UAV Collision Avoidance based on the Solution of the Suicidal Pedestrian Differential Game

Ioannis Exarchos* and Panagiotis Tsiotras †

Georgia Institute of Technology, Atlanta, GA, 30332-0150

Meir Pachter‡

Air Force Institute of Technology, Wright-Patterson A.F.B., OH 45433

We consider the following differential game of pursuit and evasion involving two unmanned aerial vehicle agents (UAVs): an evading UAV ("evader"), which has limited maneuverability, and a pursuing UAV ("pursuer"), which is assumed to be completely agile. The two UAVs move on the Euclidean plane with different but constant speeds. Whereas the pursuer can change the orientation of its velocity vector arbitrarily fast, that is, she is a "pedestrian" á la Isaacs, the evader cannot make turns having a radius smaller than a specified minimum turning radius. This problem can be seen as a reversed roles Homicidal Chauffeur game, hence the name "Suicidal Pedestrian" differential game. The aim of this paper is to derive the optimal strategies of the two players and characterize the initial conditions that lead to capture if the pursuer acts optimally, along with the states that guarantee evasion regardless of the pursuer's strategy. Both proximity-capture and point-capture are considered. This is a rare instance where a complete solution of a differential game can be obtained in closed-form. Utilizing the connection to Zermelo's Navigation Problem, the two-player solution is generalized under some assumptions for the case of multiple pursuer UAVs. The results are directly applicable to collision avoidance, in the sense that safety regions are delineated in which no collision is possible, even in the worst case of malicious intent by one of the pursuers.

I. Introduction

Collision avoidance, in particular for Air Traffic Control (ATC) applications, have received considerable attention in the literature during the last decade, due to the ever increasing demand for air travel. Two different approaches that address the problem of conflict resolution during collision avoidance have been proposed. The first one, termed cooperative collision avoidance, assumes that all agents involved in conflict wish to avoid collision and therefore cooperate with each other, possibly in a decentralized manner.[2,27] The second approach, termed uncooperative collision avoidance, assumes no coordination between the agents. This lack of information on the intent of the opposing agents leads to a zero-sum differential game formulation, in which each aircraft optimizes its course based on the worst case scenario among all possible actions taken by the opposing aircraft.[31] The first approach typically does not offer any guarantees if an agent decides to deviate from cooperative behavior, but leads to multi-agent protocols that are practical and efficient in cases of several participating agents. The second approach, treated within the framework of the theory of differential games, offers solutions (each depending on the model assumptions), which guarantee collision avoidance against any possible action of the opponent, but the results are typically limited to the case of two agents, because of the inherent difficulty in solving the game if more than two players are involved.

The literature on differential/dynamic games of pursuit and evasion is quite extensive. An interesting discussion on the historical background of problems of pursuit and evasion can be found in Ref. [22]. Isaacs,[12] in his seminal work on the extension of game theory to the framework of differential games, studied several examples of pursuit and evasion. Two-player pursuit-evasion with curvature constraints were addressed

*Ph.D. candidate, D. Guggenheim School of Aerospace Engineering, Room MK-414, 270 Ferst Dr.

†Professor, D. Guggenheim School of Aerospace Engineering, 270 Ferst Dr., AIAA Fellow.

‡Professor, Department of Electrical and Computer Engineering, AIAA Associate Fellow.

in Isaacs' Homicidal Chauffeur problem[12,15] and the Game of Two Cars.[16] For the stochastic versions of these games see Refs. [25] and [33], respectively. A general solution for problems of restricted player maneuverability was presented in Ref. [5], wherein necessary and sufficient conditions for capture, regardless of the initial conditions of the players, were derived. Reference [5] states that a pursuer is guaranteed to capture the evader regardless of initial conditions only if she is faster than the evader, and does not have a major maneuverability disadvantage against the evader. In the original formulation of the Game of Two Cars, both players have the same speed and the same maneuverability restrictions, i.e., they are identical, and capture occurs when the distance between the players becomes less than a specified constant, which is known as the "kill zone".

As with any pursuit-evasion game in which different initial conditions lead to different game outcomes, an essential part of the solution of the game is the determination of the *barrier*.[12] Simply put, the barrier is the surface that separates initial states of the game that lead to capture under optimal play by both players, from states in which capture is impossible, as long as the evader plays optimally, and evasion is guaranteed.

The ramifications of the results that emerged from the analysis of the Game of Two Cars in applications of collision avoidance have been recognized in Ref. [21]. Their usefulness as analytic solutions to validate numerical algorithms was also highlighted in Ref. [20]. Several extensions and generalizations of the Game of Two Cars appear in the literature under the name *maritime collision avoidance*.[1,17–19,23,24,32] An equivalent model, expressed in polar coordinates is also used extensively in missile engagement literature.[8–11,30] In the Game of Two Cars setting, the two agents have different constant speeds and different minimum turning radii. Analytic expressions for the barriers do exist,[19] but practical implementation is problematic because of the underlying assumptions: the exact maneuvering capability of the opponent is assumed to be known a priori, and a continuous measurement of his instantaneous orientation is also necessary.

When several pursuers are introduced in a game with maneuverability constraints, however, the shortage of theoretical results is striking. To the best of our knowledge, apart from an extension to the Game of Two Cars which features three identical agents (one evader and two pursuers) in Ref. [29], only very few heuristics in the form of sufficient conditions for capture are available, such as Ref. [3]. Reference [3] proposes a strategy for a team of fast but less maneuverable pursuers to capture a slower but agile evader by creating a tight "chain" around the evader.

Motivated by these technical difficulties, as well as the dearth of results on multi-player cases, the authors of the present paper investigated the following variation of the Game of Two Cars, termed herein the Suicidal Pedestrian Game (SPG): two agents move with different constant speeds, and the evader is subject to maneuverability restrictions, while the pursuer is assumed to be completely agile. This problem can also be viewed as an inverse to the classical Homicidal Chauffeur problem, wherein the pursuer is faster but has maneuverability restrictions, and attempts to intercept a completely agile but slower evader by bringing her into her "kill zone." Preliminary results published in Ref. [6] demonstrate that this approach leads to a game of reduced dimensionality that utilizes the least amount of information on the characteristics of the pursuing agent. Furthermore, a simple analytic expression of the barrier surface can be obtained. Further relevant work can also be found in Refs. [14, 28].

The present paper aims at generalizing the results obtained from the previous work by the authors.[6] The contributions of the work presented herein are threefold; first, it offers an alternative – and perhaps more elegant– solution to the game by directly applying the mathematical framework for games of kind, instead of considering an equivalent game of degree as proposed in Ref. [6]. Furthermore, realizing that in all practical applications of collision avoidance, point-capture is not a realistic concept since it reduces the agents to a point, the present paper alleviates this weakness by considering proximity capture instead. Note that proximity capture results cannot be obtained from the previous formulation in Ref. [6]. On the other hand, the converse is true, as point-capture is merely a limiting case of proximity capture. Finally, the paper proposes an approach towards generalizing these results for cases of several opponents. It is shown that a different metric (namely the Time to Capture – TtC) is more suitable for assessing the threat imposed by the pursuers, rather than Euclidean distance. This leads to a decomposition of the Euclidean plane into three regions that, once the initial positions of the pursuers are specified in relation to these regions, the game outcome is uniquely determined.

Besides of their intrinsic theoretical merit, the results of the present paper are directly applicable to collision avoidance and to Air Traffic Control, in the sense that a safe region around ownship is delineated such that it is guaranteed that, even in the worst case scenario of malicious behavior, a collision is avoidable. Furthermore, the presented framework assumes the least amount of information available for the pursuing

UAV, since it considers the worst case scenario of complete opponent agility, in lieu of a fixed and a priori known maneuverability characteristics which, in turn, render necessary the availability of the pursuer's heading measurements.

The rest of the paper is organized as follows: In Section II we formally define the problem to be investigated. Next, in Section III we cast the problem within a differential game framework. The solution of the game provides us with an analytic expression for the barrier. In the same section we also derive the optimal strategy of the evader, a strategy which, unlike in the case of the Homicidal Chauffeur game, will turn out to be independent of the problem parameters. Section IV is dedicated towards generalizing the results obtained for two participating agents for cases of several opponents, utilizing the connection to Zermelo's Navigation Problem.[7,34] Finally, Section V summarizes the results and contributions of the paper.

II. Problem Statement

Consider two UAVs, a pursuer and an evader, moving on the Euclidean plane. The subscripts p and e will be used for the "Pursuer" (P) and the "Evader" (E), respectively. The pursuer's objective is *capture*, that is, interception of the evader in finite time, whereas the evader's objective is *evasion*, a state in which she avoids interception indefinitely. Interception occurs when the separation of the agents becomes smaller than a constant, ℓ, known as the radius of the pursuer's kill zone. The special case $\ell = 0$ corresponds to *point-capture*. The agents have different constant speeds, namely v_e, v_p, for the evader and the pursuer, respectively. We define $\alpha = v_p/v_e$ to be the *speed ratio*. The pursuer is assumed to be agile, in the sense that she can change the heading of her velocity vector instantaneously. On the other hand, the evader is less agile and cannot make turns that have a radius smaller than her minimum turning radius R. In this setup, it is a well-known fact that if the pursuer moves with greater speed than the evader, then she will always be able to capture her opponent regardless of the initial relative positioning of the agents.[5] We will therefore restrict our analysis to the interesting case in which $\alpha \leq 1$, that is, $v_p \leq v_e$.

The equations of motion for the pursuer and the evader, written in an inertial frame of reference with coordinates x and y, are given by

$$\dot{x}_p = v_p \cos\phi_p, \tag{1}$$
$$\dot{y}_p = v_p \sin\phi_p, \tag{2}$$
$$\dot{x}_e = v_e \cos\phi_e, \tag{3}$$
$$\dot{y}_e = v_e \sin\phi_e, \tag{4}$$
$$\dot{\phi}_e = -\frac{v_e}{R}u, \qquad u \in [-1, 1]. \tag{5}$$

We wish to investigate the conditions under which capture is possible, by obtaining a characterization of initial conditions that lead to capture, as opposed to initial conditions that lead to evasion under optimal play of both agents, and derive the corresponding optimal state feedback strategies for both P and E.

III. Differential Game Formulation and Solution

In order to determine which initial states lead to capture and which initial states lead to evasion under optimal play by both UAVs, we turn to the theory of differential games.[12] The answer to this question is obtained through the solution of a *game of kind*. Whenever the state space of a game is comprised of both types of initial conditions, that is, starting points that lead to evasion and starting points that lead to capture, under optimal play of both agents, there exists a surface which separates these two regions, called the barrier. This barrier is therefore obtained by solving a game of kind. In a game of kind, the game outcome is essentially an event (in our case, capture or evasion), and, corresponding to whether or not this event occurs, the game payoff assumes discrete values. In contrast, in a *game of degree*, the payoff assumes a continuum of numerical values (e.g., how much time P needs in order to intercept E). For a more detailed presentation, as well as several examples for both types of games, the reader is referred to Ref. [12]. In this paper, we propose to cast the problem as a differential game of kind. For an alternative characterization of the problem as one of degree, see Ref. [6].

A. Differential Game

We first transform the problem from the previous fifth-dimensional *realistic game space* to a two-dimensional *reduced game space*,[12] by fixing the origin of a rotating coordinate frame at E's current position and by aligning the y-axis with E's velocity vector; see Figure 1. The evader action then consists of choosing her center of curvature at a point $C = (R/u, 0)$ on the x-axis as shown in Figure 1. Consequently, the reduced game space has only two states, namely the (x, y) position of P relative to E in the evader's fixed, velocity-aligned rotating frame. The equations of motion of P in this rotating frame are given by

$$\dot{x} = -\frac{v_e}{R} yu - v_p \sin\phi, \tag{6}$$

$$\dot{y} = \frac{v_e}{R} xu - v_e - v_p \cos\phi, \qquad u \in [-1, 1], \tag{7}$$

where ϕ is the pursuer's relative heading in this new reference frame, given by $\phi \triangleq \pi - \phi_p + \phi_e$.

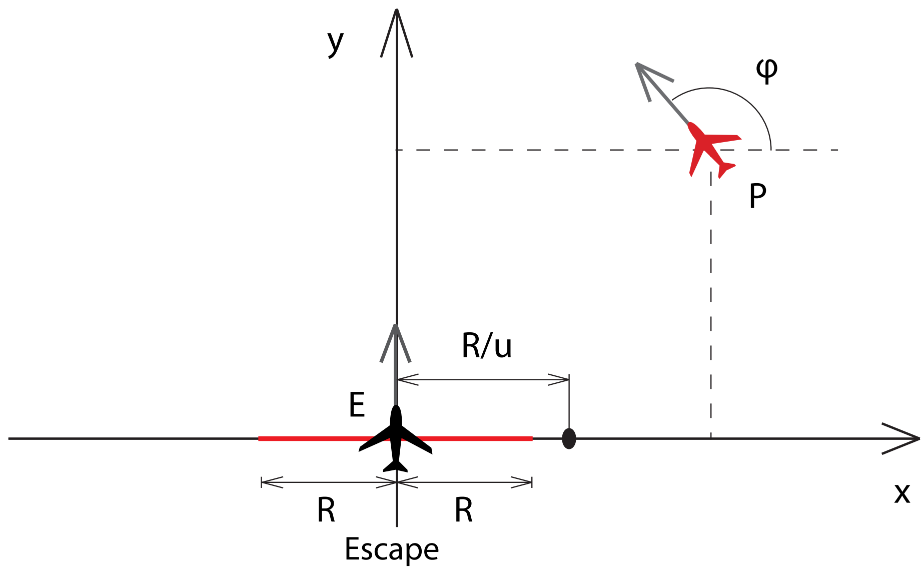


Figure 1. The reduced state space. The reference frame is fixed to E's current position with the y-axis aligned along E's velocity vector. The evader's control action is equivalent to choosing her center of rotation $C = (R/u, 0)$ on the x-axis. A rotation of E around C has the same effect as a rotation of P around C with the same angular velocity, but in the opposite direction.

We define the state vector of the game to be $\mathbf{x} = (x, y)^\mathsf{T} \in \mathbb{R}^2$. The game terminates when capture occurs, that is, when the distance between the evader and the pursuer becomes less than ℓ. The manifold contained within the game space which, once penetrated, signals the game termination, is called the *terminal surface*. The terminal surface for our game is a circle with radius ℓ centered at the origin, i.e., the evader's position in the reduced space. We may thus formally define the terminal surface by $\mathcal{C} \triangleq \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| = \ell\}$. Initializing the game within this circle leads to the trivial case when capture has already been accomplished. Hence, we focus our investigation on initial conditions that lie outside the circle, defining the (reduced) game space as $\mathcal{E} \triangleq \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| \geq \ell\}$.

Retaining the concept of the payoff as it appears in the theory of zero-sum games, we assign the values $+1$ for escape (termination does not occur) and -1 for capture (termination occurs, i.e., the circle is penetrated by the pursuer's trajectory). A necessary additional step is to distinguish the critical case in which the state $\mathbf{x}$ reaches the terminal surface but does *not* cross it, ultimately returning back into the interior of $\mathcal{E}$. The case when the terminal surface is reached without penetration, is referred to as *neutral* outcome,[12] and the corresponding payoff is assigned zero value. Thus, given an initial state $\mathbf{x}(t_0)$ at $t = t_0$ and the pursuer and evader control histories, $\phi(\cdot)$ and $u(\cdot)$ respectively, the payoff formally reads as

$$J(\mathbf{x}(t_0), \phi(\cdot), u(\cdot)) = \begin{cases} +1, & \text{for no termination (escape),} \\ 0, & \text{neutral outcome,} \\ -1, & \text{for termination (capture).} \end{cases} \tag{8}$$

We thus seek to solve the problem of conflicting actions represented by u (maximizing control) and ϕ (minimizing control) that maximize/minimize the payoff (8) under the dynamic equations (6) and (7). Our goal is to obtain an analytic expression for the barrier surface, which consists of all starting points that lead to the neutral outcome.

B. Solution of the Game

In order to solve the game defined above, we apply the framework developed in Ref. [12]. Since the game is clearly symmetric with respect to the y-axis, we will focus our analysis on the right-half plane $\{\mathbf{x} \in \mathcal{E} : x \geq 0\}$. The first step is to obtain the *usable part* of the terminal surface $\mathcal{C}$. It is not uncommon for a terminal surface of a game to be divided into two regions: the *Usable Part (UP)* and the *Nonuseable Part*, which are separated by what is known in the literature as the *Boundary of the Usable Part (BUP)*. The usable part is the subset of the terminal surface on which the pursuer can enforce capture, namely, penetration of the terminal surface. The nonusable part is the remaining part of the terminal surface. On the latter the game would end only if the evader does not play optimally. Essentially, no retrograde optimal paths exist emanating from the nonusable part (see Ref. [12] for more details) The BUP separates the points on $\mathcal{C}$ (rather, infinitesimally close to $\mathcal{C}$) where immediate capture ensues, from those leading to immediate escape.

In order to identify the usable part of the terminal surface, let $\gamma \triangleq [\gamma_1 \ \gamma_2]^\mathsf{T}$ be the unit vector normal to $\mathcal{C}$ from the point $\mathbf{x}$ on $\mathcal{C}$, pointing into the interior of $\mathcal{E}$. Then, the usable part of $\mathcal{C}$ is the region in which the following (strict) inequality holds:[12]

$$\min_{\phi} \max_{u} \sum_{i=1}^{2} \gamma_i f_i(\mathbf{x}, u, \phi) < 0, \qquad \mathbf{x} \in \mathcal{C}, \tag{9}$$

where f_i ($i = 1, 2$) denotes the right-hand-side of the differential equations (6) and (7) respectively (for our problem, $\mathbf{x}_1 = x$ and $\mathbf{x}_2 = y$). The nonusable part has the inequality sign reversed, and the BUP satisfies (9) as an equality. Parameterizing the terminal surface with the variable s as shown in Figure 2, one readily obtains

$$(x, y) = (\ell \sin s, \ell \cos s), \qquad \mathbf{x} \in \mathcal{C}. \tag{10}$$

Since we restrict the analysis in the right-half plane, the angle s assumes values between zero and π. The expression for γ therefore becomes

$$\gamma = (\sin s, \cos s). \tag{11}$$

In light of the above, and by virtue of the dynamics (6), (7), equation (9) yields, for all $\mathbf{x} \in \mathcal{C}$

$$\min_{\phi} \max_{|u| \leq 1} \left\{ (-\frac{v_e}{R} y \sin s + \frac{v_e}{R} x \cos s) u + (-v_p \sin\phi \sin s - v_e \cos s - v_p \cos\phi \cos s) \right\} < 0. \tag{12}$$

Substitution of the parameterization (10) in equation (12) causes the coefficient of u to vanish. It follows that the usable part of the terminal surface is the same, regardless of the evader's control strategy. This is merely a manifestation of the fact that the evader's dynamics are non-holonomic.

We are therefore left with the expression

$$\min_{\phi} \{-v_p \sin\phi \sin s - v_e \cos s - v_p \cos\phi \cos s\} < 0, \tag{13}$$

which may be rewritten as

$$\min_{\phi} \{-\cos(\phi - s)\} < \frac{v_e}{v_p} \cos s. \tag{14}$$

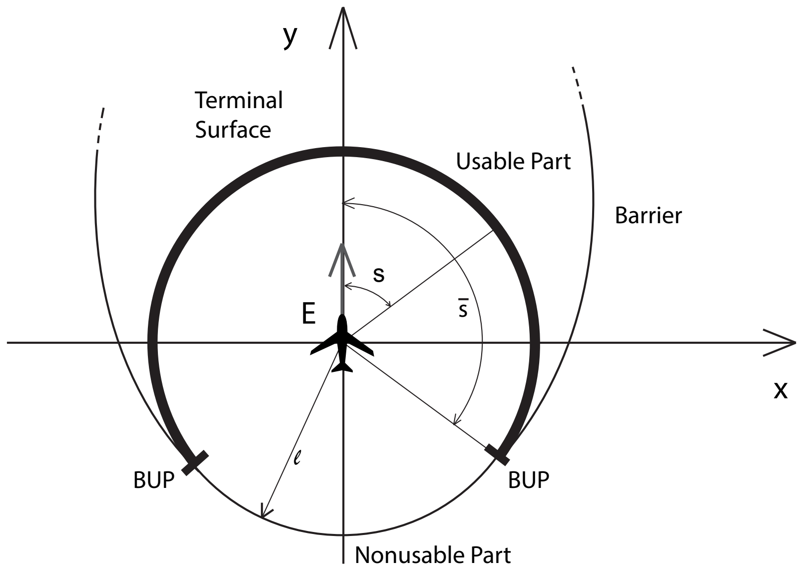


Figure 2. The terminal surface $\mathcal{C}$ of the game, which is a circle of radius ℓ. It is divided into the usable part (the dark line), and the nonusable part, separated by the BUP. The BUP connects to the barrier, which meets the terminal surface at the BUP tangentially.

The left-hand-side of the above equation is minimized for $\phi^* = s$, having the value -1. The usable part of the terminal surface is therefore specified by

$$\cos s > -\frac{v_p}{v_e}, \qquad s \in (\frac{\pi}{2}, \pi], \tag{15}$$

and the BUP is thus determined through

$$\bar{s} = \arccos(-\frac{v_p}{v_e}), \qquad \bar{s} \in (\frac{\pi}{2}, \pi]. \tag{16}$$

An illustration of the usable part, the BUP and the nonusable part is given in Figure 2.

Having identified the BUP, we turn our attention to the construction of the barrier. The barrier is a *semipermeable surface*,[12] that is, optimal play by both agents starting from any point will generate a trajectory that does not penetrate this surface. Let $\mathcal{S}$ be such a surface in $\mathcal{E}$ and assume it is smooth, and at each of its points let $\nu \triangleq [\nu_1 \ \nu_2]^\mathsf{T}$ be its normal vector of unit length, extending into the escape zone.

The *Isaacs equation* for games of kind then formally reads:

$$\min_\phi \max_u \left[\sum_{i=1}^{2} \nu_i f_i(\mathbf{x}, u, \phi)\right] = 0, \qquad \mathbf{x} \in \mathcal{S}. \tag{17}$$

The Isaacs equation essentially states that on a semipermeable surface, such as the barrier, the vector field of the dynamics, after optimal controls have been applied, is tangent to that surface. Therefore, no penetration of that surface can occur under optimal play. Equation (17) can be rewritten, for the problem at hand, as follows

$$\min_\phi \max_{|u|\le 1} \left[-\frac{v_e}{R}(y\nu_1 - x\nu_2)u - v_p(\nu_1 \sin\phi + \nu_2 \cos\phi) - v_e \nu_2 \right] = 0, \quad \mathbf{x} \in \mathcal{S}. \tag{18}$$

Introducing the reverse time variable $\tau = t_f - t$, where t_f is the time of game termination, we define the following functions of the retrograde time:

$$A(\tau) \triangleq y\nu_1 - x\nu_2, \tag{19}$$

$$\sigma(\tau) \triangleq \operatorname{sign}(A), \tag{20}$$

$$c(\tau) \triangleq \frac{v_e}{R}\sigma(\tau), \tag{21}$$

and proceed to the calculation of the optimal controls from (18). Since $u \in [-1, 1]$, it follows from (18) that

$$u^*(\tau) = -\text{sign}(A(\tau)) = -\sigma(\tau), \tag{22}$$

which implies that E's optimal control is *bang-bang*. Furthermore, applying the Lemma on Circular Vectograms[12] in (18) for the minimization of the term $-(\nu_1 \sin\phi + \nu_2 \cos\phi)$ in terms of ϕ, yields the optimal action for the pursuer, as follows:

$$\cos\phi^*(\tau) = \nu_2, \qquad \sin\phi^*(\tau) = \nu_1. \tag{23}$$

Thus, equation (18) becomes

$$c(\tau)A(\tau) - v_p - v_e\nu_2(\tau) = 0, \qquad \mathbf{x} \in \mathcal{S}. \tag{24}$$

The next step is to derive the *Retrogressive Path Equations*.[12] These are the equations arising when one solves the game backwards in time, starting from the usable part of the terminal surface $\mathcal{C}$. Denoting with $(\circ)$ the derivative with respect to τ, the retrograde evolution of the vector ν can be calculated as

$$\overset{\circ}{\nu}_j = \sum_{i=1}^{2} \nu_i \frac{\partial f_i(\mathbf{x}, \phi^*, u^*)}{\partial \mathbf{x}_j}, \qquad j = 1, 2. \tag{25}$$

One readily concludes from (25) and (6), (7) that

$$\overset{\circ}{\nu}_1 = -c\nu_2 \tag{26}$$

$$\overset{\circ}{\nu}_2 = c\nu_1. \tag{27}$$

The retrogressive path equations for the game states can be established if one applies the optimal controls u^* and ϕ^* in equations (6) and (7) and switches the sign to reverse the time flow:

$$\overset{\circ}{x} = -cy + v_p\nu_1, \tag{28}$$

$$\overset{\circ}{y} = cx + v_e + v_p\nu_2. \tag{29}$$

A critical point now is to connect the barrier to the terminal surface. Recalling the definitions of the barrier and the BUP, it is evident that they are connected;[12] the barrier extends from the BUP of $\mathcal{C}$ into $\mathcal{E}$. Furthermore, since the vector field is tangential to the barrier, and no penetration of $\mathcal{C}$ occurs at the BUP, the two surfaces meet tangentially (see Figure 2) . This last statement essentially translates into ν being parallel to γ, and thus we may take

$$\nu = \gamma, \qquad \mathbf{x} \in \mathcal{C} \cap \mathcal{S}. \tag{30}$$

Recall that the function $A(\tau)$ in (19) determines the sign of the optimal evader control. The inverse evolution of A is given by direct application of the $(\circ)$ operator in the definition of A

$$\overset{\circ}{A} = y\overset{\circ}{\nu}_1 + \overset{\circ}{y}\nu_1 - x\overset{\circ}{\nu}_2 - \overset{\circ}{x}\nu_2,$$

which, after applying the expressions for $\overset{\circ}{\nu}_1$, $\overset{\circ}{\nu}_2$, $\overset{\circ}{x}$ and $\overset{\circ}{y}$, simplifies to

$$\overset{\circ}{A} = \nu_1 v_e. \tag{31}$$

Therefore, and since $A \equiv 0$ on $\mathcal{C}$, the sign of A sufficiently close to the terminal surface is determined by $\overset{\circ}{A}$, or

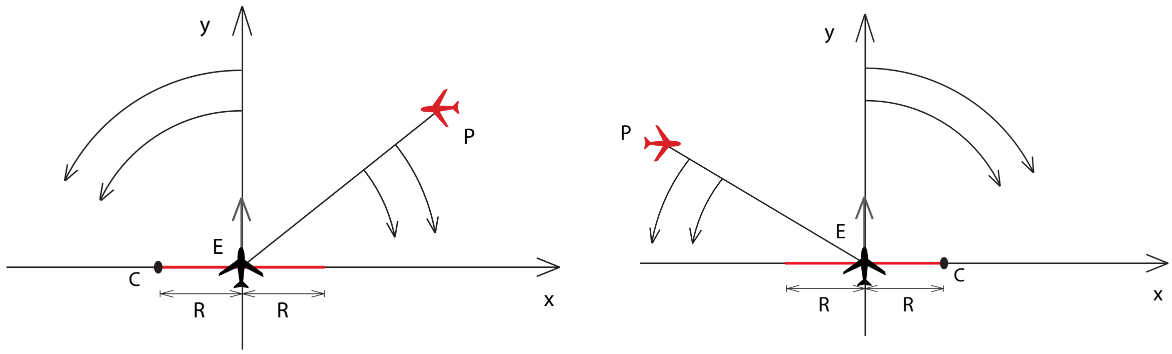
$$\sigma = \text{sign}(\nu_1 v_e) = \text{sign}(\gamma_1) = \text{sign}(\sin s) = \text{sign}(s) = 1, \tag{32}$$

where we have used the parameterization of γ_1 given by (11). This implies that

$$c = \frac{v_e}{R}, \tag{33}$$

and

$$u^* = -1. \tag{34}$$



(a) $u^* = -1$

(b) $u^* = 1$

Figure 3. Optimal evader strategies: (a) for $x > 0$ and (b) for $x < 0$, by virtue of the symmetry of the problem. The arrows indicate the corresponding rotation of the Line of Sight and the reference frame.

The evader's optimal strategy when P is close to $\mathcal{C}$ is therefore established: E will try to steer away from P with his maximum turning capability $u^* = -1$, in an attempt to eliminate the velocity vector component pointing towards P as fast as possible. The evader's strategy is depicted in Figure 3.

To obtain the boundary conditions for the retrogressive equations for the barrier, we investigate their values on the BUP. Recalling the s-parameterization of $\mathcal{C}$ and the particular value $\bar{s}$ for the BUP given by (16), we obtain the boundary conditions

$$x(\tau = 0) = \ell \sin \bar{s}, \tag{35}$$
$$y(\tau = 0) = \ell \cos \bar{s}, \tag{36}$$
$$\nu_1(\tau = 0) = \sin \bar{s}, \tag{37}$$
$$\nu_2(\tau = 0) = \cos \bar{s}. \tag{38}$$

Thus, we are now able to integrate the system of equations (26)-(29) subject to the above boundary conditions, and readily obtain

$$\nu_1(\tau) = \sin(\bar{s} - c\tau), \tag{39}$$
$$\nu_2(\tau) = \cos(\bar{s} - c\tau), \qquad \tau \in [0, \tau_{\max}]. \tag{40}$$

The above equations remain valid up until the time instant $\tau_{\max}$ is reached, defined in equation (43) below. The analytic expression of the barrier curve is:

$$x(\tau) = -R + R\cos(c\tau) + (\ell + v_p\tau)\sin(\bar{s} - c\tau), \tag{41}$$
$$y(\tau) = R\sin(c\tau) + (\ell + v_p\tau)\cos(\bar{s} - c\tau), \qquad \tau \in [0, \tau_{\max}], \tag{42}$$

where $\bar{s}$ is given by (16). Equations (41) and (42) define the *barrier* of the game, that is, they separate the game space into two regions; a region in which optimal play of the pursuer leads to capture and a region in which optimal play of the evader leads to evasion. To obtain $\tau_{\max}$, it is important to note that the barrier expression is invalidated as soon as two barrier branches intersect – the part of the barrier arc beyond the point of intersection is then no longer valid and is therefore discarded. In our case, the two branches of the barrier intersect on the y-axis, because of the inherent symmetry of the problem at hand. Thus, we may obtain $\tau_{\max}$ as the root of $x(\tau) = 0$, i.e., $\tau_{\max}$ is the solution of the transcendental equation

$$(\ell + v_p\tau_{\max})\sin(\bar{s} - c\tau_{\max}) = R - R\cos(c\tau_{\max}). \tag{43}$$

Figure 4 depicts the barrier for $v_e = 1$, $v_p = 0.6$, $R = 0.7$, $\ell = 0.5$. Notice that the barrier meets the terminal surface at the BUP tangentially, as explained earlier.

The solutions to the special cases of equal speeds as well as point-capture can be directly obtained from the barrier expressions (41) and (42). In the case of point-capture one has simply to set $\ell = 0$, whereas the equal speeds case implies $v_p = v_e = v$, $\bar{s} = \pi$.

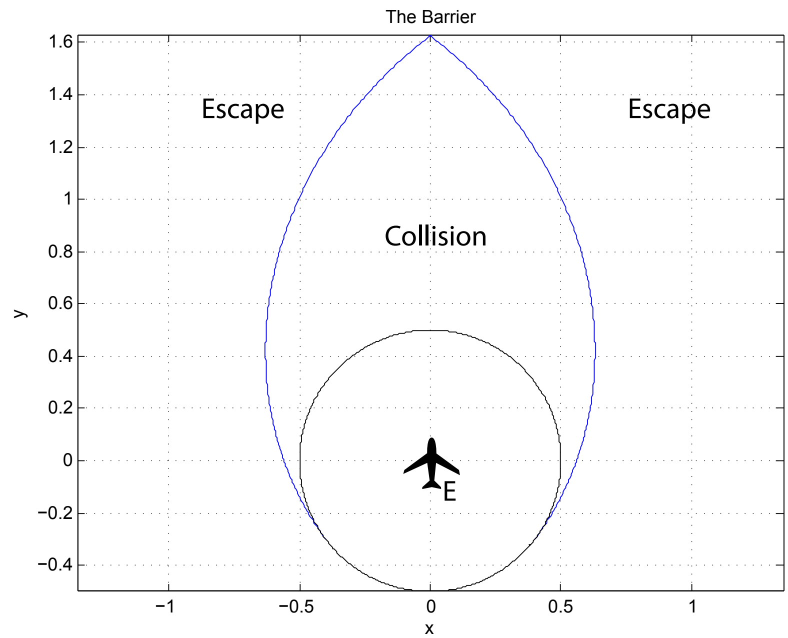


Figure 4. The barrier, given by equations (41) and (42), for $v_e = 1$, $v_p = 0.6$, $R = 0.7$, $\ell = 0.5$. Notice that the barrier meets the terminal surface tangentially at the BUP.

IV. Extension to the Case of Multiple UAVs

We now turn our attention to the case in which the evader faces several pursuers. To this end, consider the game involving one evader and an arbitrary number – say, $N-$ of pursuers, moving on a plane. The subscripts p_i and e will be reserved to designate the "i-th pursuer" and the "evader," respectively. As before, the pursuers' objective is *capture*, that is, interception of the evader in finite time, whereas the evader's objective is *evasion*, a state in which she avoids interception indefinitely. Here, interception is to be understood in the context of coincidence of the Cartesian coordinates between the evader and at least one of the pursuers (point-capture). This will make the subsequent analysis a bit easier, but the results obtained can be modified to account for proximity capture as well. For this scenario, and for the sake of simplicity, it will be assumed that all agents have the same constant speed, that is, $v_e = v_{p_i} = v$ for all $i = 1, \ldots, N$. As before, the pursuers are assumed to be agile, in the sense that they can change the direction of their velocity vector instantaneously. On the other hand, the evader is less agile and cannot make turns that have a radius smaller than a minimum turning radius R. The dynamic equations consist of N copies of the dynamics (6) and (7), i.e.,

$$\dot{x}_i = -\frac{v}{R} y_i u - v \sin\phi_i, \tag{44}$$

$$\dot{y}_i = \frac{v}{R} x_i u - v \cos\phi_i - v, \qquad i = 1, \ldots, N, \quad u \in [-1, 1], \tag{45}$$

where $(x_i, y_i) \in \mathbb{R}^2$ are the coordinates of the i-th pursuer in this new reference frame (see Figure 1), and ϕ_i is the i-th pursuer's control, given by $\phi_i \triangleq \pi - \phi_{p_i} + \phi_e$. We will now make the following assumption:

Assumption 1. *For all $i = 1, \ldots, N$, we have $y_i(0) > 0$.*

This assumption essentially translates into all the pursuers populating at $t = 0$ only the upper half plane; in other words, the evader faces all the pursuers at the beginning of the game, rather than having pursuers at her back (or, alternatively, that all UAV's at her back are sufficiently far away to be neglected). Recall that, if pursuers are located both in front of the evader as well as at her back, and the evader is in the interior of the convex hull generated by the pursuer positions, then a completely agile evader will have no strategy leading to evasion.[26] Since the set of evader strategies in our problem is just a subset of the set of strategies of a completely agile evader, capture is guaranteed in our case as well. Furthermore, since we are interested primarily in coordinated collision avoidance (such as in ATC applications), it is reasonable to assume that once a head-on collision has been avoided by the evader against all incoming UAVs, these will not turn around to pursue the evader.

A. Game Termination

We will now establish the conditions under which the game terminates. Game termination occurs if the position of at least one of the pursuers becomes identical with that of the evader (collision), or if all pursuers enter the lower half plane (evasion). The latter condition guarantees evasion because no velocity vector component of the evader points towards a pursuer, so the evader is able to escape without changing her heading (see Theorem 1 in Ref. [6]). This condition for evasion is equivalent to the evader being able to construct a hyperplane separating her from the pursuers and orient her velocity vector perpendicular to that hyperplane (see Refs. [13, 26]), as depicted in Figure 5. In our game setup, since the evader velocity vector is at all times aligned with the y-axis, we may take that hyperplane to be the x-axis, arriving at the same conclusion.

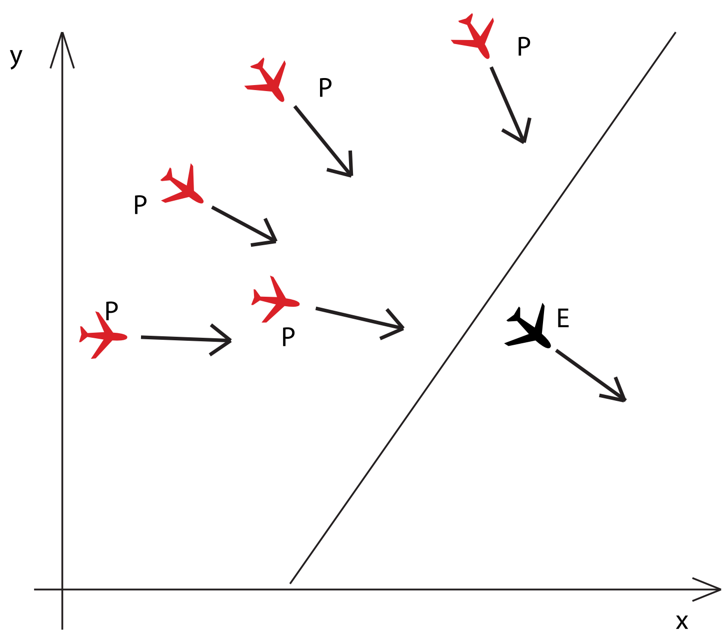


Figure 5. Evasion is guaranteed if the evader is able to construct a hyperplane separating him from the pursuers and apply his velocity vector perpendicular to that hyperplane.

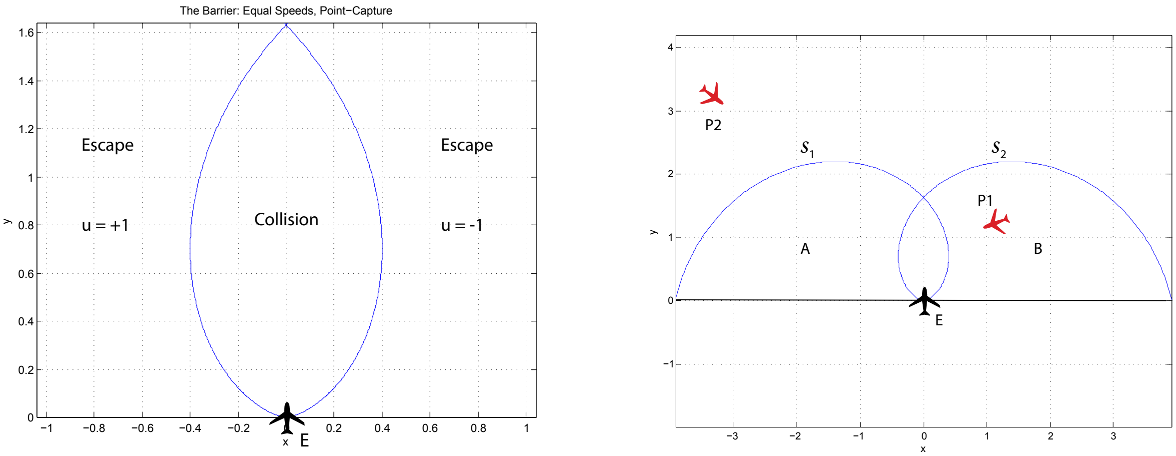
B. Extending the Barrier of the Suicidal Pedestrian Game

When the two agents have equal speed capabilities, the expression for the barrier is

$$x(\tau) = -R + R\cos(c\tau) + v\tau\sin(c\tau), \tag{46}$$

$$y(\tau) = R\sin(c\tau) - v\tau\cos(c\tau), \qquad \tau \in [0, \tau_{\max}], \tag{47}$$

where $\tau = t_f - t$ is the retrograde time variable starting from game termination, say at t_f, and $c = v/R$. To obtain $\tau_{\max}$, recall that the barrier expression in the two player game is invalidated as soon as two barrier branches intersect – the part of the barrier arc beyond the point of intersection is then no longer valid and is therefore discarded. If the evader is able to terminate the game without capture, then the evader is able to induce a trajectory that leads to the lower half plane in the evader fixed reference frame. This is equivalent to the evader being able to successfully eliminate her velocity vector component pointing towards the pursuer



(a)

(b)

Figure 6. (a) The barrier for the case of one evader against one pursuer, given by equations (46) and (47) for $v = 1$ and $R = 0.7$. (b) The barrier surfaces if multiple pursuers are present. $\mathcal{S}_1$ separates the states that lead to capture from those that lead to evasion if the evader chooses $u = -1$, while $\mathcal{S}_2$ corresponds to $u = +1$.

before capture occurs. The states for which the evader is able to do so, as opposed to the states she is not, are separated by the barrier.

In the two-player game, the extensions of the barrier surfaces beyond $\tau_{\max}$, which corresponds to intersection on the y-axis, are of no particular significance; indeed, changing the evader strategy from $u = -1$ to $u = 1$ in the case in which the pursuer is positioned in the left quadrant instead of the right quadrant, greatly reduces the capture area (see Figure 6(a). However, in the case of multiple pursuers, and if both quadrants are populated with pursuers, these extensions do offer some important information. In Figure 6(b), consider only $\mathcal{S}_1$, the barrier if the evader chooses $u = -1$ to respond to pursuer 1. Then if pursuer 2 is outside the capture area delineated by that barrier branch, the evader has enough time to eliminate the velocity vector component pointing towards her, although she initially increases it as she turns towards pursuer 2, in order to avoid pursuer 1. By choosing $u = -1$ or $u = +1$, the evader essentially chooses the capture surface to be either the one provided by $\mathcal{S}_1$, or $\mathcal{S}_2$, respectively. We therefore retain the parts of $\mathcal{S}_1$ and $\mathcal{S}_2$ beyond $\tau_{\max}$ and thus redefine a new maximum retrograde time, say τ_m, for which equations (46) and (47) are valid, to be the first nonzero time in which the barrier curve hits the x-axis (see Figure 6(b)). Then, τ_m is the unique, non-zero solution of

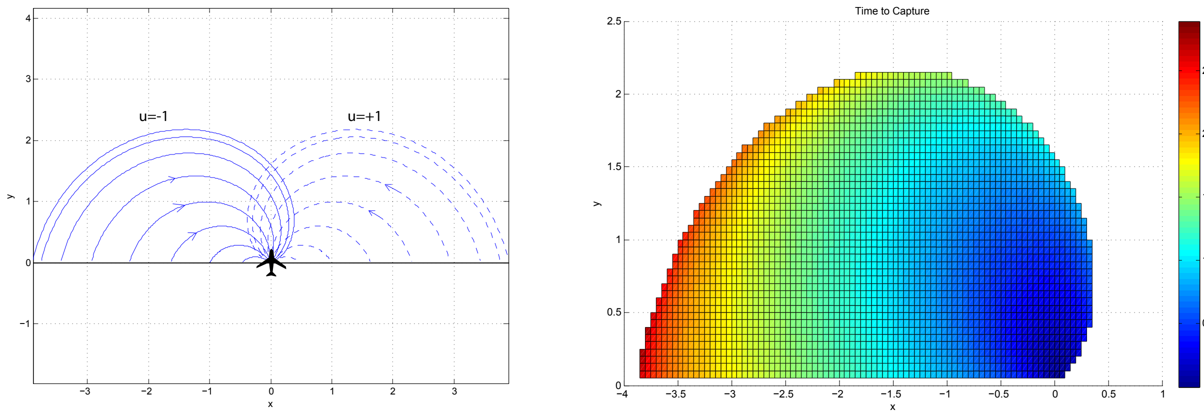
$$R \sin(c\tau_m) = v\tau_m \cos(c\tau_m), \qquad \tau_m > 0. \tag{48}$$

C. The ZNP Solution and Time-to-Capture

The connection between the Suicidal Pedestrian game and the well-known Zermelo's Navigation Problem[4, 34] (ZNP) of optimal control has been highlighted in Ref. [7]. Noticing that the dynamics given by equations (6) and (7) exhibit the same form as those considered in ZNP, one may apply Zermelo's formula to calculate the rate of the optimal control of the pursuers:

$$\dot{\phi}^* = -\frac{v}{R}u, \tag{49}$$

for any fixed, constant value u of the evader control, that is, whenever u is not a function of time or state. In the two-player case, the evader's control is *bang-bang* and without switching until game termination. Thus, for $u = 1$ or $u = -1$, one may integrate the resulting system (6), (7) and (49) of ordinary differential equations, subject to initial conditions (x, y) and $\phi^*(0)$ that will lead to a trajectory passing through the origin $(0, 0)$. However, it is more convenient to perform this integration backwards in time, by switching the signs of the ODE's, using the origin as boundary condition and a variable retrograde boundary condition $\phi_f^* \in [0,\ 2\pi]$ for (49). This will yield a parametric family of curves, and it remains to locate the one that



(a)

(b)

Figure 7. (a) Several time optimal trajectories leading to capture, members of the parametric family of curves obtained by application of Zermelo's navigation law. Solid line trajectories correspond to the evader's control being $u = -1$ while dashed to $u = +1$. (b) Time to Capture (TtC), obtained by storing the τ-variable values across the optimal trajectories, and then interpolating these values over a selected grid of choice.

passes through the original point (x, y) of interest. In fact, this integration can be performed analytically to obtain the following parametric family of curves

$$x(\phi_f^*; \tau) = -R + R\cos(c\tau) + v\tau\sin(\phi_f^* - c\tau), \tag{50}$$

$$y(\phi_f^*; \tau) = R\sin(c\tau) + v\tau\sin(\phi_f^* - c\tau), \qquad \tau \in [0, \tau_m], \tag{51}$$

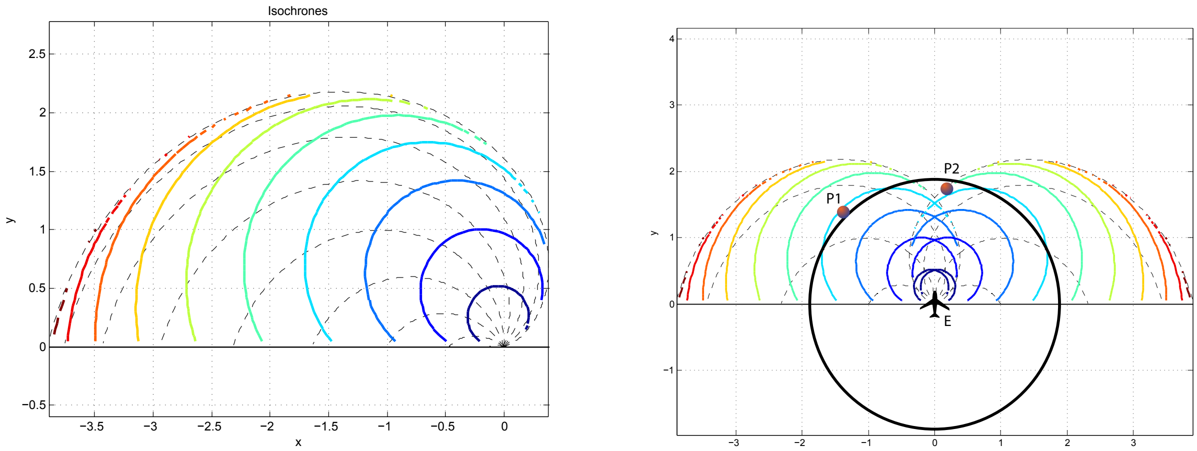
where $u = -1$ and ϕ_f^* is the free parameter. A mere reflection on the y-axis yields the optimal trajectories if $u = +1$ is used instead. Several time optimal trajectories leading to capture are depicted in Figure 7(a), for various values of ϕ_f^*.

A further interesting fact of this connection between the SPG and the ZNP is that the parametric family of curves allows for a simple calculation of the *Time to Capture* (TtC) over the entire domain in which the evader will be intercepted, given her choice of fixed control input. To obtain a characterization of TtC as a function of the evader's fixed control input u and space, one simply has to store the τ values across the optimal trajectories, and then interpolate these values over a selected grid of choice. The result of this procedure is depicted in Figure 7(b). Note that in the two-player game, when the evader faces only one pursuer, the TtC function corresponds to the Value of the game, and can be obtained in the same manner by storing the capture time of trajectories within the two-player capture region, delineated by the barrier in Figure 6(a).

An important remark concerning collision avoidance when multiple agents are in the vicinity, and a worst-case scenario of agent hostility is assumed, is the following. By comparing the TtC functions that correspond to $u = -1$ and $u = +1$, the evader is able to prioritize between the threats posed by the surrounding adversaries. The function level sets, depicted in Figure 8(a), indicate that the Euclidean distance of the pursuer relative to the evader is not the most suitable metric for the purposes of identifying which pursuer poses the most imminent threat. An example of assessing threat is shown in Figure 8(b). Although P2 is positioned closer to the evader compared to P1, as indicated by the circle centered at the evader, the evader has the incentive to react against P1 instead of P2, in order to further increase the time to capture.

D. Necessary and Sufficient Conditions for Evasion

Consider the multi-pursuer game described so far, in which all agents have the same constant speed and the initial positioning is such that the evader is facing all pursuers. We formally define the sets A and B as the sets of all points enclosed between $\mathcal{S}_1$ and the x-axis and $\mathcal{S}_2$ and the x-axis, respectively (see Figure 6(b)).



(a) (b)

Figure 8. (a) Isochrones –the value function level sets– when the evader's control is $u = -1$; plotted with colored lines and time optimal trajectories with dashed. (b) Although P2 is closer to the evader than P1 (indicated by a circle centered on E), P1 poses a more imminent threat as can be seen by the isochrones. Thus the evader can increase his time to capture by reacting to P1 with $u = +1$ instead of reacting to P2 with $u = -1$.

To this end, let $\rho(\tau) \triangleq \sqrt{x^2(\tau) + y^2(\tau)}$ and substitute (46) and (47) to obtain

$$\rho(\tau) = \sqrt{2R^2(1 - \cos(c\tau)) + v^2\tau^2 - 2R\tau v \sin(c\tau)}, \qquad \tau \in [0, \tau_m], \tag{52}$$

The change of variables

$$\theta = g_1(\tau) \triangleq \arctan(y(\tau)/x(\tau)), \qquad \tau \in [0, \tau_m],\ \theta \in [0, \pi]. \tag{53}$$

yields $(\rho \circ g_1^{-1})(\theta)$ as the expression of $\mathcal{S}_1$ in polar coordinates, while the expression of $\mathcal{S}_2$ is similarly obtained by setting

$$\theta = g_2(\tau) \triangleq \arctan(-y(\tau)/x(\tau)), \qquad \tau \in [0, \tau_m],\ \theta \in [0, \pi], \tag{54}$$

to account for the reflection on the y-axis. Then, keeping in mind that the Cartesian coordinates (x, y) are transformed to polar coordinates (r, θ) through $x = r\cos\theta$, $y = r\sin\theta$, we define

$$A \triangleq \{(x, y) : x = r\cos\theta,\ y = r\sin\theta,\ \theta \in [0, \pi],\ 0 \le r < (\rho \circ g_1^{-1})(\theta)\}, \tag{55}$$

while B is similarly defined

$$B \triangleq \{(x, y) : x = r\cos\theta,\ y = r\sin\theta,\ \theta \in [0, \pi],\ 0 \le r < (\rho \circ g_2^{-1})(\theta)\}. \tag{56}$$

We now proceed to obtain necessary and sufficient conditions for evasion. To this end, for the sets A and B defined by (55) and (56) respectively, let

$$I_A = \begin{cases} 1, & \text{if at least one pursuer is in } A \text{, that is, } (x_i, y_i) \in A \text{ for some i, } i = 1, \dots, N \\ 0, & \text{otherwise.} \end{cases} \tag{57}$$

$$I_B = \begin{cases} 1, & \text{if at least one pursuer is in } B \text{, that is, } (x_i, y_i) \in B \text{ for some i, } i = 1, \dots, N \\ 0, & \text{otherwise.} \end{cases} \tag{58}$$

and define $\Sigma = I_A + I_B$. Notice that $\Sigma \in \{0, 1, 2\}$, and that every possible positioning of pursuing agents on the upper half of the Euclidean plane corresponds to a value in Σ.

Proposition 1. *If $\Sigma \in \{0, 1\}$, the evader escapes.*

Proof. We investigate the two cases separately:

- *Case* $\Sigma = 0$. This value corresponds to an arbitrary number of pursuers being placed outside of A and B. In this case, the evader may choose either $u = 1$ or $u = -1$ since in both cases, a time optimal trajectory leading to capture, that passes through the position of at least one of the pursuers, does not exist. This means that the evader has enough time to perform her evading maneuver. The trajectories in the evader fixed reference frame, for both cases of evader control, are depicted in Figure 9.

- *Case* $\Sigma = 1$. This implies that an arbitrary number of pursers are in *either* $A \setminus (A \cap B)$ *or* $B \setminus (A \cap B)$, but not in the intersection $A \cap B$. Similarly to the previous case, the evader has enough time to complete her turning maneuver, i.e., has a control action for which no time-optimal trajectory leading to capture passing through a pursuer location exists. The difference however is that this control action is uniquely determined ($u = 1$ for pursuers in $A \setminus (A \cap B)$ or $u = -1$ for pursuers in $B \setminus (A \cap B)$). An example of this case is depicted in Figure 6(b). The evader chooses $u = -1$ to avoid P1 and, by doing so, initially turns towards P2. However, she has enough time to complete the turn without interception by P2, and thus escapes.

□

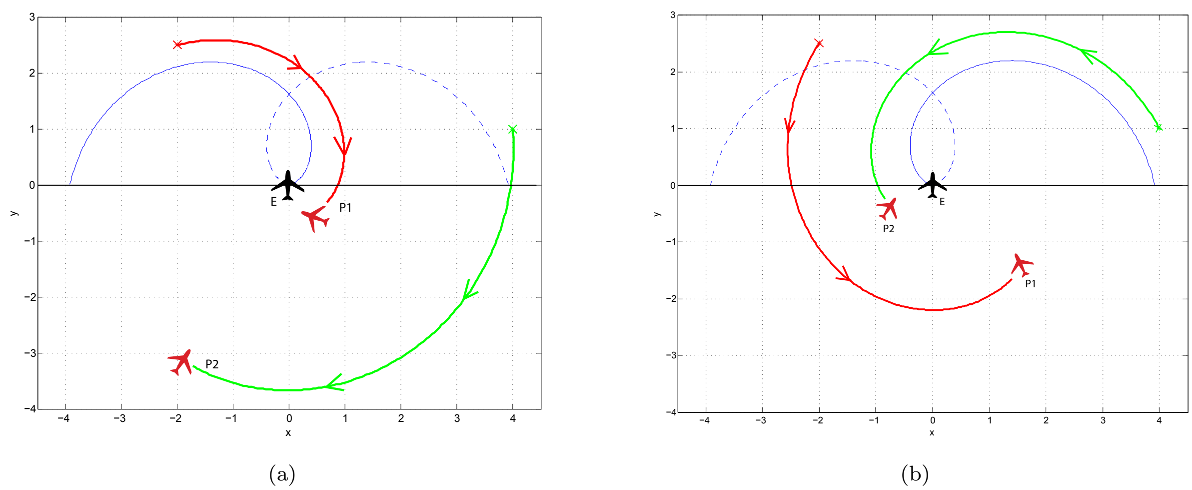


Figure 9. The case of $\Sigma = 0$. All pursuers are located outside of A and B, no ZNP solution passes through their location, and the evader can use either one of his extremal controls to escape: (a) u=-1 and (b) u=+1. In both cases, the barrier surface with a solid line corresponds to the selected control, while the one with the dashed line corresponds to the alternative choice. Both cases lead to escape. Since a policy for capture (be it optimal or not) does not exist, a pure pursuit policy is assigned to both pursuers, for the purposes of simulation.

We now proceed the next statement

Proposition 2. *If there is at least one pursuer in $A \cap B$, collision occurs.*

Proof. This follows immediately from the SPG solution. Indeed, picking an arbitrary pursuer in $A \cap B$ and disregarding all the rest of the pursuers of the game, capture is guaranteed because the pursuer is located in the capture zone delineated by the SPG, and thus the evader has no strategy that leads to evasion against that particular pursuer. Notice that this initial condition corresponds to $\Sigma = 2$. □

Before obtaining necessary and sufficient conditions for evasion, the following Lemma will be useful:

Lemma 1. *If there is at least one purser both in $A \setminus (A \cap B)$ and in $B \setminus (A \cap B)$, but none in $A \cap B$, capture occurs simultaneously by at least one pursuer from $A \setminus (A \cap B)$ and one pursuer from $B \setminus (A \cap B)$.*

Proof. It is obvious that a constant evader control $u = -1$ or $u = +1$ will lead to capture by one of the pursuers, since for both control inputs $u = -1$ and $u = +1$, a time optimal trajectory leading to capture generated by the ZNP solution and passing through pursuer positions, exists. Any intermediate constant value $u \in [-1, 1]$ will also yield capture, since the surface on the Euclidean plane covered by family of time optimal trajectories generated by the ZNP formula, is minimal for the extreme evader control values; any intermediate constant value of u will yield sets A' and B' which are supersets of A and B respectively. The singular case $u = 0$ is no exception, since Zermelo's law for the pursuers then reduces to *parallel navigation*, which again leads to interception.

Simultaneous capture occurs because of the symmetry of the problem at hand. To this end, assume, *ad absurdum* that the evader gets intercepted by only one pursuer, namely the one on his right (without loss of generality). Then, just before interception occurs, the evader could prolong his time to capture by turning to the left, towards the other pursuer. Thus, an optimal evader maneuver leads to simultaneous capture by at least one pursuer of each domain. $\square$

Noticing that the initial condition described in Lemma 1 corresponds to $\Sigma = 2$ and combining Proposition 1, Proposition 2 and Lemma 1, the following corollary is obtained immediately.

Corollary 1. *Collision occurs if and only if* $\Sigma = 2$.

Corollary 1 summarizes the necessary and sufficient conditions for evasion, under Assumption 1, in terms of purser initial positioning on the plane. It demonstrates that, once the initial positions of all pursuers have been specified, the game outcome is determined. Furthermore, Corollary 1 defines which areas of the Euclidean plane are considered "danger zones", in the sense that placing at least one pursuer at that location leads to a different game outcome.

V. Conclusions

In this paper we have investigated the pursuit and evasion differential game between an agile pursuer and an evader having maneuverability restrictions. The agents are assumed to have constant speeds. There are two problem parameters, namely the speed ratio $\alpha = v_p/v_e$ and the turn and capture radius ratio ℓ/R. Using the framework of differential game theory, it was proven that for a speed ratio $\alpha \leq 1$, the capture region is bounded. The solution of the game entails a characterization of the barrier that separates states that lead to capture under optimal play, and states that lead to evasion regardless of the pursuer's actions. Furthermore, a generalization of these results to cases in which the evading agent faces several opponents has been presented. As a first step, a metric has been proposed towards assessing the individual threat posed by each of the pursuing agents. Finally, necessary and sufficient conditions have been derived that determine whether collision occurs or not. It is shown that the Euclidean plane is decomposed into three regions that, once the initial positions of the pursuers are specified in relation with those regions, the game outcome is uniquely determined. Apart from their own intrinsic differential game theoretic merit, the results of this paper have immediate application to collision avoidance problems for Air Traffic Control, in the sense that a safe region around ownship is delineated such that it is guaranteed that, even in the worst case scenario of malicious behavior, a collision is avoidable.

Future work may include generalizations of the results of Section IV to the case of proximity capture, pursuers of different speed than the evader, as well as an extension of Lemma 1 for any choice of piecewise continuous control u for the evader. Finally, an extension of all results to a three-dimensional setting may be possible.

Acknowledgments

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