Simultaneous Position and Attitude Control Without Linear and Angular Velocity Feedback Using Dual Quaternions*

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Abstract— **In this paper, we suggest a new representation for the combined translational and rotational dynamic equations of motion of a rigid body in terms of dual quaternions. We show that with this representation it is relatively straightforward to extend existing attitude controllers based on quaternions to combined position and attitude controllers based on dual quaternions. We show this by developing setpoint nonlinear controllers for the position and attitude of a rigid body with and without linear and angular velocity feedback based on existing attitude-only controllers with and without angular velocity feedback. The combined position and attitude velocity-free controller exploits the passivity of the rigid body dynamics and can be used when no linear and angular velocity measurements are available.**

I. INTRODUCTION

Dual quaternions are built on, and are an extension of, classical quaternions. They provide a compact way to represent not only the attitude but also the position of a rigid body. They have been successfully applied to inertial navigation [1], rigid body control [2], [3], [4], [5], [6], [7], spacecraft formation flying [8], inverse kinematic analysis [9], computer vision [10], [11] and animation [12]. It has been argued that dual quaternions are the most compact and efficient way to simultaneously express the translation and rotation of robotic kinematic chains [13], [14]. Moreover, it has been shown that combined position and attitude control laws based on dual quaternions automatically take into account the natural coupling between the rotational and translational motion [5], [6]. Additionally, dual quaternions allow combined position and attitude control laws to be written compactly as a single control law.

However, the property that makes dual quaternions most attractive and useful is that, as it will be shown, the combined translational and rotational kinematic and dynamic equations of motion written in terms of dual quaternions have the same form as the translational kinematic and dynamic equations of motion written in terms of quaternions.

In this paper, we demonstrate, and take advantage of, this analogy between quaternions and dual quaternions to develop a combined position and attitude setpoint controller that does not require linear and angular velocity measurements from an existing attitude setpoint controller that does not require angular velocity measurements [15], [16].

*This work was supported by the International Fulbright Science and Technology Award sponsored by the Bureau of Educational and Cultural Affairs (ECA) of the U.S. Department of State and by AFRL through research award FA9453-13-C-0201.

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The technique proposed in this paper for developing combined position and attitude controllers from existing attitude controllers has some advantages over techniques based on the special Euclidean group $\mathrm{SE}(3)$, where rotations are represented directly by rotation matrices [17], [18], [19]. In the latter, asymptotically stability of the combined rotational and translational motion is proven by either defining two different error functions for the position and attitude error [19] or, in two steps, by first proving the asymptotical stability of the rotational motion before the asymptotical stability of the translational motion can be proven [17] (note that the translational motion depends on the rotational motion). In our approach, a single error function, the *error dual quaternion* (defined by analogy to the classical rotation error quaternion), is used to represent the combined position and attitude error. Moreover, the asymptotic stability of the combined rotational and translational motion is proven in one step by using a Lyapunov function with the same form as the Lyapunov function used to prove the asymptotic stability of the rotational controller. On the other hand, whereas quaternions produce two closed-loop equilibrium points (since quaternions cover $\mathrm{SO}(3)$ twice [20]) both representing the identity rotation matrix, rotation matrices produce a minimum of four closed-loop equilibrium points [18], [17], only one of which is the identity rotation matrix. On the bad side, dual quaternions inherit the so-called *unwinding phenomenon* from classical quaternions [21]. Known solutions for this problem are suggested.

II. MATHEMATICAL PRELIMINARIES

A. Quaternions

The classical definition of quaternion is $q = q_1 i + q_2 j + q_3 k + q_4$, where q_1, q_2, q_3, $q_4 \in \mathbb{R}$ and i, j, and k satisfy the following properties [5]: $i^2 = j^2 = k^2 = -1$, $i = jk = -kj$, $j = ki = -ik$, and $k = ij = -ji$. An alternative, and more convenient, representation of a quaternion is as an ordered pair $q = (\bar{q}, q_4)$, where $\bar{q} = [q_1 \; q_2 \; q_3]^\top \in \mathbb{R}^3$ and $q_4 \in \mathbb{R}$ are the *vector part* and the *scalar part* of the quaternion, respectively. Hereafter, quaternions with zero scalar part will be referred to as *vector quaternions*, whereas quaternions with zero vector part will be referred to as *scalar quaternions*.

The set of quaternions will be denoted by $\mathbb{H} = \{q : \; q = q_1 i + q_2 j + q_3 k + q_4, \; q_1, q_2, q_3, q_4 \in \mathbb{R}\}$. Likewise, the set of vector quaternions and the set of scalar quaternions will be denoted by $\mathbb{H}^v = \{q \in \mathbb{H} : \; q_4 = 0\}$ and $\mathbb{H}^s = \{q \in \mathbb{H} : q_1 = q_2 = q_3 = 0\}$, respectively.

By representing a quaternion as $q = (\bar{q}, q_4)$, the following basic operations on quaternions can be defined:

Addition: $a+b=(\bar{a}+\bar{b}, a_4+b_4)$,
Multiplication by a scalar: $\lambda a=(\lambda\bar{a}, \lambda a_4)$,
Multiplication: $ab{=}(a_4\bar{b}+b_4\bar{a}+\bar{a}\times\bar{b}, a_4b_4-\bar{a}\cdot\bar{b})$,
Conjugation: $a^*=(-\bar{a}, a_4)$,
Dot product: $a\cdot b=\frac{1}{2}(a^*b+b^*a)=\frac{1}{2}(ab^*+ba^*)$
$\qquad=(\bar{0}, a_4b_4+\bar{a}\cdot\bar{b})$,
Cross product: $a\times b=\frac{1}{2}(ab-b^*a^*)$
$\qquad=(b_4\bar{a}+a_4\bar{b}+\bar{a}\times\bar{b}, 0)$,
Norm: $\|a\|^2=aa^*=a^*a=a\cdot a=(\bar{0}, a_4^2+\bar{a}\cdot\bar{a})$,
Scalar part: $\mathrm{sc}(a)=(\bar{0}, a_4)\in\mathbb{H}^s$,
Vector part: $\mathrm{vec}(a)=(\bar{a}, 0)\in\mathbb{H}^v$,

where $a,b\in\mathbb{H}$, $\lambda\in\mathbb{R}$, and $\bar{0}=[0\ 0\ 0]^\mathsf{T}$. Under the natural isomorphism between $\mathbb{H}^s$ and $\mathbb{R}$, we will often identify, with a slight abuse of notation, q_4 with $(\bar{0}, q_4)$, when this is clear from the context.

We also define the multiplication of a 4-by-4 matrix with a quaternion as $M*q=(M_{11}\bar{q}+M_{12}q_4, M_{21}\bar{q}+M_{22}q_4)\in\mathbb{H}$, where

$$M=\begin{bmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{bmatrix}\in\mathbb{R}^{4\times4},$$

$q\in\mathbb{H}$, $M_{11}\in\mathbb{R}^{3\times3}$, $M_{12}\in\mathbb{R}^{3\times1}$, $M_{21}\in\mathbb{R}^{1\times3}$, and $M_{22}\in\mathbb{R}$. The following properties follow from these definitions: $a\cdot(bc)=b\cdot(ac^*)=c\cdot(b^*a)$ and $(M*a)\cdot b=a\cdot(M^\mathsf{T}*b)$ for $a,b,c\in\mathbb{H}$ and $M\in\mathbb{R}^{4\times4}$.

B. Rotational Kinematic Equations in terms of Quaternions

The relative orientation between the body frame and the inertial frame can be represented by the unit quaternion q. More precisely, if $\bar{v}^I$ is a vector expressed in the inertial frame, then the coordinates of that same vector in the body frame can be calculated from q, and vice-versa, through $v^B=q^*v^Iq$ and $v^I=qv^Bq^*$, respectively, where $v^B=(\bar{v}^B, 0)$ and $v^I=(\bar{v}^I, 0)$.

In quaternion algebra, the rotational kinematic equations have the following form [1]:

$$\dot{q}=\tfrac{1}{2}q\omega^B=\tfrac{1}{2}q\omega^B(q^*q)=\tfrac{1}{2}(q\omega^Bq^*)q=\tfrac{1}{2}\omega^Iq, \qquad (1)$$

where $\omega^B=(\bar{\omega}^B, 0)$ and $\omega^I=(\bar{\omega}^I, 0)$, and $\bar{\omega}^B$ and $\bar{\omega}^I$ are the angular velocity of the rotating body frame with respect to the inertial frame expressed in the body frame and in the inertial frame, respectively. The error between two attitudes represented by q and q_D is the unit quaternion $q_e=q_\mathrm{D}^*q$. If q_D is constant, then the error quaternion kinematic equations are given by

$$\dot{q}_e=\tfrac{1}{2}q_e\omega^B. \qquad (2)$$

C. Dual Quaternions

Dual quaternions are defined as $\hat{q}=q_r+\epsilon q_d$, where $q_r\in\mathbb{H}$ is the *real part* of the dual quaternion, $q_d\in\mathbb{H}$, is the *dual part* of the dual quaternion, and ϵ is the *dual unit*. The dual unit ϵ is defined as $\epsilon^2=0$ and $\epsilon\neq0$.

Hereafter, dual quaternions formed from vector quaternions, i.e., $q_r,q_d\in\mathbb{H}^v$, will be referred to as *dual vector quaternions*. Likewise, dual quaternions formed from scalar quaternions, i.e., $q_r,q_d\in\mathbb{H}^s$, will be referred to as *dual scalar quaternions*. In the sequel, the set of dual quaternions, dual vector quaternions, and dual scalar quaternions will be denoted by $\mathbb{H}_d=\{\hat{q}:\ \hat{q}=q_r+\epsilon q_d,\ q_r,q_d\in\mathbb{H}\}$, $\mathbb{H}_d^v=\{\hat{q}:\ \hat{q}=q_r+\epsilon q_d,\ q_r,q_d\in\mathbb{H}^v\}$, and $\mathbb{H}_d^s=\{\hat{q}:\ \hat{q}=q_r+\epsilon q_d,\ q_r,q_d\in\mathbb{H}^s\}$, respectively. We will also define the set of dual scalar quaternions with zero dual part as $\mathbb{H}_d^r=\{\hat{q}:\ \hat{q}=q_r+\epsilon(\bar{0},0),\ q_r\in\mathbb{H}^s\}$.

The basic operations on dual quaternions are defined as follows [6], [8]:

Addition: $\hat{a}+\hat{b}=(a_r+b_r)+\epsilon(a_d+b_d)$,
Multiplication by a scalar: $\lambda\hat{a}=(\lambda a_r)+\epsilon(\lambda a_d)$,
Multiplication: $\hat{a}\hat{b}=(a_rb_r)+\epsilon(a_rb_d+a_db_r)$,
Conjugation: $\hat{a}^*=a_r^*+\epsilon a_d^*$,
Swap: $\hat{a}^\mathsf{s}=a_d+\epsilon a_r$,
Dot product: $\hat{a}\cdot\hat{b}{=}\frac{1}{2}(\hat{a}^*\hat{b}+\hat{b}^*\hat{a}){=}\frac{1}{2}(\hat{a}\hat{b}^*+\hat{b}\hat{a}^*){=}a_r\cdot b_r$
$\qquad+\epsilon(a_d\cdot b_r+a_r\cdot b_d)\in\mathbb{H}_d^s$,
Cross product: $\hat{a}\times\hat{b}=\frac{1}{2}(\hat{a}\hat{b}-\hat{b}^*\hat{a}^*)=a_r\times b_r$
$\qquad+\epsilon(a_d\times b_r+a_r\times b_d)\in\mathbb{H}_d^v$,
Dual norm: $\|\hat{a}\|_d^2=\hat{a}\hat{a}^*=\hat{a}^*\hat{a}=\hat{a}\cdot\hat{a}$
$\qquad=(a_r\cdot a_r)+\epsilon(2a_r\cdot a_d)\in\mathbb{H}_d^s$,
Scalar part: $\mathrm{sc}(\hat{a})=\mathrm{sc}(a_r)+\epsilon\,\mathrm{sc}(a_d)\in\mathbb{H}_d^s$,
Vector part: $\mathrm{vec}(\hat{a})=\mathrm{vec}(a_r)+\epsilon\,\mathrm{vec}(a_d)\in\mathbb{H}_d^v$,

where $\hat{a},\hat{b}\in\mathbb{H}$ and $\lambda\in\mathbb{R}$. Note that $\hat{a}\hat{b}\neq\hat{b}\hat{a}$, in general.

We define the following *dual quaternion norm*[1] $\|\hat{a}\|^2=\hat{a}\circ\hat{a}$, where $\circ$ is the *dual quaternion circle product*, defined as $\hat{a}\circ\hat{b}=a_r\cdot b_r+a_d\cdot b_d$, for all $\hat{a},\hat{b}\in\mathbb{H}_d$. Under the isomorphism between $\mathbb{H}_d^r$ and $\mathbb{R}$, we will often identify, with a slight abuse of notation, q_4 with $(\bar{0}, q_4)+\epsilon(\bar{0},0)$.

We also define the multiplication of a 8-by-8 matrix with a dual quaternion as

$$M\star\hat{q}=(M_{11}*q_r+M_{12}*q_d)+\epsilon(M_{21}*q_r+M_{22}*q_d), \quad (3)$$

where

$$M=\begin{bmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{bmatrix},\ M_{11},M_{12},M_{21},M_{22}\in\mathbb{R}^{4\times4}.$$

The following properties follow directly from the previous definitions:

$$\hat{a}\circ(\hat{b}\hat{c})=\hat{b}^\mathsf{s}\circ(\hat{a}^\mathsf{s}\hat{c}^*)=\hat{c}^\mathsf{s}\circ(\hat{b}^*\hat{a}^\mathsf{s}),\ \hat{a},\hat{b},\hat{c}\in\mathbb{H}_d, \quad (4)$$

$$\hat{a}\circ(\hat{b}\times\hat{c})=\hat{b}^\mathsf{s}\circ(\hat{c}\times\hat{a}^\mathsf{s})=\hat{c}^\mathsf{s}\circ(\hat{a}^\mathsf{s}\times\hat{b}),\ \hat{a},\hat{b},\hat{c}\in\mathbb{H}_d^v, \quad (5)$$

$$\hat{a}\times\hat{a}=0,\quad \hat{a}\in\mathbb{H}_d^v, \quad (6)$$

$$(M\star\hat{a})\circ\hat{b}=\hat{a}\circ(M^\mathsf{T}\star\hat{b}),\ \hat{a},\hat{b}\in\mathbb{H}_d,\ M\in\mathbb{R}^{8\times8}. \quad (7)$$

D. Unit Dual Quaternions for Attitude and Position Representation

A compact way to represent the relationship between the body frame and the inertial frame when they are related by a rotation quaternion q and a translation vector $\bar{t}^B$ (expressed in the body frame) is to use the dual quaternion [1] $\hat{q}=q_r+\epsilon q_d=q+\epsilon\frac{1}{2}t^Iq=q+\epsilon\frac{1}{2}qt^B$, where $t^B=(\bar{t}^B, 0)$ and $t^I=(\bar{t}^I, 0)$.

[1]Similar norms have been defined in [22], [8].

Lemma 1. *The dual quaternion $\hat{q} = q + \epsilon\frac{1}{2}t^I q = q + \epsilon\frac{1}{2}qt^B$ is a unit dual quaternion, i.e., $\hat{q}\cdot\hat{q} = \hat{q}\hat{q}^* = \hat{q}^*\hat{q} = 1$*[2].

Proof. Note that $\hat{q}\cdot\hat{q} = q_r + \epsilon q_d = q_r q_r^* + \epsilon(q_r q_d^* + q_d q_r^*) = 1 + \epsilon(2q_r\cdot q_d)$ and that $q_r \cdot q_d = 0$ since $q_r\cdot q_d = q\cdot(\frac{1}{2}t^I q) = \frac{1}{2}t^I\cdot(qq^*) = \frac{1}{2}t^I\cdot 1 = 0$. □

The set of unit dual quaternions will be denoted as $\mathbb{H}_d^u = \{\hat{q}\in\mathbb{H}_d :\ \hat{q}\cdot\hat{q} = 1\}$.

E. Error Dual Quaternions and Attitude and Position Control

Assume that the desired orientation of the body (which might be a function of time) is given with respect to the inertial frame by the unit quaternion $q_{\rm D}$. Now assume also that the desired position of the center of mass of the body with respect to the inertial frame (which might also be a function of time) is given by the translation vector $\bar{t}_{\rm D}$. Then, we define the *desired dual quaternion* as $\hat{q}_{\rm D} = q_{\rm D} + \epsilon\frac{1}{2}t^I_{\rm D}q_{\rm D} = q_{\rm D} + \epsilon\frac{1}{2}q_{\rm D}t^{\rm D}_{\rm D}$, where $t^{\rm D}_{\rm D} = (\bar{t}^{\rm D}_{\rm D}, 0)$ and $\bar{t}^{\rm D}_{\rm D}$ are the coordinates of the desired translation vector $\bar{t}_{\rm D}$ expressed in the desired frame.

By direct analogy to the quaternion case, the *error dual quaternion* [8], [5] between the desired dual quaternion and the current dual quaternion is defined as $\hat{q}_e = \hat{q}^*_{\rm D}\hat{q} = q_e + \epsilon\frac{1}{2}q_e(t^B - t^B_{\rm D})$. Hence, the error dual quaternion $\hat{q}_e$ represents the rotation (q_e) and the translation ($\bar{t} - \bar{t}_{\rm D}$) necessary to align the desired frame with the body frame. Since $\hat{q}_e\cdot\hat{q}_e = (\hat{q}^*_{\rm D}\hat{q})(\hat{q}^*_{\rm D}\hat{q})^* = \hat{q}^*_{\rm D}\hat{q}\hat{q}^*\hat{q}_{\rm D} = \hat{q}^*_{\rm D}\hat{q}_{\rm D} = 1$ the dual error quaternion $\hat{q}_e = \hat{q}^*_{\rm D}\hat{q}$ is a unit dual quaternion.

F. Rigid Body Kinematics in terms of Dual Quaternions

The combined translational and rotational kinematic equations of a rigid body expressed in terms of dual quaternions are [1]

$$\dot{\hat{q}} = \tfrac{1}{2}\hat{\omega}^I\hat{q} = \tfrac{1}{2}\hat{q}\hat{q}^*\hat{\omega}^I\hat{q} = \tfrac{1}{2}\hat{q}\hat{\omega}^B, \tag{8}$$

where $\hat{\omega}^I = \omega^I + \epsilon(v^I + t^I\times\omega^I)$ and $\hat{\omega}^B = \omega^B + \epsilon v^B$ are the so-called *dual velocity* of the body expressed in inertial coordinates and body coordinates, respectively, $v^B = (\bar{v}^B, 0)$, and $\bar{v}^B$ are the body coordinates of the velocity vector of the body's center of mass with respect to the inertial frame. If $\dot{\hat{q}}_{\rm D} = 0$, then it can be shown that the kinematic equations in terms of error dual quaternions are

$$\dot{\hat{q}}_e = \tfrac{1}{2}\hat{q}_e\hat{\omega}^B. \tag{9}$$

Note that (8) and (9) have the same form as (1) and (2), respectively.

III. RIGID BODY DYNAMICS IN TERMS OF DUAL QUATERNIONS

Whereas much as been published about dual quaternions and rigid body kinematics [2], [6], [3], [1], the formulation of rigid body dynamics in terms of dual quaternions has been seldom addressed. In [5] and [7], the rigid body dynamics are written component-wise in terms of the real and dual parts of $\dot{\hat{\omega}}^B$ and not as an operation on dual objects (as expression (8) for the rigid body kinematics). In [4] the rigid body dynamics are written in terms of the second derivative of the dual quaternion ($\ddot{\hat{q}}$). Although this formulation is mathematically correct, in most cases $\bar{\omega}^B$ and $\bar{v}^B$ are directly measured by on-board sensors, and it is thus easier to implement feedback control laws based on $\hat{\omega}^B$ rather than on $\dot{\hat{q}}$.

In [8], the authors write the combined rotational and translational dynamics using the *dual inertia operator* defined in [22] as $\hat{M}^B = m\frac{\rm d}{{\rm d}\epsilon}I_4 + \epsilon I^B$, where m is the mass of the body,

$$I^B = \begin{bmatrix}\bar{I}^B & 0_{3\times1}\\ 0_{1\times3} & 1\end{bmatrix}, \tag{10}$$

and $\bar{I}^B\in\mathbb{R}^{3\times3}$ is the mass moment of inertia of the body about its center of mass written in the body frame. The operator $\frac{\rm d}{{\rm d}\epsilon}$ is defined by $\frac{\rm d}{{\rm d}\epsilon}\hat{a} = \frac{\rm d}{{\rm d}\epsilon}(a_r + \epsilon a_d) = a_d$ and $\left(\frac{\rm d}{{\rm d}\epsilon}\right)^2 = 0$, where $\hat{a} = a_r + \epsilon a_d$ is a dual object. In this work, we replace the dual inertia operator by, what we call, the *dual inertia matrix*. We define the *dual inertia matrix* as the following 8-by-8 symmetric matrix:

$$M^B = \begin{bmatrix} mI_3 & 0_{3\times1} & 0_{3\times3} & 0_{3\times1}\\ 0_{1\times3} & 1 & 0_{1\times3} & 0\\ 0_{3\times3} & 0_{3\times1} & \bar{I}^B & 0_{3\times1}\\ 0_{1\times3} & 0 & 0_{1\times3} & 1\end{bmatrix}. \tag{11}$$

We can then write the rigid body dynamics as

$$M^B\star(\dot{\hat{\omega}}^B)^{\sf s} = \hat{f}^B - \hat{\omega}^B\times(M^B\star(\hat{\omega}^B)^{\sf s}). \tag{12}$$

Note that like in [8], [22], we assume that the mass and inertia matrix are constant. In (12), $\hat{f}^B$ are the body coordinates of the total external *dual force* applied to the body about its center of mass, $\hat{f}^B = f^B + \epsilon\tau^B$, $f^B = (\bar{f}^B, 0)$, $\bar{f}^B$ are the body coordinates of the total external force vector applied to the body, $\tau^B = (\bar{\tau}^B, 0)$, and $\bar{\tau}^B$ are the body coordinates of the total external moment vector applied to the body about the center of mass of the body.

Our formulation based on the dual inertia matrix has two advantages over the formulation used in [8]. First, the inverse of M^B is simply $(M^B)^{-1}$, i.e., the matrix inverse of M^B. This is not the case with the inverse of the dual inertia operator. Note that the inverse of $\hat{M}^B$ is defined in [8] as $(\hat{M}^B)^{-1} = (I^B)^{-1}\frac{\rm d}{{\rm d}\epsilon} + \epsilon\frac{1}{m}I_4$. Second, the multiplication of a 8-by-8 matrix with a dual quaternion is a more general operation than the multiplication of operator $\frac{\rm d}{{\rm d}\epsilon}$ with a dual quaternion. In particular, multiplication (3) will allow us to operate on general Linear Time-Invariant (LTI) systems whose input, output, and state are dual quaternions, something which is not straightforward with the use of the $\frac{\rm d}{{\rm d}\epsilon}$ operator.

IV. POSITION AND ATTITUDE SETPOINT CONTROL WITH ERROR DUAL QUATERNION AND DUAL VELOCITY FEEDBACK

In [5] and [7], regulation and tracking laws are suggested based on the feedback of the dual velocity and of the logarithm of the error dual quaternion. However, the control law is not written in terms of the dual force ($\hat{f}^B$), but in terms of a dual quaternion defined component-wise in terms of its real and dual parts as a function of $\bar{f}$ and $\bar{\tau}$. More similar to our approach, the authors of [8] design a tracking law for a leader-follower spacecraft formation written in terms of the dual force. They propose an adaptive

[2]Note that 1 here represents the dual number $(\bar{0}, 1) + \epsilon(\bar{0}, 0)$.

Terminal Sliding Mode (TSM) control law based on the special operator $\frac{\mathrm{d}}{\mathrm{d}\epsilon}$. Below, we propose an alternative control law in terms of the dual force. The proposed control law does not involve the special operator $\frac{\mathrm{d}}{\mathrm{d}\epsilon}$ and can be readily extended to a control law that does not need (dual) velocity feedback, thus extending the results of [15], [16] for the case of combined translational and rotational motion.

Theorem 1. *Consider the rigid body kinematic and dynamic equations* (9) *and* (12). *If the input dual force is defined by the feedback control law*

$$\hat{f}^B = -k_p \mathrm{vec}\big(\hat{q}_e^*(\hat{q}_e^{\mathsf{s}} - \epsilon)\big) - k_d (\hat{\omega}^B)^{\mathsf{s}}, \quad k_p, k_d > 0, \tag{13}$$

then $\hat{q}_e \to \pm 1$ (i.e., $q_e \to \pm 1$ and $t^B - t^B_{\mathrm{D}} \to 0$) and $\hat{\omega}^B \to 0$ (i.e., $\omega^B \to 0$ and $v^B \to 0$) as $t \to +\infty$ for any initial condition.

Proof. First, note that $\hat{q}_e = \pm 1$ and $\hat{\omega}^B = 0$ are, in fact, the equilibrium conditions for the closed-loop system formed by (12), (9), and (13). Then, consider the following candidate Lyapunov function for the equilibrium point $\hat{q}_e = +1$ and $\hat{\omega}^B = 0$ (or equivalently, $(\hat{\omega}^B)^{\mathsf{s}} = 0$) motivated by Eq. (7) of Ref. [15]:

$$V(\hat{q}_e, \hat{\omega}^B) = k_p(\hat{q}_e - 1) \circ (\hat{q}_e - 1) + \tfrac{1}{2}(\hat{\omega}^B)^{\mathsf{s}} \circ (M^B \star (\hat{\omega}^B)^{\mathsf{s}}).$$

It can be easily shown that V is a valid candidate Lyapunov function since $V(\hat{q}_e = 1, \hat{\omega}^B = 0) = 0$ and $V(\hat{q}_e, \hat{\omega}^B) > 0$ for $(\hat{q}_e, \hat{\omega}^B) \in \mathbb{H}_d^u \times \mathbb{H}_d^v \backslash \{1, 0\}$. The time derivative of V is equal to $\dot{V}(\hat{q}_e, \hat{\omega}^B) = k_p 2(\hat{q}_e - 1) \circ \dot{\hat{q}}_e + (\hat{\omega}^B)^{\mathsf{s}} \circ (M^B \star (\dot{\hat{\omega}}^B)^{\mathsf{s}})$. Then, by plugging in (12) and using (5) and (6), it follows that

$$\begin{aligned}\dot{V}(\hat{q}_e, \hat{\omega}^B) &= k_p\, (\hat{\omega}^B) \overset{\mathsf{s}}{\circ} (\hat{q}_e^*((\hat{q}_e)^{\mathsf{s}} - \epsilon)) + (\hat{\omega}^B) \overset{\mathsf{s}}{\circ} \hat{f}^B \\ &= (\hat{\omega}^B) \overset{\mathsf{s}}{\circ} \big(k_p \mathrm{vec}\big(\hat{q}_e^*((\hat{q}_e)^{\mathsf{s}} - \epsilon)\big)\big) + (\hat{\omega}^B) \overset{\mathsf{s}}{\circ} \hat{f}^B \\ &= (\hat{\omega}^B) \overset{\mathsf{s}}{\circ} (k_p \mathrm{vec}\big(\hat{q}_e^*((\hat{q}_e)^{\mathsf{s}} - \epsilon)\big) + \hat{f}^B).\end{aligned} \tag{14}$$

Introducing the feedback control law (13), we get $\dot{V}(\hat{q}_e, \hat{\omega}^B) = -k_d\, (\hat{\omega}^B)^{\mathsf{s}} \circ (\hat{\omega}^B)^{\mathsf{s}} \leq 0$, for $(\hat{q}_e, \hat{\omega}^B) \in \mathbb{H}_d^u \times \mathbb{H}_d^v \backslash \{1, 0\}$. Since V is continuously differentiable, radially unbounded, positive definite, and $\dot{V} \leq 0$ over the entire state space, by using LaSalle's invariance principle, all trajectories must converge to the largest invariant set $\mathcal{M}$ inside $\{(\hat{q}_e, \hat{\omega}^B) : \dot{V} = 0\} = \{(\hat{q}_e, \hat{\omega}^B) : \hat{\omega}^B = 0\}$. In this invariant set, we have that $\mathrm{vec}\big(\hat{q}_e^*((\hat{q}_e)^{\mathsf{s}} - \epsilon)\big) = 0$ from (12) and (13). This can be rewritten as

$$\begin{aligned}\hat{q}_e^*((\hat{q}_e)^{\mathsf{s}} - \epsilon) &= \big(q_e^* + \tfrac{1}{2}\epsilon(t^B - t^B_{\mathrm{D}})^* q_e^*\big)\big(\tfrac{1}{2}q_e(t^B - t^B_{\mathrm{D}}) + \epsilon(q_e - 1)\big) \\ &= \tfrac{1}{2}(t^B - t^B_{\mathrm{D}}) + \epsilon\big(1 - q_e^* + \tfrac{1}{4}\|(t^B - t^B_{\mathrm{D}})\|^2\big).\end{aligned}$$

Retrieving only the vector part of the previous equation yields $\mathrm{vec}\big(\hat{q}_e^*((\hat{q}_e)^{\mathsf{s}} - \epsilon)\big) = \mathrm{vec}\big(\frac{1}{2}(t^B - t^B_{\mathrm{D}})\big) + \epsilon \mathrm{vec}\big(1 - q_e^* + \frac{1}{4}\|(t^B - t^B_{\mathrm{D}})\|^2\big) = \mathrm{vec}\big(\frac{1}{2}(t^B - t^B_{\mathrm{D}})\big) + \epsilon\, \mathrm{vec}\big(-q_e^*\big) = \frac{1}{2}(t^B - t^B_{\mathrm{D}}) + \epsilon \mathrm{vec}\big(q_e\big)$. Hence, in $\mathcal{M}$, we have that $\mathrm{vec}\big(\hat{q}_e^*((\hat{q}_e)^{\mathsf{s}} - \epsilon)\big) = \frac{1}{2}(t^B - t^B_{\mathrm{D}}) + \epsilon\, \mathrm{vec}\big(q_e\big) = 0$, which is satisfied if and only if $t^B - t^B_{\mathrm{D}} = 0$ and $\mathrm{vec}\big(q_e\big) = 0$. The latter condition is equivalent to $q_e = \pm 1$, which, along with the previous condition $t^B - t^B_{\mathrm{D}} = 0$, finally yields $\hat{q}_e = \pm 1$. It follows that $\mathcal{M} = \{(\hat{q}_e, \hat{\omega}^B) : \hat{q}_e = \pm 1, \hat{\omega}^B = 0\}$. Hence, $\hat{q}_e \to \pm 1$ (i.e., $q_e \to \pm 1$ and $t^B - t^B_{\mathrm{D}} \to 0$) and $\hat{\omega}^B \to 0$ (i.e., $\omega^B \to 0$ and $v^B \to 0$) as $t \to +\infty$ for any initial condition. □

Remark 1. The proof of Theorem 1 shows that $\hat{q}_e$ converges to either $+1$ or $\hat{q}_e = -1$. As a matter of fact, all solutions converge to $\hat{q}_e = +1$ except for the solution starting at $\hat{q}_e = -1$, in which case the system remains in $\hat{q}_e = -1$. Note, however, that $\hat{q}_e = +1$ and $\hat{q}_e = -1$ represent the same physical relative orientation and position between frames so either equilibrium is acceptable. This creates the annoyance however that for initial conditions close to $\hat{q}_e = -1$, a large rotation (larger than 180 degrees) will be performed, despite the fact that a shorter rotation (less than 180 degrees) to the equilibrium exists. This is a well-known issue when dealing with quaternions to describe the attitude [21], [24] and can be easily solved by switching the gain in (13) in order to follow the "short" path to the equilibrium. For details, see [6], [21], [24].

V. POSITION AND ATTITUDE CONTROL WITHOUT DUAL VELOCITY FEEDBACK

The feedback law given in Section IV for position and attitude setpoint control assumes that the error dual quaternion ($\hat{q}_e$) and the dual velocity ($\hat{\omega}^B$) are known. Theorem 2 below shows that position and attitude setpoint control can also be performed without linear and angular velocity measurements. This result follows naturally from the passivity properties [16], [15] of the systems represented inside the dashed and dotted boxes in Figure 1, which are proven in Proposition 1 and Proposition 2.

Proposition 1. *Consider the system (12)-(9) with the feedback control $\hat{f}^B = -k_p \mathrm{vec}\big(\hat{q}_e^*((\hat{q}_e)^{\mathsf{s}} - \epsilon)\big) + \hat{u}$, where $k_p > 0$. Then the map $\hat{u} \mapsto (\hat{\omega}^B)^{\mathsf{s}}$ is passive.*

Proof. By using (14), it follows that $\int_0^T (\hat{\omega}^B)^{\mathsf{s}} \circ \hat{u}\, \mathrm{d}t = \int_0^T (\hat{\omega}^B) \overset{\mathsf{s}}{\circ} (k_p \mathrm{vec}\big(\hat{q}_e^*((\hat{q}_e)^{\mathsf{s}} - \epsilon)\big) + \hat{f}^B)\, \mathrm{d}t = \int_0^T \dot{V}(\hat{q}_e, \hat{\omega}^B)\, \mathrm{d}t = V(\hat{q}_e(T), \hat{\omega}^B(T)) - V(\hat{q}_e(0), \hat{\omega}^B(0)) \geq -V(\hat{q}_e(0), \hat{\omega}^B(0))$, for all $T \geq 0$. Since the integral $\int_0^T (\hat{\omega}^B)^{\mathsf{s}} \circ \hat{u}\, \mathrm{d}t$ is bounded from below for all $T \geq 0$, the map from $\hat{u}$ to $(\hat{\omega}^B)^{\mathsf{s}}$ is passive [25]. □

Proposition 2. *Let the feedback system (12)-(9) have the feedback control of Proposition 1 and let $\hat{u} = 2\mathrm{vec}\big(\hat{q}_e^* \hat{v}^{\mathsf{s}}\big)$. Then the map $\hat{v} \mapsto \dot{\hat{q}}_e$ is passive.*

Proof. Note that $\int_0^T \dot{\hat{q}}_e \circ \hat{v}\, \mathrm{d}t = \int_0^T (\frac{1}{2}\hat{q}_e \hat{\omega}^B) \circ \hat{v}\, \mathrm{d}t = \int_0^T (\hat{\omega}^B) \overset{\mathsf{s}}{\circ} (\frac{1}{2}\hat{q}_e^* \hat{v}^{\mathsf{s}})\, \mathrm{d}t = \int_0^T (\hat{\omega}^B) \overset{\mathsf{s}}{\circ} (\frac{1}{2}\mathrm{vec}\big(\hat{q}_e^* \hat{v}^{\mathsf{s}}\big))\, \mathrm{d}t = \frac{1}{4}\int_0^T (\hat{\omega}^B) \overset{\mathsf{s}}{\circ} \hat{u}\, \mathrm{d}t$, and hence, by Proposition 1, the map from $\hat{v}$ to $\dot{\hat{q}}_e$ is passive. □

The previous propositions implies that a stabilizing controller can then be designed by ensuring that it creates a passive (or strictly passive) map from $\dot{\hat{q}}_e$ to $\hat{z}$ (see Figure 1). This observation is formalized by the next theorem.

Theorem 2. *Consider the rigid body kinematic and dynamic equations* (12) *and* (9). *Let the input dual force be defined by the feedback control law*

$$\hat{f}^B = -k_p \mathrm{vec}\big(\hat{q}_e^*((\hat{q}_e)^{\mathsf{s}} - \epsilon)\big) - 2\, \mathrm{vec}\big(\hat{q}_e^* \hat{z}^{\mathsf{s}}\big), \quad k_p > 0, \tag{15}$$

where $\hat{z}$ is the output of the following LTI system $\dot{\hat{x}}_p = A \star \hat{x}_p + B \star \dot{\hat{q}}_e$, $\hat{z} = (CA) \star \hat{x}_p + (CB) \star \dot{\hat{q}}_e$, where (A, B, C) is a minimal realization of a strictly positive real transfer

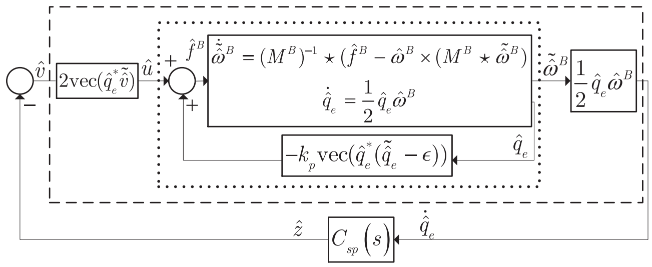

Fig. 1. Feedback interconnection.

matrix $C_{sp}(s)$ with B a full rank matrix. Then, $\hat{q}_e \to \pm 1$, $\hat{\omega}^B \to 0$, and $\hat{x}_{sp} = \dot{\hat{x}}_p \to 0$ as $t \to +\infty$ for any initial condition.

Proof. First, rewrite the LTI system as follows

$$\dot{\hat{x}}_{sp} = A \star \hat{x}_{sp} + B \star \dot{\hat{q}}_e, \;\; \hat{z} = C \star \hat{x}_{sp}, \tag{16}$$

Notice that the closed-loop system is the negative feedback interconnection between the strictly positive real system $C_{sp}(s)$ and the system inside the dashed box in Figure 1. Furthermore, note that $\hat{q}_e = \pm 1$, $\hat{\omega}^B = 0$, and $\hat{x}_{sp} = 0$ is the equilibrium condition for the closed-loop system formed by (12), (9), (16), and (15). Consider the candidate Lyapunov function motivated by Eq. 19 of Ref. [15], $V(\hat{q}_e, \hat{\omega}^B, \hat{x}_{sp}) = k_p(\hat{q}_e - 1) \circ (\hat{q}_e - 1) + \frac{1}{2}(\hat{\omega}^B)^{\mathsf{s}} \circ (M^B \star (\hat{\omega}^B)^{\mathsf{s}}) + 2\hat{x}_{sp} \circ (P \star \hat{x}_{sp})$, for the equilibrium point $\hat{q}_e = 1$, $\hat{\omega}^B = 0$, and $\hat{x}_{sp} = 0$, where $P > 0$ satisfies $A^{\mathsf{T}}P + PA = -Q$, $PB = C^{\mathsf{T}}$, and $Q > 0$. By the Kalman-Yakubovich-Popov (KYP) conditions [25], there always exist matrices P and Q satisfying these conditions. The time derivative of V is equal to $\dot{V}(\hat{q}_e, \hat{\omega}^B, \hat{x}_{sp}) = k_p(\hat{q}_e - 1) \circ (\hat{q}_e \hat{\omega}^B) + (\hat{\omega}^B)^{\mathsf{s}} \circ (M^B \star (\dot{\hat{\omega}}^B)^{\mathsf{s}}) + 4\dot{\hat{x}}_{sp} \circ (P \star \hat{x}_{sp})$. By plugging in (12), (16), and (15), and applying (4), (5), and the KYP conditions, and after some tedious algebraic manipulations, it follows that

$$\begin{aligned}
&\dot{V}(\hat{q}_e, \hat{\omega}^B, \hat{x}_{sp}) = (\hat{\omega}^B)^{\mathsf{s}} \circ (k_p \hat{q}_e^*((\hat{q}_e)^{\mathsf{s}} - \epsilon) + \hat{f}^B) - 2\hat{x}_{sp} \\
&\circ (Q \star \hat{x}_{sp}) + 4\dot{\hat{q}}_e \circ (C \star \hat{x}_{sp}) = (\hat{\omega}^B)^{\mathsf{s}} \circ (k_p \hat{q}_e^*((\hat{q}_e)^{\mathsf{s}} - \epsilon) \\
&+ \hat{f}^B) - 2\,\hat{x}_{sp} \circ (Q \star \hat{x}_{sp}) + 4(\tfrac{1}{2}\hat{q}_e\hat{\omega}^B) \circ \hat{z} = (\hat{\omega}^B)^{\mathsf{s}} \\
&\circ (k_p \hat{q}_e^*((\hat{q}_e)^{\mathsf{s}} - \epsilon) + \hat{f}^B + 2\,\hat{q}_e^*\hat{z}^{\mathsf{s}}) - 2\,\hat{x}_{sp} \circ (Q \star \hat{x}_{sp}) \\
&= (\hat{\omega}^B)^{\mathsf{s}} \circ (k_p \hat{q}_e^*((\hat{q}_e)^{\mathsf{s}} - \epsilon) - k_p \mathrm{vec}\big(\hat{q}_e^*((\hat{q}_e)^{\mathsf{s}} - \epsilon)\big) + 2\hat{q}_e^*\hat{z}^{\mathsf{s}} \\
&- 2\mathrm{vec}\big(\hat{q}_e^*\hat{z}^{\mathsf{s}}\big)) - 2\hat{x}_{sp} \circ (Q \star \hat{x}_{sp}) = -2\,\hat{x}_{sp} \circ (Q \star \hat{x}_{sp}) \leq 0,
\end{aligned}$$

for $(\hat{q}_e, \hat{\omega}^B, \hat{x}_{sp}) \in \mathbb{H}_d^u \times \mathbb{H}_d^v \times \mathbb{H}_d \backslash \{1, 0, 0\}$. By LaSalle's invariance principle, all trajectories converge to the largest invariant set $\mathcal{M}$ inside $\{(\hat{q}_e, \hat{\omega}^B, \hat{x}_{sp}) : \dot{V} = 0\} = \{(\hat{q}_e, \hat{\omega}^B, \hat{x}_{sp}) : \hat{x}_{sp} = 0\}$. In $\mathcal{M}$ we have that $\hat{x}_{sp} \equiv 0 \Rightarrow \dot{\hat{x}}_{sp} \equiv 0$, which implies that $B \star \dot{\hat{q}}_e \equiv 0$. Since B is full rank, this implies that $\dot{\hat{q}}_e \equiv 0$. From $\dot{\hat{q}}_e = \frac{1}{2}\hat{q}_e\hat{\omega}^B \Leftrightarrow \hat{\omega}^B = 2\hat{q}_e^*\dot{\hat{q}}_e$, we get that $\hat{\omega}^B \equiv 0$, and thus, $\dot{\hat{\omega}}^B \equiv 0$. Hence, from (12), it follows that $\hat{f}^B \equiv 0$. On the other hand, $\hat{x}_{sp} \equiv 0$ leads to $\hat{z} \equiv 0$. Finally, since $\hat{f}^B \equiv \hat{z} \equiv 0$, (15) yields $\mathrm{vec}\big(\hat{q}_e^*((\hat{q}_e)^{\mathsf{s}} - \epsilon)\big) \equiv 0$, which is equivalent to $\hat{q}_e \equiv \pm 1$. Therefore, $\mathcal{M} = \{(\hat{q}_e, \hat{\omega}^B, \hat{x}_{sp}) : \hat{q}_e \equiv \pm 1, \hat{\omega}^B \equiv 0, \hat{x}_{sp} \equiv 0\}$. Hence, $(\hat{q}_e, \hat{\omega}^B, \hat{x}_{sp}) \to (\pm 1, 0, 0)$ for any initial condition. $\square$

VI. SIMULATION RESULTS

To compare the performance of control law (15) (without dual velocity feedback) against the performance of control law (13) (with dual velocity feedback), a simple example is considered here. A rigid body with mass moment of inertia

$$\bar{I}^B = \begin{bmatrix} 1 & 0.1 & 0.15 \\ 0.1 & 0.63 & 0.05 \\ 0.15 & 0.05 & 0.85 \end{bmatrix} \text{ Kg.m}^2$$

and mass $m = 1$ Kg is positioned at $\bar{t}^I = [x\ y\ z]^{\mathsf{T}} = [2\ 2\ 1]^{\mathsf{T}}$ m and has initial attitude equal to $q_e = [q_{e1}\ q_{e2}\ q_{e3}\ q_{e4}]^{\mathsf{T}} = [0.4618\ \ 0.1917\ \ 0.7999\ \ 0.3320]^{\mathsf{T}}$. The body's initial linear and angular velocity are equal to $\bar{v}^B = [u\ v\ w]^{\mathsf{T}} = [0.1\ -0.2\ 0.3]^{\mathsf{T}}$ m/s and $\bar{\omega}^B = [p\ q\ r]^{\mathsf{T}} = [-0.1\ 0.2\ -0.3]^{\mathsf{T}}$ rad/s, respectively. The control objective is to bring the center of mass of the body to the origin of the inertial frame (i.e, $\bar{t}_{\mathrm{D}}^I = 0$) and to align the body axes with the inertial axes (i.e., $q_{\mathrm{D}} = 1$). The control gains are chosen to be $k_p = 0.2$ (both in (13) and (15)) and $k_d = 0.4$ (in (13)). For simplicity, A and B are chosen as $-k_f I_8$ and $k_f I_8$, respectively. For this choice of A and B and, by defining $Q = -k_d(B^{-T}A + (B^{-1}A)^{\mathsf{T}})$ [15], the KYP conditions yield $P = k_d B^{-T}$ and $C = k_d I_8$. Moreover, Q and P will always be symmetric positive definite matrices for this choice of A and B.

The position and attitude of the body with controller (13) (with dual velocity feedback) and with controller (15) (without dual velocity feedback) with $k_f = 0.5$ and $k_f = 10$ are compared in Figure 2. In all three cases, $q_e \to 1$ and

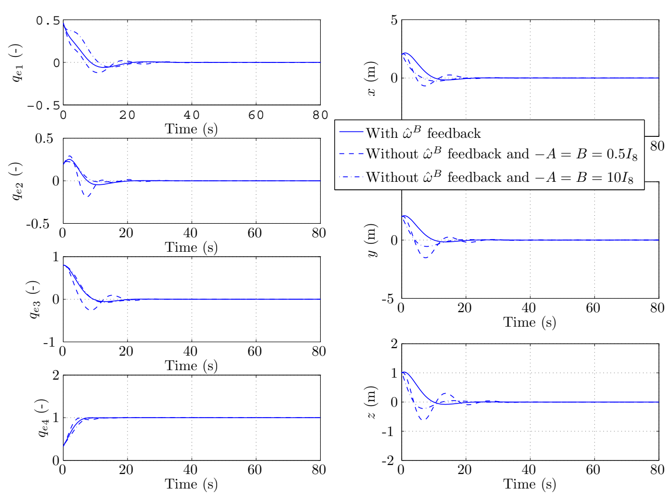

Fig. 2. Attitude and position.

$\bar{t}^I \to \bar{t}_{\mathrm{D}}^I$ as $t \to \infty$, as expected. Figure 2 also shows that by increasing k_f, the attitude transient response without dual velocity feedback can be made more similar to the attitude transient response with dual velocity feedback. Figure 3 shows the velocity and angular velocity of the body for the same three cases studied in Figure 2. As expected, $\bar{\omega}^B \to 0$ and $\bar{v}^B \to 0$ as $t \to \infty$. Moreover, increasing k_f as the same effect as in Figure 2. Finally, Figure 4 shows the control force, $\bar{f}^B$ and the control torque, $\bar{\tau}^B$ applied to the body for the same three cases.

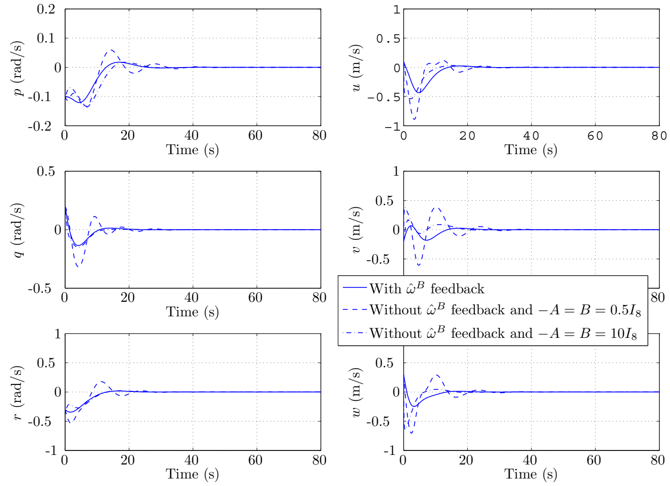

Fig. 3. Velocity and angular velocity.

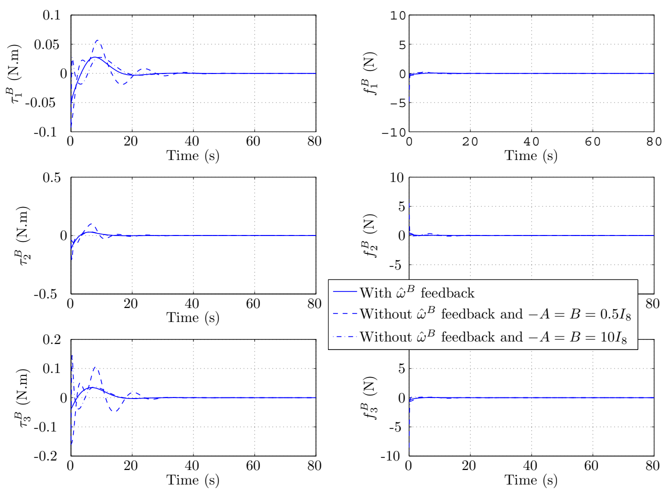

Fig. 4. Control force and torque.

VII. CONCLUSION

A velocity-free setpoint controller for the position and orientation of a rigid body is presented in this paper. It can be used in the case of a sensor malfunction or in vehicles not equipped with linear and angular velocity sensors. Also, and more importantly, this paper shows how it is relatively straightforward to extend attitude controllers based on quaternions into combined position and attitude controllers based on dual quaternions. Future work includes extending these results to the tracking case.

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