The Asymmetric Sinistral/Dextral Markov-Dubins Problem

Efstathios Bakolas and Panagiotis Tsiotras

***Abstract*— We consider a variation of the classical Markov-Dubins problem dealing with curvature-constrained, shortest paths in the plane with prescribed initial and terminal positions and tangents, when the lower and upper bounds of the curvature are not necessarily equal. The motivation for this problem stems from vehicle navigation applications when the vehicle may be biased in taking turns at a particular direction due to hardware failures or environmental conditions. We employ optimal control to characterize the structure of the shortest path and we resort to geometric techniques to provide sufficient conditions for optimality of the resulting path.**

I. Introduction

The roots of the problem regarding curvature-constrained planar paths of minimal length with prescribed positions and tangents can be traced back to the end of the nineteenth century when the Russian mathematician A. A. Markov posed the problem for the first time. It was Dubins in 1957 who solved completely the problem by characterizing the structure of the minimal-length paths using a number of constructive, geometric arguments [1]. We shall refer to the problem of finding the shortest, curvature-constrained path as the Markov-Dubins problem (MD), as suggested by Sussmann [2]. The solution of the MD is commonly interpreted as the trajectory of a car-like vehicle, known as the Dubins vehicle, which travels only forward with constant unit speed, and which is constrained to perform turns of radius equal or greater than one. Cockayne and Hall [3] characterized the accessibility sets of the Dubins vehicle, conceived as an oriented point, as a function of the travel time. Furthermore, Reeds and Shepp examined a generalization of the MD when the vehicle can move both forward and backwards, that is, the path may contain cusps [4].

All the aforementioned results were based more or less on constructive proofs. These approaches, even though sufficient for the examination of each particular optimization problem, are of limited use as tools for addressing other similar problems. A number of authors during the 1990's argued that the systematic application of optimal control techniques would provide more rigorous proofs to the MD and a more general framework for addressing similar problems in the future. In particular, Sussmann and Tang [5] and Boissonnat *et al* [6] treated both the MD and the Reeds-Shepp problem (RS) using Pontryagin's Maximum Principle along with geometric control ideas, and provided more general and rigorous proofs, refining the original results of [1] and [4]. The path *synthesis* problem, that is, the characterization of the optimal control for all possible boundary conditions, for the MD was studied by Bui et al in [7], [8], while the same problem for the RS was addressed by Souères and Laumond in [9].

Numerous extensions of the MD problem have appeared in the literature. We highlight the work by Monroy-Pérez [10] regarding the MD on a Riemannian manifold. Furthermore, the shortest-length, bounded-curvature problem in three dimensions has been investigated by Sussmann in [11]. Other interesting variations of the MD can be found in [12], [13], [14], [15], [16].

In this work we examine the curvature-constrained, shortest paths in the plane with prescribed positions and tangents when the lower and upper bounds of the curvature are not necessarily equal. The motivation for this problem stems from vehicle navigation applications when the acceleration steering capacity of the vehicle performing clockwise or counterclockwise turns is different. A typical case would be an aircraft with a damaged aileron or a missing wingtip. Our analysis shows that while the structure of the solution is similar to the standard MD, the synthesis of the shortest paths, however, is different.

The rest of the paper is organized as follows. In Section II we present the kinematic model and we formulate the minimum-time problem. In Section III we carry out a PMP analysis and derive conditions that allow us to characterize the minimum-time trajectories given arbitrary prescribed boundary conditions (synthesis problem). In Section IV we solve the synthesis problem and we compare our results with those of the standard MD. Finally, we conclude the paper with Section V.

II. Kinematic Model and Problem Formulation

In this paper we are interested in the solution of the planar, curvature-constrained, shortest-path problem with prescribed initial and final positions and tangents, when the lower and upper bounds of the path curvature are not necessarily equal. Equivalently, we can cast the problem as a minimum-time problem for the Dubins vehicle that moves in the plane subject to the following set of equations

$$\dot{x} = \cos\theta,\ \dot{y} = \sin\theta,\ \dot{\theta} = u/\rho, \tag{1}$$

where x, y are the cartesian coordinates of a reference point of the vehicle, θ is the vehicle's orientation (always tangent to the ensuing path), u is the control input and ρ a positive constant. We assume that the set of admissible control inputs $\mathcal{U}$ consists of all measurable functions u over $[0, T]$ with $u(t) \in U \triangleq [-\delta, 1]$. It follows that ρ and $\varrho \triangleq \rho/\delta$

E. Bakolas is a Ph.D. candidate at the School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0150, USA, Email: gth714d@mail.gatech.edu

P. Tsiotras is a Professor at the School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0150, USA, Email:tsiotras@gatech.edu

is the minimum turning radius for counterclockwise and clockwise turns respectively. The case $U = [-1, \delta]$ can be treated similarly. We call the system described by (1) the asymmetric, sinistral/dextral Dubins vehicle[1].

It is a well-known fact that the Dubins vehicle is a completely controllable system [5]. To establish that the asymmetric, sinistral/dextral Dubins vehicle is a completely controllable system as well, let us consider the steering problem from configuration x to y, where $\mathsf{x}, \mathsf{y} \in \mathbb{R}^2 \times \mathbb{S}^1$. To both x and y we associate two circles of radii ρ, tangent to each other, namely $\mathrm{L}_\mathsf{x}, \mathrm{R}_\mathsf{x}$ and $\mathrm{L}_\mathsf{y}, \mathrm{R}_\mathsf{y}$ respectively, as shown in Fig. 1. Let us, furthermore, restrict the input u to take values over the set $U' = \{0, 1\} \subset U$, and thus, the vehicle is not allowed to move along R_x or R_y. Furthermore, we consider the tangent lines S_i between L_x and L_y (four line segments if the two circles do not intersect and two otherwise). From these line segments only one, namely S_1 can form, along with arcs from L_x and L_y, a path concatenation that connects x and y, and which is compatible with the forward only motion requirement. As demonstrated in Fig. 1 the line segment S_1 always exists regardless of the relative position of the circles L_x and L_y with each other and thus the system is completely controllable.

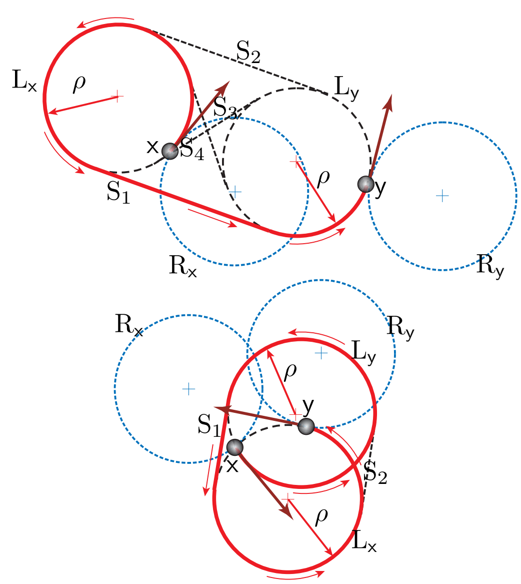


Fig. 1. The asymmetric, sinistral/dextral Dubins vehicle is completely controllable even if constrained to performing exclusively counterclockwise turns and/or following line segments.

To this end, we formulate the following minimum-time problem with fixed initial and terminal boundary conditions.

Problem 1: Given the system described by equations (1) and cost functional

$$J(u) = \int_0^{T_f} L(\mathsf{x}(t), u(t))\,\mathrm{d}t = \int_0^{T_f} 1\,\mathrm{d}t = T_f, \qquad (2)$$

where T_f is the free final time and $\mathsf{x} : [0, T_f] \mapsto \mathbb{R}^2 \times \mathbb{S}^1$ with $\mathsf{x} = (x, y, \theta)$, is the trajectory generated by the control $u \in \mathcal{U}$, determine the control input $u^* \in \mathcal{U}$ such that

1) The trajectory $\mathsf{x}^* : [0, T_f] \mapsto \mathbb{R}^2 \times \mathbb{S}^1$ generated by the control u^* satisfies the boundary conditions

$$\mathsf{x}^*(0) = (0, 0, 0), \quad \mathsf{x}^*(T_f) = (x_f, y_f, \theta_f). \qquad (3)$$

2) The control u^* minimizes the cost functional $J(u)$ given in (2).

To show the existence of an optimal solution to Problem 1 one can apply a special case of Filippov's general theorem on minimum-time problems with prescribed initial and terminal states [18], [19]. Using similar arguments as in [18] for the existence of optimal paths for the MD, the following proposition can be shown easily.

Proposition 1: The minimum-time Problem 1 from the origin $(0, 0, 0)$ to any terminal condition $\mathsf{x}_f \in \mathbb{R}^2 \times \mathbb{S}^1$ has always a solution.

III. Analysis of the Minimum-Time Problem

In this section we characterize the structure of the optimal paths using a similar approach as in [5], [13]. We shall not present a detailed analysis since the archetypes of the proofs are in most cases similar to the standard MD problem. To this end, consider the Hamiltonian $\mathcal{H} : \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R}^3 \mapsto \mathbb{R}$ of Problem 1 as follows

$$\mathcal{H}(\mathsf{x}, \mathsf{p}, u) = p_0 + p_1 \cos\theta + p_2 \sin\theta + p_3 u/\rho, \qquad (4)$$

where p_0 is some scalar and $\mathsf{p} : [0, T_f] \mapsto \mathbb{R}^3$, where $\mathsf{p} = (p_1, p_2, p_3)$, is an arbitrary continuous function. From PMP it follows that if x^* is a time-optimal trajectory generated by the control u^*, then there exists a scalar $p_0^* \in \{0, 1\}$ and an absolutely continuous function $\mathsf{p}^* : [0, T_f] \mapsto \mathbb{R}^3$, where $\mathsf{p}^* = (p_1^*, p_2^*, p_3^*)$, known as the costate, such that

1) $\|\mathsf{p}^*(t)\| + |p_0^*|$ never vanishes,
2) $\mathsf{p}^*(t)$ satisfies for almost all $t \in [0, T_f]$ the canonical equations $\dot{\mathsf{p}}^* = -\partial\mathcal{H}(\mathsf{x}^*, \mathsf{p}^*, u^*)/\partial\mathsf{x}$, equivalently,

$$\dot{p}_1^* = 0, \quad \dot{p}_2^* = 0, \quad \dot{p}_3^* = p_1^* \sin\theta^* - p_2^* \cos\theta^*, \qquad (5)$$

3) $\mathsf{p}^*(T_f)$ satisfies the transversality condition associated with the free final-time Problem 1

$$\mathcal{H}(\mathsf{x}^*(T_f), \mathsf{p}^*(T_f), u^*(T_f)) = 0. \qquad (6)$$

Because the Hamiltonian does not depend explicitly on time, it follows from (6) that $\mathcal{H}(\mathsf{x}^*(t), \mathsf{p}^*(t), u^*(t)) = 0$, for almost all $t \in [0, T_f]$, which furthermore implies, by virtue of (5), that

$$-p_0^* = p_1^*(0)\cos\theta^* + p_2^*(0)\sin\theta^* + p_3^* u^*/\rho. \qquad (7)$$

Furthermore, the optimal control u^* satisfies

$$\mathcal{H}(\mathsf{x}^*(t), \mathsf{p}^*(t), u^*(t)) = \min_{v\in[-\delta,1]} \mathcal{H}(\mathsf{x}^*(t), \mathsf{p}^*(t), v), \qquad (8)$$

for almost every $t \in [0, T_f]$. It follows that

$$u^*(t) = \begin{cases} +1 & \text{if} \quad p_3^*(t) < 0, \\ v \in [-1, 1] & \text{if} \quad p_3^*(t) = 0, \\ -\delta & \text{if} \quad p_3^*(t) > 0. \end{cases} \qquad (9)$$

Using similar arguments as in [5], [13] on can show

[1]The term sinistral (dextral) means "inclined to left (right)" [17].

Proposition 2: The optimal control u^* that solves Problem 1 belongs necessarily to U^*, where

$$\mathsf{U}^* \triangleq \{\{\mathsf{u}^\pm, 0, \mathsf{u}^\pm\}, \{\mathsf{u}^\pm, 0, \mathsf{u}^\mp\}, \{\mathsf{u}^\pm, \mathsf{u}^\mp, \mathsf{u}^\pm\}\}, \qquad (10)$$

where $\mathsf{u}^+ \triangleq 1$ and $\mathsf{u}^- \triangleq -\delta$.

Proposition 2 implies that the time-optimal paths of Problem 1 are concatenations of at most three segments, namely two bang arcs denoted as R and l for $u^* = -\delta$ and $u^* = 1$ respectively and a singular arc, denoted as s, that corresponds to $u^* = 0$. Note that a R and l segment correspond to a circular arc C of radius ϱ and ρ respectively whereas a s segment corresponds to a straight line segment S. It follows that the minimum-time paths have necessarily one of the following structures

1) $\mathrm{R}_\alpha \mathrm{s}_\ell \mathrm{R}_\gamma$, $\mathrm{l}_\alpha \mathrm{s}_\ell \mathrm{l}_\gamma$, $\mathrm{R}_\alpha \mathrm{s}_\ell \mathrm{l}_\gamma$ and $\mathrm{l}_\alpha \mathrm{s}_\ell \mathrm{R}_\gamma$ (paths of type $\mathrm{C}_\alpha \mathrm{S}_\ell \mathrm{C}_\gamma$),
2) or $\mathrm{l}_\alpha \mathrm{R}_\beta \mathrm{l}_\gamma$ and $\mathrm{R}_\alpha \mathrm{l}_\beta \mathrm{R}_\gamma$ (paths of type $\mathrm{C}_\alpha \mathrm{C}_\beta \mathrm{C}_\gamma$).

where the subscripts α, β, γ and ℓ denote the duration of motion along each path segment.

Remark 1 When $\sigma > 0$, where $\sigma \in \{\alpha, \beta, \gamma, \ell\}$ we say that the corresponding path segment is *non-trivial* and it is *trivial* otherwise.

Proposition 2 provides us with six families of paths that suffice to connect any arbitrary two configurations in $\mathbb{R}^2 \times \mathbb{S}^1$ in minimum time. However, no information regarding the switching times is yet available, something that renders the synthesis problem more complicated.

Let us consider an open interval $\mathcal{I} \subset [0, T_f]$ for which $p_3^*(t) \neq 0$ for all $t \in \mathcal{I}$. The restriction of the optimal control u^* on $\mathcal{I}$ is a piecewise constant function with at most two jumps and $u^*(t) \in \{-\delta, +1\}$ for all $t \in \mathcal{I}$. By virtue of (5) and (7) p_3^* satisfies

$$\ddot{p}_3^*(t) = -\left(\frac{u^*(t)}{\rho}\right)^2 p_3^*(t) - \frac{u^*(t) p_0^*}{\rho}, \qquad (11)$$

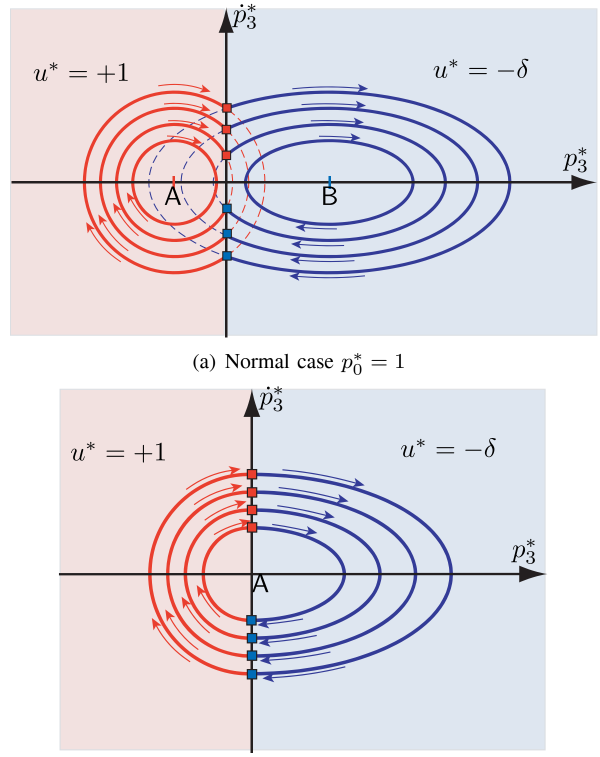
for all $t \in \mathcal{I}$. It follows readily that

$$\left(\frac{\rho \dot{p}_3^*(t)}{u^*(t)}\right)^2 + \left(p_3^*(t) + \frac{\rho p_0^*}{u^*(t)}\right)^2 = C_1^2 + \left(\frac{\rho C_2}{u^*(t)}\right)^2, \quad (12)$$

where C_1, C_2 are constants. The phase portrait of $(p_3^*, \dot{p}_3^*)$ is given in Fig. 2 and, in particular, in Fig. 2(a) for the normal case ($p_0^* = 1$) and in Fig. 2(b) for the abnormal case ($p_0^* = 0$). Contrary to the standard MD, the phase portrait of $(p_3^*, \dot{p}_3^*)$ is not symmetric w.r.t. the axis $p_3 = 0$ (compare for example, with [13]).

Proposition 3: A path $\mathrm{C}_\alpha \mathrm{C}_\beta \mathrm{C}_\gamma$ of non-trivial C segments, corresponds to an optimal trajectory for Problem 1 only if

1) $\beta \in (\pi\rho, 2\pi\rho)$, $\quad \beta \in (\pi\varrho, 2\pi\varrho)$



(a) Normal case $p_0^* = 1$

(b) Abnormal case $p_0^* = 0$

Fig. 2. Phase portrait of $(p_3^*, \dot{p}_3^*)$ when $u^* \neq 0$.

2) $\max\{\alpha, \gamma\} = \varepsilon(\delta, \beta)$, where

$$\varepsilon(\delta, \beta) = 2\pi\varrho + 2\varrho \ \mathrm{atan}\left(\delta \tan\frac{\beta}{2\rho}\right), \qquad (13)$$

$$\varepsilon(\delta, \beta) = 2\pi\rho + 2\rho \ \mathrm{atan}\left(\delta^{-1} \tan\frac{\beta}{2\varrho}\right), \qquad (14)$$

3) $\min\{\alpha, \gamma\} < (\beta - \pi)\varrho$, $\quad \min\{\alpha, \gamma\} < (\beta - \pi)\rho$,

for $\mathrm{R}_\alpha \mathrm{l}_\beta \mathrm{R}_\gamma$ and $\mathrm{l}_\alpha \mathrm{R}_\beta \mathrm{l}_\gamma$ paths respectively.

Proof: We investigate only the case when the path is of type $\mathrm{R}_\alpha \mathrm{l}_\beta \mathrm{R}_\gamma$. The case when the path is of type $\mathrm{l}_\alpha \mathrm{R}_\beta \mathrm{l}_\gamma$ can be treated similarly. First, we consider the abnormal case $p_0^* = 0$. It follows from Fig. 2(b) that a point in the $(p_3^*, \dot{p}_3^*)$ plane stays in the half plane $p_3 \leq 0$ for exactly time $\beta = \pi\rho$, which is the time required for a particle with coordinates $(p_3^*(t), \dot{p}_3^*(t))$ to travel half of the circumference of a circle centered at the origin with constant angular speed $\omega = 1/\rho$. However, using the same geometric argument as in Lemma 23 in [5] we can show that the resulting path is not optimal. Hence, all optimal extremals are normal.

We therefore let $p_0^* = 1$ in (11)-(12). In Fig. 3, we observe that the phase portrait of $(p_3^*, \rho\dot{p}_3^*/|u^*|)$ consists of clockwise circles centered at points A and B, with coordinates $(-\rho, 0)$ and $(\varrho, 0)$, and radii r and r_δ for $u^* = +1$ and $u^* = -\delta$ respectively; we denote these circles by $C(\mathsf{A}; r)$ and $C(\mathsf{B}; r_\delta)$ respectively. Note that a jump from $u^* = -\delta$ to $u^* = +1$ and vice versa occurs only if $C(\mathsf{B}, r_\delta)$ intersects $C(\mathsf{A}, r)$ along the axis $p_3^* = 0$, that is, $r \geq \rho$ and $r_d \geq \varrho$ and furthermore, $r_\delta = \sqrt{r^2 + \varrho^2 - \rho^2}$ as shown in Fig. 3.

From Fig. 3 it follows that the time β corresponds to the travel time from D to C along the circle $C(\mathsf{A};r)$ with constant angular speed $\omega = 1/\rho$. The times α and γ are upper bounded by the travel time from C to D along the circle $C(\mathsf{B};r)$ with constant angular speed $\omega = 1/\varrho$. We observe that $\pi\rho$ is a strict lower bound for β since $\rho,\ \varrho > 0$. Furthermore, $2\pi\rho$ and $2\pi\varrho$ is a strict upper bound for β and both α and γ respectively, given the vector field of the system (1) being $2\pi\rho$ and $2\pi\varrho$ periodic when $u^* = 1$ and $u^* = -\delta$ respectively.

Next, we improve the upper bound on α, γ. In particular, we observe in Fig. 3 that given β, where $\beta = 2(\pi - \widehat{\mathsf{CAO}})\rho$, either α or γ is maximized if and only if $\varepsilon \triangleq \max\{\alpha,\gamma\} \leq 2(\pi - \widehat{\mathsf{DBO}})\varrho$. Equivalently, α or γ is maximized if and only if the point $(p_3^*(\tau), \rho\dot{p}_3^*(\tau)/|u^*(\tau)|)$, for $\tau \in \{0, T_f\}$ coincides with C or D respectively. It follows from Fig. 3 that $\widehat{\mathsf{DBO}}$ and $\widehat{\mathsf{CAO}}$ belong to $[0,\pi/2)$. Furthermore, using simple geometric arguments along with $\delta \in (0,1]$, it follows that $\widehat{\mathsf{DBO}} = \mathrm{atan}\left(\delta \tan \widehat{\mathsf{CAO}}\right)$. Thus, $\varepsilon = 2\left(\pi - \mathrm{atan}\left(\delta \tan \widehat{\mathsf{CAO}}\right)\right)\varrho$, and $\beta = 2(\pi - \widehat{\mathsf{CAO}})\rho$. Equation (13) now follows readily.

Finally, the third condition of the Proposition is proved as in Lemma 3 of [8]. ■

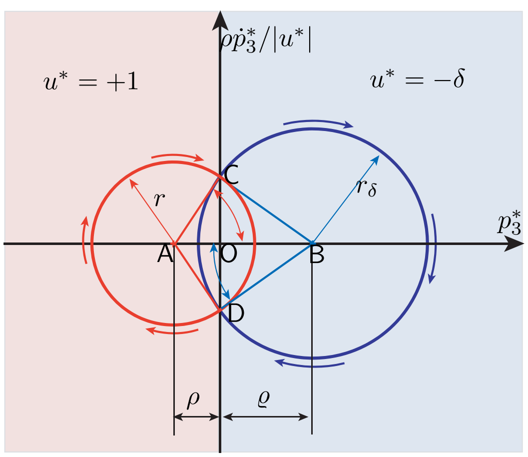


Fig. 3. Phase portrait $(p_3^*, \rho\dot{p}_3^*/u^*)$.

Proposition 4: An $\mathrm{R}_\alpha \mathrm{s}_\ell \mathrm{R}_\gamma$ path corresponds to a time-optimal trajectory of Problem 1 only if $\alpha + \gamma \leq 2\pi\varrho$.

Proof: See the proof of Lemma 5 of [8]. ■

Remark 2 Lemma 5 of [8] does not apply for lsl paths in our case. In particular, we observe in Fig. 4 that a vehicle starting from the origin reaches the terminal configuration $\mathsf{x}_f = (x_f, y_f, \theta_f)$ by traversing an lsl path along which the total change of orientation θ is strictly greater than 2π. The total elapsed time is the same as if the vehicle had traversed an RsR path of total change of orientation θ strictly less than 2π. That is, if the path RsR is time-optimal, then the lsl path is necessarily time-optimal as well. Thus, we conjecture that there exist time-optimal lsl paths along which the total change of orientation θ is strictly greater than 2π and necessarily upper bounded by $2\pi + \theta_f$. As we demonstrate in Section IV our conjecture is indeed correct.

Proposition 5: An $\mathrm{l}_\alpha \mathrm{s}_\ell \mathrm{l}_\gamma$ path corresponds to a time-optimal trajectory of Problem 1 only if $\alpha + \gamma \leq (2\pi + \theta_f)\rho$.

Finally, for Rsl and lsR paths, as in the standard MD, we simply take the most conservative bounds. In particular, we have the following proposition.

Proposition 6: An $\mathrm{l}_\alpha \mathrm{s}_\ell \mathrm{R}_\gamma$ and an $\mathrm{R}_\alpha \mathrm{s}_\ell \mathrm{l}_\gamma$ path corresponds to a time-optimal trajectory of Problem 1 only if $\max\{\alpha, \delta\gamma\} < 2\pi\rho$ and $\max\{\delta\alpha, \gamma\} < 2\pi\rho$, respectively.

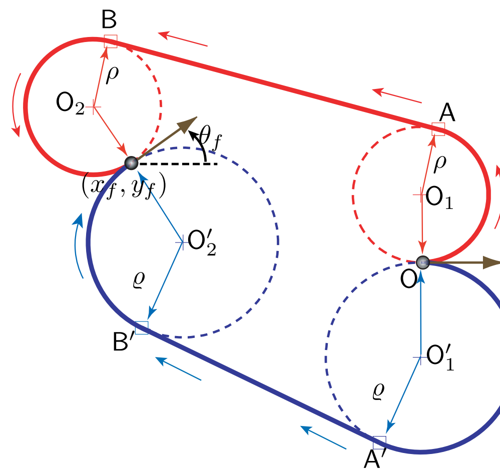
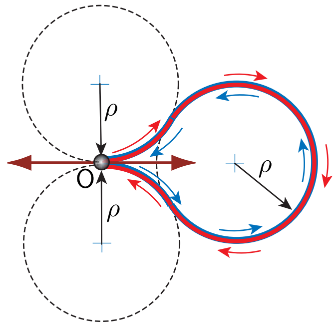


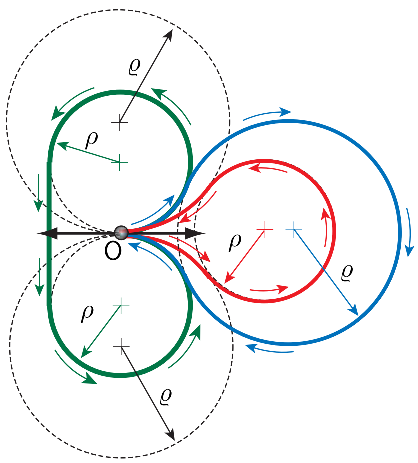
Fig. 4. Paths of the $\mathrm{l}_\alpha \mathrm{s}_\ell \mathrm{l}_\gamma$ type along which the total change of orientation θ, namely $2\pi + \theta_f = (\alpha + \gamma)/\rho$, is larger than 2π can be still optimal contrary to the standard Markov-Dubins problem.

IV. Time-Optimal Synthesis

The synthesis problem deals with the complete characterization of the optimal control/trajectory pairs when the vehicle starts from $(0,0,0)$ at time $t = 0$ and reaches any fixed terminal configuration (x_f, y_f, θ_f) at minimum time $t = T_f$. Following the approach employed by Bui *et al* [7], [8], given θ_f we characterize the optimal trajectories for all $(x_f, y_f) \in \mathbb{R}^2$; the plane $\theta = \theta_f$ in $\mathbb{R}^2 \times \mathbb{S}^1$ is known as the P_θ plane. In particular, we partition the plane P_θ in a number of domains, such that any terminal configuration in the interior of any of these domains can be reached in minimum time in a unique fashion. In our case, there exist at most six domains (not necessarily connected), one for each path type; the number of domains depends on the ratio $\delta^{-1} = \varrho/\rho$. Figure 5 demonstrates why the synthesis of an optimal feedback controller for the asymmetric, sinistral/dextral Dubins vehicle is quite different than for the standard Dubins vehicle. In particular, we consider the problem of steering the vehicle from $(0,0,0)$ to $(0,0,\pi)$. For the standard Dubins vehicle the optimal path is either an $\mathrm{l}_\alpha \mathrm{R}_\beta \mathrm{l}_\gamma$ path or an $\mathrm{R}_\alpha \mathrm{l}_\beta \mathrm{R}_\gamma$ path, where $\alpha = \gamma = \pi\rho/3$ and $\beta = 5\pi\rho/3$, as shown in Fig. 5(a). It follows from Fig. 5(b) that for the asymmetric, sinistral/dextral Dubins vehicle the optimal path is an $\mathrm{R}_\alpha \mathrm{l}_\beta \mathrm{R}_\gamma$ path, where $\alpha = \gamma = \varrho\ \mathrm{acos}\left(1/(1+\delta)\right)$ and $\beta = \pi\rho + 2\delta\alpha$, and an $\mathrm{l}_\alpha \mathrm{s}_\ell \mathrm{l}_\gamma$ path, where $\alpha = \gamma = 3\pi\rho/2$ and $\ell = 2\rho$ for $\delta \geq \tilde{\delta}$ and $\delta \leq \tilde{\delta}$ respectively, where $\tilde{\delta}$ is the solution of $1/(1+\delta) + \cos((\pi - \delta)/(1+\delta)) = 0$. In particular, for the $\mathrm{l}_\alpha \mathrm{s}_\ell \mathrm{l}_\gamma$ path we have $\alpha + \gamma = 3\pi\rho$, a case that would not be optimal for the standard Dubins vehicle problem. An $\mathrm{l}_\alpha \mathrm{R}_\beta \mathrm{l}_\gamma$ path is never optimal for $\delta \in (0,1)$.



(a) Classical Markov-Dubins case.



(b) Asymmetric sinistral/dextral Markov-Dubins case.

Fig. 5. The shortest path for the steering problem from $(0,0,0)$ to $(0,0,\pi)$ for the classical and the asymmetric sinistral/dextral Dubins vehicle.

The synthesis problem is solved in two steps, as proposed in [7], [8]. First, we construct the reachable sets for each of the six optimal candidate control sequences $u_k \in \mathsf{U}^*$, where $k \in \{1,\ldots,6\}$. In particular, for each $u_k \in \mathsf{U}^*$, and given the total time of motion $t^k \in [0,\infty)$ we integrate equations (1) from 0 to $t \in [0, t^k]$ with $(x(0), y(0), \theta(0)) = (0,0,0)$; we denote the solution as $\varphi^k : [0,\infty) \mapsto \mathbb{R}^2 \times \mathbb{S}^1$, where $\varphi^k(t) = (x^k(t), y^k(t), \theta^k(t))$. By virtue of Proposition 2, for each $k \in \{1,\ldots,6\}$ the solution φ^k depends on two path parameters, namely α and $\hat{\beta}$, where $\hat{\beta} \in \{\beta, \ell\}$ for $\mathrm{C}_\alpha \mathrm{C}_\beta \mathrm{C}_\gamma$ and $\mathrm{C}_\alpha \mathrm{S}_\ell \mathrm{C}_\gamma$ paths respectively; we write $\varphi^k(t;\alpha,\hat{\beta})$. Note that given the total time t^k, the time of motion along the third segment γ is uniquely defined by $\gamma = t^k - \alpha - \hat{\beta}$, with $t^k \geq \alpha + \hat{\beta}$, for all types of admissible paths. Using Propositions 3-6 we can readily obtain for each k the intervals $\mathcal{I}_\alpha^k$ and $\mathcal{I}_{\hat{\beta}}^k$ on which α and $\hat{\beta}$ belong to. To this end, we define the projection Π_θ from $\mathbb{R}^2 \times \mathbb{S}^1$ to P_θ with

$$\Pi_\theta\left((x^k, y^k, \theta^k)\right) \triangleq \begin{cases} (x^k, y^k), & \text{if } \theta^k = \theta, \\ \emptyset, & \text{otherwise} \end{cases}. \qquad (15)$$

The reachable set for the control sequence u_k is thus given by

$$\mathfrak{R}_{k,\theta} \triangleq \bigcup_{\substack{\alpha \in \mathcal{I}_\alpha^k, \hat{\beta} \in \mathcal{I}_{\hat{\beta}}^k \\ t^k \geq \alpha + \hat{\beta}}} \Pi_\theta\left(\varphi^k(t^k;\alpha,\hat{\beta})\right) \subset P_\theta. \qquad (16)$$

Given a point (x,y) in P_θ with $(x,y) \in \mathfrak{R}_{k,\theta}$, where $k \in \mathcal{K}$, where $\mathcal{K} \neq \emptyset$ and $\mathcal{K} \subseteq \{1,\ldots,6\}$, then $u_\kappa \in \mathsf{U}^*$ for $\kappa \in \mathcal{K}$ is a time-optimal control sequence if and only if the time t^κ for which $\Pi_\theta\left(\varphi^\kappa(t^\kappa;\alpha,\hat{\beta})\right) = (x,y)$ satisfies $t^\kappa = \min_{k\in\mathcal{K}} t^k$; we write $t^\kappa = T_f(x,y;\theta)$. Repeating the process for each $(x,y) \in P_\theta$ we construct the time-optimal partition of P_θ, that is, we divide P_θ into six domains, $\mathfrak{R}_{k,\theta}^*$ with $k \in \{1,\ldots,6\}$, not necessarily connected, such that any terminal configuration that lies in $\mathfrak{R}_{k,\theta}^*$ can be reached in minimum time by application of the optimal control sequence $u^* = u_k \in \mathsf{U}^*$. Furthermore, the terminal configurations that correspond to nonempty intersections of the boundaries of two or more domains $\mathfrak{R}_{k,\theta}^*$ can be reached in minimum time with the application of more than one of the six control sequences; we denote the union of all these nonempty intersections as $\partial\mathfrak{R}_\theta^*$. Finally, we shall write from now on, $\mathfrak{R}_{\mathrm{lsR}}^*$ to denote the domain $\mathfrak{R}_{k,\theta}^*$ for $k \in \{1,\ldots,6\}$ such that $u_k \in \mathsf{U}^*$ equals $\{+1, 0, -\delta\}$.

The partition of P_θ, for $\theta = \pi/3$, for different values of the ratio $\delta^{-1} = \varrho/\rho$, is given in Fig. 6. We observe that as the ratio ϱ/ρ increases the domains $\mathfrak{R}_{\mathrm{lsl}}^*$ $\mathfrak{R}_{\mathrm{Rsl}}^*$ and $\mathfrak{R}_{\mathrm{lsR}}^*$, primarily, and the domain $\mathfrak{R}_{\mathrm{RlR}}^*$, secondary, expand against the domain $\mathfrak{R}_{\mathrm{RsR}}^*$ as well as the disconnected components of $\mathfrak{R}_{\mathrm{lsR}}^*$ and $\mathfrak{R}_{\mathrm{Rsl}}^*$ that are close to the origin. We observe, in particular, that for $\varrho/\rho = 1.8$ (Fig 6(c)) the partition of P_θ consists of five domains since the domain $\mathfrak{R}_{\mathrm{lRl}}^*$ is reduced to the empty set. Similarly, for $\varrho/\rho = 2$ (Fig 6(d)) only four domains are non-empty since $\mathfrak{R}_{\mathrm{RsR}}^* = \mathfrak{R}_{\mathrm{lRl}}^* = \emptyset$. As observed in Fig 6(a)-6(d) the boundaries of each domain change significantly as the ratio ϱ/ρ varies.

V. Conclusion

In this article we have proposed and solved a generalization of the MD regarding the construction of time-optimal trajectories for an asymmetric Dubins' vehicle, which has a bias towards left (equivalently, right) turns, a situation that may be the result of an actuator failure. In the formulation of our problem the constraints over the curvature associated to clockwise and counterclockwise turns are not necessarily equal. Our analysis reveals that while the structure of the optimal control is qualitatively the same with the standard MD, the synthesis problem, that is, the determination of the optimal path for arbitrary prescribed boundary conditions, is, nonetheless, significantly different.

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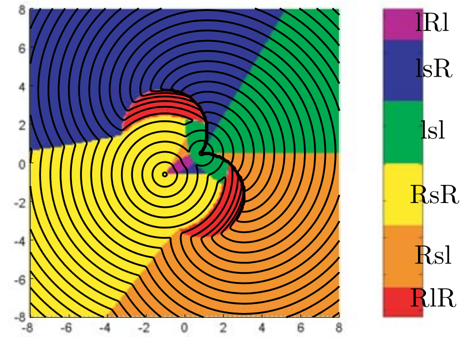
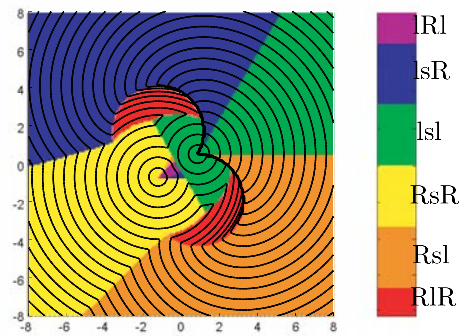
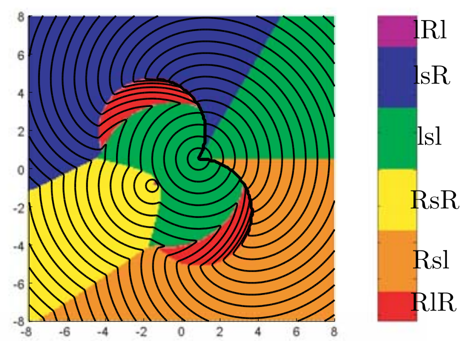


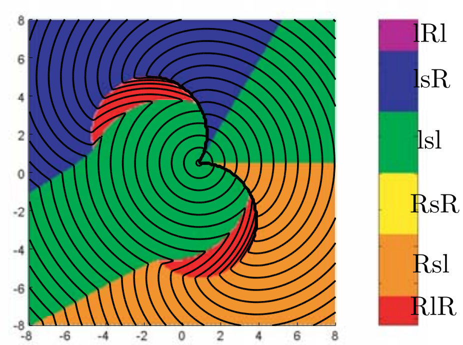
(a) $\varrho/\rho = 1.2$



(b) $\varrho/\rho = 1.4$



(c) $\varrho/\rho = 1.8$



(d) $\varrho/\rho = 2.0$

Fig. 6. Partition of $P_{\pi/3}$ and contours of $T_f = T_f(x, y)$ for different values of the ratio $\delta^{-1} = \varrho/\rho$.