Exponentially Convergent Control Laws for Nonholonomic Systems in Power Form ⋆

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Abstract

This paper introduces a method for constructing exponentially convergent control laws for n-dimensional nonholonomic systems in power form. The methodology is based on the construction of a series of nested invariant manifolds for the closed-loop system under a linear control law. A recursive algorithm is presented which uses these manifolds to construct a 3-dimensional system in power form. It is shown that the feedback controller for the original system is the one for this 3-dimensional system with proper choice of the gains.

Key words: Nonholonomic Systems, Power Form, Invariant Manifolds, Exponential Convergence.

1 Introduction

Nonholonomic control systems commonly arise from mechanical systems when non-integrable constraints are imposed on the motion, i.e., velocity constraints, which can not be integrated to generate constraints on the configuration space. Examples include rolling disks, mobile robots [15], underactuated symmetric rigid spacecraft [11,20], etc. One challenging aspect of these control systems is that they are controllable but not stabilizable by a smooth static or dynamic state feedback control laws [4]. A number of approaches have been proposed to solve the stabilization problem for nonholonomic systems. These methodologies can be broadly classified as discontinuous, time-invariant stabilization

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and time-varying (usually smooth) stabilization. The non-smoothness of time-invariant feedback controllers is a consequence of the structural properties of the system [4]. Stabilization results using non-smooth, time-invariant control laws have been proposed in [11,20,5,7,1]. References [11,20] deal with the attitude stabilization of underactuated spacecraft by developing non-smooth, time-invariant control laws. Piecewise smooth and discontinuous controllers have been reported in [5,7]. Samson in [17] showed how to asymptotically stabilize a mobile robot to a point using time-varying, smooth state feedback. Coron in [6] subsequently proved that all controllable driftless systems could be stabilized to an equilibrium point using smooth, periodic, time-varying feedback. References [14,19] and [16] deal with the construction of time-varying control laws for several nonholonomic systems. Hybrid time-varying feedback control laws are proposed in [10] for a class of cascade nonlinear systems, which could also be used to stabilize a class of nonholonomic systems, as well as for tracking problems. Reference [13] develops time-varying control laws with exponential convergence with respect to homogeneous norms. Finally, [18] develops nonsmooth, time-varying feedback control laws which guarantee global, asymptotic stability with exponential convergence about an arbitrary configuration. Of particular relevance to the work in this paper is Ref. [1], where a nonsmooth transformation is used to develop time-invariant, exponentially convergent controllers for systems in chained form. For a more comprehensive review of all the recent advances in the control of nonholonomic systems the interested reader may consult [9].

In this paper we deal with n-dimensional nonholonomic systems in power form with two inputs which are described as [7,9]

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_j &= \frac{1}{(j-2)!}\, x_1^{j-2}\, u_2 \qquad\qquad j = 2, 3, \dots, n \end{aligned} \tag{1}$$

We derive feedback control laws for the system in Eqs. (1) using a set of invariant manifolds constructed by direct integration of the closed-loop equations subject to linear feedback. This idea, originally introduced in [20], was later extended in [7] and [8]. Finite-time stabilizing and tracking controllers which use these manifolds as sliding surfaces have also been proposed in [2] for 3-dimensional power form systems. The general n-dimensional problem has not been addressed, however. In this paper we show how one can use these invariant manifolds to construct a series of generated systems in power form of reduced dimension. By repeating this process one ends up with a 3-dimensional system in power form. The proposed control law for the system in Eqs. (1) is a feedback control law for this 3-dimensional system with proper choice of the control gains.

The main contribution of the paper lies in the construction of control laws achieving exponential rates of convergence. Moreover, the rates of convergence for each state can be specified *a priori* by the designer. The resulting controllers are discontinuous at the origin and are similar *in form* to the ones recently proposed by Astolfi [1] for nonholonomic systems in chained form. The approach in [1], however, uses a singular transformation for resolving the singularity at the equilibrium (σ process). Also, the approach proposed here generates controllers with multi-time scale convergence properties, as a result of the invariant manifold method, which is not evidently present in [1]. It should be pointed out, however, that the proposed controllers (without further modification) *may* not render the closed-loop system stable (in the sense of Lyapunov) since closed-loop trajectories may move away from the origin before returning there. Nevertheless, departure from the neighborhood of the equilibrium may not be necessarily unacceptable in practice. For instance, large angular excursions from the rest position may be perfectly valid maneuvers for underactuated spacecraft. Also, common experience indicates that some parallel parking maneuvers are accomplished more efficiently, if the driver moves the car to a better posture if the initial conditions are not "good" (see Ref. [14] and [17] for a discussion on the relationship between controlling nonholonomic systems and the problem of parallel parking.)

2 Invariant Manifolds and Their Properties

Consider the system in Eqs. (1) and the following linear feedback

$$u_1 = -k\, x_1, \qquad u_2 = -k_1\, x_2, \qquad (k > 0,\ k_1 > 0) \tag{2}$$

With this linear control law, the closed-loop equations are

$$\begin{aligned} \dot{x}_1 &= -\, k x_1 \\ \dot{x}_j &= -\frac{1}{(j-2)!}\, k_1\, x_1^{j-2}\, x_2, \qquad j = 2, 3, \ldots, n \end{aligned} \tag{3}$$

Equations(3) can be explicitly integrated to obtain

$$\begin{aligned} x_1(t) &= x_{10}\, e^{-kt} \\ x_2(t) &= x_{20}\, e^{-k_1 t} \\ x_j(t) &= s_{1,j-2}(x_0) + \frac{k_1}{(j-2)!\,((j-2)k + k_1)}\, x_{10}^{j-2}\, x_{20}\, e^{-((j-2)k+k_1)t} \qquad j = 3, 4, \ldots, n \end{aligned} \tag{4}$$

where $x_0 = [x_{10}, x_{20}, x_{30}, \ldots, x_{n0}]^T \in \mathbb{R}^n$ is the initial state of the system, and where

$$s_{1,j} = x_{j+2} - \frac{k_1}{j!\,(j\,k + k_1)}\, x_1^j\, x_2, \qquad j = 1, 2, \ldots, n-2 \tag{5}$$

Equation (5) defines a series of smooth manifolds by

$$\Pi_{1,j} = \{x \in \mathbb{R}^n : s_{1,j}(x) = 0\}, \qquad j = 1, 2, \ldots, n-2 \tag{6}$$

each of dimension $(n-1)$. The manifold

$$\Pi_1 = \bigcap_{j=1}^{n-2} \Pi_{1,j} = \{x \in \mathbb{R}^n : s_{1,j}(x) = 0, \quad j = 1, 2, \ldots, n-2\} \tag{7}$$

is then a two-dimensional smooth manifold [3], since rank $\left[\frac{\partial s_{1,j}}{\partial x_i}\right] = n-2$.

Lemma 1 *Consider the system in Eqs. (1) under the feedback control in Eq. (2) and the manifold* Π_1 *as in Eq. (7). Then, for all initial conditions* $x_0 \in \Pi_1$*, the closed-loop trajectories of the system converge exponentially to the origin.*

PROOF. Since

$$\begin{aligned} \dot{s}_{1,j} &= \dot{x}_{j+2} - \frac{k_1}{j!(j\,k + k_1)}\,(j\,x_1^{j-1}\dot{x}_1 x_2 + x_1^j \dot{x}_2) \\ &= \frac{k\,k_1}{(j-1)!\,(j\,k + k_1)}\, x_1^j x_2 + \frac{k}{(j-1)!\,(j\,k + k_1)}\, x_1^j\, u_2 = 0 \end{aligned} \tag{8}$$

each manifold $\Pi_{1,j}$ is invariant for the closed-loop system. Subsequently, the manifold Π_1 is also invariant for Eqs. (3). For $x_0 \in \Pi_1$ the solutions of the closed-loop system are given by Eqs. (4) where $s_{i,j}(x_0) = 0$, for $j = 1, 2, \ldots, n-2$. The assertion of the lemma follows immediately.

3 A Recursive Algorithm for Systems in Power Form

In this section we present a recursive algorithm for generating a series of systems which will be used to construct an exponentially convergent feedback controller for the system in Eqs. (1). All the systems generated by this recursive process (herein called the *generated systems*) can be put in power form through a linear transformation. These generated systems are, however, of reduced dimension. The methodology is based on the idea that by constructing a set of $(n-2)$ manifolds for the n-dimensional system, the problem of constructing an exponentially convergent controller for the initial system becomes one of constructing an exponentially convergent controller for a similar system in power form but of dimension $(n-1)$. By repeating this process, one ends up with a 3-dimensional system in power form.

3.1 The Recursive Process

Consider the n-dimensional system given in Eqs. (1), and construct a set of $(n-2)$ invariant manifolds for this system under a linear control feedback $u_1 = -kx_1$, $u_2 = -k_1 x_2$ as in Eqs. (5).

Define the following linear transformation

$$\begin{aligned} x_{2,1} &= x_1 \\ x_{2,j} &= ((j-1)\,k + k_1)\, s_{1,j-1} \qquad j = 2, 3, \ldots, n-1 \end{aligned} \tag{9}$$

Then one can define the following system in terms of $x_{2,j}$, for $1 \leq j \leq n-1$

$$\begin{aligned} \dot{x}_{2,1} &= u_{2,1} \\ \dot{x}_{2,j} &= \frac{1}{(j-2)!}\, x_{2,1}^{j-2} u_{2,2} \qquad j = 2, 3, \ldots, n-1 \end{aligned} \tag{10}$$

where

$$u_{2,1} = u_1, \qquad u_{2,2} = k\, x_1\, u_2 - k_1\, x_2\, u_1 \tag{11}$$

The system in Eqs. (10) will be called the second *generated system* and we use the first index in the subscript of the state elements to denote this. For consistency, we define the first generated system to be simply the original system in Eqs. (1), that is, we let $x_{1,j} = x_j$ for $1 \leq j \leq n$. Notice that the system in Eqs. (10) is a system in power form of dimension $(n-1)$. The same process can be therefore repeated for this system.

After repeating this process $(i-1)$ times one obtains the ith generated system (of dimension $(n-i+1)$) given by

$$\dot{x}_{i,1} = u_{i,1}$$

$$\dot{x}_{i,j} = \frac{1}{(j-2)!}\, x_{i,1}^{j-2}\, u_{i,2} \qquad j = 2, 3, \ldots,\; n-i+1 \tag{12}$$

For the ith generated system, one can construct $(n-i-1)$ invariant manifolds using the linear feedback control,

$$u_{i,1} = -\,k\, x_{i,1}, \qquad u_{i,2} = -\,k_i\, x_{i,2} \tag{13}$$

and the methodology described earlier. The corresponding manifolds are defined by

$$\Pi_{i,j} = \{x \in \mathbb{R}^{n-i+1} : s_{i,j}(x) = 0\}, \qquad j = 1, 2, \ldots,\; n-i-1 \tag{14}$$

where

$$s_{i,j} = x_{i,j+2} - \frac{k_i}{j!\,(jk+k_i)}\, x_{i,1}^{j}\, x_{i,2} \qquad j = 1, 2, \ldots,\; n-i-1$$

Defining

$$\begin{aligned} x_{i+1,1} &= x_{i,1} \\ x_{i+1,j} &= ((j-1)k + k_i)\, s_{i,j-1} \qquad j = 2, 3, \ldots,\; n-i \end{aligned} \tag{15}$$

the $(i+1)$th generated system can be described as follows

$$\begin{aligned} \dot{x}_{i+1,1} &= u_{i+1,1} \\ \dot{x}_{i+1,j} &= \frac{1}{(j-2)!}\, x_{i+1,1}^{j-2}\, u_{i+1,2} \qquad j = 2, 3, \ldots,\; n-i \end{aligned} \tag{16}$$

where

$$u_{i+1,1} = u_{i,1}, \qquad u_{i+1,2} = k\, x_{i,1}\, u_{i,2} - k_i\, x_{i,2}\, u_{i,1} \tag{17}$$

This process can be continued until the $(n-2)$th generated system, which is the 3-dimensional system

$$\begin{aligned} \dot{x}_{n-2,1} &= u_{n-2,1} \\ \dot{x}_{n-2,2} &= u_{n-2,2} \\ \dot{x}_{n-2,2} &= x_{n-2,1} u_{n-2,2} \end{aligned} \tag{18}$$

By construction, it is immediate that if $x_{i+1,1} = x_{i+1,2} = x_{i+1,3} = \cdots = x_{i+1,n-i} = 0$ for the $(i+1)$th generated system, then for the ith generated system we have that $x_{i,1} = x_{i,3} = \cdots = x_{i,n-i+1} = 0$. Thus, any convergent feedback controller about the origin for the $(i+1)$th generated system, will also make the ith generated system converge to the (imbedded in $\mathbb{R}^{n-i+1}$) one-dimensional manifold

$$\mathcal{M}_i = \{x \in \mathbb{R}^{n-i+1} : x_{i,1} = x_{i,3} = \cdots = x_{i,n-i+1} = 0\} \tag{19}$$

In particular, if the convergent controller for the $(i+1)$th generated system is chosen such that, in addition, satisfies the property $\lim_{t\to\infty} x_{i,2} = 0$ then the same control laws will make the ith generated system converge about the *origin.*

3.2 The 3-Dimensional System

The first step in the proposed derivation of the feedback controller is to construct an exponentially convergent, static, state-feedback controller for the $(n-2)$th generated system in Eqs. (18). For notational convenience, let us redefine $z_i = x_{n-2,i}$ $(i = 1, 2, 3)$ and $v_j = u_{n-2,j}$ $(j = 1, 2)$. Then the system in Eqs. (18) can be written, equivalently, as

$$\begin{aligned} \dot{z}_1 &= v_1 \\ \dot{z}_2 &= v_2 \\ \dot{z}_3 &= z_1 v_2 \end{aligned} \tag{20}$$

Theorem 2 *Consider the system in Eqs. (20) and the feedback control*

$$v_1 = -k\, z_1, \qquad v_2 = -k_{n-2}\, z_2 - \mu \frac{s}{z_1} \tag{21}$$

with $k > 0$, $k_{n-2} > 0$ and $\mu > (k + k_{n-2})^2/k$, and where

$$s = z_3 - \left(\frac{k_{n-2}}{k_{n-2} + k}\right) z_1\, z_2 \tag{22}$$

Then the closed-loop system has the property,

$$\lim_{t\to\infty} (z_1(t), z_2(t), z_3(t)) = 0 \tag{23}$$

with exponentially decaying rate, for all initial conditions such that $z_1(0) \neq 0$. Moreover, the control law in Eq. (21) is bounded along closed-loop trajectories.

PROOF. First, notice that $z_1 = z_1(0)\, e^{-kt}$ and thus z_1 decreases exponentially with rate of decay k. Note that s represents the invariant manifold for the system in Eq. (20) under the linear feedback

$$v_1 = -\, k\, z_1, \qquad v_2 = -\, k_{n-2}\, z_2 \tag{24}$$

Using Eqs. (21), the differential equation for s is

$$\begin{aligned} \dot{s} &= \left(\frac{k}{k_{n-2}+k}\right) z_1\, v_2 - \left(\frac{k_{n-2}}{k_{n-2}+k}\right) z_2\, v_1 \\ &= -\left(\frac{k\mu}{k_{n-2}+k}\right) s \end{aligned} \tag{25}$$

and s decreases exponentially with rate of decay $k\mu/(k_{n-2}+k) > k_{n-2}+k$. By definition, $\lim_{t\to\infty} s(t) = 0$ implies that $\lim_{t\to\infty} z_3(t) = 0$. The differential equation for z_2 can be written as follows

$$\dot{z}_2 = -\, k_{n-2}\, z_2 - \mu\, \gamma_1(t) \tag{26}$$

where the function $\gamma_1(t) = s(t)/z_1(t)$ is an exponentially decaying function with rate of decay greater than k_{n-2}. One can then immediately conclude that $\lim_{t\to\infty} z_2(t) = 0$. Moreover, the rate of decay of z_2 equals k_{n-2} (see, for example, Lemma 1 in [18]). Therefore, the closed-loop trajectories of the system in Eqs. (20) with the control laws in Eqs. (24) have the property that $\lim_{t\to\infty}(z_1(t), z_2(t), z_3(t)) = 0$ with exponentially rate of convergence. The claim that the control law (21) is bounded follows immediately from the fact $\gamma_1(t)$ is bounded.

4 The Feedback Controller

The following theorem contains the main result of this paper. It shows that the control law in Eq. (21) can also be used to make the original system in Eqs. (1) converge exponentially to the origin.

Theorem 3 *Consider the system in Eqs. (1) ($n \geq 3$) and the feedback controller*

$$u_1 = u_{n-2,1} \qquad u_2 = -\sum_{\ell=1}^{n-3} \frac{k_\ell}{k^{\ell-1}} \frac{x_{\ell,2}}{x_1^{\ell-1}} + \frac{u_{n-2,2}}{k^{n-3} x_1^{n-3}} \tag{27}$$

where

$$u_{n-2,1} = -k\, x_{n-2,1} \qquad u_{n-2,2} = -\, k_{n-2}\, x_{n-2,2} -\, \mu\, \frac{s}{x_{n-2,1}} \tag{28}$$

where $k > 0,\ k_1 > 0,\ k_i > k + k_{i-1}\,(i = 2, 3, \ldots, n-2)$ *and* $\mu > (k + k_{n-2})^2/k$,

$$s = s_{n-2,1} = x_{n-2,3} - \left(\frac{k_{n-2}}{k_{n-2}+k}\right) x_{n-2,1}\, x_{n-2,2},$$

and $x_{i,j}$ $(i = 1, 2, \ldots, n-2,\ j = 1, 2, \ldots, n-i+1)$ *derived through the recursive process described in Section 3.*

Then for any nonzero initial value $x_1(0) \neq 0$, *this control laws is bounded along the trajectories of the closed-loop system and has the property that*

$$\lim_{t\to\infty}(x_1(t), x_2(t), \ldots, x_n(t)) = 0 \tag{29}$$

with exponential rate of convergence.

PROOF. We assume that $n \geq 4$ since the case when $n = 3$ has been addressed in Theorem 2. First, recall that the recursive algorithm guarantees that $x_1 = x_{2,1} = x_{3,1} = \cdots = x_{n-2,1}$, and $u_1 = u_{2,1} = u_{3,1} = \cdots = u_{n-2,1}$. From Theorem 2 we have that the control law in Eq. (28) achieves $\lim_{t\to\infty} x_{n-2,j}(t) = 0$ for $j = 1, 2, 3$. In addition, from the same theorem we have that the function

$$\gamma_1 = \frac{s}{x_1} \tag{30}$$

decays exponentially with rate greater than k_{n-2}. From Eqs. (15) we have immediately that $\lim_{t\to\infty} x_{n-3,j}(t) = 0$ for $j = 1, 2, 3, 4$.

The rest of the proof is shown by induction. To this end, let us assume that for the $(i+1)$th generated system we have that $\lim_{t\to\infty} x_{i+1,j}(t) = 0$, for $j = 1, 2, 3, \ldots, n-i$, which implies that $\lim_{t\to\infty} x_{i,j}(t) = 0$, for $j = 1, 3, \ldots, n-i+1$. It has been shown previously that, with the control law in Eq. (28), $x_{n-2,2}$ decays exponentially with rate k_{n-2}. Assume now that the functions $x_{i+\ell,2}$, where $\ell = n-2-i, n-3-i, \ldots, 1$, decay exponentially, each with corresponding rate $k_{i+\ell}$. We would also like to show that $\lim_{t\to\infty} x_{i,2}(t) = 0$ with exponential decay rate k_i.

The differential equation for $x_{i,2}$ is given by

$$\dot{x}_{i,2} = -k_i\, x_{i,2} - \sum_{\ell=1}^{n-2-i} \left(\frac{k_{i+\ell}}{k^\ell}\right) \frac{x_{i+\ell,2}}{x_1^\ell} - \left(\frac{\mu}{k^{n-2-i}}\right) \frac{s}{x_1^{n-1-i}}, \qquad i = 1, 2, \ldots, n-3 \tag{31}$$

From the requirement on the k_i one obtains

$$
\begin{aligned}
k_{n-2} &> k_i + (n-2-i)k \\
k_{n-3} &> k_i + (n-3-i)k \\
&\vdots \\
k_{i+1} &> k_i + k
\end{aligned}
$$

According to Eq. (25), $\lim_{t\to\infty} s(t) = 0$ with rate greater than $(k_{n-2}+k)$, thus the functions

$$\gamma_{n-1-i} = \frac{s}{x_1^{n-1-i}}, \qquad i = 1, 2, \ldots, n-3 \tag{32}$$

decay exponentially with rate greater than k_i. By assumption, $x_{i+\ell,2}$ ($\ell = n-2-i, n-3-i, \ldots, 1$) decay exponentially, each with corresponding rate $k_{i+\ell}$, and x_1 decays with rate k. Thus, the functions

$$\rho_\ell = \frac{x_{i+\ell,2}}{x_1^\ell}, \qquad \ell = n-2-i, n-3-i, \ldots, 1 \tag{33}$$

decay exponentially with rate greater than k_i.

Equation (31) is thus a linear differential equation in terms of $x_{i,2}$ perturbed by exponentially decaying terms of rate greater than k_i. Thus, we have that

$$\lim_{t\to\infty} x_{i,2}(t) = 0 \tag{34}$$

with rate of k_i.

The fact that the control law is bounded follows immediately from the fact that x_1 reaches the origin only asymptotically (not in finite time), and the fact that the the functions $\rho_\ell(t)$ in Eq. (33) and $\gamma_\nu(t)$ in Eq. (32) are bounded along closed-loop trajectories.

Remark 4 *It is clear that for the previous procedure to work, the attraction of the trajectories to the corresponding manifolds at each step should take place on different times scales. This is achieved by the inequalities posed on the gains k and k_i ($i = 1, 2, \ldots, n-2$). From the proof of Theorem 3 it should be clear that each state x_i of the original system in Eq. (1) converges exponentially to zero, each with rate $k_1 + (i-2)\,k$ ($i = 2, 3, \ldots, n$) and x_1 decays with rate k.*

Remark 5 *The control law in Eqs. (27) will work as long as* $x_1(0) \neq 0$*. If initially* $x_1(0) = 0$ *one can use any control law such that* x_1 *becomes nonzero. One possible choice is to use*

$$u_1 = u_{10} \qquad u_2 = 0 \tag{35}$$

where u_{10} *is some nonzero constant [1,7,20]. Similarly, this modification may be necessary in practice when* $x_1(0)$ *is very small, in order to avoid excessive values of the states and the control input* u_2*. Such an approach is given in [12].*

Although the control laws ensure boundedness and attractivity to the origin, arbitrarily small values of $x_1(0)$ will result to trajectories leaving any small neighborhood of the origin thus precluding Lyapunov stability for the closed-loop system — a consequence similar to the controllers proposed in [1]. This behaviour of the control law is not different than common driver experience indicates during a parking maneuver. Often, it is necessary to drive away from the final position, in order to get a better car posture, thus avoiding excessive effort (manifested by driving back and forth and by turning periodically the stirring wheel).

5 Numerical Example

We present numerical simulation to demonstrate the previous theoretical developments. We choose a 5-dimensional system in power form with initial conditions $x(0) = (0.1, 1, 1, 1, 1)$. The gains were chosen as $k = k_1 = 1$ $k_2 = 3$, $k_3 = 5$, and $\mu = 37$. The initial condition is close to $x_1(0) = 0$ and therefore we modify the control law in Eq. (27) according to the Remark 5 using $u_{10} = 5$ for 0.5 seconds.

Figure 1 shows the trajectories of the system. Notice that the control laws drives the states to the origin at different time scales. This is more evident in Fig. 2 where it is shown the logarithm of the norm of the states. The linear slope indicates exponential rate of convergence. Moreover, notice that the slopes of the states x_3, x_4 and x_5 are steeper than those of x_1 and x_2 which decay with the same rate ($k = k_1 = 1$). The time history of the control effort is shown in Fig. 3.

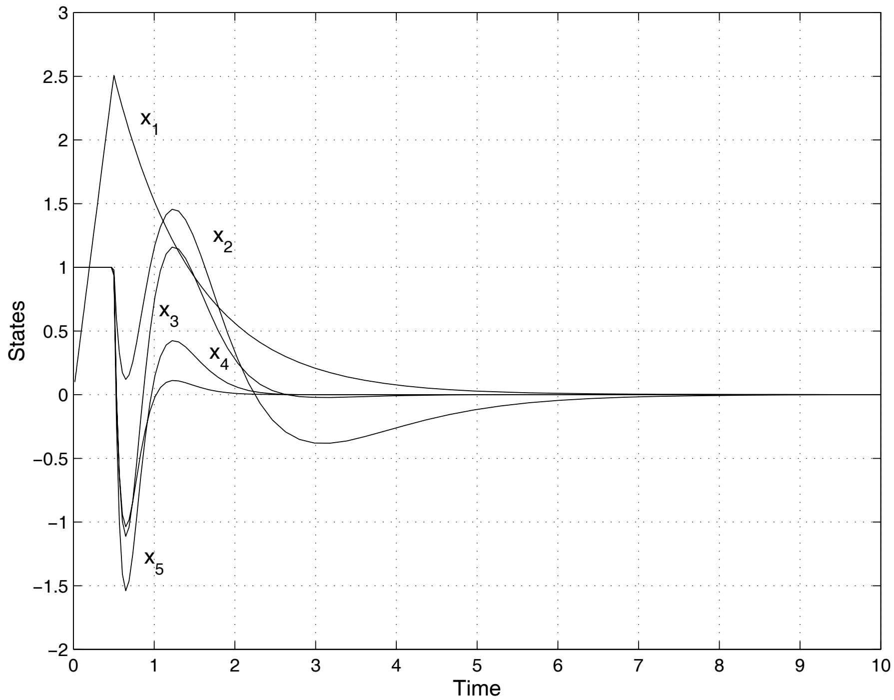

Fig. 1. Trajectories of closed-loop system.

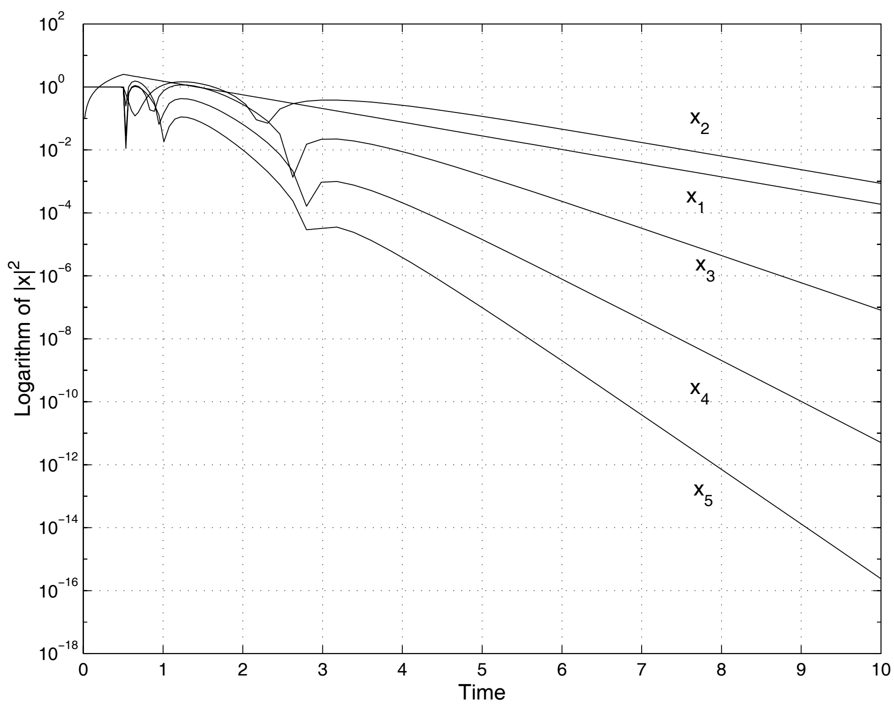

Fig. 2. Logarithmic plot of $|x_j|^2$ vs. time.

It is a well-known fact that a system in power (or chained) form can be used to describe the kinematics of a cart with trailers. Figure 4 shows the path of a

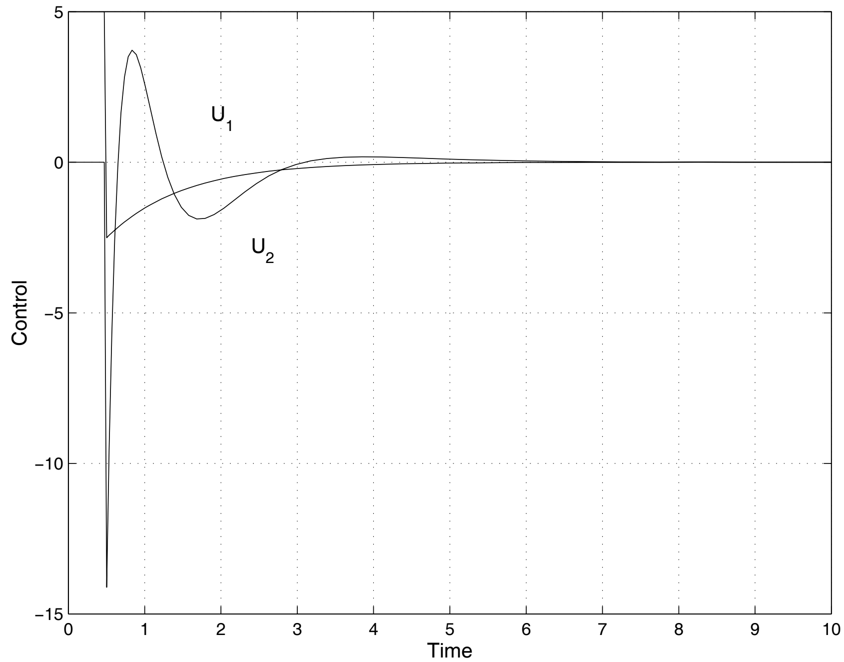

Fig. 3. Time history of control effort.

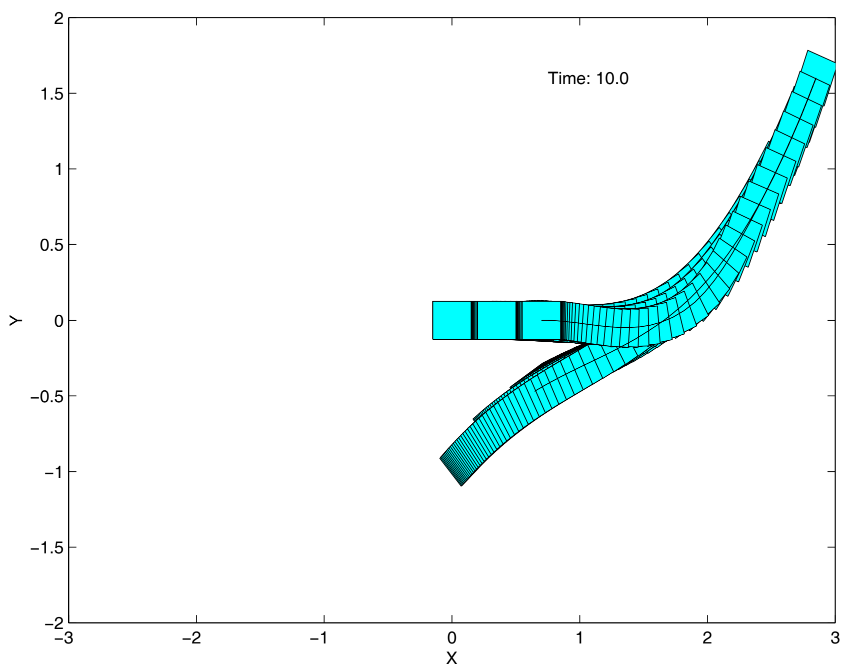

Fig. 4. Path of a cart with two trailers corresponding to a 5-dimensional system in power form.

cart with two trailers corresponding to a 5-dimensional system in power form and the results in Fig. 1. Figure 4 makes it clear how the control prefers to move the cart to a better position instead of trying to perform a sharp left turn with subsequent corrections of the final cart/trailer posture.

6 Conclusions

We use the method of invariant manifolds to construct exponentially convergent feedback control laws for n-dimensional nonholonomic systems in power form. The construction of the proposed control laws is based on a recursive algorithm which uses the invariant manifolds as new coordinates in order to construct a series of generated systems in power form of reduced dimension. The proposed controller is the one for the 3-dimensional system with proper choice of control gains. In essence, the gains are chosen such that the states of the new generated system converge to zero faster than the previous generated system. Simulation results show the exponential convergence of a 5th-dimensional system using the proposed feedback control law. Finally, because of the equivalence between chained and power form systems, the control laws proposed here can also be used for controlling nonholonomic systems in chained form.

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