Finite-Horizon Covariance Control of Linear Time-Varying Systems

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Abstract— **We consider the problem of finite-horizon optimal control of a discrete linear time-varying system subject to a stochastic disturbance and fully observable state. The initial state of the system is drawn from a known Gaussian distribution, and the final state distribution is required to reach a given target Gaussian distribution, while minimizing the expected value of the control effort. We derive the linear optimal control policy by first presenting an efficient solution for the diffusion-less case, and we then solve the case with diffusion by reformulating the system as a superposition of diffusion-less systems. We show that the resulting solution coincides with a LQG problem with particular terminal cost weight matrix.**

I. Introduction

The work in this paper is aimed at solving the problem of the optimal steering of a discrete time varying stochastic linear system, with a fully observable state, a known Gaussian distribution of the initial state, and a state and input-independent white-noise Gaussian diffusion. The goal is to find the optimal input to steer the state of the system to a pre-specified target Gaussian distribution in a given time, while minimizing the expected value of the input signal ℓ_2-norm. Unlike the classical LQG case [1], where the final state covariance appears as a by-product of the solution, here we are required to reach exactly the target covariance at the given final time.

The covariance steering problem is relevant to a wide range of control and path-planning applications, such as decentralized control of swarm robots [2], closed-loop cooling [3], and other areas, where it is more natural to specify a distribution over the state rather than a fixed set of values.

The steady-state covariance control problem, (a.k.a. the Covariance Assignment problem), has been extensively studied for both continuous and discrete-time stochastic linear systems [4], [5], [6], [7]. A finite-time optimal solution for the continuous case has been recently derived in [8], [9], and [10], with a connection to the problems of Shrödinger bridges [11] and the Optimal Mass Transfer [12]. In these works the authors showed that, if the diffusion term affects the system through all control input channels, the target probability can always be achieved in finite time, and the solution is given in state-feedback form. A more general case, in which the control input and the diffusion channels are different, can be solved using a soft constraint on the target distribution (such as using the Wasserstein distance [13]), or by numerical optimization methods [14].

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The discrete finite-time case has been addressed in [15], in which the author used a relaxed formulation for the target covariance in order to facilitate its numerical solution. In this paper we treat a similar problem as in [15], but we impose a hard equality constraint in the final distribution instead, so the relaxation imposed in [15] is not needed. In addition, the solution in [15] is based on a non-linear convex programming with a large number of variables ($\mathscr{O}\{n\times m\times N\}$, where n is a state size, m is an input size, and N is the number of time steps). The proposed method, on the other hand, requires only $n^2/2+n$ decision variables.

Another special case of linear discrete finite-time Gaussian stochastic systems was mentioned in [16], in which the author shows a relation between the relative entropy and the minimum energy LQG optimal control problems. The system discussed in [16] has a full control authority and the disturbance matrix is invertible. This paper extends the results presented in [16] to a general linear system, In addition, the conditions for the solvability presented in [16] follow naturally from the analysis presented here.

Main Contribution: In this paper we first derive the minimum-control-effort optimal steering solution for fully-observable linear time-varying discrete stochastic systems, subject to boundary conditions in terms of their Gaussian distribution. The problem considered herein can be viewed as a subset of the problems presented in [15], but with a different solution formulation. We provide necessary conditions for the existence of the solution, and proposes an efficient numerical scheme for attaining it. Furthermore, we show that the resulting controller coincides with solving a LQG [1] problem, with the particular choice of the terminal cost weight matrix.

The notation used throughout this paper is quite standard. A unit matrix is denoted as I, and $\mathbb{E}[\cdot]$ denotes the expectation operator. A random variable x with normal distribution is denoted as $x\sim\mathscr{N}(\mu,\Sigma)$, where μ is its mean, and Σ its covariance matrix. The trace of a square matrix is denoted by $\mathrm{Tr}[\cdot]$. The positive-definiteness of the square matrix R is denoted as $R\succ 0$, and semi-definiteness is denoted as $R\succeq 0$. A zero matrix with dimensions $m\times n$ is denoted as $0_{m\times n}$. An $n\times n$ diagonal matrix with $(a_1,a_2,\ldots,a_n)$ on the diagonal is denoted as $\mathrm{diag}[a_1,a_2,\ldots,a_n]$. A square root of a diagonal matrix S is denoted as $S^{\frac{1}{2}}$.

II. Problem Statement

A. Problem Formulation

Consider the discrete stochastic linear time-varying system

$$x_{k+1}=A_kx_k+B_ku_k+G_kw_k. \quad k=0,1,...,N, \tag{1}$$

where $x\in\mathbb{R}^n$ is the state, $u\in\mathbb{R}^m$ is the control input, and $w\in\mathbb{R}^r$ is a zero-mean white Gaussian noise with unit covariance.

Our objective is to steer the trajectories of system (1) from a given initial Gaussian distribution having mean $\mathbb{E}[x_0] = \mu_0$ and covariance Σ_0 to a final Gaussian distribution having mean $\mathbb{E}[x_{N+1}] = \mu_F$ and covariance Σ_F. That is, we wish the initial and final states to be distributed according to

$$x_0 \sim \mathcal{N}(\mu_0, \Sigma_0), \qquad x_{N+1} \sim \mathcal{N}(\mu_F, \Sigma_F), \tag{2}$$

with $\mu_0, \Sigma_0, \mu_F, \Sigma_F$ given, while minimizing the cost function

$$J(u_0, ..., u_N) = \mathbb{E}[\sum_{k=0}^{N} u_k^\top u_k]. \tag{3}$$

B. System Dynamics

For each step k, the system state x_k can be explicitly calculated as follows. Let A_{k_1,k_0}, B_{k_1,k_0}, G_{k_1,k_0} denote the transition matrices of the state, the input, and the diffusion term from step k_0 to step k_1+1 ($k_1 > k_0$) as follows

$$A_{k_1,k_0} = A_{k_1} A_{k_1-1} \cdots A_{k_0}, \qquad A_{k,k} \triangleq A_k, \tag{4a}$$

$$B_{k_1,k_0} = A_{k_1,k_0+1} B_{k_0}, \qquad B_{k,k} \triangleq B_k, \tag{4b}$$

$$G_{k_1,k_0} = A_{k_1,k_0+1} G_{k_0}, \qquad G_{k,k} \triangleq G_k. \tag{4c}$$

Let also U_{k_1,k_2} and W_{k_1,k_2} ($k_1 \le k_2$) be the vectors

$$U_{k_1,k_2} = \begin{bmatrix} u_{k_1} \\ u_{k_1+1} \\ \vdots \\ u_{k_2} \end{bmatrix}, \qquad W_{k_1,k_2} = \begin{bmatrix} w_{k_1} \\ w_{k_1+1} \\ \vdots \\ w_{k_2} \end{bmatrix}, \tag{5}$$

and, for simplicity, let $U_k \triangleq U_{0,k}$ and $W_k \triangleq W_{0,k}$. For convenience, define the matrices

$$\bar{B}_{k_1,k_0} \triangleq \begin{bmatrix} B_{k_1,k_0} & B_{k_1,k_0+1} & \cdots & B_{k_1,k_1} \end{bmatrix}, \tag{6a}$$

$$\bar{G}_{k_1,k_0} \triangleq \begin{bmatrix} G_{k_1,k_0} & G_{k_1,k_0+1} & \cdots & G_{k_1,k_1} \end{bmatrix}, \tag{6b}$$

and let $\bar{A}_k \triangleq A_{k,0}$, $\bar{B}_k \triangleq \bar{B}_{k,0}$, $\bar{G}_k \triangleq \bar{G}_{k,0}$. The system state at step $k+1$ is given by

$$x_{k+1} = \bar{A}_k x_0 + \bar{B}_k U_k + \bar{G}_k W_k. \tag{7}$$

Since $\mathbb{E}[W_k] = 0$, the mean of the state obeys

$$\mu_{k+1} \triangleq \mathbb{E}[x_{k+1}] = \bar{A}_k \mu_0 + \bar{B}_k \mathbb{E}[U_k]. \tag{8}$$

Defining now $\widetilde{U}_k \triangleq U_k - \mathbb{E}[U_k]$, $\widetilde{x}_k \triangleq x_k - \mu_k$. It follows that

$$\widetilde{x}_{k+1} = \bar{A}_k \widetilde{x}_0 + \bar{B}_k \widetilde{U}_k + \bar{G}_k W_k. \tag{9}$$

The state covariance is given by[1]

$$\begin{aligned} \Sigma_{k+1} &\triangleq \mathbb{E}[\widetilde{x}_{k+1} \widetilde{x}_{k+1}^\top] \\ &= \bar{A}_k \Sigma_0 \bar{A}_k^\top + \bar{B}_k \mathbb{E}[\widetilde{U}_k \widetilde{U}_k^\top] \bar{B}_k^\top + \bar{G}_k \mathbb{E}[W_k W_k^\top] \bar{G}_k^\top \\ &\quad + \bar{B}_k \mathbb{E}[\widetilde{U}_k \widetilde{x}_0^\top] \bar{A}_k^\top + \bar{A}_k \mathbb{E}[\widetilde{x}_0 \widetilde{U}_k^\top] \bar{B}_k^\top \\ &\quad + \bar{B}_k \mathbb{E}[\widetilde{U}_k W_{k-1}^\top] \bar{G}_{k-1}^\top + \bar{G}_{k-1} \mathbb{E}[W_{k-1} \widetilde{U}_k^\top] \bar{B}_k^\top, \end{aligned} \tag{10}$$

[1]A causal state-feedback controller at step k is independent of the diffusion term at step k', with $k' \ge k$.

and the cost function (3) can be written as

$$J(U_N) = \mathbb{E}[U_N^\top U_N] = \underbrace{\mathbb{E}[U_N]^\top \mathbb{E}[U_N]}_{J_\mu} + \underbrace{\mathrm{Tr}\,[\mathbb{E}[\widetilde{U}_N \widetilde{U}_N^\top]]}_{J_\Sigma}. \tag{11}$$

It will be assumed in this paper that the system (1) is *controllable*, that is, if $G_k \equiv 0$, the reachable set at $k = N+1$ is $\mathbb{R}^n$, that is, for any $x_S \in \mathbb{R}^n$ and $x_F \in \mathbb{R}^n$, there exist a set of controls $\{u_k\}_{k=0}^N$ that brings the state from $x_0 = x_S$ to $x_{N+1} = x_F$. From (7) it is straightforward to conclude that system (1) is controllable if and only if $\bar{B}_N$ is full row rank.

III. Optimal Covariance Steering

As seen from (8), (10) and (11), the problem of steering the mean and the covariance can be separated into two independent subproblems: finding an optimal $\mathbb{E}[U_N]$ that minimizes J_μ satisfying the mean constraint (8) and the boundary condition (2), and finding an optimal $\widetilde{U}_N$ that minimizes J_Σ satisfying the covariance constraint (10) and the boundary condition (2). This section presents an analytical solution to both problems.

A. Steering the Mean

Since the dynamics of the state mean are governed by (8), and the cost function that is influenced by the mean is given in (11), the optimal solution for $\mathbb{E}[U_N]$ will not influence the covariance part of the solution. The solution for the mean steering is well known in the literature, and is given below for the sake of completeness.

Proposition 1: Given the controllable system (1), the optimal control $\mathbb{E}[U_N^\star]$ that minimizes the cost

$$J_\mu = \mathbb{E}[U_N]^\top \mathbb{E}[U_N] = \sum_{k=0}^{N} \mathbb{E}[u_k]^\top \mathbb{E}[u_k],$$

subject to the constraint

$$\bar{A}_N \mu_0 + \bar{B}_N \mathbb{E}[U_N] = \mu_F, \tag{12}$$

is given by

$$\mathbb{E}[U_N^\star] = \bar{B}_N^\top (\bar{B}_N \bar{B}_N^\top)^{-1} (\mu_F - \bar{A}_N \mu_0). \tag{13}$$

Now that we have the mean steering solution, the rest of the paper will concentrate on solving the covariance steering problem, using the deviation-from-mean dynamics given by (9), and the covariance-part cost J_Σ given in (11). For simplicity, we will assume that the original system has zero-mean constraints for the initial and final states.

B. Steering the Covariance

In this section we present the covariance steering controller by first deriving a necessary condition for the solution, and then presenting a numerical scheme to find a controller that satisfies these necessary conditions.

To this end, assume a controller of the form

$$\widetilde{U}_N = L \widetilde{x}_0, \tag{14}$$

where $L \in \mathbb{R}^{(Nr)\times n}$. The covariance-related part of the cost function (11) can now be rewritten as:

$$J_\Sigma = \mathrm{Tr}\,[\mathbb{E}[\widetilde{U}_N \widetilde{U}_N^\top]] = \mathrm{Tr}\,[L\Sigma_0 L^\top]. \tag{15}$$

1) Diffusion-less Case: Suppose that $G_k = 0$ for all $k \in [0,N]$ in (1). In this case, the final state covariance (10) becomes

$$\begin{aligned}\Sigma_F = \Sigma_{N+1} = \bar{A}_N \Sigma_0 \bar{A}_N^\top + \bar{B}_N \mathbb{E}[\widetilde{U}_N \widetilde{U}_N^\top]\bar{B}_N^\top \\ + \bar{B}_N \mathbb{E}[\widetilde{U}_N \widetilde{x}_0^\top]\bar{A}_N^\top + \bar{A}_N \mathbb{E}[\widetilde{x}_0 \widetilde{U}_N^\top]\bar{B}_N^\top.\end{aligned} \tag{16}$$

Applying the controller (14) results in the final covariance given by

$$\Sigma_{N+1} = (\bar{A}_N + \bar{B}_N L)\Sigma_0 (\bar{A}_N + \bar{B}_N L)^\top = \Sigma_F. \tag{17}$$

The following proposition describes the diffusionless linear discrete covariance steering control algorithm:

Proposition 2: Let the controllable system (1), with zero diffusion, and positive definite initial state covariance $\Sigma_0 \succ 0$, and let

$$V_0 S_0 V_0^\top = \Sigma_0, \quad V_F S_F V_F^\top = \Sigma_F, \quad U_\Omega S_\Omega V_\Omega^\top = \Omega, \tag{18}$$

be the singular value decompositions (SVDs) of the respective matrices, where

$$\Omega \triangleq S_F{}^{\frac{1}{2}} V_F^\top (\bar{B}_N \bar{B}_N^\top)^{-1} \bar{A}_N V_0 S_0{}^{\frac{1}{2}}. \tag{19}$$

Then the optimal control gain $L \in \mathbb{R}^{(Nr)\times n}$ that minimizes (15) subject to a constraint $\Sigma_{N+1} = \Sigma_F$, is given by

$$L^\star = \bar{B}_N^\top (\bar{B}_N \bar{B}_N^\top)^{-1} (V_F S_F{}^{\frac{1}{2}} U_\Omega V_\Omega^\top S_0{}^{-\frac{1}{2}} V_0^\top - \bar{A}_N). \tag{20}$$

Proof: Please see the Appendix. ■

The proof of Proposition 2 reveals that the optimal control can also be obtained from

$$L^\star = -\bar{B}_N^\top \Lambda (\mathrm{I} + \bar{B}_N \bar{B}_N^\top \Lambda)^{-1} \bar{A}_N, \tag{21}$$

where Λ is the solution of a matrix Riccati equation.

Proposition 3: The matrix Λ in (21) that satisfies the constraint (17), and minimizes the cost function (15), satisfies the matrix Riccati equation

$$(\Theta\Sigma_F)\Lambda + \Lambda(\Theta\Sigma_F)^\top + \Lambda\Sigma_F\Lambda + \Theta(\Sigma_F - \bar{A}_N \Sigma_0 \bar{A}_N^\top)\Theta = 0, \tag{22}$$

where $\Theta = (\bar{B}_N \bar{B}_N^\top)^{-1}$.

Proof: Substituting L from (51) into the constraint (45), and using matrix inversion identity, yields

$$\Sigma_F = (\mathrm{I} + \bar{B}_N \bar{B}_N^\top \Lambda)^{-1} \bar{A}_N \Sigma_0 \bar{A}_N^\top (\mathrm{I} + \Lambda \bar{B}_N \bar{B}_N^\top)^{-1}, \tag{23}$$

which can be rewritten as (22). ■

Note that the previous approach can be generalized to the case where it is required that the final covariance is only partially constrained, i.e., given $D \in \mathbb{R}^{n_p \times n}$ with $n_p \le n$ and final *partial* covariance matrix $\Sigma_F \in \mathbb{R}^{n_p \times n_p}$, the boundary condition for the state covariance at step $N+1$ is defined as

$$D\mathbb{E}[\widetilde{x}_{N+1}\widetilde{x}_{N+1}^\top]D^\top = D\Sigma_{N+1}D^\top = \Sigma_F. \tag{24}$$

Rewriting the above equation for a linear controller gain yields

$$D(\bar{B}_N L + \bar{A}_N)\Sigma_0(\bar{B}_N L + \bar{A}_N)^\top D^\top = \Sigma_F, \tag{25}$$

which can be seen as the covariance-steering for diffusion-less system having transition matrices $D\bar{B}_N$ and $D\bar{A}_N$, with the solution given by Proposition 2.

2) Non-zero Diffusion Case: Consider now the complete system given by (1), including the diffusion term ($G_k \neq 0$). The system (1) at time step $N+1$ can be viewed as a sum of $N+1$ uncorrelated ($\mathbb{E}[x_k^{(i)} x_m^{(j)\top}] = 0,\ k,m,i,j \in [0,N+1],\ i \neq j$), diffusion-less sub-systems as follows

$$x_{N+1} = \sum_{i=0}^{N} x_{N+1}^{(i)} + G_N w_N, \tag{26}$$

where $x_{N+1}^{(i)}$ for all $i = 0,\dots,N$ are computed, for all $k \in [i,N]$, from

$$x_{k+1}^{(i)} = A_k x_k^{(i)} + B_k u_k^{(i)}, \quad x_i^{(i)} = \begin{cases} x_0, & \text{for } i = 0, \\ G_{i-1}w_{i-1}, & \text{otherwise,} \end{cases} \tag{27}$$

and $x^{(i)}$ and $u^{(i)}$ denote the state and the input of the i'th sub-system. The final state can therefore be expressed as

$$x_{N+1} = \bar{A}_N x_0 + \bar{B}_N U_{0,N}^{(0)} + \sum_{i=1}^{N} \bar{G}_{N,i-1} w_{i-1} + \bar{B}_{N,i} U_{i,N}^{(i)} + G_N w_N, \tag{28}$$

where,

$$U_{k_1,k_2}^{(i)} \triangleq \begin{bmatrix} u_{k_1}^{(i)} \\ u_{k_1+1}^{(i)} \\ \vdots \\ u_{k_2}^{(i)} \end{bmatrix}, \quad 0 \le k_1 \le k_2 \le N. \tag{29}$$

We assume control laws with a linear dependence on $x_i^{(i)}$, that is, similarly to (14), we let $L^{(k)} \in \mathbb{R}^{(m(N-k+1))\times n}$, $k \in [0,N]$, be a set of matrices, such that

$$U_{i,N}^{(i)} = \begin{cases} L^{(i)} x_i^{(i)}, & i \in [1,N], \\ L^{(0)} x_0 + \mathbb{E}[U_N], & i = 0. \end{cases} \tag{30}$$

Since all states $x^{(i)}$ for $i \in [1,N]$ have zero mean, the mean of x_{N+1} is governed by equation (8). The covariance of the final state derived from (28) is then given by

$$\begin{aligned}\Sigma_{N+1} = (\bar{A}_N + \bar{B}_N L^{(0)})\Sigma_0(\bar{A}_N + \bar{B}_N L^{(0)})^\top \\ + \sum_{i=1}^{N} (A_{N,i} + \bar{B}_{N,i} L^{(i)}) G_{i-1} G_{i-1}^\top (A_{N,i} + \bar{B}_{N,i} L^{(i)})^\top \\ + G_N G_N^\top.\end{aligned} \tag{31}$$

Theorem 1: Let the system (1), initial and final state means μ_0 and μ_F, and initial and final state covariance matrices $\Sigma_0 \succeq 0$ and $\Sigma_F \succeq 0$. Let $y_0 = x_0 - \mu_0$ and define, for $k \in [0,N]$,

$$y_k = x_k - (A_{k-1}x_{k-1} + B_{k-1}u_{k-1}). \tag{32}$$

Furthermore, let $\Phi_k \in \mathbb{R}^{n\times n}$ be given by

$$\Phi_k = (\mathsf{I} + \bar{B}_{N,k}\bar{B}_{N,k}^\top \Lambda)^{-1} A_{N,k}, \tag{33}$$

where $\Lambda = \Lambda^\top \in \mathbb{R}^{n\times n}$ is the solution of the matrix equation

$$\sum_{k=1}^{N} \Phi_k G_{k-1} G_{k-1}^\top \Phi_k^\top + \Phi_0 \Sigma_0 \Phi_0^\top = \Sigma_F - G_N G_N^\top. \tag{34}$$

The optimal linear control law that minimizes the cost function (15) subject to a constraints $\Sigma_{N+1} = \Sigma_F$ and $\mu_{N+1} = \mu_F$, and with the initial state mean μ_0 and covariance Σ_0, is given by

$$u_k = B_{N,k}^\top (\bar{B}_N \bar{B}_N^\top)^{-1} (\mu_F - \bar{A}_N \mu_0) + \sum_{i=0}^{k} L_k^{(i)} y_i\,, \tag{35}$$

where,

$$L_k^{(i)} = -B_{N,k}^\top \Lambda \Phi_i. \tag{36}$$

Proof: Since the mean of the state is governed by (8), the mean-steering solution $\mathbb{E}[U_N]$ is given by Proposition 1, equation (13).

The second part of the controller, $\widetilde{U}_N$, having the linear form (14) is directed to minimizing the covariance-related cost (15), while adhering to the constraint $\mathbb{E}[\widetilde{x}_{N+1}\widetilde{x}_{N+1}^\top] = \Sigma_F$.

The Lagrangian of the minimization problem (15) subject to the constraint (2) is given by

$$\mathfrak{L}(u,\Lambda) = \mathrm{Tr}\,[\mathbb{E}[\widetilde{U}_N \widetilde{U}_N^\top]] + \mathrm{Tr}\,[\Lambda(\mathbb{E}[\widetilde{x}_{N+1}\widetilde{x}_{N+1}^\top] - \Sigma_F)]. \tag{37}$$

Using (30), and (31), the Lagrangian can be rewritten in terms of $L^{(i)}$, $i \in [0,N]$ as follows

$$\begin{aligned}\mathfrak{L}(u,\Lambda) = \mathrm{Tr}\Big\{ & L^{(0)}\Sigma_0 (L^{(0)})^\top - \Lambda\Sigma_F \\ & + \Lambda(\bar{A}_N + \bar{B}_N L^{(0)})\Sigma_0(\bar{A}_N + \bar{B}_N L^{(0)})^\top \\ & + \sum_{i=1}^{N} L^{(i)} G_{i-1} G_{i-1}^\top (L^{(i)})^\top \\ & \quad + \Lambda(A_{N,i} + \bar{B}_{N,i} L^{(i)}) G_{i-1} G_{i-1}^\top (A_{N,i} + \bar{B}_{N,i} L^{(i)})^\top \Big\},\end{aligned} \tag{38}$$

yielding the first and second order necessary conditions for a minimizer $L^{(i)} + \bar{B}_{N,i}^\top \Lambda(A_{N,i} + \bar{B}_{N,i} L^{(i)}) = 0$, and $\mathsf{I} + \bar{B}_{N,i}^\top \Lambda \bar{B}_{N,i} \succ 0$, respectively. Following a similar derivation as in Proposition 2, the resulting optimal control gain is given by (36), with Φ_k given by (33). Substituting this control back into the constraint equation (31), results in the closed-loop covariance equation (34).

Therefore, the matrix Λ that satisfies the constraint (34) provides the optimal gains for the optimal controller (35). ■

Note that the controller in (35) can be efficiently calculated by updating the vector $U_{k,N}$ at every step k (starting from $k=0$) by $U_{0,N} = \bar{B}_N^\top (\bar{B}_N \bar{B}_N^\top)^{-1} (\mu_F - \bar{A}_N \mu_0)$, $U_{k,N} = U_{k,N} + L^{(k)}(x_k - \hat{x}_k)$, where $\hat{x}_0 = \mu_0$ and $\hat{x}_{i+1} = A_i \hat{x}_i + B_i u_i$. The non-negativity of the left-hand side of (34) yields

$$\Sigma_F - G_N G_N^\top \succeq 0, \tag{39}$$

which is exactly the condition for solvability for the covariance steering problem provided in [16, Proposition 5.1].

IV. Relation with LQG

The stochastic control problem formulated in Section II-A can be also viewed as a special case of the standard discrete LQG [1, p.264]. This similarity will be detailed in this section, focusing on the covariance control, thus assuming a zero-mean state.

Theorem 2: Let the system (1), with zero-mean states, and initial and final state covariance matrices Σ_0 and Σ_F. Let $Q_f \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Assume that the LQG controller that minimizes the cost function

$$J(u_0,...,u_N) = \mathbb{E}[\sum_{k=0}^{N} u_k^\top u_k \; + x_{N+1}^\top Q_f x_{N+1}], \tag{40}$$

subject to the dynamics (1), results in the final state covariance being equal to Σ_F. Then, this controller coincides with the optimal controller given by the problem described in Theorem 1, with $\Lambda = Q_f$.

Proof: The Lagrangian of the original problem can be written as

$$\mathfrak{L} = \mathbb{E}[\sum_{k=0}^{N} u_k^\top u_k \; + x_{N+1}^\top \Lambda x_{N+1}] - \mathrm{Tr}\,[\Lambda\Sigma_F]. \tag{41}$$

Given that $\Lambda = Q_f$, minimizing the Lagrangian (41) yields the same result as minimizing (40), and the optimal solution is given by the LQG controller. Since, by construction, this solution agrees with the boundary conditions, it is also a solution of the covariance steering problem. ■

Corollary 1: Assume Λ, which solves the optimal control problem given by (3), is unique. Then, the controller (35) coincides with the LQG controller that minimizes the cost function:

$$J(u_0,...,u_N) = \mathbb{E}[\sum_{k=0}^{N} u_k^\top u_k \; + x_{N+1}^\top \Lambda x_{N+1}] \tag{42}$$

Proof: Recall that the Lagrangian of the optimal control problem given by (3) can be written as (41). Since $\Lambda = \Lambda^\top$ is given,

$$\begin{aligned} U_N &= \arg\min_{U_N} \mathbb{E}[\sum_{k=0}^{N} u_k^\top u_k \; + x_{N+1}^\top \Lambda x_{N+1}] - \mathrm{Tr}\,[\Lambda\Sigma_F] \\ &= \arg\min_{U_N} \mathbb{E}[\sum_{k=0}^{N} u_k^\top u_k \; + x_{N+1}^\top \Lambda x_{N+1}], \end{aligned} \tag{43}$$

subject to the dynamics (1). The solution to (43) is given by the LQG controller,and minimizes the cost (40) with $Q_F = \Lambda$. ■

Note that the presented results coincide with the results in [16]. In fact, equation (5.5) in [16] is exactly equation (34), with the right closed-loop transition matrices.

V. Numerical Example

In this section the performance of the algorithm is tested using a simple example of a fourth-order linear time-varying system, which is derived from linearizing and discretizing a non-linear cart-pole dynamics along a particular trajectory.

Let y denote the cart's position, let u denote the force pushing the cart, and let θ denote the pole's angle measured from vertical axis so that $\theta = 0$ indicates the configuration when the pole points vertically downwards. The equations of the of the cart-pole system are

$$\ddot{\theta} = \frac{-(u + m_p l \dot{\theta}^2 \sin\theta)\cos\theta - (m_c + m_p) g \sin\theta}{l(m_c + m_p \sin^2\theta)},$$
$$\ddot{y} = \frac{u + m_p \sin\theta (l\dot{\theta}^2 + g\cos\theta)}{m_c + m_p \sin^2\theta}. \quad (44)$$

The following parameters were used in the numerical simulations: $m_p = 0.01[\text{kg}], m_c = 1[\text{kg}], \ell = 0.25[\text{m}], g = 9.81[\text{m}/\text{sec}^2]$. The equations of motion were linearized about a trajectory that brings the pole rom the downward position $\theta_0 = 0$ to the upward position $\theta_F = \pi$ in 1 second , and then discretized using Euler's method with sampling time of $T_s = 0.001$ sec, resulting in a linear discrete time-varying system with states defined as $x \triangleq \begin{bmatrix} \delta\theta & \delta\dot{\theta} & \delta y & \delta\dot{y} \end{bmatrix}^\top$, where $\delta\theta$, $\delta\dot{\theta}$, δy, and $\delta\dot{y}$ denote deviations from the nominal values of θ, $\dot{\theta}$, y, and $\dot{y}$ respectively. To this model, a disturbance noise was added, with $G = \begin{bmatrix} 0 & 0.004 & 0 & 0.008 \end{bmatrix}^\top$.

The initial and the final states are chosen as $\mu_0 = \mu_F = 0_{4\times1}, \Sigma_0 = \Sigma_F = \text{diag}[0.01, 0.01, 0.01, 0.01]$.

The results are shown in Figures 1-3. Figure 1 exhibits 10 randomly-generated closed-loop trajectories (states and control), and the 3σ bounds calculated from 20,000 Monte-Carlo runs. The controller costs are shown in Figure 3. Figure 2 depicts evaluation of state covariance singular values through time.

Similarly to the LTI example, the results show that the two algorithms give exactly the same results.

VI. Conclusions

In this work we have derived a minimum-control-effort optimal steering solution for linear time-varying discrete stochastic systems, subject to boundary conditions in the form of Gaussian distribution parameters. Having presented the influence of the diffusion at each time-step on the final covariance, we have formulated a condition for calculating the optimal control law from a class of linear-state-dependent control laws. The resulting controller set consists of "open-loop" inputs, which are recalculated at each step based on the diffusion term reconstruction from the previous step.

In addition, we have shown that the solution to the covariance steering problem coincides with the solution to a specially-formulated LQG problem. This similarity allowed an efficient calculation of the controller values using a

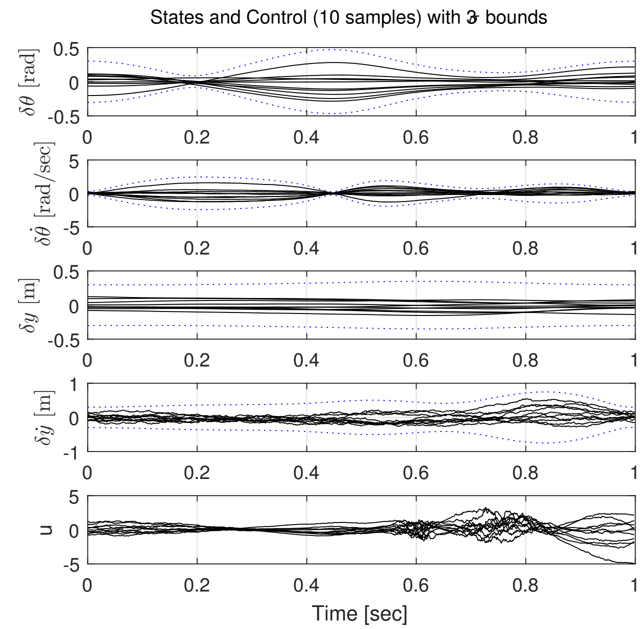

Fig. 1. Linearized Cart-pole states & controls.

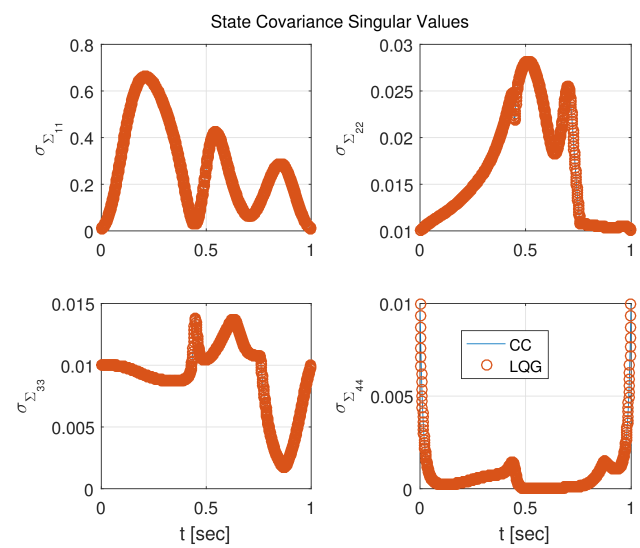

Fig. 2. Linearized Cart-pole: State covariance Singular Values.

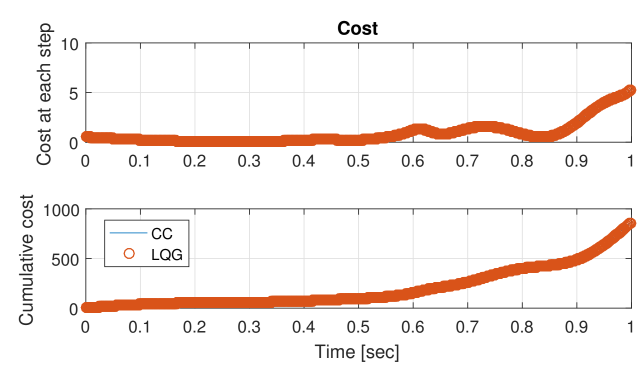

Fig. 3. Linearized Cart-pole: Control cost.

backward-propagated discrete-time dynamic Riccati equation, as well as a justification for using a linear feedback controller for the covariance steering.

Future work will address a case with a state-dependent cost function, the conditions for the existence of the solution, and the algorithm applicability for the covariance steering of non-linear systems.

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APPENDIX

PROOF OF PROPOSITION 2

Let $\Xi \triangleq \bar{A}_N + \bar{B}_N L$. Then the constraint (17) can be written as

$$\Xi \Sigma_0 \Xi^\top = \Sigma_F. \tag{45}$$

First we show feasibility. Substituting (20) into (45) yields

$$\begin{aligned}\Xi &= \bar{A}_N + \bar{B}_N \bar{B}_N^\top (\bar{B}_N \bar{B}_N^\top)^{-1} (V_F {S_F}^{\frac{1}{2}} U_\Omega V_\Omega^\top {S_0}^{-\frac{1}{2}} V_0^\top - \bar{A}_N) \\ &= V_F {S_F}^{\frac{1}{2}} U_\Omega V_\Omega^\top {S_0}^{-\frac{1}{2}} V_0^\top,\end{aligned} \tag{46}$$

and hence

$$\begin{aligned}\Xi \Sigma_0 \Xi^\top &= V_F {S_F}^{\frac{1}{2}} U_\Omega V_\Omega^\top {S_0}^{-\frac{1}{2}} V_0^\top V_0 S_0 V_0^\top V_0 {S_0}^{-\frac{1}{2}} V_\Omega U_\Omega^\top {S_F}^{\frac{1}{2}} V_F^\top \\ &= V_F S_F V_F^\top = \Sigma_F.\end{aligned} \tag{47}$$

To show optimality, introduce the Lagrangian of the equality constraint minimization problem (15) and (17)

$$\mathfrak{L}(L, \Lambda) = \operatorname{Tr}[L \Sigma_0 L^\top] + \operatorname{Tr}[\Lambda(\Xi \Sigma_0 \Xi^\top - \Sigma_F)] \tag{48}$$

where $\Lambda \in \mathbb{R}^{n \times n}$. Without loss of generality we assume that $\Lambda = \Lambda^\top$. The first-order optimality condition $\mathfrak{L}_L(L, \Lambda) = 0$ yields:

$$L + \bar{B}_N^\top \Lambda (\bar{A}_N + \bar{B}_N L) = L + \bar{B}_N^\top \Lambda \Xi = 0, \tag{49}$$

whereas the second order condition $\mathfrak{L}_{LL}(L, \Lambda) = 0$ yields

$$\mathsf{I} + \bar{B}_N^\top \Lambda \bar{B}_N \succ 0. \tag{50}$$

It follows that

$$L = -\bar{B}_N^\top \Lambda (\mathsf{I} + \bar{B}_N \bar{B}_N^\top \Lambda)^{-1} \bar{A}_N. \tag{51}$$

Substituting this value of L into the constraint (17) yields

$$\begin{aligned}\Xi &= \bar{A}_N - \bar{B}_N \bar{B}_N^\top \Lambda (\mathsf{I} + \bar{B}_N \bar{B}_N^\top \Lambda)^{-1} \bar{A}_N \\ &= (\mathsf{I} + \bar{B}_N \bar{B}_N^\top \Lambda)^{-1} \bar{A}_N.\end{aligned} \tag{52}$$

Using the SVDs (18) we can rewrite the constraint (45) as

$$\Xi V_0 S_0^{\frac{1}{2}} R^\top = V_F S_F^{\frac{1}{2}}, \tag{53}$$

where R is an orthogonal matrix. Combining (53) with (52) yields

$$\bar{B}_N \bar{B}_N^\top \Lambda = \bar{A}_N V_0^\top S_0^{\frac{1}{2}} R^\top S_F^{-\frac{1}{2}} V_F^\top - \mathsf{I}, \tag{54}$$

and the resulting optimal gain is

$$L^\star = \bar{B}_N^\top (\bar{B}_N \bar{B}_N^\top)^{-1} (V_F {S_F}^{\frac{1}{2}} R {S_0}^{-\frac{1}{2}} V_0^\top - \bar{A}_N). \tag{55}$$

In order to find R, the optimal gain equation is substituted into the cost function J_Σ, resulting in

$$\begin{aligned}J_\Sigma &= \operatorname{Tr}[\bar{B}_N^\top (\bar{B}_N \bar{B}_N^\top)^{-1} (V_F S_F^{\frac{1}{2}} R S_0^{-\frac{1}{2}} V_0^\top - \bar{A}_N) V_0 S_0 V_0^\top L^{\star\top}] \\ &= \operatorname{Tr}[(\bar{B}_N \bar{B}_N^\top)^{-1} (\Sigma_F + \bar{A}_N \Sigma_0 \bar{A}_N^\top)] - 2 \operatorname{Tr}[R^\top U_\Omega S_\Omega V_\Omega^\top]\end{aligned} \tag{56}$$

where Ω was defined in (19). The minimum of the cost (56) is attained by maximizing the term $\operatorname{Tr}[R^\top U_\Omega]$, yielding

$$R^\star = \arg\min_{R \in \mathscr{U}^n} J_\Sigma = \arg\max_{R \in \mathscr{U}^n} \operatorname{Tr}[R^\top U_\Omega S_\Omega V_\Omega^\top] = U_\Omega V_\Omega^\top, \tag{57}$$

where the last equation follows from the von Newmann trace inequality [17]. Substituting $R^\star$ into the optimal gain $L^\star$ yields (20).