A Solution of the Time-Optimal Hamilton-Jacobi-Bellman Equation on the Interval Using Wavelets

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***Abstract*— Wavelet basis functions allow efficient representation of functions with isolated singularities owing to their nice localization properties in both space/time and frequency domains. In this paper we propose a Wavelet-Extension Algorithm (WEA) for solving the Time-Optimal Hamilton-Jacobi-Bellman (TO-HJB) equation using the Daubechies wavelets and their antiderivatives as weighting and trial functions, respectively. Convergence of the proposed numerical scheme is shown. The advantage of the proposed technique in the paper is demonstrated by numerical examples.**

I. Introduction

Since Mallat developed the main algorithm for the wavelet transform [1], wavelets have played a critical role in the areas of signal processing, data and image compression, modeling of multi-scale phenomena, etc. The advantages of wavelets for solving partial differential equations (pde's) have also been noticed early on [2], [3], [4], [5], using both Galerkin and collocation methods [6], [7], [8]. Galerkin's method, in particular, is a classical method for solving pde's, and there is a vast literature on it [9], [10], [11], [12]. A survey of the literature prior to 1972 is given in [13]. A modern treatment is given in [14] for linear operators and in [15] for nonlinear operators. In these references, it has been shown that Galerkin's method converges as the number of basis functions increases to infinity if the linear operator is bounded, or symmetric, or positive and bounded from below. In [16] the authors have shown that the linear operator associated with the Generalized Hamilton-Jacobi-Bellman (GHJB) equation for an optimal regulator problem does not satisfy any of these requirements, and thus they proceeded in deriving alternative conditions that guarantee the convergence of Galerkin's scheme. However, it is known that the HJB equation associated with the time-optimal control problem may have non-smooth solutions and, hence, does not meet those conditions either.

In this paper, we use Galerkin's method to numerically solve the TO-HJB equation over the interval using wavelets. Wavelets seem particularly appropriate for solving the HJB pde for time-optimal problems, since they are especially suitable for approximating functions that are not smooth. In addition to the non-smoothness, the value function needs to satisfy a boundary condition in the *interior* of the domain. Therefore, standard numerical methods for solving pde's cannot be used for solving the HJB pde. In this paper, we follow the approach of Xu and Shann [5] and work with the antiderivatives of wavelets instead of the wavelets themselves. Working with the antiderivatives one automatically ensures that the interior boundary condition is satisfied, while at the same time one keeps some of the nice properties of wavelets (e.g., a multiresolution decomposition of the solution space). A proof of convergence of the proposed numerical scheme - the "Wavelet-Extension Algorithm" (WEA) is also provided.

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II. Preliminaries

The following notation will be used throughout the paper.

A. Nomenclature

1) Let $\Omega \subset \mathbb{R}$ be an open set. Then $C^0(\Omega)$ is the space of all functions which are continuous on Ω.
2) If $G \subset \mathbb{R}$, then $\mathrm{cl}(G)$ denotes the closure of G in $\mathbb{R}$.
3) By $\mathbf{x}_N$ we denote the vector $\mathbf{x}_N = [x_1, \ldots, x_N]^T \in \mathbb{R}^N$.
4) By $L^2(\Omega)$ we denote the space of square integrable functions on $\Omega \subset \mathbb{R}$, that is, $f \in L^2(\Omega)$ implies that $\int_\Omega |f(x)|^2 \mathrm{d}x < \infty$. The space $L^2(\Omega)$ is equipped with the inner product $\langle f, g \rangle_{L^2(\Omega)} := \int_\Omega f(x) g(x) \mathrm{d}x$, and the norm $\|f\|_{L^2(\Omega)} := (\int_\Omega |f(x)|^2 \,\mathrm{d}x)^{1/2}$.

Definition 1 (Sobolev Spaces): Let $\Omega \subset \mathbb{R}$ be an open set. Then the Sobolev space $W^{s,2}(\Omega)$ is defined by

$$W^{s,2}(\Omega) := \left\{ f \in L^2(\Omega) : \ \frac{\mathrm{d}^\alpha f}{\mathrm{d}x^\alpha} \in L^2(\Omega), \ \forall\ 0 \le \alpha \le s \right\}$$

with norm $\|f\|_{W^{s,2}(\Omega)} := \sum_{\alpha=0}^{s} \left\| \frac{\mathrm{d}^\alpha f}{\mathrm{d}x^\alpha} \right\|_{L^2(\Omega)}$.
The semi-norm of $f \in W^{s,2}(\Omega)$ is given by $|f|_{W^{s,2}(\Omega)} := \left\| \frac{\mathrm{d}^s f}{\mathrm{d}x^s} \right\|_{L^2(\Omega)}$. For simplicity, we will denote the norm $\|\cdot\|_{W^{s,2}(\Omega)}$ by $\|\cdot\|_{s,2,\Omega}$ and the semi-norm $|\cdot|_{W^{s,2}(\Omega)}$ by $|\cdot|_{s,2,\Omega}$.

In the following, we assume that Ω is an open subset of $\mathbb{R}$ with the origin in its interior. The space of interest in this paper is the Sobolev space $W^{1,2}(\Omega)$. In addition, we will deal with functions in $W^{1,2}(\Omega)$ which are zero at the origin, that is, $W_0^{1,2}(\Omega) := \{f \in W^{1,2}(\Omega) : f(0) = 0\}$. It can be readily shown that $W_0^{1,2}(\Omega)$ is a subspace of $W^{1,2}(\Omega)$. It is a standard result from Sobolev space theory [17] that $W^{1,2}(\Omega)$ is the completion of $C^1(\Omega)$ with respect to the norm $\|\cdot\|_{1,2,\Omega}$.

B. Wavelet Theory Overview

In this section, we give a brief introduction to wavelets. For more details the interested reader is referred to [18], [19].

The classical wavelets are defined on $\mathbb{R}$ and form two-parameter families of basis functions, which induce a multi-resolution decomposition of $L^2(\mathbb{R})$. This is the main property that makes wavelets attractive in applications. Specifically, wavelets induce the following nested sequence of subspaces

$$\mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \cdots \subset \mathcal{V}_j \subset \mathcal{V}_{j+1} \subset \cdots \subset L^2(\mathbb{R})$$

such that $\bigcup_{j=0}^{\infty} \mathcal{V}_j$ is dense in $L^2(\mathbb{R})$, i.e., $\mathrm{cl}\big(\bigcup_{j=0}^{\infty} \mathcal{V}_j\big) = L^2(\mathbb{R})$. The space $L^2(\mathbb{R})$ can then be decomposed as

$$L^2(\mathbb{R}) = \mathcal{V}_0 \bigoplus_{j=0}^{+\infty} \mathcal{W}_j = \bigoplus_{j=-1}^{+\infty} \mathcal{W}_j = \lim_{j\to\infty} \mathcal{V}_j$$

where the space $\mathcal{W}_j$ is the orthogonal complement of $\mathcal{V}_j$ in the larger subspace $\mathcal{V}_{j+1}$, that is, $\mathcal{V}_{j+1} = \mathcal{V}_j \oplus \mathcal{W}_j$. Moreover, $\mathcal{W}_j = \mathrm{cl}(\mathrm{span}\{\sqrt{2^j}\psi(2^j x - k)\}_{k\in\mathbb{Z}, j\geq 0})$ and $\mathcal{V}_j = \mathrm{cl}(\mathrm{span}\{\sqrt{2^j}\phi(2^j x - k)\}_{k\in\mathbb{Z}, j\geq 0})$, where, ψ is the *mother wavelet* and ϕ is the *scaling function*.

For notational convenience we define the two-parameter family of functions

$$\psi_{j,k}(x) := \begin{cases} \phi(x-k), & \text{for } j=-1, \\ \sqrt{2^j}\psi(2^j x - k), & \text{for } j \geq 0, \end{cases} \tag{1}$$

where $k \in \mathbb{Z}$.

C. Wavelets as frames for $L^2(\Omega)$

Definition 2 ([5]): Let $\{\varphi_n\}_{n=1}^{\infty}$ be a subset of a Banach Space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$. Let $\mathrm{span}\{\varphi_n\}$ be the set of all elements of the form $\sum \alpha_n \varphi_n,\ (\alpha_n \in \mathbb{R})$ which converge (strongly) in $\mathcal{B}$. Then $\{\varphi_n\}_{n=1}^{\infty}$ is said to be a *frame* of $\mathcal{B}$ if $\mathrm{span}\{\varphi_n\} = \mathcal{B}$. □

In the following we use the notation $\Omega = (-R, +R)$ and $\Omega_+ = (0, R)$. The following theorem extends the results of [5] in order to introduce a frame for $L^2(\Omega)$.

Theorem 1: Let $\mathrm{supp}\,\psi_{-1,0} = \mathrm{supp}\,\psi_{0,0} = [0, R]$. Then the set $\{\psi_{j,k}|_\Omega : j \geq -1, k \in \mathcal{I}_j\}$ where

$$\mathcal{I}_j = \left\{ k \in \mathbb{Z} \;\middle|\; \begin{array}{ll} 1-2R \leq k \leq R-1, & j=-1 \\ k_{\min} \leq k \leq k_{\max}, & j \geq 0 \end{array} \right\}$$

$k_{\min} = 1 - (2^j + 1)R$, $k_{\max} = 2^j R - 1$, forms a frame for $L^2(\Omega)$. □

Proof: The proof is similar to the one in [5], where only the case Ω_+ was treated. It hinges on the obvious fact that all wavelets such that $\mathrm{supp}\,\psi_{j,k} \cap \Omega \neq \emptyset$ will be sufficient for a frame. The index set $\mathcal{I}_j$ for $j \geq -1$ selects all these wavelets. □

The following is immediate from Theorem 1.

Corollary 1: Let the space $\mathcal{V}_J(\Omega) = \mathrm{span}\{\psi_{j,k}|_\Omega : -1 \leq j < J,\ k \in \mathcal{I}_j\}$. Then $\mathcal{V}_J(\Omega) \subset \mathcal{V}_{J+1}(\Omega)$ and $\bigcup_{J=0}^{\infty} \mathcal{V}_J(\Omega)$ is dense in $L^2(\Omega)$. □

III. Antiderivatives as frames for $W_0^{1,2}(\Omega)$

In this section, we construct a frame for $W_0^{1,2}(\Omega)$ using the antiderivatives of the wavelets. First we state the following theorem, which will be used in the sequel.

Theorem 2: The semi-norm $|\cdot|_{1,2,\Omega}$ is equivalent to the norm $\|\cdot\|_{1,2,\Omega}$ in $W_0^{1,2}(\Omega)$. □

Proof: For any $v \in W_0^{1,2}(\Omega)$ we have, trivially, that $v(x) = \int_0^x \frac{\mathrm{d}}{\mathrm{d}t} v(t)\mathrm{d}t$. Note that, for $x \geq 0$,

$$\begin{aligned} |v(x)| &= \left| \int_0^x \frac{\mathrm{d}}{\mathrm{d}t} v(t)\mathrm{d}t \right| \leq \int_0^x \left| \frac{\mathrm{d}}{\mathrm{d}t} v(t) \right| \mathrm{d}t \\ &\leq \left(\int_0^x \left| \frac{\mathrm{d}}{\mathrm{d}t} v(t) \right|^2 \mathrm{d}t \right)^{\frac{1}{2}} x^{\frac{1}{2}} \end{aligned}$$

using the Cauchy-Schwarz inequality. Similarly, for $x \leq 0$,

$$|v(x)| \leq \left(\int_x^0 \left| \frac{\mathrm{d}}{\mathrm{d}t} v(t) \right|^2 \mathrm{d}t \right)^{\frac{1}{2}} (-x)^{\frac{1}{2}}.$$

Moreover, for $x \in \Omega_+$, $\int_0^x \left|\frac{\mathrm{d}}{\mathrm{d}t}v(t)\right|^2 \mathrm{d}t \leq \int_0^R \left|\frac{\mathrm{d}}{\mathrm{d}t}v(t)\right|^2 \mathrm{d}t$ and for $x \in \Omega \setminus \Omega_+$, $\int_x^0 \left|\frac{\mathrm{d}}{\mathrm{d}t}v(t)\right|^2 \mathrm{d}t \leq \int_{-R}^0 \left|\frac{\mathrm{d}}{\mathrm{d}t}v(t)\right|^2 \mathrm{d}t$. It now follows that

$$\begin{aligned} &\|v\|_{0,2,\Omega}^2 = \int_{-R}^0 |v(x)|^2 \,\mathrm{d}x + \int_0^R |v(x)|^2 \,\mathrm{d}x \\ &\leq -\int_{-R}^0 x \int_x^0 \left| \frac{\mathrm{d}}{\mathrm{d}t} v(t) \right|^2 \mathrm{d}t\mathrm{d}x + \int_0^R x \int_0^x \left| \frac{\mathrm{d}}{\mathrm{d}t} v(t) \right|^2 \mathrm{d}t\mathrm{d}x \\ &\leq \int_0^{-R} x\mathrm{d}x \int_{-R}^0 \left| \frac{\mathrm{d}}{\mathrm{d}t} v(t) \right|^2 \mathrm{d}t + \int_0^R x\mathrm{d}x \int_0^R \left| \frac{\mathrm{d}}{\mathrm{d}t} v(t) \right|^2 \mathrm{d}t \\ &= \frac{R^2}{2} \int_{-R}^R \left| \frac{\mathrm{d}}{\mathrm{d}t} v(t) \right|^2 \mathrm{d}t = \frac{R^2}{2} |v|_{1,2,\Omega}^2. \end{aligned}$$

Therefore,

$$|v|_{1,2,\Omega}^2 \leq \|v\|_{0,2,\Omega}^2 + |v|_{1,2,\Omega}^2 \leq \left(1 + \frac{R^2}{2}\right) |v|_{1,2,\Omega}^2$$

which implies

$$|v|_{1,2,\Omega}^2 \leq \|v\|_{1,2,\Omega}^2 \leq \left(1 + \frac{R^2}{2}\right) |v|_{1,2,\Omega}^2.$$

□

Lemma 1: Let $\{\psi_{j,k}|_\Omega : j \geq -1,\ k \in \mathcal{I}_j\}$, be a frame for $L^2(\Omega)$. Then, for $j \geq -1$ and $k \in \mathcal{I}_j$, the set

$$\left\{ \Psi_{j,k}(x) = \int_0^x \psi_{j,k}(s)\,\mathrm{d}s,\ \forall x \in \Omega \right\} \subset W_0^{1,2}(\Omega)$$

forms a frame for $W_0^{1,2}(\Omega)$. □

Proof: Clearly, $\Psi_{j,k} \in W_0^{1,2}(\Omega)$. Given any $v \in W_0^{1,2}(\Omega)$, let $w = v'$. Since $v \in W_0^{1,2}(\Omega)$ and $w = v'$, it follows that $w \in L^2(\Omega)$. Hence,

$$\left\| w - \sum_{j=-1}^{\infty} \sum_{k\in\mathcal{I}_j} \alpha_{j,k} \psi_{j,k} \right\|_{0,2,\Omega} = 0, \tag{2}$$

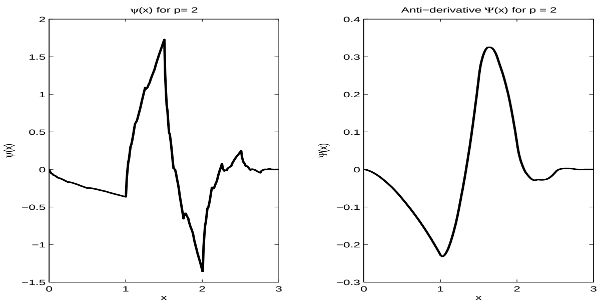

Fig. 1. Daubechies wavelet ψ $(p = 2)$ and its antiderivative Ψ.

equivalently,

$$\left| v - \sum_{j=-1}^{\infty} \sum_{k \in \mathcal{I}_j} \alpha_{j,k} \Psi_{j,k} \right|_{1,2,\Omega} = 0$$

From Theorem 2, we know that the semi-norm $|\cdot|_{1,2,\Omega}$ is equivalent to the norm $\|\cdot\|_{1,2,\Omega}$ in $W_0^{1,2}(\Omega)$, therefore,

$$\left\| v - \sum_{j=-1}^{\infty} \sum_{k \in \mathcal{I}_j} \alpha_{j,k} \Psi_{j,k} \right\|_{1,2,\Omega} = 0.$$

It follows that the set $\{\Psi_{j,k}\}$, for $j \geq -1$, $k \in \mathcal{I}_j$, forms a frame for $W_0^{1,2}(\Omega)$. □

Figure 1 shows the Daubechies wavelet of order $p = 2$ along with its antiderivative.

IV. Time-Optimal Control

Consider an optimal control problem whose dynamics are given by the nonlinear differential equation

$$\dot{x} = f[x(t)] + g[x(t)]u, \tag{3}$$

with the boundary conditions $x(t_0) = x_0,\ x(t_f) = 0$ where, $x(t) \in \mathbb{R},\ f : \mathbb{R} \to \mathbb{R},\ g : \mathbb{R} \to \mathbb{R}$ and t_f is free. In addition, we assume that

A1: $f(|x|) \neq g(|x|)$,
A2: $f(-|x|) \neq -g(-|x|)$,
A3: $g(x) = 0 \Rightarrow x = 0$.

It is also assumed that the control input is constrained by $|u| \leq 1$. Without loss of generality we can assume that $g(x) \geq 0$ for all $x \in \mathbb{R}$, since we can always write $g(x)u = |g(x)| \operatorname{sgn}[g(x)]\, u = |g(x)|v$, for a new control $v := \operatorname{sgn}[g(x)]\, u$, which satisfies $|v| \leq 1$. We will deal with time-optimal control problems. Therefore, the cost function to be minimized is

$$\min_{|u| \leq 1} \int_{t_0}^{t_f} \mathrm{d}t \tag{4}$$

The optimal control is computed from the Hamilton-Jacobi-Bellman equation

$$\frac{\partial V}{\partial t} + \min_{|u| \leq 1} H\left(x, \frac{\partial V}{\partial x}, u\right) = 0 \tag{5}$$

with boundary condition $V(x(t_f), t_f) = 0$, which for the time-optimal problem reduces to

$$1 + \min_{|u| \leq 1} \left(\frac{\partial V}{\partial x} \big(f(x) + g(x)u \big) \right) = 1 + f(x) \frac{\partial V}{\partial x}(x) - g(x) \left| \frac{\partial V}{\partial x}(x) \right| = 0 \tag{6}$$

subject to the boundary condition $V(0) = 0$. Once V is known, the optimal control is given in a feedback form (since $g(x) \geq 0$ for all $x \in \mathbb{R}$) as follows

$$u = -\operatorname{sgn}\left(\frac{\partial V}{\partial x} \right). \tag{7}$$

In the following section we propose an algorithm for solving the HJB equation (6).

V. Wavelet-Extension Algorithm (WEA)

Let $V_1(x)$ denote the value function satisfying the HJB equation (6) for all $x \in \Omega_+$. Using the result of [20], one obtains from (7) that $\frac{\partial V_1}{\partial x}(x) > 0,\ \forall\, x \in \Omega_+$. Therefore, the HJB equation (6) for $x \in \Omega_+$ can be written as

$$1 + f(x) \frac{\partial V_1}{\partial x}(x) - g(x) \frac{\partial V_1}{\partial x}(x) = 0, \quad x \in \Omega_+ \tag{8}$$

with boundary condition $V_1(0) = 0$, or

$$1 + f(|x|) \frac{\partial V_1}{\partial x}(|x|) - g(|x|) \frac{\partial V_1}{\partial x}(|x|) = 0, \quad x \in \Omega_+ \tag{9}$$

with boundary condition $V_1(0) = 0$.

We now extend V_1 from $W_0^{1,2}(\Omega_+) \to W_0^{1,2}(\Omega)$ as follows

$$\bar{V}_1(x) := \begin{cases} V_1(x), & x \in \Omega_+, \\ -V_1(-x), & x \in \Omega \setminus \Omega_+. \end{cases} \tag{10}$$

Note that $\bar{V}_1$ is continuous and $\bar{V}_1(0) = 0$. Moreover,

$$\frac{\partial \bar{V}_1}{\partial x}(x) = \frac{\partial V_1}{\partial x}(x), \text{ or } \frac{\partial \bar{V}_1}{\partial x}(x) = \frac{\partial V_1}{\partial x}(|x|), \quad \forall\, x \in \Omega_+.$$

Similarly, it follows that

$$\frac{\partial \bar{V}_1}{\partial x}(x) = \frac{\partial V_1}{\partial x}(-x), \text{ or } \frac{\partial \bar{V}_1}{\partial x}(x) = \frac{\partial V_1}{\partial x}(|x|), \ \forall\, x \in \Omega \backslash \Omega_+.$$

Therefore, for all $x \in \Omega$, (9) can be written as

$$1 + \frac{\partial \bar{V}_1}{\partial x}(x) \Big(f(|x|) - g(|x|) \Big) = 0, \quad \forall\, x \in \Omega, \tag{11}$$

with boundary condition $\bar{V}_1(0) = 0$.

For simplicity, let us denote $r_1(x) := f(|x|) - g(|x|)$. We can then rewrite (11) as

$$\mathrm{HJB}_{\mathrm{mod}}(\bar{V}_1, r_1) = 1 + \frac{\partial \bar{V}_1}{\partial x} r_1(x) = 0, \quad x \in \Omega. \tag{12}$$

Let now V_2 denote the value function which satisfies the HJB equation for $x \in \Omega \setminus \Omega_+$. Therefore, from (7) we have that $\frac{\partial V_2}{\partial x}(x) < 0,\ \forall\, x \in \Omega \setminus \Omega_+$. Hence, the HJB equation (6) can be written as

$$1 + f(x) \frac{\partial V_2}{\partial x}(x) + g(x) \frac{\partial V_2}{\partial x}(x) = 0, \quad x \in \Omega \setminus \Omega_+,$$

or

$$1 + f(-|x|)\frac{\partial V_2}{\partial x}(-|x|) + g(-|x|)\frac{\partial V_2}{\partial x}(-|x|) = 0, \quad (13)$$

for all $x \in \Omega \setminus \Omega_+$ and with boundary condition $V_2(0) = 0$.

We now extend V_2 as follows:

$$\bar{V}_2(x) = \begin{cases} V_2(x), & x \in \Omega \setminus \Omega_+, \\ -V_2(-x), & x \in \Omega_+. \end{cases} \quad (14)$$

Hence,

$$\frac{\partial \bar{V}_2}{\partial x}(x) = \frac{\partial V_2}{\partial x}(x), \text{ or } \frac{\partial \bar{V}_2}{\partial x}(x) = \frac{\partial V_2}{\partial x}(-|x|), \; \forall\, x \in \Omega \backslash \Omega_+.$$

Similarly,

$$\frac{\partial \bar{V}_2}{\partial x}(x) = \frac{\partial V_2}{\partial x}(-x), \text{ or } \frac{\partial \bar{V}_2}{\partial x}(x) = \frac{\partial V_2}{\partial x}(-|x|) \; \forall\, x \in \Omega_+.$$

Therefore, for all $x \in \Omega$, (13) can be written as

$$1 + \frac{\partial \bar{V}_2}{\partial x}(x) \cdot (f(-|x|) + g(-|x|)) = 0, \quad \forall\, x \in \Omega \quad (15)$$

with boundary condition $\bar{V}_2(0) = 0$.

As before, for simplicity, we will denote $r_2(x) := f(-|x|) + g(-|x|)$ and rewrite (15) as follows

$$\mathrm{HJB}_{\mathrm{mod}}(\bar{V}_2, r_2) = 1 + \frac{\partial \bar{V}_2}{\partial x} r_2(x) = 0, \quad x \in \Omega. \quad (16)$$

Assume that the solutions to the equations (12) and (16) exist and are unique. We seek approximate solutions, V_{WEA_1} and V_{WEA_2}, to the equations (12) and (16) respectively, using the method of weighted residuals [13].

To this end, we assume that the solution V_{WEA_i}, $(i = 1, 2)$, is of the form

$$V_{\mathrm{WEA}_i} = \sum_{j=-1}^{J-1} \sum_{k \in \mathcal{I}_j} c^i_{j,k} \Psi_{j,k}, \quad i = 1, 2 \quad (17)$$

where, $\{\Psi_{j,k}\}$ is as in Lemma 1 and $c^i_{j,k}$ are the corresponding coefficients. Substituting expression (17) into equation $\mathrm{HJB}_{\mathrm{mod}}(\bar{V}_i, r_i) = 0$ results in an error

$$\mathrm{Err}_i := \mathrm{HJB}_{\mathrm{mod}}\left(\sum_{j=-1}^{J-1} \sum_{k \in \mathcal{I}_j} c^i_{j,k} \Psi_{j,k}, r_i\right), \quad i = 1, 2. \quad (18)$$

The unknown coefficients $c^i_{j,k}$ are determined by setting the projection of the error (18) on each element that spans the subspace $\mathcal{V}_J(\Omega)$ (i.e. $\{\psi_{j,k}|_\Omega\}$, where, $-1 \le j < J$ and $k \in \mathcal{I}_j$) to zero, namely,

$$\langle \mathrm{Err}_i, \psi_{j,k} \rangle_{L^2(\Omega)} = 0, \quad i = 1, 2. \quad (19)$$

Once we have found V_{WEA_1} and V_{WEA_2}, the solution of the HJB equation (6) can be computed for all $x \in \Omega$ (from (10) and (14)) as follows

$$V_{\mathrm{WEA}}(x) = \begin{cases} V_{\mathrm{WEA}_1}(x), & x \in \Omega_+, \\ V_{\mathrm{WEA}_2}(x), & x \in \Omega \setminus \Omega_+. \end{cases} \quad (20)$$

VI. Convergence

For the solution of the HJB equation to be defined in the classical sense, it must be differentiable over the whole domain of interest. Unfortunately, solutions to the HJB equation often have discontinuous derivatives even if all the problem data are smooth. The theory of viscosity solutions has been developed to interpret as solutions to the HJB equation value functions which are only continuous [21], [22]. There always exists a unique viscosity solution $V^\star \in C^0(\Omega)$ to the HJB equation (6).

Next, we show that the solution (V_{WEA}) obtained using the WEA algorithm converges to $V^\star$ as $J \to \infty$. To prove this claim, we first select a set of linearly independent functions from the constructed frame of Theorem 1. Specifically, let $k_i(R, j) : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ $(i = 1, 2)$ be such that the set $\{\psi_{j,k}|_\Omega\}$ where $\tilde{\mathcal{I}}_j = \{k \in \mathbb{Z} : k_1 \le k \le k_2\} \subset \mathcal{I}_j$ and for $-1 \le j < J$, $k \in \tilde{\mathcal{I}}_j$ is linearly independent over the interval Ω.

Lemma 2: Let $-1 \le j < J$ and $k \in \tilde{\mathcal{I}}_j$. Then the set $\{\Psi_{j,k}\}$ is linearly independent over Ω if and only if the set $\{\psi_{j,k}\}$ is linearly independent over Ω. □

Lemma 3: Let $-1 \le j < J$ and $k \in \tilde{\mathcal{I}}_j$. Define the matrix $\mathbf{A}_J(\mathbf{\Psi}_J, r_i, \mathbf{d\Psi}_J)$ as below. Then $\mathrm{rank}\{A_J(\mathbf{\Psi}_J, r_i, \mathbf{d\Psi}_J)\} = N$, where, $\mathbf{\Psi}_J = [\Psi_{-1,k_1(R,-1)}, \cdots, \Psi_{J-1,k_2(R,J-1)}]^T$, $\mathbf{d\Psi}_J = [\psi_{-1,k_1(R,-1)}, \cdots, \psi_{J-1,k_2(R,J-1)}]^T$, and N is the number of elements in the set $\{\psi_{j,k}\}$, given by $N = \sum_{j=-1}^{J-1} k_2 - k_1 + 1$. □

Proof: The proof is a direct consequence of Lemma 14 in [16]. □

To reduce the complexity of the expressions in the following two lemmas we reorder the functions $\psi_{j,k}$ and $\Psi_{j,k}$ (for $j = 1, \ldots, J-1$ and $k \in \mathcal{I}_j$) and rewrite them simply as ψ_n and Ψ_n, where, $n = 1, \ldots, M$ and $M = 3R - 1 + \sum_{j=0}^{J-1}\left((2^{j+1} + 1)R - 1\right)$, for any given J.

Lemma 4: Suppose the set $\{\psi_n\}_{n=1}^N$ for $N < M$ is linearly independent over Ω and the set $\{\psi_n\}_{n=1}^{N+1}$ is linearly dependent over Ω. Let $\mathbf{\Psi}_N = [\Psi_1, \Psi_2, \ldots, \Psi_N]^T$ and let, for $i = 1, 2$, $W_{\mathrm{WEA}_i} = (\mathbf{a}^i_N)^T \mathbf{\Psi}_N$ and $V_{\mathrm{WEA}_i} = (\mathbf{b}^i_{N+1})^T \mathbf{\Psi}_{N+1}$ satisfy the equations:

$$\langle \mathrm{HJB}_{\mathrm{mod}}(W_{\mathrm{WEA}_i}, r_i), \psi_n \rangle_{L^2(\Omega)} = 0, \quad n = 1, \ldots, N,$$

$$\langle \mathrm{HJB}_{\mathrm{mod}}(V_{\mathrm{WEA}_i}, r_i), \psi_n \rangle_{L^2(\Omega)} = 0, \quad n = 1, \ldots, N+1,$$

respectively, where, $\mathbf{a}^i_N = [a^i_1, a^i_2, \ldots, a^i_N]^T$ and $\mathbf{b}^i_{N+1} = [b^i_1, b^i_2, \ldots, b^i_{N+1}]^T$. Then $V_{\mathrm{WEA}_i} = W_{\mathrm{WEA}_i}$. □

Suppose now that the set $\{\psi_n\}_{n=1}^N$ is linearly independent over Ω and suppose these functions are orthonormalized to form the set $\{\tilde{\psi}_n\}_{n=1}^N$. Then there exist constants γ_{ij} such that $\tilde{\psi}_1 = \gamma_{11}\psi_1$, $\tilde{\psi}_2 = \gamma_{21}\psi_1 + \gamma_{22}\psi_2, \ldots, \tilde{\psi}_N = \gamma_{N1}\psi_1 + \cdots + \gamma_{NN}\psi_N$. Let $\mathbf{d}\tilde{\mathbf{\Psi}}_N = [\tilde{\psi}_1, \tilde{\psi}_2, \ldots, \tilde{\psi}_N]^T$ and $(\mathbf{\Upsilon}_N)_{ij} = \gamma_{ij}$. It follows that we can write $\mathbf{d}\tilde{\mathbf{\Psi}}_N = \mathbf{\Upsilon}_N \mathbf{d\Psi}_N$. Similarly, if the set $\{\Psi_n\}_{n=1}^N$ is orthonormalized to form the set $\{\tilde{\Psi}_n\}_{n=1}^N$, then there exists a lower triangular matrix Γ_N such that $\tilde{\mathbf{\Psi}}_N = \mathbf{\Gamma}_N \mathbf{\Psi}_N$. For $i = 1, 2$ and $n = 1, \ldots, N$,

$$\mathbf{A}_J(\mathbf{\Psi}_J, r_i, \mathbf{d\Psi}_J) = \begin{pmatrix} \langle \frac{\partial \Psi_{-1,k_1}}{\partial x} r_i, \psi_{-1,k_1} \rangle_{L^2(\Omega)} & \cdots & \langle \frac{\partial \Psi_{J-1,k_2}}{\partial x} r_i, \psi_{-1,k_1} \rangle_{L^2(\Omega)} \\ \vdots & \ddots & \vdots \\ \langle \frac{\partial \Psi_{-1,k_1}}{\partial x} r_i, \psi_{J-1,k_2} \rangle_{L^2(\Omega)} & \cdots & \langle \frac{\partial \Psi_{J-1,k_2}}{\partial x} r_i, \psi_{J-1,k_2} \rangle_{L^2(\Omega)} \end{pmatrix}, \quad i = 1, 2$$

Lemma 5: Let $V_{\mathrm{WEA}_i} = (\mathbf{c}_N^i)^T \mathbf{\Psi}_N$ be the solution to

$$\langle \mathrm{HJB}_{\mathrm{mod}}(V_{WEA_i}, r_i), \psi_n \rangle_{L^2(\Omega)} = 0, \quad n = 1, \ldots, N$$

and let $W_{\mathrm{WEA}_i} = (\tilde{\mathbf{c}}_N^i)^T \tilde{\mathbf{\Psi}}_N$ be the solution to

$$\langle \mathrm{HJB}_{\mathrm{mod}}(W_{\mathrm{WEA}_i}, r_i), \tilde{\psi}_n \rangle_{L^2(\Omega)} = 0, \quad n = 1, \ldots, N$$

Then $V_{\mathrm{WEA}_i} = W_{\mathrm{WEA}_i}$ for $i = 1, 2$. □

Theorem 3: Let $\tilde{\mathbf{c}}_J^i$ be found by solving $\mathbf{A}_J(\tilde{\mathbf{\Psi}}_J, r_i, \mathbf{d}\tilde{\mathbf{\Psi}}_J)\tilde{\mathbf{c}}_J^i = \tilde{\mathbf{b}}_J$, where, $\tilde{\mathbf{b}}_J = [\tilde{b}_{-1,k_1(R,-1)}, \cdots, \tilde{b}_{J-1,k_2(R,J-1)}]^T$ with $\tilde{b}_{j,k} = -\langle 1, \tilde{\psi}_{j,k} \rangle_{L^2(\Omega)}$, $\forall \; -1 \leq j < J$, $k \in \tilde{\mathcal{I}}_j$ and $\tilde{\mathbf{c}}_J^i = [\tilde{c}^i_{-1,k_1(R,-1)}, \cdots, \tilde{c}^i_{J-1,k_2(R,J-1)}]^T$. Then $\|\tilde{\mathbf{c}}_J^i\|_{\ell^2} < \infty$, $J \geq 0$ and $i = 1, 2$. □

Proof: The proof is a direct consequence of Lemma 20 in [16]. □

Corollary 2: Let $c^i_{j,k}$ $(-1 \leq j < J$ and $k \in \mathcal{I}_j)$ be the coefficients found by solving the set of linear equations (19). Let $\mathbf{c}_J^i = [c^i_{-1,1-2R}, \ldots, c^i_{J-1,k_{\max}}]^T$, where, $k_{\max} = 2^{J-1}R - 1$. Then $\|\mathbf{c}_J^i\|_{\ell^2} < \infty$, $J \geq 0$ and $i = 1, 2$. □

Theorem 4: Let $V^\star \in C^0(\Omega)$ be the unique viscosity solution and V_{WEA} be the approximate solution (using WEA) of the time-optimal HJB equation (6). Then $\|V^\star - V_{\mathrm{WEA}}\|_{1,2,\Omega} \to 0$ as $J \to \infty$. □

VII. Numerical Examples

In this section we give two numerical examples to demonstrate the proposed algorithm. The examples are purposely simple so that analytical solutions are available for comparison purposes.

Example 1: Consider the system (3) with $f(x) = \frac{-3x^2+2|x|-2}{2(2|x|+3)(3x^2+5)}$, $g(x) = \frac{3x^2+2|x|+8}{2(2|x|+3)(3x^2+5)}$. For this system, the optimal value function is

$$V^\star(x) = \begin{cases} x^2 + 3x, & x > 0, \\ -x^3 - 5x, & x < 0. \end{cases}$$

Using the WEA algorithm with Daubechies wavelets of order $p = 2$ (and scale $J = 4$) provides the results shown in Fig. 2.

Example 2: Consider the system (3) with $f(x) = \frac{|x|}{2|x|+1}$, $g(x) = \frac{1+|x|}{2|x|+1}$. For this system the optimal value function is

$$V^\star(x) = \begin{cases} x^2 + x, & x > 0, \\ -x, & x < 0. \end{cases}$$

Using the WEA algorithm with Daubechies wavelets of order $p = 2$ (and scale $J = 4$) provides the results shown in Fig. 3.

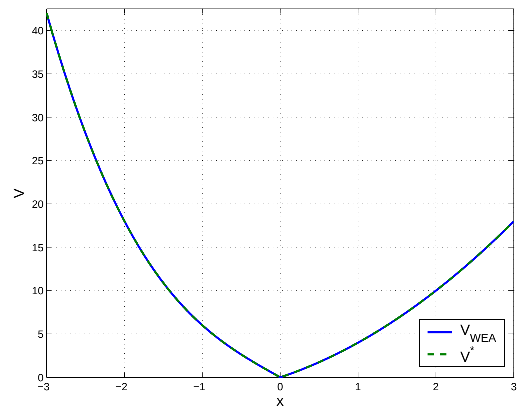

Fig. 2. Example 1: Value Function using Daubechies wavelets with $p = 2, J = 4$.

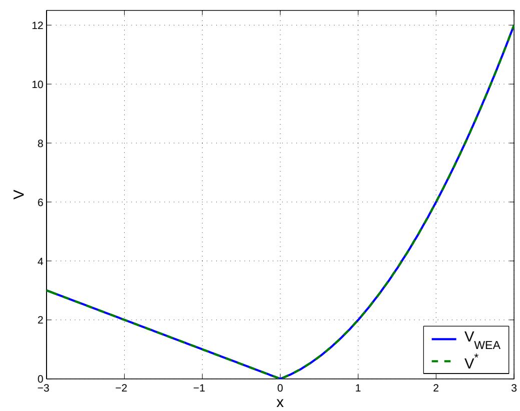

Fig. 3. Example 2: Value Function using Daubechies wavelets with $p = 2, J = 4$.

VIII. Comparison with Polynomial Expansions

We have also used polynomials as the underlying bases functions for solving the HJB equation for the previous two examples. The numerical results showed that the polynomials were able to approximate the solution for Example 1, although the minimum error achieved was only 0.4118. This error was achieved using polynomials of degree 70, after which increasing the polynomial degree did not give better results. Similar observations hold for Example 2. The minimum error ($E = 0.3911$) for this case was achieved using polynomials of degree 75, after which increasing the polynomial degree did not improve the error. It is reminded that using wavelets, 45 coefficients were used for Example 1, resulting in an error of 0.0554. For Example 2, 54 wavelet coefficients were used, resulting in an error of 0.034. Hence, using the proposed algorithm the approxima-

TABLE I
APPROXIMATION ERROR FOR EXAMPLE 1: CASE I VS. CASE II

	Ω		$\Omega_- \cup \Omega_+$	
p	no. of trial fcns	E	no. of trial fcns	E
2	30	0.2405	$18 + 18 = 36$	2.8466
4	74	0.2247	$46 + 46 = 92$	2.7845

TABLE II
APPROXIMATION ERROR FOR EXAMPLE 1: CASE I VS. CASE III

		Case I	Case III
p	J	E	E
2	2	0.2405	0.1735
4	2	0.2247	0.0522

tion error was reduced by roughly an order of magnitude while also using a smaller number of trial functions.

An alternative method for solving the time-optimal control problem is to initially solve the problem separately for the domains Ω_+ and Ω_- and then put the two solutions together. A frame for $W_0^{1,2}(\Omega_+)$ has been constructed in Xu and Shan [5]. A frame for $W_0^{1,2}(\Omega_-)$ can be constructed in a similar fashion. Specifically, it can be verified that the functions $\Psi_{j,k}(x) = \int_0^x \psi_{j,k}(s)\,\mathrm{d}s, \ \forall\ x \in (\Omega_-)$, where, $\psi_{j,k}$ is given by (1) with $-1 \leq j < J$ and $k \in \mathcal{I}_j^-$, where, $\mathcal{I}_j^-$ is given by

$$\mathcal{I}_j^- = \left\{ k \in \mathbb{Z} \;\middle|\; \begin{array}{ll} 1 - 2R \leq k \leq -1, & j = -1 \\ 1 - (2^j + 1)R \leq k \leq -1, & j \geq 0 \end{array} \right\}$$

form a frame for $W_0^1(\Omega_-)$. A comparison of this methods with the WEA shows the advantage of the latter. To confirm this claim, solutions using both these methods were obtained using the antiderivatives of the Daubechies wavelets of order $p = 2$ and $p = 4$ at scale $J = 2$ both as weighting and as trial functions for the value function. Table I shows the approximation error for Example 1 for both cases; in Case I Example 1 was solved for the whole domain Ω and in Case II Example 1 was solved for $\Omega_- \cup \Omega_+$. In the table E denotes the approximation error, calculated from $E = \|V^\star - V_{\mathrm{WEA}}\|_{0,2,\Omega}$. As seen from Table I the approximation error in Case I is significantly less than the approximation error of Case II, even though the total number of trial functions used in the first case is less than the number of trial functions used in the second case. This indicates the superiority of the WEA algorithm.

Now we further show the advantage of the use the wavelets as weighting functions, along with the use of their antiderivatives as trial functions (Case III). Table II shows that the approximation error for Case III is less when compared to that of Case I when the antiderivatives were used both as weighting functions as well as trial functions.

IX. CONCLUSIONS

In this paper, we have proposed the use of antiderivatives of wavelets as trial functions for solving the time-optimal Hamilton-Jacobi-Bellman equation which is known to often exhibit non-differentiable solutions. The motivation for using wavelets stems from the fact that wavelets can approximate efficiently non-smooth functions with a small number of wavelet coefficients. The use of the antiderivatives also ensures that the solution to the HJB automatically satisfies the boundary condition at the origin.

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