(1+x_3^2+x_4^2) \\ (b-\Omega)x_1 + cx_4/(1+x_3^2+x_4^2) \\ \Omega x_4 + x_2x_3x_4 + x_1(1+x_3^2-x_4^2)/2 \\ -\Omega x_3 + x_1x_3x_4 + x_2(1-x_3^2+x_4^2)/2 \end{bmatrix} \quad (38a)$$

$$g(x) \triangleq \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (38b)$$

where $x = \mathrm{col}(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$, $g_1(x), g_2(x)$ are the column vectors of $g(x)$ and $u = \mathrm{col}(u_1, u_2) \in \mathbb{R}^2$. Clearly, f, g_1 and g_2 are C^∞ vector fields and $f(0) = 0$.

Let the output function be

$$y = h(x) = \begin{bmatrix} h_1(x) \\ h_2(x) \end{bmatrix} \triangleq \begin{bmatrix} x_1 + k_1x_3 \\ x_2 + k_2x_4 \end{bmatrix} \quad (39)$$

where $k_1 > 0$ and $k_2 > 0$. Defining $z = \mathrm{col}(z_1, z_2) \triangleq \mathrm{col}(x_3, x_4)$, we have

$$\dot{z} = f_0(z) + r(z,y)y \quad (40)$$

where

$$f_0(z) \triangleq \begin{bmatrix} -\frac{1}{2}k_1z_1 + \Omega z_2 - \frac{1}{2}k_1z_1^3 - (k_2 - \frac{k_1}{2})z_1z_2^2 \\ -\frac{1}{2}k_2z_2 - \Omega z_1 - \frac{1}{2}k_2z_2^3 - (k_1 - \frac{k_2}{2})z_1^2z_2 \end{bmatrix} \quad (41)$$

$$r(z,y) = \begin{bmatrix} \frac{1}{2}(1+z_1^2-z_2^2) & z_1z_2 \\ z_1z_2 & \frac{1}{2}(1-z_1^2+z_2^2) \end{bmatrix} \quad (42)$$

It can be shown that the zero dynamics corresponding to $\dot{z} = f_0(z)$ is globally asymptotically stable and the corresponding Lyapunov function is

$$V_0(z) = p_3(z_1^2 + z_2^2), \qquad p_3 > 0. \quad (43)$$

Furthermore, $\mathrm{L}_g h(x) = \mathrm{I}_2$, where I_2 is the 2×2 identity matrix. Hence (38) and (39) form a minimum phase system with relative degree $\{1, 1\}$. Next by taking

$$P_0 = \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}, \qquad R = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} \quad (44)$$

with $p_1, p_2 > 0$ and $r_1, r_2 > 0$, and applying Theorem 4.4 to the system (38) and (39) the optimal control law $\phi(x) = \mathrm{col}(\phi_1(x), \phi_2(x))$ is computed from (33) to be

$$\begin{aligned} \phi_1(x) &= (b-\Omega)x_2 - cx_3/(1+x_3^2+x_4^2) \\ &- k_1[\Omega x_4 + x_2x_3x_4 + \frac{x_1}{2}(1+x_3^2-x_4^2)] \quad (45a) \\ &- (p_3x_3/2p_1)(1+x_3^2+x_4^2) - (p_1/r_1)(x_1+k_1x_3) \\ \phi_2(x) &= -(b-\Omega)x_1 - cx_4/(1+x_3^2+x_4^2) \\ &- k_2[-\Omega x_3 + x_1x_3x_4 + \frac{x_2}{2}(1-x_3^2+x_4^2)] \quad (45b) \\ &- (p_3x_4/2p_2)(1+x_3^2+x_4^2) - (p_2/r_2)(x_2+k_2x_4) \end{aligned}$$

The Lyapunov function that guarantees asymptotic stability of the closed-loop system with the previous control law is given from (32)

$$V(x) = p_3(x_3^2+x_4^2) + p_1(x_1+k_1x_3)^2 + p_2(x_2+k_2x_4)^2 \quad (46)$$

while the performance integrand (36) is

$$\begin{aligned} L(x,u) &= [u + \frac{1}{2}R^{-1}L_2^{\mathrm{T}}(x)]^{\mathrm{T}}R[u + \frac{1}{2}R^{-1}L_2^{\mathrm{T}}(x)] \\ &+ p_3[k_1x_3^2(1+x_3^2) + k_2x_4^2(1+x_4^2) \\ &+ (k_1+k_2)x_3^2x_4^2] \\ &+ \frac{p_1^2}{r_1}(x_1+k_1x_3)^2 + \frac{p_2^2}{r_2}(x_2+k_2x_4)^2 \end{aligned} \quad (47)$$

where

$$L_2^{\mathrm{T}}(x) = \begin{bmatrix} r_1\{p_3x_3(1+x_3^2+x_4^2)/p_1 - 2(b-\Omega)x_2 \\ +2cx_3/(1+x_3^2+x_4^2) \\ +2k_1[\Omega x_4 + x_2x_3x_4 + x_1(1+x_3^2-x_4^2)/2]\} \\ r_2\{p_3x_4(1+x_3^2+x_4^2)/p_2 + 2(b-\Omega)x_1 \\ +2cx_4/(1+x_3^2+x_4^2) \\ +2k_2[-\Omega x_3 + x_1x_3x_4 + x_2(1-x_3^2+x_4^2)/2]\} \end{bmatrix}$$

The equation for $\phi(x)$ provides a *family* of feedback stabilizing control laws for the system (38), which are optimal with respect to the performance functional (47). This 7-parameter family $(k_1, k_2, r_1, r_2, p_1, p_2, p_3)$ allows for great flexibility in the design of optimal feedback control laws for the spinning top. It can be easily checked that by taking $k_1 = k_2 = \kappa, p_1/r_1 = p_2/r_2 = \alpha$ and $p_3/p_1 = p_3/p_2 = 2$, the control law (45) reduces to (23). Notice, however, that the control laws (22) and (24) cannot be derived form (45) by any admissible choice of the parameters.

Remark: It should be pointed out that the control laws obtained above are globally asymptotically stabilizing for all $x \in \mathbb{R}^4$. Physically, this implies global asymptotic stability for the closed loop system from all initial configurations, except in the case when $\eta = \infty$. This case corresponds to direction cosines $(\gamma_1, \gamma_2, \gamma_3) = (0, 0, -1)$ in equation (3), that is, the top symmetry axis is along the downward direction. Therefore, global stability here implies stability from all initial conditions except the initial condition corresponding to this singular "upside-down" configuration. (Note that by the global stabilizing nature of the control laws of the system in (ω, η) coordinates, one has that $\eta(t) < \infty$ for all $t \geq 0$ as long as $\eta(0) \neq \infty$.) If the top is *initially* upside down, then one can apply an arbitrary input to drive the top to any nonsingular orientation. The stabilizing control laws obtained above can then be applied from this new orientation. Thus, the top can be globally asymptotically stabilized to the sleeping motion, including the singular one.

5. Numerical Example

We apply the control laws obtained in Section 4 to stabilize the spinning top to the sleeping motion. We assume that the top parameters are $J_1 = \ell = 1$, $mg = 3$ and $J_3 = 0.2$. The gain and control parameters in these simulations were all taken equal to unity. If $\Omega = 1$ then $b = 0.2, c = 6$, which corresponds to a unstable top. If the initial conditions are $x(0) = (0, 0, 0.01, 0.01)^T$ which implies initially the slowly spinning top has zero transverse angular velocity with tilt angle $\Theta = 1.62$ deg. Obviously without external control inputs the top will fall toward the downward position. To demonstrate the effect of the control laws, we apply the control laws (45) at $t = 3.1$ sec when the tilt angle is about 150 deg. Figure 1 shows the time history of the tilt angle Θ. It is seen from these figures that before the controls are applied ($t < 3.1$ second), the tilt angle Θ grows rapidly and after the controls are applied the tilt angle Θ (and the states x_3, x_4) are driven to zero asymptotically, as required.

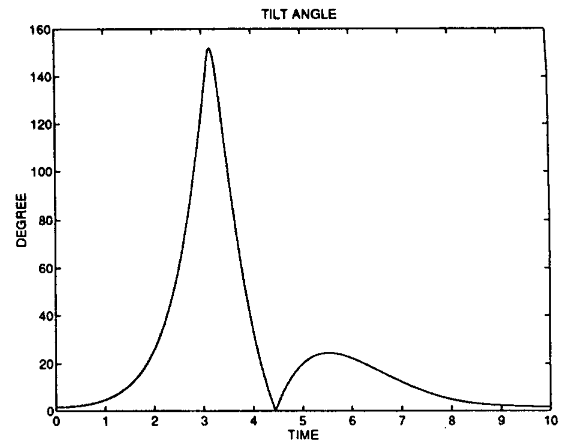

Figure 1: Tilt angle Θ vs. time.

6. Conclusions

The stability and stabilization of a spinning top were examined, using a new formulation of the kinematics based on stereographic coordinates, which facilitates the design of feedback control laws. The stabilizing control laws, using only two torque actuators, were synthesized using the Hamilton-Jacobi-Bellman theory with zero dynamics. Although the methodology is demonstrated using the spinning top example, it is believed that this theory will be helpful for a broad class of problems encountered in rotational dynamics and kinematics.

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