Systems & Control Letters 23 (1994) 395-402
North Holland

Spin-axis stabilization of symmetric spacecraft with two control torques

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Received 24 August 1993

Abstract: It is a well known fact that a symmetric spacecraft with two control torques supplied by gas jet actuators is not controllable, if the two control torques are along axes that span the two-dimensional plane orthogonal to the axis of symmetry. However, feedback control laws can be derived for a restricted problem corresponding to attitude stabilization about the symmetry axis. In this configuration, the final state of the system is a uniform revolute motion about the symmetry axis. The purpose of this paper is to present a new methodology for constructing feedback control laws for this problem, based on a new formulation for the attitude kinematics.

Keywords: Rigid body, Stabilization, Feedback, Angular velocities, Eulerian angles.

1. Introduction

The problem of stabilization of a rigid body has recently received renewed attention in the literature. Most recent results are concerned, however, with the problem of the stabilization of the angular velocities [1, 3, 4, 5, 6, 8, 16, 20], although a few important results have also been derived for the more difficult problem of complete attitude stabilization [9, 11, 14]. Moreover, the majority of the results in the literature are for the case of rigid body stabilization using three independent control torques. On the contrary, the problem of attitude stabilization with less than three independent control torques has only recently been dealt with [9, 11, 14].

A complete mathematical description of the attitude stabilization problem was first given by Crouch [11], where he provided necessary and sufficient conditions for controllability of a rigid body in the case of one, two, or three independent acting torques. Byrnes and Isidori in [9] showed that a rigid spacecraft controlled by two pairs (couples) of gas jet actuators cannot be asymptotically stabilized to an equilibrium using a continuously differentiable, i.e. $\mathcal{C}^1$, feedback control law. In [14] the problem of attitude stabilization of a symmetric spacecraft was treated, using control torques supplied by two pairs of gas jet actuators about axes spanning a two dimensional plane orthogonal to the axis of symmetry. The complete dynamics of the spacecraft system fail to be controllable or even accessible in these cases, thus the methodologies of [9] and [11] are not applicable. However, the spacecraft dynamics is strongly accessible and small time locally controllable in a restricted sense, namely when the spin rate remains zero. It is shown in [14] that the restricted (non-spinning spacecraft) dynamics cannot be asymptotically stabilized using *smooth* $\mathcal{C}^1$ feedback. A *nonsmooth* control strategy was also developed in [14] for the restricted spacecraft dynamics which achieves arbitrary reorientation of the spacecraft. This nonsmooth control law is based on previous results on stabilization of nonholonomic mechanical systems [7].

In this paper the problem of attitude stabilization of a rigid body (spacecraft) is revisited. Specifically, we consider the stabilization of a symmetric spacecraft about its axis of symmetry using two control torques supplied by a pair of gas jets about axes spanning a two-dimensional plane orthogonal to the axis of symmetry. Without loss of generality, we can assume that the torques act along the principal axes. We derive globally asymptotically stabilizing (GAS) and globally exponentially stabilizing (GES) feedback controls using the *new formulation* of the kinematic equations developed in [22]. The final configuration corresponds to spin-axis stabilization, i.e., to a uniform revolute motion about the axis of symmetry. This is of prime practical importance, since spin stabilization is often utilized during deployment and station-keeping of modern satellites in orbit.

The control laws achieve global stabilization of the complete spacecraft dynamics to a circular attractor rather than to an isolated equilibrium. Feedback stabilization on reduced equilibrium manifolds or about attractors has received attention recently, since it appears to be an important extension of stabilization about an equilibrium, yielding bounded trajectories [9]. A major purpose of this paper is also to illustrate the novel new formulation of the kinematics used here, which promises to be extremely useful in the design of control laws for problems in rigid body attitude dynamics.

2. System Dynamics and Kinematics

2.1. Euler's Equations of Motion

Let $\omega_1, \omega_2, \omega_3$ denote the angular velocity components along a body-fixed reference frame located at the center of mass and aligned along the principal axes of a rotating rigid body. The dynamics of the rotational motion of a rigid body are described by the celebrated Euler's equations. For a symmetric body ($I_1 = I_2$), subject to two control torques along the principal axes perpendicular to the symmetry axis, they take the form

$$\dot{\omega}_1 = a_1\omega_2\omega_3 + u_1 \tag{1a}$$
$$\dot{\omega}_2 = a_2\omega_3\omega_1 + u_2 \tag{1b}$$
$$\dot{\omega}_3 = 0 \tag{1c}$$

where $a_1 \triangleq (I_2 - I_3)/I_1$, $a_2 \triangleq (I_3 - I_1)/I_2$, $u_1 \triangleq M_1/I_1$ and $u_2 \triangleq M_2/I_2$. Here M_1, M_2 are the acting torques and I_1, I_2, I_3 denote the principal moments of inertia. Introducing the complex variables $\omega \triangleq \omega_1 + i\,\omega_2$ and $u \triangleq u_1 + i\,u_2$, we rewrite (1a)-(1b) in the compact form

$$\dot{\omega} = -i\,a_1\omega_{30}\omega + u \tag{2}$$

where $\omega_{30} = \omega_3(0)$ and dot represents differentiation with respect to time. A complete formulation of the attitude problem requires the description of the kinematics, in addition to the dynamics introduced here. In contrast to the dynamics formalism, there is more than one way to describe the kinematics of a rotating body. In the next section we give a brief overview of the attitude kinematics question and we derive a new formulation for describing the kinematics of a rotating body.

2.2. Kinematics

The kinematic equations relate the components of the angular velocity vector with the rates of a set of parameters, that describe the relative orientation of two reference frames (commonly the inertial and the body-fixed frames). Any two reference frames are related by a rotation matrix R. The set of all such matrices form what is commonly known as the (three-dimensional) rotation group, consisting of all matrices which are orthogonal and have determinant $+1$, denoted by $SO(3)$. That is, $SO(3)$ is the subgroup of all invertible 3×3 matrices, defined by $SO(3) = \{R \in Gl(3, \mathbb{R}) : RR^t = I, \quad det R = +1\}$, where $Gl(n, \mathbb{R})$ is the general linear group of all $n \times n$ invertible matrices with real entries. Henceforth, we will refer to $SO(3)$ simply as the rotation group. In fact, $SO(3)$ carries an inherent smooth manifold structure, and thus, forms a (continuous) Lie group. The attitude history of the moving reference frame with respect to the constant (inertial) reference frame can therefore be described by a curve traced by the corresponding rotation $R(t) \in SO(3)$, with $SO(3)$ taken with its manifold structure. The differential equation satisfied while $R(t)$ is moving along this trajectory is given by Poisson's system of equations

$$\dot{R} = S(\omega_1, \omega_2, \omega_3)R \tag{3}$$

where $S(\omega_1, \omega_2, \omega_3)$ is the skew-symmetric matrix

$$S(\omega_1, \omega_2, \omega_3) \triangleq \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}$$

There is more than one way to parameterize the rotation group, i.e., to specify a set of parameters such that every element $R \in SO(3)$ is uniquely and unambiguously determined [21]. The commonly used three-dimensional parameterization of the rotation group leads to the familiar Eulerian angle formulation of the kinematics of a rotating rigid body.

Introducing, for example, the three-dimensional parameterization of $SO(3)$, based on a 3-2-1 Eulerian angle sequence [13], one has that the rotation matrix $R = R(\psi, \theta, \phi)$ is given by

$$R = \begin{bmatrix} c\psi c\theta & s\psi c\theta & -s\theta \\ -s\psi c\phi + c\psi s\theta s\phi & c\psi c\phi + s\psi s\theta s\phi & c\theta s\phi \\ s\psi s\phi + c\psi s\theta c\phi & -c\psi s\phi + s\psi s\theta c\phi & c\theta c\phi \end{bmatrix} \tag{4}$$

where c and s denote cos and sin, respectively. The associated kinematic equations are

$$\dot{\phi} = \omega_1 + (\omega_2 \sin\phi + \omega_3 \cos\phi)\tan\theta \tag{5a}$$

$$\dot{\theta} = \omega_2 \cos\phi - \omega_3 \sin\phi \tag{5b}$$

$$\dot{\psi} = (\omega_2 \sin\phi + \omega_3 \cos\phi)\sec\theta \tag{5c}$$

Using this parameterization of $SO(3)$, the orientation of the local body-fixed reference frame with respect to the inertial reference frame is found by first rotating the body about its 3-axis through an angle ψ, then rotating about its 2-axis by an angle θ and finally rotating about its 1-axis by an angle ϕ. With this choice of Eulerian angles, ϕ and θ determine the orientation of the local body-fixed 3-axis (the symmetry axis) with respect to the inertial 3-axis, and ψ determines the relative rotation about this axis [13]. The manifold $SO(3)$ is an imbedded submanifold of $\mathbb{R}^{3\times 3}$ of dimension three, and the Eulerian angles provide a local coordinate system for this submanifold, when considered as taking values from $\mathbb{R}^3$. In the present analysis we will also often consider the Eulerian angles as taking values from an appropriate submanifold of $S^1 \times S^1 \times S^1$, since we don't want to distinguish between orientations corresponding to angles that differ by an integer multiple of 2π. Another parametrization of $SO(3)$ (in terms of quaternions) is provided by $SU(2)$, the unitary group of complex 2×2 matrices with unit determinant, which also gives the double (universal) covering of $SO(3)$. In the subsequent discussion, and for the purpose of exposition, we will restrict ourselves to the Eulerian angle parameterization, although any other parameterization of $SO(3)$ is equally valid.

3. Alternative Formulation of the Kinematics

In this section we present a reformulation of the kinematics that will simplify the ensuing analysis significantly. This new formulation is based on an idea by Darboux [12], and was initially applied to the problem of attitude dynamics in [22], although it appears that Leimanis [15] was also aware of this possibility. Let $[a,b,c]^t$ denote any column vector of the matrix representation of R having entries r_{ij}, for $i,j = 1,2,3$ (where superscript t denotes the transpose). That is, $[a,b,c]^t = [r_{1j}, r_{2j}, r_{3j}]^t$, for some $j = 1,2,3$. Clearly from (3) one has that

$$\begin{bmatrix} \dot{a} \\ \dot{b} \\ \dot{c} \end{bmatrix} = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{6}$$

Note that these three parameters a, b, c do not provide another three-dimensional parameterization of the rotation group, as one might expect. In order to do this, one needs at least two columns of the matrix R (the third column being just the cross product of the first two). This choice would lead to a global, nonsingular, six-dimensional parameterization of $SO(3)$ [21]. Nevertheless, the entries of each column of R denote the direction cosines of the corresponding local body-fixed axis with respect to the inertial axes, and this information will be very useful in the sequel.

Because of the constraint $a^2 + b^2 + c^2 = 1$ between the elements of each column of R we can eliminate one of the three parameters a, b, c, to get a system of two first order differential equations. The most natural and elegant way to reduce the third order system (6) to a second order system is by the use of *stereographic projection* [10]. That is, if we let a, b, and c represent coordinates on the unit sphere $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$ in $\mathbb{R}^3$, then, for $(a,b,c) \in S^2$, the stereographic projection $\sigma : S^2 \backslash \{(0,0,-1)\} \to \mathbf{C}$ defined by

$$w = \sigma(a,b,c) \stackrel{\triangle}{=} \frac{b - ia}{1 + c} = \frac{1 - c}{b + ia} \tag{7}$$

leads to the following differential equation for the complex quantity w

$$\dot{w} = -i\,\omega_3 w + \frac{\omega}{2} + \frac{\bar{\omega}}{2} w^2 \tag{8}$$

where $\omega = \omega_1 + i\,\omega_2$ and the bar denotes complex conjugate. Equation (8) is a scalar Riccati equation with time-varying coefficients. The real and imaginary parts of $w = w_1 + i\,w_2$ satisfy the differential equations

$$\dot{w}_1 = \omega_3 w_2 + \omega_2 w_1 w_2 + \frac{\omega_1}{2}(1 + w_1^2 - w_2^2)$$

$$\dot{w}_2 \quad = \quad -\omega_3 w_1 + \omega_1 w_1 w_2 + \frac{\omega_2}{2}(1 + w_2^2 - w_1^2)$$

In (7) we have chosen the base point (pole) of the projection to be the point $(0, 0, -1) \in S^2$. Recall that the stereographic projection σ establishes a one-to-one correspondence between the unit sphere S^2 and the extended complex plane $\mathbf{C}_\infty \triangleq \mathbf{C} \cup \{\infty\}$. It can be easily verified that the inverse map $\sigma^{-1} : \mathbf{C} \to S^2 \backslash \{(0, 0, -1)\}$, $w \mapsto (a, b, c)$ is given by

$$a = \frac{i\,(w - \bar{w})}{|w|^2 + 1}, \quad b = \frac{w + \bar{w}}{|w|^2 + 1}, \quad c = -\frac{|w|^2 - 1}{|w|^2 + 1}$$

and can be used to find a, b, c once w is known. Here $|\cdot|$ denotes the absolute value of a complex number, i.e., $z\bar{z} = |z|^2$, $z \in \mathbf{C}$.

In order to establish the relationship between w and the particular parametrization of $SO(3)$ used, notice that we can, in principle, identify $[a, b, c]^t$ with any column vector of the rotation matrix R, where R can be expressed in terms of *any* of the parameterizations of $SO(3)$. By nulling the appropriate elements of a given column of the rotation matrix we can align the corresponding body-fixed axis with any inertial axis. If, for example, one needs to stabilize the $3rd$-body axis to a uniform rotation about the jth inertial axis ($j = 1, 2, 3$), one suffices to make $r_{1j} = r_{2j} = 0$ and $r_{3j} = 1$. This gives a great deal of flexibility in the analysis and design of control laws for attitude stabilization.

In the case of the three-dimensional 3-2-1 Eulerian angle parameterization the matrix $R = R(\psi, \theta, \phi)$ is given by (4). Since we are interested in the stabilization of the symmetry axis (the body 3-axis) we identify $[a, b, c]^t$ with the third column of R, establishing the following correspondence between w and (θ, ϕ)

$$w = \frac{\sin\phi\cos\theta + i\sin\theta}{1 + \cos\phi\cos\theta}$$

or in terms of real and imaginary parts of w,

$$w_1 = \frac{\sin\phi\cos\theta}{1 + \cos\phi\cos\theta}, \qquad w_2 = \frac{\sin\theta}{1 + \cos\phi\cos\theta} \tag{9}$$

4. Feedback Control Strategy

In this section we present a methodology for constructing feedback control laws for the system of equations (1) and (5a)-(5b), which depends on the alternative formulation of the kinematic equations presented in Section 3. Asymptotic stability of the closed-loop system is easily demonstrated via Lyapunov's direct method.

It is clear from equation (1c) that no control can affect the value of the component of the angular velocity ω_3 along the symmetry axis. In fact, the value of ω_3 remains constant for all $t \geq 0$. As already mentioned, the complete system of equations (1)-(5) is not controllable. Therefore, if the initial condition $\omega_3(0)$ is not zero, no control can drive the system to the origin ($\omega_1 = \omega_2 = \omega_3 = \phi = \theta = \psi = 0$). Of course, if $\omega_3(0) \neq 0$ then it is meaningless to require $\psi = 0$, but we may require a control law such that $\omega_1 = \omega_2 = \phi = \theta = 0$. From equation (7) notice that $w = 0$ (with the previous identification with the third column of the rotation matrix) implies that the body-fixed 3-axis is aligned with the inertial 3-axis. However, we have no *a priori* information about the relative rotation of the body about its symmetry axis. In fact, the stabilizing control laws for the four equations involving $\omega_1, \omega_2, \phi, \theta$ induce a spinning motion about the symmetry axis, i.e., spin-axis stabilization which is of important practical interest. The closed-loop trajectories asymptotically approach the set

$$\mathcal{N} = \{(\omega_1, \omega_2, \omega_3, \phi, \theta, \psi) \in \mathbb{R}^3 \times S^3 : \omega_1 = \omega_2 = \phi = \theta = 0\}$$

That is, stabilization is achieved about the one-dimensional submanifold $\phi = \theta = 0$ of S^3 — the unit sphere in $\mathbb{R}^4$. On this submanifold, the angle ψ can have any value. The problem is closely related with the definition of "stabilization about an attractor" of Byrnes and Isidori [9]. Recalling that ψ is an ignorable variable for the system (5), in the subsequent analysis we tacitly discard the equation for ψ.

Using (2) and the kinematic equation (8), the attitude equations for a symmetric body can be written as

$$\dot{\omega} = -i\, a_1 \omega_{30} \omega + u \tag{10a}$$

$$\dot{w} = -i\, \omega_{30} w + \frac{\omega}{2} + \frac{\bar{\omega}}{2} w^2 \tag{10b}$$

This system of differential equations is in one-to-one correspondence with the system of equations (1a)-(1b) and (5a)-(5b). The system of equations (10) falls within the general class of nonlinear systems of the form

$$\dot{y} = h(y) + u \tag{11a}$$

$$\dot{x} = f(x, y) \tag{11b}$$

where $h : \mathbb{R}^m \to \mathbb{R}^m$ and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are smooth vector fields, with $f(0,0) = 0$. Equations (11) represent a system in cascade form and it is a well-known result [2, 18, 19] that for systems of this form, if the subsystem $\dot{x} = f(x, y)$ is smoothly stabilizable (regarding y as a control-like variable), then the extended system (11) is also smoothly stabilizable. In other words, if in (11) the subsystem (11b) is smoothly stabilizable, then adding an integrator (up to the new control $u = -h(y) + v$, if necessary) does not change this property. We will use this result in order to derive asymptotically stabilizing control laws for the system (10). In the sequel, $|\cdot|$ denotes the norm (magnitude) of a complex number and $||\cdot||$ denotes the norm of a vector in $\mathbf{C} \times \mathbf{C}$ defined by $||(z_1, z_2)|| = |z_1| + |z_2|$ for $z_1,\, z_2 \in \mathbf{C}$.

We have the following theorems concerning asymptotic stabilization of the system (10).

Theorem 4.1. *The choice of the feedback control law*

$$\omega = -\kappa w \tag{12}$$

with $\kappa > 0$ globally exponentially stabilizes (10b) with rate of decay $\kappa/2$.

Proof. With the choice of feedback (12) the closed-loop system becomes

$$\dot{w} = -i\, \omega_{30} w - \frac{\kappa}{2}(1 + |w|^2) w \tag{13}$$

The positive definite function $V : \mathbf{C} \to \mathbb{R}$ defined by $V(w) = w\bar{w} = |w|^2$ is a Lyapunov function for (13). Indeed, differentiating along the closed-loop trajectories one can easily verify that

$$\dot{V}(w) = -\kappa(1 + |w|^2)|w|^2 \leq 0$$

Since $\dot{V}(w) = 0$ if and only if $w = 0$, the closed-loop system (13) is asymptotically stable. Global asymptotic stability follows from the facts that these statements hold for all $w \in \mathbf{C}$ and V is radially unbounded, i.e., $V(w) \to \infty$, for $|w| \to \infty$. Notice that since $\dot{V} \leq -\kappa V$ one, in fact, guarantees *exponential stability* for the closed loop system with rate of decay $\kappa/2$. □

The control law for (10) in terms of u is given by the next theorem.

Theorem 4.2. *The choice of the feedback control law*

$$u = i\, a_1 \omega_{30} \omega + \kappa(i\, \omega_{30} w - \frac{\omega}{2} - \frac{\bar{\omega}}{2} w^2) - \alpha(\omega + \kappa w) \tag{14}$$

with $\kappa > 0$ and $\alpha > 0$, globally asymptotically stabilizes system (10).

Proof. With this choice of feedback, the closed-loop system becomes

$$\dot{\omega} = -\kappa(-i\, \omega_{30} w + \frac{\omega}{2} + \frac{\bar{\omega}}{2} w^2) - \alpha(\omega + \kappa w) \tag{15a}$$

$$\dot{w} = -i\, \omega_{30} w + \frac{\omega}{2} + \frac{\bar{\omega}}{2} w^2 \tag{15b}$$

The set $\mathcal{E} = \{(\omega, w) \in \mathbf{C} \times \mathbf{C} : \omega + \kappa w = 0\}$ is a positively invariant set and a global asymptotic attractor for (15). To see this, let $z \triangleq \omega + \kappa w$. Then the system equations become

$$\dot{z} = -\alpha z \tag{16a}$$

$$\dot{w} = -i\, \omega_{30} w - \frac{\kappa}{2} w + \frac{z}{2} - \frac{\kappa}{2} w |w|^2 + \frac{\bar{z}}{2} w^2 \tag{16b}$$

LaSalle's theorem guarantees the global asymptotic stability of (15), if the trajectories of (15), or equivalently of (16) remain bounded [17]. To this end, let V be the positive definite function of Theorem 4.1, i.e., let $V(w) = |w|^2$. We will show that V is nonincreasing outside a bounded set that contains the origin; in particular, we claim that $\dot{V}(w) \leq 0$ on the set $\mathcal{D} = \{w \in \mathbf{C} : |w| \geq |z(0)|/\kappa\}$. This will imply boundedness of solutions of w, hence of (16). Differentiating along trajectories of (16b) one obtains

$$\begin{aligned} \dot{V}(w) &= -\kappa |w|^2 - \kappa |w|^4 + Re(z\bar{w})(1 + |w|^2) \\ &\leq -\kappa |w|^2 - \kappa |w|^4 + |z||w|(1 + |w|^2) \end{aligned}$$

where $Re(.)$ denotes the real part of a complex number and where we made use of the fact that $Re(z) \leq |z|$ for all $z \in \mathbf{C}$. From (16a) one has that $z(t) = z(0)e^{-\alpha t}$ and in particular $|z(t)| \leq |z(0)|$. Thus,

$$\begin{aligned} \dot{V}(w) &\leq -\kappa |w|^2 - \kappa |w|^4 + |z(0)||w|(1 + |w|^2) \\ &= -(1 + |w|^2)|w|(\kappa |w| - |z(0)|) \end{aligned}$$

For $|w| \geq |z(0)|/\kappa$ one has $\dot{V}(w) \leq 0$ as claimed. This completes the proof. □

The argument of the previous theorem is equivalent to the results of Sontag [18, 19] about global asymptotic stability of interconnected systems of the form (11). Indeed, subsystem (16b) with $z = 0$ is GAS according to Theorem 4.1, while subsystem (16a) is obviously also GAS. Boundedness of solutions of (16b) imply that this subsystem satisfies the Convergent Input Bounded State (CIBS) condition, according to the terminology of [18], hence also the Convergent Input Convergent State (CICS) condition. The global asymptotic stability of the whole system is then a consequence of a cascade connection of a GAS with a CICS system. In general, for cascade systems of the form (11) it suffices to show that the control-independent subsystem (11b) satisfies some kind of existence of solutions property for small inputs (e.g. a global Lipschitz condition as in [24], or an Input to State Stability (ISS) condition as in [19]) and then make the control-driven system GAS.

The control law of Theorem 4.2 is not the only choice of stabilizing feedback for the system (10). The especially simple structure of equations (10) allows the derivation of several other globally asymptotically stabilizing control laws.

Theorem 4.3. *The choice of the feedback control law*

$$u = i\,a_1\omega_{30}\omega - \kappa(-i\,\omega_{30}w + \frac{\omega}{2} + \frac{\bar{\omega}}{2}w^2) - \alpha(\omega + \kappa w) - w(1 + |w|^2) \tag{17}$$

with $\kappa > 0$ *and* $\alpha > 0$*, globally exponentially stabilizes system (10) with rate of decay* $\beta/2$*, where* $\beta = \min\{2\alpha, \kappa\}$*.*

Proof. The positive definite function $V : \mathbf{C}\times\mathbf{C} \to \mathbb{R}$ defined by $V(\omega, w) = |w|^2 + |\omega + \kappa w|^2/2$ is a Lyapunov function for the closed loop system. Indeed, by differentiating along the closed loop trajectories of the system, one can easily verify that

$$\dot{V}(\omega, w) = -\alpha|\omega + \kappa w|^2 - \kappa|w|^2(1 + |w|^2)$$

Since $\dot{V}(\omega, w) \leq 0$ for all $(\omega, w) \in \mathbf{C}\times\mathbf{C}\backslash\{(0, 0)\}$, the closed loop system is asymptotically stable. Global asymptotic stability follows from the facts that the previous statements hold for all $(\omega, w) \in \mathbf{C}\times\mathbf{C}$ and V is radially unbounded, i.e., $V(\omega, w) \to \infty$, for $||(\omega, w)|| \to \infty$. In fact, since $\dot{V} \leq -\beta V$, where $\beta = \min\{2\alpha, \kappa\}$ the closed loop system is *globally exponentially stable* with rate of decay $\beta/2$. □

Surprising enough, there is also a *linear* feedback control which globally asymptotically stabilizes system (10) (we owe this observation to Prof. M. Corless). This is the result of the following theorem.

Theorem 4.4. *The choice of the linear feedback control law*

$$u = -\kappa_1\omega - \kappa_2 w \tag{18}$$

with $\kappa_1 > 0$ *and* $\kappa_2 > 0$*, globally asymptotically stabilizes the system (10).*

Proof. With this choice of feedback, the closed-loop system becomes

$$\dot{\omega} = -\kappa_1\omega - \kappa_2 w \tag{19a}$$

$$\dot{w} = -i\,\omega_{30}w + \frac{\omega}{2} + \frac{\bar{\omega}}{2}w^2 \tag{19b}$$

Choose the following positive definite function $V : \mathbf{C}\times\mathbf{C} \to \mathbb{R}$

$$V(\omega, w) = \frac{1}{2}|\omega|^2 + \kappa_2\ln(1 + |w|^2) \tag{20}$$

Differentiating along trajectories of (19) one has that

$$\dot{V}(\omega, w) = -\kappa_1|\omega|^2 \leq 0 \tag{21}$$

According to LaSalle's theorem, the trajectories will approach the largest invariant subset of the set $\mathcal{D} = \{(\omega, w) \in \mathbf{C} \times \mathbf{C} : \dot{V}(\omega, w) = 0\}$. If however $\dot{V}(\omega, w) \equiv 0$, then $\omega = \dot{\omega} \equiv 0$ and from (19a) $w \equiv 0$. Therefore, the only trajectory that maintains $\dot{V} \equiv 0$ is the zero trajectory $\omega = w \equiv 0$. The system (19) is therefore asymptotically stable. Since V is radially unbounded, the system is, in fact, globally asymptotically stable. □

Corollary 4.1. *The choice of the feedback control law*

$$u = -\kappa_1 \omega - \kappa_2 (1 + |w|^2) w \tag{22}$$

with $\kappa_1 > 0$ *and* $\kappa_2 > 0$, *globally asymptotically stabilizes the system (10).*

One word of caution should be mentioned here, as far as our terminology of "global" stabilization is concerned. Although the previous control laws provide global asymptotic stability of the system in (ω, w) coordinates, our main interest in practice is the configuration of the system in terms of Eulerian angles, which provide a physical description of the orientation. It is clear that any control law claiming global asymptotic stability in (ω, w) coordinates has to exclude initial conditions that correspond to $w = \infty$ i.e. with $(a, b, c) = (0, 0, -1)$ in (7). However, as a result of the stabilization of the closed-loop system we have boundedness of the solutions of w for all $t > 0$, and thus we avoid the singular orientation corresponding to direction cosines $(0, 0, -1)$ in (7). Therefore, global stability here implies stability from all initial conditions except the initial condition corresponding to this singular "upside-down" configuration along the body symmetry axis. The easiest way to remedy this problem is simply to turn the thrusters on to move (even infinitesimally) away from this singular orientation and then use the results of this section. For all other initial conditions, the closed-loop trajectories converge to the origin.

As a last comment, we note that Theorems 4.1-4.4 and Corollary 4.1 give asymptotically stabilizing control laws for the $(\omega_1, \omega_2, \phi, \theta)$ subsystem of equations. Stabilization of this subsystem corresponds, in the complete system, to asymptotic stabilization about the axis of symmetry. The problem of also stabilizing $\psi = 0$ (assuming of course that $\omega_3(0) = 0$) is more difficult. In fact, in [14] it was shown that any stabilizing feedback control law of the complete system, i.e., for $(\omega_1, \omega_2, \phi, \theta, \psi)$, must be necessarily *nonsmooth*. The stabilization of the complete system (1)-(5) with $\omega_3 = 0$ using the kinematic equation (8) is the subject of a forthcoming paper [23]. The control laws thus derived are especially simple and elegant.

5. Conclusions

The problem of stabilization of a symmetric spacecraft with two gas jet actuators aligned about the principal axes of equal moments of inertia is investigated. Using a new formulation for the kinematic equations, asymptotically stabilizing controls have been derived for the restricted problem of spin-axis stabilization. The asymptotic stability of the closed-loop system is proved by construction of appropriate Lyapunov functions. The stabilizing control laws thus derived are especially simple and elegant. Moreover, since the control laws are given directly in terms of the elements of the rotation matrix, they do not depend on the particular choice of the Eulerian angle set used to describe the attitude orientation in inertial space (or the particular parameterization of the rotation group, for that matter). This provides a great deal of freedom in the analysis and design of attitude control laws.

Acknowledgements

The authors would like to thank Professor M. Corless for helpful discussions during the preparation of this work. This research has been supported by the National Science Foundation under Grant No. MSS-9114388.

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