An Exact Stability Analysis Test for Single-Parameter Polynomially-Dependent Linear Systems

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***Abstract*— We provide a new condition for testing the stability of a single-parameter, polynomially-dependent linear system of polynomial degree N of the form**

$$\dot{x} = A(\rho)x, \quad A(\rho) = \sum_{i=0}^{N} \rho^i A_i \tag{1}$$

over a compact interval. The test is nonconservative and can be cast as a convex feasibility problem in terms of a pair of linear matrix inequalities (LMI's).

I. Introduction

We consider single-parameter dependent linear systems where the state matrix A depends polynomially on the parameter ρ, as in (1). It is assumed that ρ belongs to a compact set Ω, taken here without loss of generality as $\Omega = [-1, +1]$. The stability of (1) over a compact interval has been investigated, for example, in [1], [2] where the maximum interval of stability around the origin was given. A few years later, Saydy et al. [3], [4] gave necessary and sufficient conditions for the stability of (1) using guardian maps. This method was later extended in [5] and [6] to LTI systems with many parameters of the form:

$$\dot{x} = \left(A_0 + \sum_{i=1}^{m} \rho_i A_i \right) x$$

However, the stability conditions in [5] and [6] are only sufficient. Despite the results of [3], [4], nonconservative *Lyapunov-based* conditions for stability for (1) have remained elusive thus far. The main obstacle in deriving sufficient and *necessary* conditions using Lyapunov function theory is the absence of the knowledge of the correct functional dependence of the parameter-dependent quadratic Lyapunov function $V(x) = x^{\intercal} P(\rho) x$ that will lead to necessary conditions. One is therefore content with the derivation of sufficient conditions only, based on an a priori postulate of the a Lyapunov matrix $P(\rho)$; see Refs. [7], [8], [9], [10]. In fact, by choosing a constant P one ensures the so-called quadratic stability for (1). This notion of stability is restrictive for most applications of interest, since it ensures stability against arbitrary fast variations of ρ.

Recently, Bliman [11], [12], [13] developed Lyapunov-based necessary and sufficient conditions for multi-variable, *affinely-dependent* parameter linear systems. The results of [11], [12] show that the search for Lyapunov functions for such systems can be restricted, without loss of generality, within the class of *polynomially-dependent* quadratic Lyapunov functions. The upper bound on the degree of the Lyapunov function is not known a priori, however. Moreover, the stability conditions can be formulated and solved in terms of convex feasibility problems (LMI's). Similar results were reported in [14], [15] where also an upper bound on the degree of the polynomial dependence of the Lyapunov function was provided. Closely related to this line of research is also the work of Chesi et al [16]. Therein the authors show that homogeneous polynomially parameter-dependent quadratic Lyapunov functions can be used to investigate the robust stability of polytopic linear models. For systems affected by *polynomial* time-invariant uncertainty Chesi [17] has also provided computationally attractive LMI conditions, albeit these conditions are not Lyapunov based.

In this paper we derive Lyapunov-type necessary and sufficient conditions for the stability of the single-parameter *polynomially-dependent* system in (1). Specifically, we first show that the stability of (1) over $[-1, +1]$ is *equivalent* to the stability of an associated parameter-dependent system with an *affine* (as opposed to polynomial) dependence on ρ. In [14] necessary and sufficient conditions for the stability of the system (1) where $N = 1$ are given in terms of linear matrix inequalities. We may therefore use the results of [14] to provide a necessary and sufficient condition for stability for (1).

II. Auxiliary System

Consider the matrix $A(\rho)$ of polynomial degree N in (1)

$$A(\rho) = A_0 + \rho A_1 + \rho^2 A_2 + \cdots + \rho^N A_N \tag{2}$$

Let $\deg A$ denote the degree of A in terms of ρ. We thus have from (2) that $\deg A = N$. Decompose the matrix $A(\rho)$ in its even and odd polynomial parts

$$A(\rho) = A_0^a(\rho^2) + \rho A_1^a(\rho^2) \tag{3}$$

where $A_0^a(\cdot)$ and $A_1^a(\cdot)$ are polynomial matrices in terms of ρ^2. For example, if N is odd, then

$$A_0^a(\rho^2) := A_0 + \rho^2 A_2 + \rho^4 A_4 \cdots + \rho^{N-1} A_{N-1} \tag{4a}$$
$$A_1^a(\rho^2) := A_1 + \rho^2 A_3 + \rho^4 A_5 \cdots + \rho^{N-1} A_N \tag{4b}$$

whereas if N is even,

$$A_0^a(\rho^2) := A_0 + \rho^2 A_2 + \rho^4 A_4 + \cdots + \rho^N A_N \tag{5a}$$
$$A_1^a(\rho^2) := A_1 + \rho^2 A_3 + \rho^4 A_5 + \cdots + \rho^{N-2} A_{N-1} \tag{5b}$$

Let $\bar{\mathbb{C}}^+$ denote the closed right-half of the complex plane.

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Lemma 1: Let the matrix $A(\rho)$ in (2). Then the matrix $A(\rho)$ is Hurwitz for all $\rho \in [-1,+1]$ if and only if the matrix

$$\begin{pmatrix} A_0^a(r) & rA_1^a(r) \\ A_1^a(r) & A_0^a(r) \end{pmatrix} \tag{6}$$

is Hurwitz for all $r \in [0,1]$, where $A_0^a(r)$, $A_1^a(r)$ as in (4) or (5).

Proof: The matrix $A(\rho)$ is Hurwitz for all $\rho \in [-1,+1]$ if and only if $\det(sI - A_0^a(\rho^2) - \rho A_1^a(\rho^2)) \neq 0$ for all $\rho \in [-1,+1]$ and for all $s \in \bar{\mathbb{C}}^+$, which holds if and only if $\det(sI - A_0^a(\rho^2) \pm \rho A_1^a(\rho^2)) \neq 0$ for all $\rho \in [0,1]$ and for all $s \in \bar{\mathbb{C}}^+$.

From the identity

$$\begin{pmatrix} I & -\rho I \\ 0 & I \end{pmatrix} \begin{pmatrix} sI - A_0^a(\rho^2) & -\rho^2 A_1^a(\rho^2) \\ -A_1^a(\rho^2) & sI - A_0^a(\rho^2) \end{pmatrix} \begin{pmatrix} I & \rho I \\ 0 & I \end{pmatrix}$$
$$= \begin{pmatrix} sI - A_0^a(\rho^2) + \rho A_1^a(\rho^2) & 0 \\ -A_1^a(\rho^2) & sI - A_0^a(\rho^2) - \rho A_1^a(\rho^2) \end{pmatrix}$$

it follows that $\det(sI - A_0^a(\rho^2) \pm \rho A_1^a(\rho^2)) \neq 0$ for all $\rho \in [0,1]$ and for all $s \in \bar{\mathbb{C}}^+$ if and only if

$$\det \begin{pmatrix} sI - A_0^a(\rho^2) & -\rho^2 A_1^a(\rho^2) \\ -A_1^a(\rho^2) & sI - A_0^a(\rho^2) \end{pmatrix} \neq 0 \tag{7}$$

for all $\rho \in [0,1]$ and for all $s \in \bar{\mathbb{C}}^+$. The last condition is equivalent to the statement that the matrix in (6) is Hurwitz for all $r \in [0,1]$. ■

The following theorem is immediate.

Theorem 1: The matrix $A(\rho) = A_0^a(\rho^2) + \rho A_1^a(\rho^2)$ is Hurwitz for all $\rho \in [-1,+1]$ if and only the matrix

$$A^{(1)}(\rho) := \begin{pmatrix} A_0^a\left(\frac{\rho+1}{2}\right) & \frac{\rho+1}{2} A_1^a\left(\frac{\rho+1}{2}\right) \\ A_1^a\left(\frac{\rho+1}{2}\right) & A_0^a\left(\frac{\rho+1}{2}\right) \end{pmatrix} \tag{8}$$

is Hurwitz for all $\rho \in [-1,+1]$.

Proof: The result follows directly from Lemma 1 by setting $\rho = (r+1)/2$. Then $\rho \in [-1,+1]$ if and only if $r \in [0,1]$. ■

Notice that the matrix in (8) depends polynomially on the parameter ρ. Hence the same procedure can be repeated for this matrix as well. Moreover, from (4) and (5) it follows that $\deg A_0^a \leq (N-1)/2$ and $\deg A_1^a \leq (N-1)/2$ if N is odd, and $\deg A_0^a \leq N/2$ and $\deg A_1^a \leq (N-2)/2$ if N is even. Hence, $\deg A^{(1)} \leq \max\{\deg A_0^a, \deg A_1^a + 1\} = \lfloor (N+1)/2 \rfloor$. Therefore, the polynomial dependence of the new matrix $A^{(1)}(\rho)$ has been reduced by a factor of two.

Specifically, one can rewrite explicitly the matrix in (8) as

$$A^{(1)}(\rho) = A_0^{(1)} + \rho A_1^{(1)} + \rho^2 A_2^{(1)} + \cdots + \rho^{N_1} A_{N_1}^{(1)} \tag{9}$$

where $N_1 = \lfloor (N+1)/2 \rfloor$ for some matrices $A_j^{(1)} \in \mathbb{R}^{2n \times 2n}$ and $j = 0,1,2,\cdots,N_1$. This procedure will lead after at most $q_{\max} := \lfloor \log_2 N \rfloor + 1$ steps to *an affine* system

$$A^{(q_{\max})}(\rho) = A_0^{(q_{\max})} + \rho A_1^{(q_{\max})} \tag{10}$$

for some constant matrices $A_0^{(q_{\max})}, A_1^{(q_{\max})} \in \mathbb{R}^{2^{q_{\max}} n \times 2^{q_{\max}} n}$.

The following result is thus immediate from the previous iterative procedure.

Corollary 1: The matrix $A(\rho)$ in (2) is Hurwitz for all $\rho \in [-1,+1]$ if and only the matrices $A^{(q)}(\rho)$ are Hurwitz for all $\rho \in [-1,+1]$ and all $q = 1,2,\ldots,q_{\max}$. Specifically, the polynomial matrix $A(\rho)$ in (2) is Hurwitz for all $\rho \in [-1,+1]$ if and only if the *affine* matrix $A^{(q_{\max})}(\rho)$ in (10) is Hurwitz for all $\rho \in [-1,+1]$.

More to the point, the previous corollary allows one to check the stability of the polynomial matrix $A(\rho)$ for all $\rho \in [-1,+1]$ by checking the stability of the *affine* matrix $A^{(q_{\max})}(\rho)$ for all $\rho \in [-1,+1]$. To this end, we make use of the following recent result from [14]. The result provides Lyapunov-based, necessary and sufficient conditions to test the stability of a single-parameter affine matrix of the form (10).

III. LMI Stability Condition

Theorem 2 ([14]): Given the matrices $\mathcal{A}_0$, $\mathcal{A}_1 \in \mathbb{R}^{n \times n}$ with $\operatorname{rank} \mathcal{A}_1 = \ell$, let

$$m := \begin{cases} \frac{1}{2}(2n\ell - \ell^2 + \ell), & \text{if} \quad \ell < n, \\ \frac{1}{2}n(n+1) - 1, & \text{if} \quad \ell = n. \end{cases} \tag{11}$$

Then the following two statements are equivalent:

(i) $\mathcal{A}_0 + \rho \mathcal{A}_1$ is Hurwitz for all $\rho \in [-1,+1]$.

(ii) There exists a set of $m+1$ matrices $\{\mathcal{P}_i\}_{0 \leq i \leq m}$, such that the following two matrix inequalities are satisfied for all $\rho \in [-1,+1]$.

$$\mathcal{R}(\rho) := (\mathcal{A}_0 + \rho \mathcal{A}_1)^\intercal \mathcal{P}(\rho) + \mathcal{P}(\rho)(\mathcal{A}_0 + \rho \mathcal{A}_1) < 0,$$
$$\mathcal{P}(\rho) = \sum_{i=0}^{m} \rho^i \mathcal{P}_i > 0.$$

In order to find a nonconservative way of checking the previous matrix inequalities we first notice that the parameter-dependent matrix $\mathcal{P}(\rho)$ can be written as

$$\mathcal{P}(\rho) = \left(\rho^{[k]} \otimes I_n\right)^\intercal \mathcal{P}_\Sigma \left(\rho^{[k]} \otimes I_n\right) \tag{12}$$

for some constant matrix $\mathcal{P}_\Sigma = \mathcal{P}_\Sigma^\intercal \in \mathbb{R}^{nk \times nk}$, where $k = \lceil \frac{m}{2} \rceil + 1$ and where $\rho^{[k]} \in \mathbb{R}^k$ is defined as $\rho^{[k]} := (1 \ \rho \ \rho^2 \ \ldots \ \rho^{k-1})^\intercal \in \mathbb{R}^k$. Similarly, the matrix $\mathcal{R}(\rho)$ can be expressed as

$$\mathcal{R}(\rho) \quad = \quad \left(\rho^{[k']} \otimes I_n\right)^\intercal \mathcal{R}_\Sigma \left(\rho^{[k']} \otimes I_n\right) \tag{13}$$

where $k' = k$ if m is odd or $k' = k+1$ if m is even, and where $\mathcal{R}_\Sigma \in \mathbb{R}^{nk' \times nk'}$ is given by

$$\mathcal{R}_\Sigma := \mathcal{H}_\Sigma^\intercal \mathcal{P}_\Sigma \mathcal{F}_\Sigma + \mathcal{F}_\Sigma^\intercal \mathcal{P}_\Sigma \mathcal{H}_\Sigma \tag{14}$$

where $\mathcal{H}_\Sigma := \hat{J}_k \otimes I_n$, $\mathcal{F}_\Sigma := \hat{J}_k \otimes \mathcal{A}_0 + \check{J}_k \otimes \mathcal{A}_1$, and $\hat{J}_k := [I_k \ 0_{k \times 1}]$ and $\check{J}_k := [0_{k \times 1} \ I_k]$. Specifically, note that $\mathcal{R}_\Sigma$ depends linearly on $\mathcal{P}_\Sigma$.

The following result can then be used to test the matrix inequalities of Theorem 2.

Lemma 2 ([14],[18]): Let $\Theta \in \mathbb{R}^{nk\times nk}$. Then the matrix inequality

$$\left(\rho^{[k]} \otimes I_n\right)^{\intercal} \Theta \left(\rho^{[k]} \otimes I_n\right) < 0 \tag{15}$$

holds for all $\rho \in [-1,\, 1]$ if and only if there exist matrices $D \in \mathbb{R}^{n(k-1)\times n(k-1)}$ and $G \in \mathbb{R}^{n(k-1)\times n(k-1)}$ such that

$$D = D^{\intercal} > 0, \quad G + G^{\intercal} = 0,$$
$$\Theta < \begin{bmatrix} \hat{J}_{k-1} \otimes I_n \\ \check{J}_{k-1} \otimes I_n \end{bmatrix}^{\intercal} \begin{bmatrix} -D & G \\ G^{\intercal} & D \end{bmatrix} \begin{bmatrix} \hat{J}_{k-1} \otimes I_n \\ \check{J}_{k-1} \otimes I_n \end{bmatrix}.$$

The following is a direct consequence of Lemma 2 and Theorem 2.

Corollary 2: Let the parameter-dependent matrix $\mathcal{A}(\rho) = \mathcal{A}_0 + \rho\mathcal{A}_1$, where $\mathcal{A}_0, \mathcal{A}_1 \in \mathbb{R}^{n\times n}$ with $\operatorname{rank} \mathcal{A}_1 = \ell$ and let $k = \lceil \frac{m}{2} \rceil + 1$ where m as in (11). Then, $\mathcal{A}(\rho)$ is Hurwitz for all $\rho \in [-1,\, 1]$ if and only if there exist symmetric matrices $\mathcal{P}_\Sigma \in \mathbb{R}^{nk\times nk}$, $D_1 \in \mathbb{R}^{n(k-1)\times n(k-1)}$ and $D_2 \in \mathbb{R}^{nk\times nk}$ and skew-symmetric matrices $G_1 \in \mathbb{R}^{n(k-1)\times n(k-1)}$, $G_2 \in \mathbb{R}^{nk\times nk}$, such that

$$D_1 = D_1^{\intercal} > 0, \quad G_1 + G_1^{\intercal} = 0, \tag{16}$$
$$-\mathcal{P}_\Sigma < \begin{bmatrix} \hat{J}_{k-1} \otimes I_n \\ \check{J}_{k-1} \otimes I_n \end{bmatrix}^{\intercal} \begin{bmatrix} -D_1 & G_1 \\ G_1^{\intercal} & D_1 \end{bmatrix} \begin{bmatrix} \hat{J}_{k-1} \otimes I_n \\ \check{J}_{k-1} \otimes I_n \end{bmatrix}, \tag{17}$$

and

$$D_2 = D_2^{\intercal} > 0, \quad G_2 + G_2^{\intercal} = 0, \tag{18}$$
$$\mathcal{R}_\Sigma < \begin{bmatrix} \hat{J}_{k} \otimes I_n \\ \check{J}_{k} \otimes I_n \end{bmatrix}^{\intercal} \begin{bmatrix} -D_2 & G_2 \\ G_2^{\intercal} & D_2 \end{bmatrix} \begin{bmatrix} \hat{J}_{k} \otimes I_n \\ \check{J}_{k} \otimes I_n \end{bmatrix}, \tag{19}$$

where $\mathcal{R}_\Sigma = \mathcal{R}_\Sigma(\mathcal{P}_\Sigma)$ as in (14).

Remark 1 Notice that when $\mathcal{A}(\rho)$ is nominally stable, i.e., when the matrix $\mathcal{A}_0$ is Hurwitz, the inequality (17) is not necessary. This is due to the fact that $\mathcal{A}_0$ Hurwitz along with inequality (19) guarantees that $\mathcal{P}(0) > 0$. Also, (19) ensures that $\mathcal{P}(\rho) > 0$ for all $\rho \in [-1,\, 1]$; see [19]. Assuming therefore nominal stability, one can discard the inequality (17), thus reducing considerably the number of variables in the convex feasibility problem of Corollary 2.

IV. Numerical Example

Example 1: Consider the following polynomial matrix

$$A(\rho) = \begin{bmatrix} 1-\rho^2-2(\rho+1)^4 & -1+\rho^2+(\rho+1)^4 \\ 2-2\rho^2-2(\rho+1)^4 & -2+2\rho^2+(\rho+1)^4 \end{bmatrix} \tag{20}$$

The eigenvalues of $A(\rho)$ can be easily computed as $\lambda_1(\rho) = -1+\rho^2$ and $\lambda_2(\rho) = -(1+\rho)^4$. Therefore the matrix $A(\rho)$ is Hurwitz if and only if $\rho \in (-1, +1)$. We will Corollary 2 to verify the stability domain[1] of (20).

For the matrix in (20), we can compute according to (5),

$$A_0^a(\rho^2) = A_0 + \rho^2 A_2 + \rho^4 A_4, \quad A_1^a(\rho^2) = A_1 + \rho^2 A_3,$$

[1]Strictly speaking, we cannot use Corollary 2 for this example since the domain is not compact. Nonetheless, the purpose of this example is to show that the result of Corollary 2 is "tight". That is, the LMI's are feasible but not strictly feasible on $(-1, +1)$.

where,

$$A_0 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -8 & 4 \\ -8 & 4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -13 & 7 \\ -14 & 8 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} -8 & 4 \\ -8 & 4 \end{bmatrix}, \quad A_4 = \begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix}$$

Then, $A(\rho)$ can be written as

$$A(\rho) = A_0^a(\rho^2) + \rho A_1^a(\rho^2).$$

According to equation (8) let

$$\begin{aligned} A^{(1)}(\rho) &= \begin{bmatrix} A_0^a\left(\frac{\rho+1}{2}\right) & \frac{\rho+1}{2} A_1^a\left(\frac{\rho+1}{2}\right) \\ A_1^a\left(\frac{\rho+1}{2}\right) & A_0^a\left(\frac{\rho+1}{2}\right) \end{bmatrix} \\ &= A_0^{(1)} + \rho A_1^{(1)} + \rho^2 A_2^{(1)}, \end{aligned}$$

where,

$$A_0^{(1)} = \begin{bmatrix} -8 & 15/4 & -6 & 3 \\ -15/2 & 13/4 & -6 & 3 \\ -12 & 6 & -8 & 15/4 \\ -12 & 6 & -15/2 & 13/4 \end{bmatrix},$$
$$A_1^{(1)} = \begin{bmatrix} -15/2 & 4 & -8 & 4 \\ -8 & 9/2 & -8 & 4 \\ -4 & 2 & -15/2 & 4 \\ -4 & 2 & -8 & 9/2 \end{bmatrix},$$
$$A_2^{(1)} = \begin{bmatrix} -1/2 & 1/4 & -2 & 1 \\ -1/2 & 1/4 & -2 & 1 \\ 0 & 0 & -1/2 & 1/4 \\ 0 & 0 & -1/2 & 1/4 \end{bmatrix}.$$

Rewriting $A^{(1)} = A_0^{(1),a}(\rho^2) + \rho^2 A_1^{(1),a}(\rho^2)$ where $A_0^{(1),a}(\rho^2) = A_0^{(1)} + \rho^2 A_2^{(1)}$ and $A_1^{(1),a}(\rho^2) = A_1^{(1)}$ one obtains,

$$\begin{aligned} A^{(2)}(\rho) &= \begin{bmatrix} A_0^{(1),a}\left(\frac{\rho+1}{2}\right) & \frac{\rho+1}{2} A_1^{(1),a}\left(\frac{\rho+1}{2}\right) \\ A_1^{(1),a}\left(\frac{\rho+1}{2}\right) & A_0^{(1),a}\left(\frac{\rho+1}{2}\right) \end{bmatrix} \\ &= \begin{bmatrix} A_0^{(1)} + \left(\frac{\rho+1}{2}\right) A_2^{(1)} & \left(\frac{\rho+1}{2}\right) A_1^{(1)} \\ A_1^{(1)} & A_0^{(1)} + \left(\frac{\rho+1}{2}\right) A_2^{(1)} \end{bmatrix} \\ &= A_0^{(2)} + \rho A_1^{(2)}, \end{aligned}$$

where the numerical values of $A_0^{(2)}$ and $A_1^{(2)}$ are given below,

$$A_0^{(2)} = \begin{bmatrix} -33/4 & 31/8 & -7 & 7/2 & -15/4 & 2 & -4 & 2 \\ -31/4 & 27/8 & -7 & 7/2 & -4 & 9/4 & -4 & 2 \\ -12 & 6 & -33/4 & 31/8 & -2 & 1 & -15/4 & 2 \\ -12 & 6 & -31/4 & 27/8 & -2 & 1 & -4 & 9/4 \\ -15/2 & 4 & -8 & 4 & -33/4 & 31/8 & -7 & 7/2 \\ -8 & 9/2 & -8 & 4 & -31/4 & 27/8 & -7 & 7/2 \\ -4 & 2 & -15/2 & 4 & -12 & 6 & -33/4 & 31/8 \\ -4 & 2 & -8 & 9/2 & -12 & 6 & -31/4 & 27/8 \end{bmatrix}$$

$$A_1^{(2)} = \begin{bmatrix} -1/4 & 1/8 & -1 & 1/2 & -15/4 & 2 & -4 & 2 \\ -1/4 & 1/8 & -1 & 1/2 & -4 & 9/4 & -4 & 2 \\ 0 & 0 & -1/4 & 1/8 & -2 & 1 & -15/4 & 2 \\ 0 & 0 & -1/4 & 1/8 & -2 & 1 & -4 & 9/4 \\ 0 & 0 & 0 & 0 & -1/4 & 1/8 & -1 & 1/2 \\ 0 & 0 & 0 & 0 & -1/4 & 1/8 & -1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1/4 & 1/8 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1/4 & 1/8 \end{bmatrix}$$

According to Corollary 1 the condition that $A(\rho)$ in (20) is Hurwitz for all $\rho \in [-1, +1]$ is equivalent to the condition that the matrix $A^{(2)}(\rho) = A_0^{(2)} + \rho A_1^{(2)}$ is Hurwitz for all $\rho \in [-1, +1]$. Applying the method suggested in [20], the whole stability domain Ω_S for $A^{(2)}(\rho)$ can be calculated as

$$\Omega_S = (-32.891477,\, -4.907828) \cup (-1.226272,\, +1)$$

It can be readily checked that $\operatorname{rank} A_1^{(2)} = 6$. Applying Corollary 2 to $A^{(2)}$ with $n = 8$, $\ell = 6$, $m = \frac{1}{2}(2n\ell - \ell^2 + \ell) = 33$ and $k = \lceil \frac{m}{2} \rceil + 1 = 18$ it can be verified that the two LMI's (17) and (19) are satisfied for all $\rho \in (-1, +1)$. In fact, a feasible solution exists even for $k = 3$. The LMI's (17) and (19) are not *strictly* feasible in this case, however due to the loss of *asymptotic* stability at $\rho = \pm 1$; see also footnote at the bottom of the previous page. Nonetheless, it can be verified that the LMI's (17) and (19) are strictly feasible for all $\eta > 0$ such that $[-\eta, +\eta] \subset (-1, +1)$.

V. Conclusions

A new necessary and sufficient condition for checking robust stability of a linear time-invariant system with polynomial dependence of a single parameter over a compact interval is proposed. It is shown that robust stability of the original polynomially-dependent matrix is equivalent to the robust stability of an auxiliary system (of increased dimension) that depends only affinely on the parameter. The stability of the latter can be checked exactly by solving a feasibility problem in terms of LMI's. The proposed condition competes with the recent result of Chesi [17] for the scalar case. A direct comparison of computational complexity with the approach in [17] is not straightforward, since the dimension of the LMI conditions in [17] is not known a priori. This situation is similar to the one in [11]. Our methodology does not suffer from this drawback, as it provides an explicit bound on the LMI dimensions of the problem. On the other hand, the results in [17] encompass the multi-parameter case. The generalization of our results to the multi-parameter case is not readily evident however.

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