Trajectory Desensitization in Optimal Control Problems

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***Abstract*— The efficacy of the so-called sensitivity function in developing desensitized optimal control schemes is studied. A sensitivity function provides information about the first order variation of the state under parameter variations at a given time instant along a trajectory. It is demonstrated that the sensitivity function can be employed to effectively desensitize either an optimal trajectory or the state at a particular time instant (for example, the final state) along the optimal trajectory. Zermelo's path optimization problem is chosen to test the theory. Monte-Carlo simulations are carried out, validating the key idea. The limitations of the proposed approach are identified and the possibilities for future work are discussed.**

I. Introduction

Typical trajectory optimization techniques, as prominently used in optimal control theory, require accurate knowledge of the parameters of the dynamical system [1] of interest. Robustness to parametric uncertainty is desirable in many safety critical applications, such as aerospace system and high-precision robotics. Additionally, in the investment sector, a risk averse strategy can be useful for obtaining gains for a system suffering from high parametric uncertainty. Many such problems boil down to minimizing the dispersion in either the optimal trajectory or the state at some time along the trajectory, given the performance criterion and the dynamics. Methods from robust optimal control [2]–[5] and feedback control synthesis [6] address this issue, with an inherent trade-off between cost and robustness to be decided. Indeed, the increased cost is incurred due to additional control effort, in magnitude or over time. It would be desirable to alleviate the additional effort induced onto the control feedback by, instead, picking a trajectory which is less sensitive to variations under parametric uncertainty. This is the main goal of *desensitized optimal control* (DOC).

Prior work on of *trajectory sensitivity design*, restricted to analyzing linear systems and linear feedback controller gains. Winsor and Roy [7] employ a technique to design controllers that provide assurance of system performance under modeling inaccuracy and demonstrate the feasibility of their procedure. In [8], an increased-order augmented system is used to obtain a sensitivity-reduced design for linear regulators. A brief study on the modification of the weighing matrix for sensitivity reduction is conducted in [9]. In Refs. [10], [11], the idea of sensitivity reduction by feedback in the frequency domain has been explored. Tang [12] proposed using an augmented cost function with trajectory and cost sensitivities added to the original cost to derive a controller. The approach was further tested on the linear quadratic regulator (LQR) problem, which was later applied to active suspension control for passenger cars [13]. A more detailed discussion on methods for desensitized control is presented in [14].

Seywald and Kumar [15] are probably the first authors to propose a systematic approach to obtain an optimal *open-loop* trajectory that is insensitive to perturbations in the parameters for general nonlinear systems. In this approach, one elevates the parameters of interest to system states and reformulates the problem with an added *sensitivity cost*. The proposed method exploits *sensitivity matrices*, whose components are also treated as states, along with the uncertain parameters. The approach is verified using Zermelo's navigation problem with an uncertain parameter. With the motivation to solve the Mars pinpoint landing problem, the solution is extended to optimal control problems with control constraints [16], [17]. Some extensions of the work by Seywald and Kumar include analyzing the landing problem with uncertainties in atmospheric density and aerodynamic characteristics [18] and using *direct collocation and non-linear programming* [19]. Performing DOC with sensitivity matrices has close connections to the technique of *covariance trajectory-shaping*, owing to the fact that both sensitivity and covariance matrices measure the variation in states under plant uncertainties [20], [21].

DOC, in its current form, has to deal with an optimization problem of $(n + \ell)^2 + n + \ell$ number of states, where n denotes the number of states for the original system, and ℓ denotes the number of targeted system parameters. One of the main objectives of this work is to reduce the computational complexity of existing DOC formulations. To this end, we first recall the concept of differentiability of solutions with respect to a parameter and the corresponding sensitivity equations [22]. The sensitivity functions provide first-order estimates of the effect of parameter variations on solutions (obtained from a given state equation with appropriate initial conditions). In this paper, the entries in the sensitivity function are elevated to states which are then utilized to formulate desensitized optimal control schemes using augmented cost functions and state constraints. The proposed schemes lead to fewer states for the augmented system, when compared with the approach proposed by Seywald and Kumar in [15]. The theory is demonstrated using two versions of Zermelo's path optimization problem.

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II. Motivation

Consider the standard optimal control problem of minimizing the cost

$$\mathcal{J}(u) = \phi(x(t_f), t_f) + \int_{t_0}^{t_f} L(x(t), u(t), t)\,\mathrm{d}t, \tag{1}$$

subject to

$$\dot{x} = f(x, p, u, t), \quad x(t_0) = x_0, \tag{2}$$

$$\psi(x(t_f), t_f) = 0, \tag{3}$$

where $t \in [t_0,\, t_f]$ denotes time, with t_0 being the fixed initial time and t_f being the final time, $x(t) \in \mathbb{R}^n$ denotes the state, with x_0 being the fixed state at t_0. The control $u \in \mathcal{U} = \{\text{Piecewise Continuous (PWC)}, u(t) \in U\ \forall\ t \in [t_0, t_f]\}$, with $U \subseteq \mathbb{R}^m$, the set of allowable values of $u(t)$, $\phi : \mathbb{R}^n \times [t_0, t_f] \to \mathbb{R}$, the terminal cost function, and $L : \mathbb{R}^n \times \mathbb{R}^m \times [t_0, t_f] \to \mathbb{R}$, the running cost. Finally, $\psi : \mathbb{R}^n \times [t_0, t_f] \to \mathbb{R}^k$ is a function representing k-number of constraint equations at the final time. The above problem is to be solved by finding the optimal control $u^* \in \mathcal{U}$ that minimizes the cost function in (1). The solution involves the optimal path $x^*(t)$, $t \in [t_0, t_f]$, determined from $\dot{x}^*(t) = f(x^*(t), p, u^*(t), t)$ subject to $x^*(t_0) = x_0$.

The system dynamics represented by the function $f(x, p, u, t)$ contains the model parameters, $p \in \mathcal{P} \subset \mathbb{R}^\ell$, which are assumed to be constant. It is understood that the optimal solution $(x^*(t), u^*(t))$ is model sensitive and, if changes in the parameters p occur at any time $t \in [t_0, t_f]$, then the optimality of the obtained solution is not guaranteed. Consequently, the optimal control problem has to be resolved for each new value of the parameter vector. If the optimal solution u^* is used despite the parameter variations, one can expect a dispersion in the trajectories from the nominal x^*. With a motivation to minimize the dispersion of the final state of the optimal solution, under parameter uncertainties, Seywald and Kumar constructed an augmented cost function using sensitivity matrices [15]. It should be noted that the sensitivity matrix, from Refs. [15], [16], differs from the standard sensitivity matrix defined in Ref. [22].

The approach goes as follows. First, the parameters of interest and the corresponding entries in sensitivity matrix are elevated to states, and the augmented state $[\tilde{x}^\top\ (\mathrm{vec}\,\tilde{S})^\top]^\top$, where $\tilde{x} = [x\ p]^\top$, along with the corresponding dynamics and initial conditions are derived as follows

$$\dot{\tilde{x}} = [f^\top(x, p, u, t)\ \ 0_{1\times\ell}]^\top, \quad \tilde{x}(t_0) = [x_0^\top\ \ p_0^\top]^\top, \tag{4}$$

and

$$\dot{\tilde{S}}(t|t_0, x_0) = \frac{\partial f}{\partial \tilde{x}}(x, p, u, t), \quad \tilde{S}(t_0|t_0, x_0) = I_{(n+p)}, \tag{5}$$

where $p(t) \in \mathcal{P}$ denotes the ℓ parameters of interest and p_0 is the nominal value of these parameters, and $\tilde{S}(t|t_0, x_0)$ represents the sensitivity of the vector $\tilde{x}(t)$ at time t with respect to perturbations in the initial state vector $\tilde{x}(t_0)$. That is,

$$\tilde{S}(t|t_0, x_0) = \frac{\partial \tilde{x}(t)}{\partial \tilde{x}(t_0)}. \tag{6}$$

The augmented cost function, given in (7) below, is then minimized to obtain an optimal solution with final state being "desensitized" with respect to the parameter variations.

$$\begin{aligned}\mathcal{J}_a(u) &= \mathcal{J}(u) \\ &+ \int_{t_0}^{t_f} \|\, \mathrm{vec}\left(\tilde{S}(t_f|t_0, x_0)\tilde{S}(t|t_0, x_0)^{-1}\right) \|^2_{Q(t)}\Big)\mathrm{d}t,\end{aligned} \tag{7}$$

with $Q(t) \geq 0$, for all $t \geq 0$. Note that the sensitivity matrix of Seywald in (6) is a state transition matrix and its properties are exploited to construct the sensitivity of the final state with respect to the variations in the state at time $t \in [0, t_f]$, which is then plugged into the running cost. This is elaborated upon in [15]. However, this approach requires propagating the original states, the targeted parameters, and the elements in the sensitivity matrix, resulting to a total of $(n+\ell)^2 + n + \ell$ number of states.

In this paper, we use the traditional sensitivity function and develop an alternative scheme for optimal trajectory/state desensitization with respect to parameter variations with improved computational efficiency.

III. Desensitized Optimal Control Schemes

A. Sensitivity Equation

Consider the dynamics in (2), and assume variations in the model parameters $p \in \mathcal{P}$, with $p = p_0$ being the nominal value of the parameter vector. Furthermore, assume that $f(x, p, u, t)$ is continuous in (x, p, u, t), and continuously differentiable with respect to x and p for all $(x, p, u, t) \in \mathbb{R}^n \times \mathcal{P} \times U \times [t_0, t_f]$. The solution to the differential equation from the initial condition x_0 with control input $u \in \mathcal{U}$ is given by

$$x(p, t) = x_0 + \int_{t_0}^{t} f(x(p, \tau), p, u(\tau), \tau)\,\mathrm{d}\tau. \tag{8}$$

Since $f(x, p, u, t)$ is differentiable with respect to p,

$$\begin{aligned}\frac{\partial x}{\partial p}(p, t) = \int_{t_0}^{t} \bigg[&\frac{\partial f}{\partial x}(x(p, \tau), p, u(\tau), \tau)\frac{\partial x}{\partial p}(p, \tau) \\ &+ \frac{\partial f}{\partial p}(x(p, \tau), p, u(\tau), \tau)\bigg]\,\mathrm{d}\tau.\end{aligned} \tag{9}$$

Taking the derivative with respect to t, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial x}{\partial p}(p, t)\right] =& \left.\frac{\partial f(x, p, u(t), t)}{\partial x}\right|_{x = x(p,t)} \frac{\partial x}{\partial p}(p, t) \\ &+ \left.\frac{\partial f(x, p, u(t), t)}{\partial p}\right|_{x = x(p,t)}.\end{aligned} \tag{10}$$

Evaluating (10) at the nominal conditions ($p = p_0$), the dynamics for the *parameter sensitivity function*

$$S(t) = \left.\frac{\partial x(p, t)}{\partial p}\right|_{x = x(p_0, t)} \tag{11}$$

is given by

$$\dot{S}(t) = A(t)S(t) + B(t), \quad S(t_0) = 0_{n\times\ell}, \tag{12}$$

where

$$A(t) = \left.\frac{\partial f(x,p,u(t),t)}{\partial x}\right|_{x=x(p_0,t),\,p=p_0}, \tag{13}$$

$$B(t) = \left.\frac{\partial f(x,p,u(t),t)}{\partial p}\right|_{x=x(p_0,t),\,p=p_0}. \tag{14}$$

Note that the initial condition for the sensitivity function is the zero matrix, since the initial state is given (fixed). Equation (12) is called the *sensitivity equation* in the literature [22]. To propagate the sensitivity function, the state x has to be propagated along the dynamics under the nominal conditions,

$$\dot{x} = f(x, p_0, u, t), \quad x(t_0) = x_0. \tag{15}$$

From the properties of continuous dependence with respect to the parameters and the differentiability of solutions of ordinary differential equations, for sufficiently small variations in p_0, the solution $x(p,t)$ can be approximated by

$$x(p,t) \approx x(p_0,t) + S(t)(p - p_0). \tag{16}$$

This is a first-order approximation of $x(p,t)$ about the nominal solution $x(p_0,t)$, given $\|p-p_0\|$ is sufficiently small.

Remark 1: The difference between the sensitivity function and the sensitivity matrix lies in the fact that the former cannot accommodate time-varying parameters which is evident in (16). On the other hand, sensitivity matrices can be used to investigate variation in the state $x(t)$ (at time t) with respect to a variation in the parameter $p(t')$ (at any other time $t_0 \leq t' \leq t$) i.e., the variations can be different along the trajectory and the parameter can be time-varying. In this regard, though sensitivity matrices can handle more complexity, this comes with an added cost in terms of computation. In addition, for most problems, the model parameters are some constants whose values are prone to changes from the nominal.

The sensitivity function in (11) can be penalized/constrained to desensitize optimal control solutions, as demonstrated next.

B. Optimal Trajectory Desensitization

Trajectory desensitization allows one to find a robust path that is relatively immune to parameter variations, thereby mitigating the requirements on the feedback controller that may be used to track the obtained optimal trajectory. Revisiting the standard optimal control problem discussed in Section II, namely, (1)-(3), consider minimizing the augmented cost

$$\begin{aligned}\mathcal{J}_a(u) = \phi(x(t_f),t_f) + \int_{t_0}^{t_f} \Big(& L(x(t),u(t),t) \\ & + \|\operatorname{vec} S(t)\|^2_{Q(t)}\Big)\mathrm{d}t,\end{aligned} \tag{17}$$

with an augmented state $[x^\top \ (\operatorname{vec} S)^\top]^\top$, whose dynamics is obtained from (15), (12), and the terminal condition (3), that minimizes the original cost function in (1), while penalizing the sensitivity of the state with respect to the parameters along the optimal trajectory. The weighting factor for the sensitivity cost, $Q(t)$, can be tuned to balance between minimizing the original cost and the sensitivity cost. The proposed approach is demonstrated using the Zermelo's path optimization problem in Section IV-A.

C. Final State Desensitization

Final state desensitization is critical, especially among problems involving unmanned vehicles, where they have to reach their goal state precisely under external disturbances. If one is interested only in the variation of the final state (or state at a particular time instant $t > t_0$) with respect to parameter variations, then the corresponding sensitivity terms alone can be penalized by adding an extra term to the terminal cost of the original cost function as follows.

$$\begin{aligned}\mathcal{J}_a(u) = \phi(x(t_f),t_f) + \|\operatorname{vec} S(t_f)\|^2_Q & \\ + \int_{t_0}^{t_f} L(x(t),u(t),t)\,\mathrm{d}t&,\end{aligned} \tag{18}$$

where $Q \geq 0$ is the weighing factor for the terminal sensitivity cost. An alternative to this approach is to directly constrain the sensitivity function at the final time by adding additional state constraints of the form

$$\tilde{\psi}(S(t_f), t_f) = 0. \tag{19}$$

The two approaches for final state desensitization are demonstrated using a minimum time Zermelo's problem, discussed in Section IV-B.

IV. Numerical Examples

In this section, two versions of the Zermelo's navigation problem are employed to demonstrate the efficacy of the proposed schemes for trajectory and final state desensitization.

A. Trajectory Desensitization

Consider the Zermelo's problem [15] with currents parallel to the shore (assumed to be the x_1-axis) as a function of x_2 such that

$$v_{\text{current}} = p x_2, \tag{20}$$

where p is a parameter which is uncertain and its nominal value is p_0. The dynamics for a boat traveling in the currents can be written as

$$\dot{x}_1 = \cos(u) + p x_2, \tag{21}$$

$$\dot{x}_2 = \sin(u), \tag{22}$$

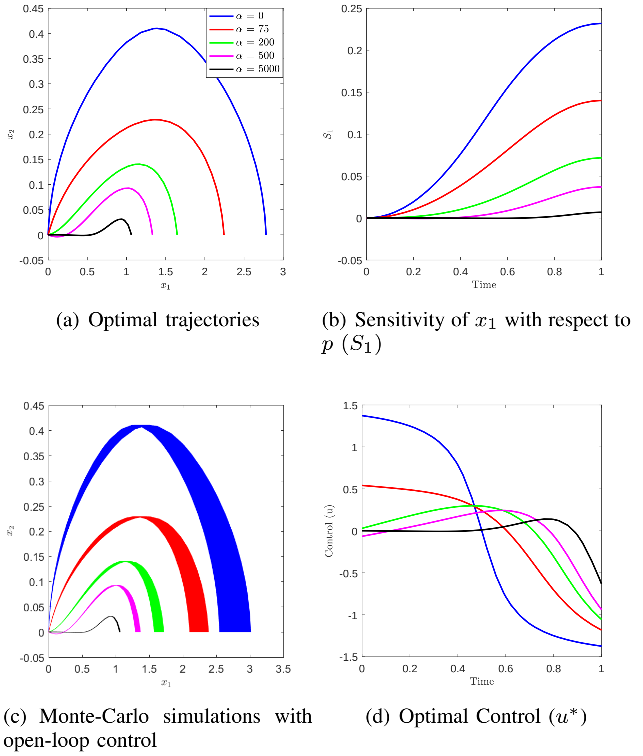
where u is its heading control, $u \in \mathcal{U} = \{\text{PWC}, u(t) \in (-\pi, \pi],\ \forall\, t \in [0, t_f]\}$, for some $t_f > 0$. In this example, the cost function that has to be minimized is expressed in Mayer form as

$$\mathcal{J}(u) = -x_1(t_f), \tag{23}$$

and subject to the boundary conditions

$$x_1(0) = 0, \quad x_2(0) = 0, \quad x_2(t_f) = 0. \tag{24}$$

For this example, we let $t_f = 1$, and $p_0 = 10$. Apart from maximizing the length that the boat can traverse along the shore while meeting the boundary conditions, the optimal trajectory has to be desensitized with respect to the variations



(a) Optimal trajectories

(b) Sensitivity of x_1 with respect to p (S_1)

(c) Monte-Carlo simulations with open-loop control

(d) Optimal Control (u^*)

Fig. 1: Results obtained for the Zermelo's path optimization problem with trajectory desensitization

in p. The goal can be facilitated by obtaining the sensitivity equation under nominal conditions ($p_0 = 10$), using (12),

$$\dot{S} = \begin{bmatrix} \dot{S}_1 \\ \dot{S}_2 \end{bmatrix} = \begin{bmatrix} p_0 S_2 + x_2 \\ 0 \end{bmatrix}, \quad S(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{25}$$

where

$$S_i(t) = \frac{\partial x_i(p,t)}{\partial p}\bigg|_{x=x(p_0,t)}, \; i = 1, 2, \tag{26}$$

and then by penalizing the terms S_1 and S_2 in an augmented cost function, as shown in (17). Note that $S_2(t) = 0$, for all $t \geq 0$, which means that the state x_2 is not affected by the uncertainty of the currents along a nominal trajectory. Therefore, in this particular example, just penalizing S_1 is sufficient. The augmented cost function can then be written as

$$\mathcal{J}_a(u) = -x_1(t_f) + \int_{t_0}^{t_f} \alpha S_1^2(t)\,\mathrm{d}t. \tag{27}$$

The weighting factor α (which is a constant in this case) is chosen by the designer. For $\alpha = 0$, the optimal solution for the original cost function (23) is obtained. Several test cases were run for $\alpha = \{0, 75, 200, 500, 5000\}$, using GPOPS-II [23], and the results are shown in Figs. 1 and 2.

The levels of desensitization can be clearly observed in Fig. 1. As the value of α increases, the trajectory becomes more conservative by staying closer to the shore, while trying to maximize $x_1(t_f)$; see Fig. 1(a). At the same time, the magnitude of the sensitivity of x_1 with respect to p (that is, S_1) also decreases along the trajectory which is observed in Fig. 1(b). Monte-Carlo simulations, shown in Fig. 1(c), illustrate the idea of trajectory desensitization. It can be seen that with high weight on α, the trajectories obtained using open-loop control under parameter variations stay closer to the corresponding optimal trajectory. For the Monte-Carlo simulations that are shown in Fig. 1(c) and in the rest of the paper, the corresponding parameter p was chosen randomly between $\pm 10\%$ about its nominal value and it is kept constant for one Monte-Carlo run.

For the simulations with feedback controller in Fig. 2, a linear quadratic regulator is constructed by minimizing the cost

$$\mathcal{J}_f = \frac{1}{2}\|\Delta x(t_f)\|^2 + \int_0^{t_f} \Delta u^2(t)\,\mathrm{d}t, \tag{28}$$

where $\Delta x(t) = x(t) - x^*(t)$, $\Delta u(t) = u(t) - u^*(t)$, and by linearizing the dynamics along the reference trajectory as

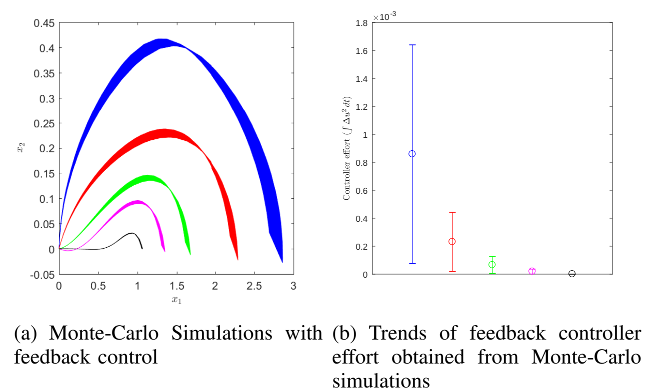
$$\Delta\dot{x} = \mathcal{A}(t)\Delta x + \mathcal{B}(t)\Delta u, \quad \Delta x(t_0) = 0_{n\times 1}, \tag{29}$$

where

$$\mathcal{A}(t) = \frac{\partial f(x, p_0, u, t)}{\partial x}\bigg|_{x=x^*(t),\, u=u^*(t)}, \tag{30}$$

$$\mathcal{B}(t) = \frac{\partial f(x, p_0, u, t)}{\partial u}\bigg|_{x=x^*(t),\, u=u^*(t)}. \tag{31}$$

Consequently, the control effort required to track the optimal



(a) Monte-Carlo Simulations with feedback control

(b) Trends of feedback controller effort obtained from Monte-Carlo simulations

Fig. 2: Results for trajectory desensitization with feedback control

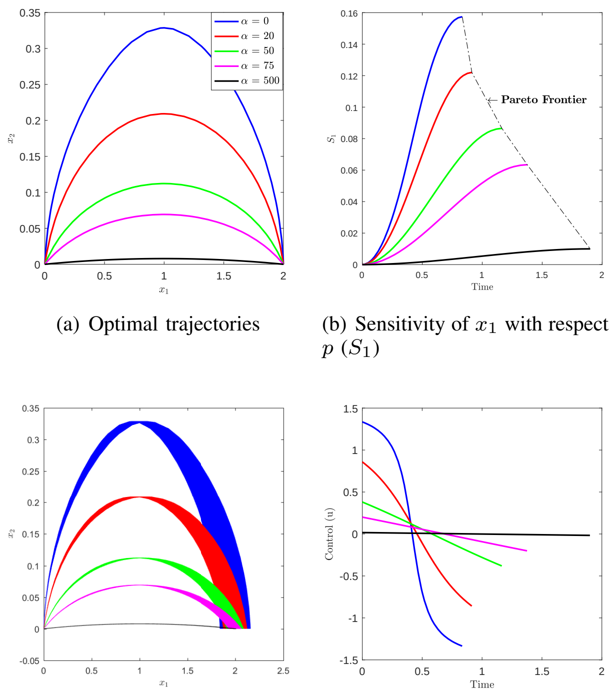
trajectory (for some α) under parameter variations, using the feedback controller, reduces as the value of α increases, as can be observed in Fig. 2(b). These results further corroborate the proposed approach for desensitized optimal control that deals with the trade-off between optimality and tracking effort using a feedback controller.

B. *Final State Desensitization*

For this example, the dynamics presented in the previous subsection (22), and the initial conditions (24) are retained. The vehicle, starting from the origin, has to reach a point along the shore $(2, 0)$ in minimum time. However, the parameter p is uncertain, and the goal state $(2, 0)$ has to be reached as accurately as possible i.e., the sensitivity of final state with respect to the parameter variation is of concern. For this purpose, the augmented cost function, equivalent to (18), can be constructed as

$$\mathcal{J}_a(u) = t_f + \alpha S_1^2(t_f). \tag{32}$$

The sensitivity equation (25) too remains the same for this example, as it only depends on the dynamics of the problem and the targeted parameters, but not on the original cost function. The results obtained from the simulations for $\alpha = \{0, 25, 75, 1000\}$, while minimizing the cost function (32) are presented in Fig. 3.



(a) Optimal trajectories

(b) Sensitivity of x_1 with respect to p (S_1)

(c) Monte-Carlo simulations with open-loop control

(d) Optimal Control (u^*)

Fig. 3: Results obtained for the Zermelo's path optimization problem with final state desensitization

All optimal trajectories meet the boundary conditions at the initial and final times, but they differ in their final state sensitivity with respect to the uncertain parameter. A clear trade-off between the time taken to reach the goal state (t_f) and final state sensitivity ($S_1(t_f)$) can be observed from Figs. 3(b) and 3(c). A Pareto frontier can be drawn to quantitatively establish the trade-off between $S_1(t_f)$ and t_f, which is also shown in Fig. 3(b). The Monte-Carlo simulations, in which the parameter is again randomly varied between $\pm 10\%$ about the nominal value, further strengthen the claim that the dispersion in the final state can be reduced at the cost of the time it takes to reach the goal state. Note the similarity in the trends of the sensitivity term S_1 with the trajectory desensitization by comparing Figs. 1(b) and 3(b). The reason behind the similarity could be because of the fact that since $S_2(t) = 0,$ for all $t \geq 0$, $\dot{S}_1 = x_2$. This implies that S_1 is almost always increasing along the trajectory, as the slope is mostly positive, and penalizing S_1 at the final state indirectly penalizes its slope at every point along the trajectory, thus desensitizing the entire trajectory in this example. The results can also be compared with Seywald's in Ref. [15]. The effect of desensitization remains the same, while achieving the goal with fewer number of states in our approach. The number of states in the augmented model is 4 with the proposed approach, whereas it is 21 in the case of Seywald's approach.

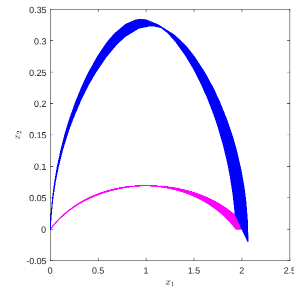
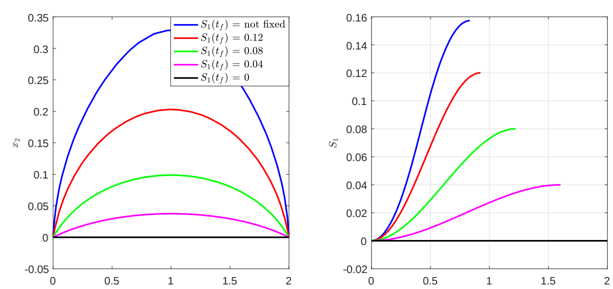


Fig. 4: Monte-Carlo simulations for non-desensitized trajectory with feedback controller and desensitized trajectory with open-loop

In Fig. 4, a comparison between the non-desensitized trajectory with feedback controller and a desensitized trajectory ($\alpha = 75$, the color code remains the same as in Fig. 3) with open-loop control is made again using Monte-Carlo simulations. The feedback controller design is same as the one developed in Section IV-A. It can be observed that the dispersion in x_1 at final time ($x_1(t_f)$) is almost the same in both cases. However, the energy of the overall control signal, $\int_{t_0}^{t_f} (u(t) + \Delta u(t))^2 \mathrm{d}t = 0.08$, on an average is significantly higher in the non-desensitized case, as opposed to 0.02 in the case of desensitization (open-loop). This indicates that by employing desensitization to work, the dispersion in the final state can be reduced with lesser control effort by compromising on the original cost, which is travel time in this example.



(a) Optimal trajectories

(b) Sensitivity of x_1 with respect to p (S_1)

Fig. 5: Results obtained for the Zermelo's path optimization problem - Sensitivity of the final state with respect to the uncertain parameter is fixed

The results obtained for Zermelo's problem while directly fixing the value of S_1 at final time for different cases,

$S_1(t_f) = \{\text{not fixed}, 0.12, 0.08, 0.04, 0\}$, are presented in Fig. 5. It is interesting to observe that the optimal trajectory for the case $S_1(t_f) = 0$ goes along the shore without venturing into the currents, as to be expected from the physics of this problem.

C. Discussion

The proposed scheme outperforms the approach presented in Ref. [15] in terms of the computation complexity with just $n + n\ell$ number of states, as opposed to $(n+\ell)^2 + n + \ell$ number of states in the latter approach. Also, the freedom in formulating desensitized optimal control schemes, especially for trajectory desensitization, is significant with sensitivity functions. However, given the dynamics and the constraints, (2)-(3), there exists a set of trajectories, and each trajectory has an associated sensitivity cost $\left(\int_{t_0}^{t_f} \| \operatorname{vec} S(t) \|^2_{Q(t)} \mathrm{d}t\right)$. Given the original cost function (1), the optimal trajectory that minimizes the cost function also has an associated sensitivity cost. In the proposed approached with an augmented cost function, by penalizing the sensitivity cost, one expects to find another trajectory that is less sensitive, while satisfying all the constraints. Such a trajectory, however, may not exist. There is an implicit assumption here, namely, that there exists a path that is less sensitive to parametric variations compared to the optimal one, given the original cost function along with state and control constraints. In some problems, it could be the case where the solution to the original optimal control problem can no longer be desensitized. Analysis on "desensitizability" of an optimal control problem is a potential direction for future work.

Similarly, in the case of final state desensitization where the sensitivities at the final time are fixed, it is assumed that the control u is able to drive the augmented state, with the sensitivity terms added to the original state vector x, given the constraint in the original optimal control problem. In that case, in order to be able to desensitize a given system, there is an implicit assumption that the system with the augmented state $[x^\top \ (\operatorname{vec} S)^\top]^\top$ is controllable. Otherwise, the control input u may not have enough authority to drive the additional states introduced via the sensitivity function. In such cases, an additional feedback term is the only available option to handle the parameter variations. Finally, a close connection between the state covariance matrix and the sensitivity matrix has been established [21]. This offers another interesting direction for future work.

V. Conclusion

The idea of desensitized optimal control is explored using sensitivity functions, and the corresponding sensitivity equation. Various schemes for trajectory/state desensitization are proposed which compete with the existing approaches in terms of the computational complexity, and it is realized that the sensitivity function is more tractable compared to the Seywald's sensitivity matrix. The proposed schemes are demonstrated using fixed final time, and minimum time Zermelo's optimal control problems.

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