Optimal Pursuer and Moving Target Assignment using Dynamic Voronoi Diagrams

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***Abstract*— We consider a Voronoi-like partitioning problem for a team of pursuers distributed in the plane. Each element of the partition is uniquely associated with a pursuer in the following sense: if a moving target at a given instant of time resides inside a particular member of the partition, then the pursuer associated with this set can intercept this moving target faster than any other pursuer. In our problem formulation, the moving target does not necessarily travel along prescribed trajectories, as it is typically assumed in the literature but, instead, it can apply an "evading" strategy in response to the actions of its pursuer. It is further assumed that the structure of the evading strategy of the target is only partially known to the pursuers. We characterize an approximate solution to this problem by associating it with a standard Voronoi partitioning problem. Simulation results are presented to highlight the theoretical developments.**

I. Introduction

We address a Voronoi-like partitioning problem for a set of pursuers (moving generators) whose objective is to capture moving targets in the plane. The solution of this problem furnishes a scheme that assigns a pursuer from a given team of pursuers to a moving target with respect to a generalized proximity metric, namely the minimum capture time (rather than with the Euclidean distance metric as in the standard Voronoi diagram problem). The problem considered in this work can be put under the umbrella of dynamic Voronoi diagram problems, that is, Voronoi-like partitioning problems where the generators are moving points in the plane [1]–[8]. Specifically, we consider the following partitioning problem: Given a team of n vehicles (pursuers), which are distributed over n distinct locations in the plane, partition the plane into n "capture zones," such that each pursuer is assigned to a unique capture zone. The rule that assigns each pursuer to a capture zone is the following: a pursuer associated with a particular capture zone, can capture a moving target traveling within the same zone at a given instant of time, faster than any other pursuer from the given set of pursuers. In our problem formulation, we do not constraint the moving target to follow a prescribed trajectory, as it is usually assumed in the literature [3], [4]. Instead, the target can apply an "evading" strategy in response to the actions of its pursuer. The target's strategy is a feedback control law that depends only on the relative position between the moving target and its pursuer.

In the special case, when the "evading" strategy of the target is perfectly known to the pursuers, one deals with a problem of *pursuit-with-anticipation* [9]. It turns out that in this case, the locally optimal control strategy of each pursuer can be derived from the solution of the classic Zermelo's navigation problem (ZNP for short). The partitioning problem for this pursuit-with-anticipation scenario was addressed in our previous work in [10]. In contrast to the approach presented in [10], in the current framework, we assume that the pursuers have only partial knowledge of the evading strategy of the target. The standing assumption of the proposed approach is that the projection of the target's velocity on the relative position vector of the moving target from its pursuer is only a function of the relative distance between the target and its pursuer. Under the previous assumptions, it is shown that the globally optimal control strategy for each pursuer can be characterized in feedback form by making use of the results presented in [11], [12]. It turns out in this case that the feedback control law that solves the optimal pursuit problem is completely independent of the evading strategy of the target. Furthermore, it is demonstrated that the minimum capture time is a monotone function of the relative distance between the pursuer and the target, thus allowing us to associate the solution of the partitioning problem with the standard Voronoi diagram generated by the initial positions of the pursuers.

The rest of the paper is organized as follows. Sections II and III present the formulation and the feasibility of the optimal pursuit problem, respectively. Subsequently, Sections V and IV present the formulation and an approximate solution of the dynamic partitioning problem, respectively. Section VI gives a short comparison of the proposed scheme and the approach followed in our previous work [10]. Simulation results are presented in Section VII. Finally, Section VIII concludes the paper with a summary of remarks.

II. Formulation of the Optimal Pursuit Problem

Consider a team of n pursuers located at time $t=0$ at n distinct points in the plane, denoted by $\mathcal{P} := \{\bar{\mathsf{x}}_{\mathcal{P}}^i \in \mathbb{R}^2, i \in \mathcal{I}\}$, where $\mathcal{I} := \{1,\ldots,n\}$. It is assumed that the kinematics of the i^{th} pursuer starting at point $\bar{\mathsf{x}}_{\mathcal{P}}^i \in \mathcal{P}$ are given by

$$\dot{\mathsf{x}}_{\mathcal{P}}^i = u_{\mathcal{P}}^i, \quad \mathsf{x}_{\mathcal{P}}^i(0) = \bar{\mathsf{x}}_{\mathcal{P}}^i, \tag{1}$$

where $\mathsf{x}_{\mathcal{P}}^i := (x_{\mathcal{P}}^i, y_{\mathcal{P}}^i) \in \mathbb{R}^2$ and $\bar{\mathsf{x}}_{\mathcal{P}}^i := (\bar{x}_{\mathcal{P}}^i, \bar{y}_{\mathcal{P}}^i) \in \mathbb{R}^2$ denote the position vectors of the i^{th} pursuer at time t and $t=$

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0, respectively, and $u^i_{\mathcal{P}}$ is the control input (velocity vector) of the i^{th} pursuer. We assume that $u^i_{\mathcal{P}} \in \mathcal{U}_{\mathcal{P}}$, where $\mathcal{U}_{\mathcal{P}}$ consists of all piece-wise continuous functions taking values in the set $U_{\mathcal{P}} = \{\mathsf{z} \in \mathbb{R}^2 : |\mathsf{z}| \leq \bar{u}_{\mathcal{P}}\}$, where $\bar{u}_{\mathcal{P}}$ is a positive constant (maximum allowable speed of the pursuers). The goal of each pursuer, which is initially located at a point in $\mathcal{P}$, is to capture a moving target detected in its vicinity. It is assumed that the kinematics of such a moving target are described by

$$\dot{\mathsf{x}}_{\mathcal{T}} = u_{\mathcal{T}}, \quad \mathsf{x}_{\mathcal{T}}(0) = \bar{\mathsf{x}}_{\mathcal{T}}, \tag{2}$$

where $\mathsf{x}_{\mathcal{T}} := (x_{\mathcal{T}}, y_{\mathcal{T}}) \in \mathbb{R}^2$ and $\bar{\mathsf{x}}_{\mathcal{T}} := (\bar{x}_{\mathcal{T}}, \bar{y}_{\mathcal{T}}) \in \mathbb{R}^2$ denote the target's position vectors at time t and $t = 0$, respectively, and $u_{\mathcal{T}}$ is the control input (velocity vector) of the target. It is further assumed that the moving target can employ an evading strategy in response to the pursuer's actions. In particular, $u_{\mathcal{T}}$ is a feedback control law, which depends on the relative position of the target from the i^{th} pursuer, that is, $u_{\mathcal{T}} = u_{\mathcal{T}}(\mathsf{x}_{\mathcal{T}} - \mathsf{x}^i_{\mathcal{P}})$.

Assumption 1: There exists a Lipschitz continuous function $f : \mathbb{R}_+ \mapsto \mathbb{R}$ such that the evading strategy $u_{\mathcal{T}}$ of the target satisfies the following condition

$$\langle u_{\mathcal{T}}, \mathsf{x}_{\mathcal{T}} - \mathsf{x}^i_{\mathcal{P}} \rangle = f(|\mathsf{x}_{\mathcal{T}} - \mathsf{x}^i_{\mathcal{P}}|). \tag{3}$$

The interpretation of Assumption 1 is as follows: The projection of the velocity vector of the moving target on the relative position vector of the moving target from the i^{th} pursuer depends only on the relative distance between the target and its pursuer. Furthermore, in this work, we do not explicitly assume that the maximum allowable speed of the target is strictly less than $\bar{u}_{\mathcal{P}}$. In order, however, to avoid situations where the maneuvering target can always escape capture if it is faster than its pursuer, it is assumed that the structure of the evading strategy of the target is partially known to each pursuer. Specifically, we assume that

$$f(\mathsf{z}) \leq \bar{f}(\mathsf{z}), \quad \text{for all } \mathsf{z} \geq 0, \tag{4}$$

where $\bar{f}(\cdot)$ is a continuous function, which is known to all of the pursuers. The function $\bar{f}$ provides a bound on the rate at which the target can move away from its pursuer. As it will be shown in the sequel, condition (4) will allow us to approximate the *winning set* of the i^{th} pursuer, that is, the set of initial positions of a moving target from which the i^{th} pursuer can capture the target in finite time.

To this end, let $\mathsf{x}_{\mathcal{T}}(\cdot; u_{\mathcal{T}}, \bar{\mathsf{x}}_{\mathcal{T}})$ and $\mathsf{x}^i_{\mathcal{P}}(\cdot; u^i_{\mathcal{P}}, \bar{\mathsf{x}}^i_{\mathcal{P}})$ denote the trajectories of the target and the i^{th} pursuer generated by $u_{\mathcal{T}}$ and $u^i_{\mathcal{P}}$ and originating from $\bar{\mathsf{x}}_{\mathcal{T}}$ and $\bar{\mathsf{x}}^i_{\mathcal{P}}$, respectively. The objective of each pursuer is to determine an admissible pursuit strategy that minimizes the time T_{f} such that $|\mathsf{x}_{\mathcal{T}}(t; u_{\mathcal{T}}, \bar{\mathsf{x}}_{\mathcal{T}}) - \mathsf{x}^i_{\mathcal{P}}(t; u^i_{\mathcal{P}}, \bar{\mathsf{x}}^i_{\mathcal{P}})| > \epsilon_c$ for all $t < T_{\mathsf{f}}$ (*time of first capture*), for a sufficiently small $\epsilon_c > 0$, where ϵ_c is the *capturability radius* of the pursuit problem.

To this end, let us consider the state transformation $\mathsf{y}^i := \mathsf{x}_{\mathcal{T}} - \mathsf{x}^i_{\mathcal{P}}$. Equation (1) can then be written in the following compact form

$$\dot{\mathsf{y}}^i = \mathsf{u}^i + u_{\mathcal{T}}(\mathsf{y}^i), \quad \mathsf{y}^i(0) = \bar{\mathsf{y}}^i := \bar{\mathsf{x}}_{\mathcal{T}} - \bar{\mathsf{x}}^i_{\mathcal{P}}, \tag{5}$$

where $\mathsf{u}^i := -u^i_{\mathcal{P}}$. Thus, the optimal pursuit strategy of the i^{th} pursuer follows from the solution of the following minimum-time problem.

Problem 1 (i^{th} MTP): Let the system described by equation (5), and let $u_{\mathcal{T}}$ satisfy Assumption 1. Determine the control input $\mathsf{u}^i \in \mathcal{U}_{\mathcal{P}}$ such that

i) The trajectory $\mathsf{y}^i_* : [0, T_{\mathsf{f}}] \mapsto \mathbb{R}^2$ generated by the control u^i_* satisfies the boundary conditions

$$\mathsf{y}^i_*(0) = \bar{\mathsf{y}}^i, \quad |\mathsf{y}^i_*(T_{\mathsf{f}})| \leq \epsilon_c. \tag{6}$$

ii) The control u^i_* minimizes, along the trajectory y^i_*, the cost functional $J(\mathsf{u}^i) := T_{\mathsf{f}} = T_{\mathsf{f}}(\bar{\mathsf{y}}^i)$.

Problem 1 can be interpreted as a problem of steering an integrator from $\bar{\mathsf{y}}^i$ to a ball of radius ϵ_c centered at the origin, in the presence of a spatially-varying drift $u_{\mathcal{T}}(\mathsf{y}^i)$ in minimum-time. If the function $u_{\mathcal{T}}$ is perfectly known to the pursuers, then Problem 1 can be reduced to a special case of Zermelo's navigation problem. Here we employ, however, a different approach that will allow us to characterize the unique, global solution of Problem 1 in closed form, which does not follow directly from the solution of the ZNP. The following proposition gives the solution of Problem 1.

Proposition 1: If Problem 1 is feasible, then its solution is unique, and it is given in feedback form as follows

$$\mathsf{u}^i_* = -\bar{u}_{\mathcal{P}} \frac{\mathsf{y}^i_*}{|\mathsf{y}^i_*|}. \tag{7}$$

Proof: Let $|\mathsf{y}^i|^2 = \langle \mathsf{y}^i, \mathsf{y}^i \rangle$ and suppose that y^i is a trajectory generated from some admissible control u^i on $[0, T_{\mathsf{f}}]$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathsf{y}^i|^2 = \frac{\mathrm{d}}{\mathrm{d}t}\langle \mathsf{y}^i, \mathsf{y}^i \rangle = 2\langle \mathsf{y}^i, \mathsf{u}^i + u^i_{\mathcal{T}}(\mathsf{y}^i) \rangle. \tag{8}$$

In light of Assumption 1, and equations (5) and (8), it follows that, for all $t \in [0, T_{\mathsf{f}}]$,

$$\dot{\eta}^i = \frac{f(\eta^i)}{\eta^i} + v^i, \quad \eta^i(0) = \bar{\eta}^i := |\bar{\mathsf{y}}^i|, \tag{9}$$

where $\eta^i := |\mathsf{y}^i|$ and v^i is a new scalar control input given by

$$v^i := \frac{\langle \mathsf{u}^i, \mathsf{y}^i \rangle}{\eta^i}. \tag{10}$$

First, we show that $\eta^i(t) = |\mathsf{y}^i(t)| > 0$ for all $t \in [0, T_{\mathsf{f}}]$. Indeed, let us assume that $|\bar{\mathsf{y}}^i| > \epsilon_c$ (if $|\bar{\mathsf{y}}^i| \leq \epsilon_c$, then the i^{th} MTP admits a trivial solution and $T_{\mathsf{f}} = 0$). By continuity, if $\eta^i(t_1) = 0$ for some $t_1 > 0$, then there exists $t_2 < t_1$ such that $\eta^i(t_2) = \epsilon_c$. By definition, $T_{\mathsf{f}} = \inf\{\tau : \eta^i(\tau) = \epsilon_c\}$. It follows that $T_{\mathsf{f}} \leq t_2 < t_1$, and hence $\eta^i(t) \geq \epsilon_c > 0$, for all $t \in [0, T_{\mathsf{f}}]$. It follows that the rhs of equation (9) is well-defined, and $\dot{\eta}^i(t)$ exists for all $t \in [0, T_{\mathsf{f}}]$.

By virtue of the Cauchy-Schwartz inequality, it follows from (10) that $|v^i| \leq \bar{u}_{\mathcal{P}}$. Therefore, Problem 1 reduces to the problem of determining a scalar control v^i_* with $|v^i_*| \leq \bar{u}_{\mathcal{P}}$ that will steer the scalar system described by equation (9) to the interval $[0, \epsilon_c]$ in minimum time. In [11], it is shown that the solution of this scalar min-time problem is given by $v_*^i = -\bar{u}_{\mathcal{P}}$. Therefore, (10) implies that

$$\langle \mathsf{u}_*^i, \mathsf{y}_*^i \rangle = -\bar{u}_{\mathcal{P}} \eta_*^i, \tag{11}$$

which implies that u_*^i is a vector of length $\bar{u}_{\mathcal{P}}$ parallel to the unit vector $-\mathsf{y}_*^i/|\mathsf{y}_*^i|$, thus completing the proof. ■

Proposition 1 implies, in particular, that the solution of the optimal control Problem 1 is independent of the evading strategy of the target, $u_{\mathcal{T}}$. However, as we shall see next, the characterization of the winning set of the i^{th} pursuer depends on the evading strategy of the target, hence on f as well.

III. The Winning Sets of the Pursuers

Next, we examine the feasibility of Problem 1 for a given $\bar{\mathsf{y}}^i \in \mathbb{R}^2$. This will allow us to characterize the winning set of the i^{th} pursuer, that is, the set of the initial positions of the target from which it can be captured by the i^{th} pursuer in finite time. In other words, the winning set of the i^{th} pursuer is given by

$$\mathcal{W}_f(\bar{\mathsf{x}}_{\mathcal{P}}^i) := \{\mathsf{x} \in \mathbb{R}^2 : T_{\mathsf{f}}(\mathsf{x} - \bar{\mathsf{x}}_{\mathcal{P}}^i) < \infty\}, \tag{12}$$

where $T_{\mathsf{f}}(\mathsf{x} - \bar{\mathsf{x}}_{\mathcal{P}}^i)$ is the time of capture of the target by the i^{th} pursuer, when the target resides initially at x. First, note that if $|\bar{\mathsf{y}}^i| \le \epsilon_c$, then capture occurs trivially at $t = 0$. Hence, the set $\{\mathsf{y} \in \mathbb{R}^2 : |\mathsf{y}| \le \epsilon_c\}$ is necessarily a subset of the winning set for each pursuer, regardless of the dynamics of the pursuer or the target. Next, we compute the winning set for the non-trivial case $|\bar{\mathsf{y}}^i| > \epsilon_c$.

Proposition 2: Let $\epsilon_c > 0$. Then Problem 1 is feasible for the i^{th} pursuer for all $|\bar{\mathsf{y}}^i| > \epsilon_c$ if and only if

$$f(\mathsf{z}) < \bar{u}_{\mathcal{P}} \mathsf{z}, \quad \text{for all} \quad \epsilon_c \le \mathsf{z} \le |\bar{\mathsf{y}}^i|. \tag{13}$$

Proof: Proposition 1 implies that the closed loop dynamics of (5) can be written in terms of $\eta^i = |\mathsf{y}^i|$ as follows

$$\dot{\eta}^i = \frac{f(\eta^i)}{\eta^i} - \bar{u}_{\mathcal{P}}, \quad \eta^i(0) = \bar{\eta}^i. \tag{14}$$

Condition (13) implies that

$$\dot{\eta}^i = \frac{f(\eta^i)}{\eta^i} - \bar{u}_{\mathcal{P}} < 0, \quad \text{for all} \quad \epsilon_c \le \eta^i \le |\bar{\mathsf{y}}^i|. \tag{15}$$

From (15) it follows that the set $\{\mathsf{z} : 0 < \mathsf{z} \le \epsilon_c\}$ is an attractive invariant set for (14) for all initial conditions $\eta^i(0) > \epsilon_c$. Furthermore, $\dot{\eta}^i < 0$ for $\eta^i = \epsilon_c$. It follows that there exists $T = T(\epsilon_c)$, such that $\eta^i(t) \le \epsilon_c$ for $t \ge T(\epsilon_c)$, thus showing feasibility of the Problem 1.

Conversely, suppose there exists $\tilde{\eta}^i = |\tilde{\mathsf{y}}|$, where $\tilde{\mathsf{y}} \in \mathbb{R}^2$, such that $\epsilon_c \le \tilde{\eta}^i \le |\bar{\mathsf{y}}^i|$ and

$$f(\tilde{\eta}^i) \ge \bar{u}_{\mathcal{P}} \tilde{\eta}^i. \tag{16}$$

Notice that the set $S := \{\mathsf{z} : \mathsf{z} \ge \tilde{\eta}^i\}$ is invariant for (14) since $f(\mathsf{z})/\mathsf{z} - \bar{u}_{\mathcal{P}} \ge 0$ for all $\mathsf{z} \in \mathrm{bd}S$. Since $\eta^i(0) \in S$, it follows that $\eta^i(t) \ge \tilde{\eta}^i$, for all $t \ge 0$, which implies that the Problem 1 is not feasible for $\epsilon_c < \tilde{\eta}^i$. If, on the other hand, $\epsilon_c = \tilde{\eta}^i$ then either $f(\epsilon_c) > \bar{u}_{\mathcal{P}}\epsilon_c$ or $f(\epsilon_c) = \bar{u}_{\mathcal{P}}\epsilon_c$. In the first case, any trajectory starting from $\eta^i(0) > \epsilon_c$ can never reach the set $\{\mathsf{z} : \ 0 \le \mathsf{z} \le \epsilon_c\}$. In the second case, $\eta^i = \epsilon_c$ is an equilibrium solution for (14). Since the right hand side of (14) is Lipschitz continuous at $\eta^i = \epsilon_c$, this equilibrium can only be reached asymptotically [13]. In both cases, Problem 1 is infeasible. ■

Henceforth, we refer to (13) as the *capturability condition* of Problem 1. In order to characterize the winning set of the i^{th} pursuer, let

$$\bar{\eta}_f := \inf\{\mathsf{z} \in [\epsilon_c, \infty) : f(\mathsf{z}) \ge \bar{u}_{\mathcal{P}} \mathsf{z}\}. \tag{17}$$

Note that $\bar{\eta}_f \ge \epsilon_c$. If $f(\mathsf{z}) < \bar{u}_{\mathcal{P}}\mathsf{z}$ for all $\mathsf{z} \in [\epsilon_c, \infty)$, we take $\bar{\eta}_f := \infty$, and hence $\mathcal{W}_f(\bar{\mathsf{x}}_{\mathcal{P}}^i) = \mathbb{R}^2$. If $f(\mathsf{z}) \ge \bar{u}_{\mathcal{P}}\mathsf{z}$ for all $\mathsf{z} \in [\epsilon_c, \infty)$, then $\bar{\eta}_f = \epsilon_c$, and hence $\mathcal{W}_f(\bar{\mathsf{x}}_{\mathcal{P}}^i) = \{\mathsf{x} \in \mathbb{R}^2 : \ |\bar{\mathsf{x}}_{\mathcal{P}}^i - \mathsf{x}| \le \epsilon_c\}$. Finally, if $\epsilon_c < \bar{\eta}_f < \infty$, then it follows readily from (17) that $f(\mathsf{z}) < \bar{u}_{\mathcal{P}}\mathsf{z}$ for all $\epsilon_c \le \mathsf{z} < \bar{\eta}_f$ and hence, in light of Proposition 2, $\mathcal{W}_f(\bar{\mathsf{x}}_{\mathcal{P}}^i) := \{\mathsf{x} \in \mathbb{R}^2 : \ |\bar{\mathsf{x}}_{\mathcal{P}}^i - \mathsf{x}| < \bar{\eta}_f\}$. For all cases the winning set of the i^{th} pursuer can be defined compactly as

$$\mathcal{W}_f(\bar{\mathsf{x}}_{\mathcal{P}}^i) := \{\mathsf{x} : \ |\bar{\mathsf{x}}_{\mathcal{P}}^i - \mathsf{x}| < \bar{\eta}_f\} \cup \{\mathsf{x} : \ |\bar{\mathsf{x}}_{\mathcal{P}}^i - \mathsf{x}| \le \epsilon_c\}. \tag{18}$$

Note, however, that the i^{th} pursuer does not know exactly its winning set, since it has only partial knowledge of f, and consequently of $\bar{\eta}_f$ as well. As a result, each pursuer can only compute an approximation of its actual winning set. To this end, let

$$\bar{\eta}_{\bar{f}} := \inf\{\mathsf{z} \in [\epsilon_c, \infty) : \bar{f}(\mathsf{z}) \ge \bar{u}_{\mathcal{P}} \mathsf{z}\}. \tag{19}$$

In light of (4), it follows that $\bar{\eta}_{\bar{f}} \le \bar{\eta}_f$. Let

$$\mathcal{W}_{\bar{f}}(\bar{\mathsf{x}}_{\mathcal{P}}^i) := \{\mathsf{x} : \ |\bar{\mathsf{x}}_{\mathcal{P}}^i - \mathsf{x}| < \bar{\eta}_{\bar{f}}\} \cup \{\mathsf{x} : |\bar{\mathsf{x}}_{\mathcal{P}}^i - \mathsf{x}| \le \epsilon_c\}. \tag{20}$$

Clearly, $\mathcal{W}_{\bar{f}}(\bar{\mathsf{x}}_{\mathcal{P}}^i) \subseteq \mathcal{W}_f(\bar{\mathsf{x}}_{\mathcal{P}}^i)$. Hence, $\mathcal{W}_{\bar{f}}(\bar{\mathsf{x}}_{\mathcal{P}}^i)$ is a conservative approximation of the winning set $\mathcal{W}_f(\bar{\mathsf{x}}_{\mathcal{P}}^i)$. Note that, contrary to $\mathcal{W}_f(\bar{\mathsf{x}}_{\mathcal{P}}^i)$, the i^{th} pursuer has perfect knowledge of $\mathcal{W}_{\bar{f}}(\bar{\mathsf{x}}_{\mathcal{P}}^i)$. Furthermore, the closeness of the approximation of the winning set of the i^{th} pursuer with $\mathcal{W}_{\bar{f}}(\bar{\mathsf{x}}_{\mathcal{P}}^i)$ depends on the difference $\bar{\eta}_f - \bar{\eta}_{\bar{f}}$.

IV. The Dynamic Voronoi Partitioning Problem

Next, we formulate a dynamic Voronoi-like partitioning problem based on the minimum time-to-go of the i^{th} MTP, which will allow us to assign a pursuer starting from a point in $\mathcal{P}$ to a moving target traveling in the plane. The space we wish to partition, denoted henceforth as $\mathcal{W}$, is the union of all $\mathcal{W}_{\bar{\eta}_f}(\bar{\mathsf{x}}_{\mathcal{P}}^i)$, where $i \in \mathcal{I}$.

Problem 2: Given a collection of n pursuers, initially located at distinct points in $\mathcal{P}$, and the cost function

$$c^i(\mathsf{x}, \bar{\mathsf{x}}_{\mathcal{P}}^i) := T_{\mathsf{f}}(\mathsf{x} - \bar{\mathsf{x}}_{\mathcal{P}}^i), \tag{21}$$

where T_{f} is the minimum time from the solution of Problem 1, determine a partition $\mathcal{V} = \{\mathcal{V}^i : i \in \mathcal{I}\}$ of $\mathcal{V}$ such that

i) $\mathcal{W} = \bigcup_{i \in \mathcal{I}} \mathcal{V}_i$
ii) for all $\mathsf{x} \in \mathcal{V}^i$, $c(\bar{\mathsf{x}}_{\mathcal{P}}^i, \mathsf{x}) < \infty$

iii) $c(\bar{\mathsf{x}}_{\mathcal{P}}^i,\mathsf{x}) \le c(\bar{\mathsf{x}}_{\mathcal{P}}^j,\mathsf{x})$ for $i,j\in\mathcal{I}$ with $j\neq i$.

Henceforth, we shall refer to the solution of Problem 2 as the Optimal Pursuit Dynamic Voronoi Diagram (OP-DVD). The set $\mathcal{V}^i$, constitutes a Voronoi cell (Dirichlet domain) of the OP-DVD. We say that the i^{th} and j^{th} pursuers, where $i,\ j\in\mathcal{I}$, are neighbors if and only if the set $\mathcal{V}^i\cap\mathcal{V}^j$ is neither non-empty nor a singleton. Because the evading strategy of any moving target is not perfectly known, we can only provide approximate solutions to Problem 2, as it is discussed next.

V. Construction of an Approximate OP-DVD

In order to construct an approximate OP-DVD, we will first investigate whether the minimum time-to-go of Problem 1 belongs to a class of generalized metrics that are associated with Voronoi-like partitions, for which efficient computational techniques exist in the literature [1].

To this end, observe that direct integration of equation (14) yields

$$T_{\mathsf{f}}(\bar{\mathsf{y}}^i) := \begin{cases} 0, & \text{if } 0\le|\bar{\mathsf{y}}^i|\le\epsilon_c,\\ \displaystyle\int_{\epsilon_c}^{|\bar{\mathsf{y}}^i|}\frac{\mu\,\mathrm{d}\mu}{\bar{u}_{\mathcal{P}}\mu-f(\mu)}, & \text{if } \epsilon_c<|\bar{\mathsf{y}}^i|<\bar{\eta}_f,\\ \infty, & \text{otherwise.}\end{cases} \tag{22}$$

The following result will be useful in the subsequent analysis

Proposition 3: Let $\bar{\eta}_f>\epsilon_c$. Given two points $\xi,\psi\in\mathbb{R}^2$, with $|\xi|,|\psi|\in(\epsilon_c,\bar{\eta}_f)$, the minimum-time of Problem 1 satisfies

$$\epsilon_c<|\xi|<|\psi|<\bar{\eta}_f \Leftrightarrow 0<T_{\mathsf{f}}(\xi)<T_{\mathsf{f}}(\psi)<\infty, \tag{23}$$

and, furthermore,

$$\epsilon_c<|\xi|=|\psi|<\bar{\eta}_f \Leftrightarrow 0<T_{\mathsf{f}}(\xi)=T_{\mathsf{f}}(\psi)<\infty. \tag{24}$$

Proof: First, notice that the minimum-time of Problem 1 satisfies

$$T_{\mathsf{f}}(\psi)-T_{\mathsf{f}}(\xi)=\int_{|\xi|}^{|\psi|}\phi(\mu)\,\mathrm{d}\mu,\quad \phi(\mu):=\frac{\mu}{\bar{u}_{\mathcal{P}}\mu-f(\mu)}.$$

The function $\phi:(\epsilon_c,\bar{\eta}_f)\mapsto\mathbb{R}$ is continuous and strictly positive on $(\epsilon_c,\bar{\eta}_f)$. From the mean value theorem for Riemann integrals [14], it follows that there exists $\epsilon_c<|\xi|\le\zeta\le|\psi|<\bar{\eta}_f$ such that

$$T_{\mathsf{f}}(\psi)-T_{\mathsf{f}}(\xi)=\int_{|\xi|}^{|\psi|}\phi(\mu)\,\mathrm{d}\mu=\phi(\zeta)(|\psi|-|\xi|). \tag{25}$$

Since $\phi(\zeta)>0$ for all $\epsilon_c<\zeta<\bar{\eta}_f$, the result follows readily. ■

Corollary 1: Let $\bar{\eta}_f>\epsilon_c$ and let ξ,ψ be two given points in $\mathbb{R}^2$. Then the minimum-time of Problem 1 satisfies

$$|\xi|\le|\psi|\Rightarrow T_{\mathsf{f}}(\xi)\le T_{\mathsf{f}}(\psi). \tag{26}$$

Proof: The statement of the corollary for the case when $\epsilon_c<|\xi|\le|\psi|<\bar{\eta}_f$ has already been proved in Proposition 3. The proof for the other cases, namely, when $|\xi|\le\epsilon_c<|\psi|<\bar{\eta}_f$, or $|\xi|\le|\psi|\le\epsilon_c$, or $\epsilon_c<|\xi|<\bar{\eta}_f\le|\psi|$, and $\bar{\eta}_f\le|\xi|\le|\psi|$ follows trivially from (22). ■

Next, we present the solution of Problem 2.

Theorem 1: Let $V:=\{V^i, i\in\mathcal{I}\}$ be the standard Voronoi partition generated by the set $\mathcal{P}$, and assume that $\bar{\eta}_f>\epsilon_c$. The solution of Problem 2 is given by

$$\mathcal{V}^i=V^i\cap\mathcal{W}_f(\bar{\mathsf{x}}_{\mathcal{P}}^i),\quad i\in\mathcal{I}, \tag{27}$$

where $\mathcal{W}_f(\bar{\mathsf{x}}_{\mathcal{P}}^i)$ is the winning set of the i^{th} pursuer, given by (18).

Proof: Let $\mathsf{x}\in V^i\cap\mathcal{W}_f(\bar{\mathsf{x}}_{\mathcal{P}}^i)$. In particular, $\mathsf{x}\in V^i$ if and only if $|\mathsf{x}-\bar{\mathsf{x}}_{\mathcal{P}}^i|\le|\mathsf{x}-\bar{\mathsf{x}}_{\mathcal{P}}^j|$, for all $j\neq i$, which implies, in light of Corollary 1, that $T_{\mathsf{f}}(\mathsf{x}-\bar{\mathsf{x}}_{\mathcal{P}}^i)\le T_{\mathsf{f}}(\mathsf{x}-\bar{\mathsf{x}}_{\mathcal{P}}^j)$ for all $i\neq j$. Furthermore, if $\mathsf{x}\in\mathcal{W}_f(\bar{\mathsf{x}}_{\mathcal{P}}^i)$ then $T_{\mathsf{f}}(\mathsf{x}-\bar{\mathsf{x}}_{\mathcal{P}}^i)<\infty$. It follows that $\mathsf{x}\in\mathcal{V}^i$ and hence $V^i\cap\mathcal{W}_f(\bar{\mathsf{x}}_{\mathcal{P}}^i)\subseteq\mathcal{V}^i$ for all $i\in\mathcal{I}$.

Next, assume $\mathsf{x}\in\mathcal{V}^i$. By the definition of $\mathcal{V}^i$, it follows that $T_{\mathsf{f}}(\mathsf{x}-\bar{\mathsf{x}}_{\mathcal{P}}^i)<\infty$ and $T_{\mathsf{f}}(\mathsf{x}-\bar{\mathsf{x}}_{\mathcal{P}}^i)\le T_{\mathsf{f}}(\mathsf{x}-\bar{\mathsf{x}}_{\mathcal{P}}^j)$, for all $j\neq i$. If $0<T_{\mathsf{f}}(\mathsf{x}-\bar{\mathsf{x}}_{\mathcal{P}}^i)\le T_{\mathsf{f}}(\mathsf{x}-\bar{\mathsf{x}}_{\mathcal{P}}^j)<\infty$, it follows from Proposition 3 that $|\mathsf{x}-\bar{\mathsf{x}}_{\mathcal{P}}^i|\le|\mathsf{x}-\bar{\mathsf{x}}_{\mathcal{P}}^j|$, for all $j\neq i$. In addition, it follows readily that $T_{\mathsf{f}}(\mathsf{x}-\bar{\mathsf{x}}_{\mathcal{P}}^i)\le T_{\mathsf{f}}(\mathsf{x}-\bar{\mathsf{x}}_{\mathcal{P}}^j)$ implies that $|\mathsf{x}-\bar{\mathsf{x}}_{\mathcal{P}}^i|\le|\mathsf{x}-\bar{\mathsf{x}}_{\mathcal{P}}^j|$, for all $j\neq i$ and $\mathsf{x}\in\mathcal{V}^i$, when $T_{\mathsf{f}}(\mathsf{x}-\bar{\mathsf{x}}_{\mathcal{P}}^j)=0$ and $T_{\mathsf{f}}(\mathsf{x}-\bar{\mathsf{x}}_{\mathcal{P}}^j)=\infty$ as well. Thus $\mathsf{x}\in V^i$. Furthermore, since $T_{\mathsf{f}}(\mathsf{x}-\bar{\mathsf{x}}_{\mathcal{P}}^i)<\infty$, then $\mathsf{x}\in\mathcal{W}_f(\bar{\mathsf{x}}_{\mathcal{P}}^i)$. Hence $\mathsf{x}\in V^i\cap\mathcal{W}_f(\bar{\mathsf{x}}_{\mathcal{P}}^i)$ and $\mathcal{V}^i\subseteq V^i\cap\mathcal{W}_f(\bar{\mathsf{x}}_{\mathcal{P}}^i)$ for $i\in\mathcal{I}$. ■

Theorem 1 suggests that the i^{th} element of the partition that solves Problem 2 is the intersection of the winning set of the i^{th} pursuer with the cell of the standard Voronoi diagram generated by the set $\mathcal{P}$ that is associated with the generator $\bar{\mathsf{x}}_{\mathcal{P}}^i$. Note that the OP-DVD encodes the proximity relations between a target and the pursuers with respect to time of capture, for all pursuers in $\mathcal{P}$.

The following proposition deals with the neighboring relations between the set of pursuers in $\mathcal{V}$.

Proposition 4: Let $V:=\{V^i : i\in\mathcal{I}\}$ be the standard Voronoi partition generated by the set $\mathcal{P}$ and let $i,j\in\mathcal{I}$ with $i\neq j$. Then the i^{th} pursuer is a neighbor of the j^{th} pursuer in the OP-DVD if and only if

i) The generators $\bar{\mathsf{x}}_{\mathcal{P}}^i$ and $\bar{\mathsf{x}}_{\mathcal{P}}^j$ correspond to two neighboring nodes of the dual Delaunay graph of V.

ii) $|\bar{\mathsf{x}}_{\mathcal{P}}^i-\bar{\mathsf{x}}_{\mathcal{P}}^j|\le 2\bar{\eta}_f$.

Proof: The proof follows immediately from Theorem 1 and the definition of $\bar{\eta}_f$, and it is thus omitted. ■

Theorem 1 provides an efficient way for the construction of the exact OP-DVD provided, however, that the sets $\mathcal{W}_{\bar{\eta}_f}(\bar{\mathsf{x}}_{\mathcal{P}}^i)$, where $i\in\mathcal{I}$, are perfectly known. The following corollary, which follows readily from Theorem 1, furnishes an approximate solution to Problem 2.

Corollary 2: Let $V:=\{V^i : i\in\mathcal{I}\}$ be the standard Voronoi partition generated by the set $\mathcal{P}$. An approximate solution of Problem 2 is given by

$$\widetilde{\mathcal{V}} := \{\widetilde{\mathcal{V}}^i,\ i \in \mathcal{I}\}, \quad \widetilde{\mathcal{V}}^i = V^i \cap \mathcal{W}_{\bar{\eta}_{\bar{f}}}(\bar{\mathsf{x}}^i_{\mathcal{P}}), \quad i \in \mathcal{I}. \tag{28}$$

One important question that arises in the context of the previous discussion is whether the approximate OP-DVD can provide us with reliable information regarding the actual proximity relations among the pursuers in $\mathcal{P}$ (this information is encoded in the exact OP-DVD).

Proposition 5: Let $V := \{V^i : i \in \mathcal{I}\}$ be the standard Voronoi partition generated by the set $\mathcal{P}$. The i^{th} pursuer is a neighbor of the j^{th} pursuer if

i) the generators $\bar{\mathsf{x}}^i_{\mathcal{P}}$ and $\bar{\mathsf{x}}^j_{\mathcal{P}}$ correspond to two neighboring nodes of the dual Delaunay graph of V
ii) $|\bar{\mathsf{x}}^i_{\mathcal{P}} - \bar{\mathsf{x}}^j_{\mathcal{P}}| \leq 2\bar{\eta}_{\bar{f}}$.

Proof: The proof follows readily from Proposition 4 and the definition of $\bar{\eta}_{\bar{f}}$ and $\bar{\eta}_f$. ■

VI. The Solution of the i^{th} MTP and its Relation to the Zermelo's Navigation Problem

In this section, we highlight the advantages of the scheme for addressing Problem 2 presented in Section V, by comparing it with the approach introduced in our previous work [10]. Specifically, in [10], we assumed that the evading strategy $u_{\mathcal{T}}$ was perfectly known to the pursuers (pursuit-with-anticipation), thus reducing the i^{th} MTP to the Zermelo's navigation problem (ZNP). The extremal control u^i_* of the ZNP has necessarily the following structure: $\mathsf{u}^i_* = \bar{u}_{\mathcal{P}}(\cos\theta^i_*, \sin\theta^i_*)$, where θ^i_* satisfies the following differential equation, known as the *navigation formula* (for more details see for example [15, pp. 239-247, pp. 370-373])

$$\dot{\theta}^i_* = (\mu_1 - \nu_2)\cos\theta^i_*\sin\theta^i_* + \nu_1\sin^2\theta^i_* - \mu_2\cos^2\theta^i_*, \tag{29}$$

where $\mu := \langle(1,0), u_{\mathcal{T}}\rangle$, $\nu := \langle(0,1), u_{\mathcal{T}}\rangle$, and $\mu_1, \mu_2, \nu_1, \nu_2$ denote partial spatial derivatives. It follows that the optimal control u^i_* is determined up to a single parameter, namely $\bar{\theta}^i = \theta_*(0) \in [0, 2\pi)$; we subsequently write $\mathsf{u}^i_*(t; \bar{\theta}^i)$. One key observation here is that the solution of the ZNP depends explicitly on $u_{\mathcal{T}}$ and its partial derivatives through the navigation formula (29), in contrast to the solution of the i^{th} MTP which is independent of $u_{\mathcal{T}}$ under Assumption 1. Furthermore, the control u^i_* solving the ZNP is not expressible, in general, in closed form, given that (5) along with (29) form a coupled system of three nonlinear equations, which does not admit, in general, an analytic solution.

Additionally, the navigation formula (29) does not necessarily furnish a global optimal solution to the ZNP. In particular, the pursuit strategy $\mathsf{u}^i_*(t; \bar{\theta}^i)$ may either be: 1) maximizing (locally or globally) the time of capture, 2) minimizing (locally or globally) the time of capture or 3) be an abnormal control law (that is, an extremal law that corresponds to an abnormal extremal curve of the ZNP [16]). The following proposition provides a sufficient condition for determining whether an extremal control u^i_* maximizes or minimizes locally the time of capture of the moving target or it is an abnormal control law [15]–[17].

Proposition 6: Let $\mathsf{y}^i_*(\tau)$ be the extremal curve generated by $\mathsf{u}^i_*(\tau; \bar{\theta}^i)$, for $\tau \in [0, t]$. If the functional

$$\mathsf{I}[\mathsf{y}^i_*, \mathsf{u}^i_*] := \bar{u}_{\mathcal{P}} + \langle u_{\mathcal{T}}(\mathsf{y}^i_*), \mathsf{u}^i_*\rangle, \tag{30}$$

satisfies $\mathsf{I}[\mathsf{y}^i_*, \mathsf{u}^i_*] > 0$ (< 0) for all $\tau \in [0, t]$, then the control $\mathsf{u}^i_*(\tau; \bar{\theta}^i)$ minimizes (maximizes) locally or globally the final time of Problem 1. Furthermore, if $\mathsf{I}[\mathsf{y}^i_*, \mathsf{u}^i_*] = 0$ for all $\tau \in [0, t]$, then u^i_* is an abnormal control law of Problem 1.

The main caveat here is that Proposition 6 does not allow one to characterize the global minimizing extremals of the ZNP, whereas the optimality (local or global) of any abnormal extremals is still inconclusive. Therefore, in general, there does not exist a straightforward method to conclude global optimality of the solution of ZNP without either resorting to exhaustive numerical techniques or restricting our attention to particular classes of drift terms, as those examined in detail in [10].

VII. Simulation Results

In this section, we present simulation results to illustrate the previous developments. We consider a scenario where the maneuvering target is faster than the i^{th} pursuer, but the winning set of the i^{th} pursuer is non-empty as a result of the information pattern employed in Section II. In particular, it is assumed that the target has a constant speed and its evading strategy is given by

$$u_{\mathcal{T}}(\mathsf{y}^i) = \begin{cases} \alpha\mathsf{y}^i + \rho(\mathsf{y}^i)\mathsf{S}\mathsf{y}^i, & \text{for } \epsilon_c \leq |\mathsf{y}^i| \leq \dfrac{M}{\alpha}, \\ M\dfrac{\mathsf{y}^i}{|\mathsf{y}^i|}, & \text{for } |\mathsf{y}^i| > \dfrac{M}{\alpha}, \end{cases} \tag{31}$$

where M and α are some positive constants with $M > \max\{\bar{u}_{\mathcal{P}}, \alpha\}$, S is a nonzero skew symmetric matrix in $\mathbb{R}^{2\times 2}$, and $\rho(\mathsf{y}^i) := \sqrt{M^2 - \alpha^2|\mathsf{y}^i|^2}/|\mathsf{S}\mathsf{y}^i|$. Note that

$$f(\mathsf{y}^i) := \langle u_{\mathcal{T}}, \mathsf{y}^i\rangle = \begin{cases} \alpha|\mathsf{y}^i|^2, & \text{for } \epsilon_c \leq |\mathsf{y}^i| \leq \dfrac{M}{\alpha}, \\ M|\mathsf{y}^i| & \text{for } |\mathsf{y}^i| > \dfrac{M}{\alpha}, \end{cases} \tag{32}$$

satisfies Assumption 1.

The intuition behind the evading strategy (31) is as follows: Let $\mathbf{e}_1(\mathsf{y}^i) := \mathsf{y}^i/|\mathsf{y}^i|$ be the unit vector along the line connecting the target and the i^{th} pursuer ("line-of-sight" direction), and let $\mathbf{e}_2(\mathsf{y}^i)$ be the unit vector orthogonal to $\mathbf{e}_1(\mathsf{y}^i)$ ("tangential" direction). The strategy of the target is to allocate its velocity vector, which has a constant magnitude $M > u_{\mathcal{P}}$, along the directions $\mathbf{e}_1(\mathsf{y}^i)$ and $\mathbf{e}_2(\mathsf{y}^i)$ so that it moves with constant speed M along the line-of-sight direction when it is sufficiently far away from the pursuer, and it uses an increasingly larger tangential component as its distance from the pursuer decreases, in an effort to maneuver away or confuse its pursuer.

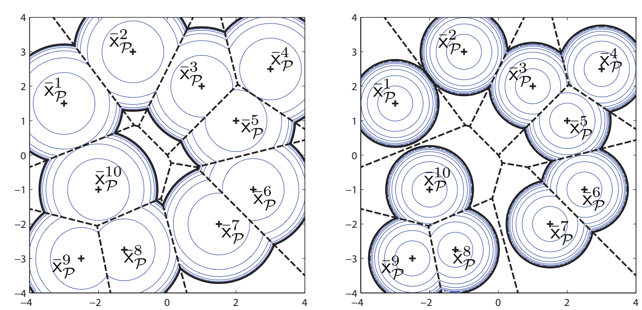
Assume for this example that the set $\mathcal{P}$ consists of ten locations, and let $\bar{f}$ be defined as f modulo the replacement of α by $\bar{\alpha}$, where $\bar{\alpha}$ is a positive scalar with $\alpha \leq \bar{\alpha} < M$. In this case, the capturability condition (13) reduces to

$\eta^i(0) < \bar{u}_{\mathcal{P}}/\alpha$, which implies that $\bar{\eta}_f = \bar{u}_{\mathcal{P}}/\alpha < M/\alpha$ and $\bar{\eta}_{\bar{f}} = \bar{u}_{\mathcal{P}}/\bar{\alpha} < M/\bar{\alpha}$. Furthermore, it is easy to show that for $\epsilon_c < |\bar{\mathrm{y}}^i| < \bar{\eta}_f$ the minimum-time to capture for Problem 1 is given by

$$T_{\mathsf{f}}(\bar{\mathrm{y}}^i) = -\frac{1}{\alpha}\ln\left(\frac{\bar{u}_{\mathcal{P}} - \alpha|\bar{\mathrm{y}}^i|}{\bar{u}_{\mathcal{P}} - \alpha\epsilon_c}\right). \tag{33}$$

Figure 1(a) illustrates the exact OP-DVD along with the level sets of $T_{\mathsf{f}}(\bar{\mathrm{y}}^i)$ for $\alpha = 0.7$, $\epsilon_c = 0.05$ and $\bar{u}_{\mathcal{P}} = 1.2$. An approximation of the OP-DVD for $\bar{\alpha} = 0.95$ is illustrated in Fig. 1(b).

Next, we examine the discrepancies between the neighboring relations among the pursuers of the exact and the approximate OP-DVDs. In light of Proposition 5, given $i, j \in \mathcal{I}$ with $i \neq j$, the i^{th} and j^{th} pursuers are neighbors provided that the generators $\bar{\mathrm{x}}^i_{\mathcal{P}}, \bar{\mathrm{x}}^j_{\mathcal{P}} \in \mathcal{P}$ correspond to two neighboring nodes of the dual Delaunay graph of the standard Voronoi diagram generated by the set $\mathcal{P}$ and $|\bar{\mathrm{x}}^i_{\mathcal{P}} - \bar{\mathrm{x}}^j_{\mathcal{P}}| < 2\bar{\eta}_{\bar{f}} = 2\bar{u}_{\mathcal{P}}/\bar{\alpha}$. For this particular example, we can explicitly compute a lower bound of $\Delta\bar{\eta} := \bar{\eta}_f - \bar{\eta}_{\bar{f}}$ as a function of the error $\Delta\alpha := \bar{\alpha} - \alpha$. Specifically, $\Delta\bar{\eta} = \bar{u}_{\mathcal{P}}/\alpha - \bar{u}_{\mathcal{P}}/\bar{\alpha} = \bar{u}_{\mathcal{P}}\Delta\alpha/\bar{\alpha}\alpha$, which implies that $\Delta\bar{\eta} \geq \bar{u}_{\mathcal{P}}\Delta\alpha/\bar{\alpha}^2$. It follows readily from Propositions 4 and 5 that if $|\bar{\mathrm{x}}^i_{\mathcal{P}} - \bar{\mathrm{x}}^j_{\mathcal{P}}| < 2\bar{\eta}_{\bar{f}} + 2\bar{u}_{\mathcal{P}}\Delta\alpha/\bar{\alpha}^2 \leq 2\bar{\eta}_f$, then the i^{th} and j^{th} pursuers are neighbors of the exact OP-DVD although they may not be neighbors of the approximate OP-DVD. Consequently, the accuracy of the knowledge about the neighboring relations between the pursuers of the exact OP-DVD is contingent upon the smallness of the error $\Delta\alpha$. The situation is illustrated in Fig. 1, where the approximate OP-DVD conceals the fact that the 1^{st} and the 10^{th}, and the 7^{th} and the 8^{th} are neighboring pursuers of the exact OP-DVD.



(a) Minimum-time wave fronts for Problem 1.

(b) Approximate minimum-time wave fronts for Problem 1.

Fig. 1. The exact and an approximate solution of Problem 2 for a team of ten pursuers.

VIII. Conclusion

In this article, we have formulated a new dynamic partitioning problem that deals with the characterization of the sets of initial conditions from which a pursuer, from a given team of pursuers, can capture a moving target faster than any other pursuer from the same team. It is assumed that the target can employ a feedback "evading" strategy in response to the pursuers' actions, which is only partially known to the pursuers. We have presented an efficient scheme for the construction of an approximate solution of this partitioning problem by associating it with a standard Voronoi diagram.

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