Optimal Feedback Velocity Profile Generation for a Vehicle with Given Acceleration Limits: A Level Set Implementation

S. Jain,* P. Tsiotras† and E. Velenis‡

Abstract— **In this paper, we use level set methods to numerically generate the minimum-time optimal velocity profiles for a vehicle with given acceleration limits driving along a specified path. The proposed approach solves the Hamilton-Jacobi-Bellman (HJB) equation associated with the given optimal control problem by computing the level sets of the value function. Once the optimal cost is found, the optimal feedback control can be computed online thus generating the velocity profile quickly. The results are compared to a semi-analytic approach that was developed recently for the same problem by the last two authors.**

I. Introduction

In a series of recent papers the last two authors of this paper have developed a semi-analytic solution to the problem of optimal velocity generation of a point-mass vehicle on a prescribed path with a given, elliptical acceleration envelope [1], [2], [3]. The work of [1], [2], [3] provides an alternative approach for the trajectory optimization problem of ground vehicles, which, typically, is dealt with in the literature using numerical methods [4], [5], [6]. These numerical optimization approaches are computationally intensive, and they cannot be readily applied in cases where the environment changes unpredictably. As a result, they are not suitable for real-time path-planning optimization. Motivated by the requirement to reduce the on-board computational cost, references [1], [2], [3] have exploited the potential of solving the trajectory planning problem (or at least part of this problem) analytically or semi-analytically. Indeed, for a point mass model moving on a given path, the problem is essentially one-dimensional (using the path arc length as the independent variable) with double-integrator dynamics. Hence, it can be readily solved. However, several complications arise from additional state and control constraints that make the problem more intricate than the standard time-optimal problem of the double-integrator [7]. Although "intuitive" solutions for this problem were known for sometime [8], [9], [10] (see also [11], [12], [13]), nonetheless, the work in [1], [3] was the first to offer a rigorous proof of optimality using optimal control theory. The necessary optimality conditions were explicitly derived, allowing one to determine the number and type of control switchings. In particular, the switching points can be found semi-analytically instead of numerically (as it was done in [11], [12] and [13]).

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Since open-loop trajectories are non-robust, a receding horizon implementation of the optimal velocity profile of [1] was proposed in [2] to account for changes in the mid-point and terminal boundary conditions. The receding horizon algorithm ensures that at the end of each executed subarc the vehicle can reach a "safe state" (for example, complete stop) regardless of the (a priori unknown) changes in the environment outside the planning horizon. This is achieved by designing a dynamic scheme that determines appropriate planning and execution horizons.

An alternative solution to the problem posed in [1], [2], [3] is provided in this paper. In particular, the optimal control is constructed by solving the Hamilton-Jacobi-Bellman equation. This allows the construction of the whole field of extremals that ensures optimality with respect to any initial condition. Hence, using this approach, a certain degree of robustness is guaranteed against perturbations of the state. Note that this differs significantly from previous numerical solutions that dealt with *open-loop* optimal controls only [8], [9], [10], [1], [4], [5], [6], [11], [12], [13]

The solution of the Hamilton-Jacobi-Bellman partial differential equation is notoriously difficult. Apart from the possibility of non-smooth solutions, the potential high dimensionality of the state vector hinders the development of universally efficient numerical methods. Nonetheless, recently, efficient numerical methods have been proposed for the solution of certain types of Hamilton-Jacobi-Bellman equations, even for moderately high dimensions (five or six). These methods are based on the computation of the level sets of the associated value function of the Hamilton-Jacobi-Bellman equation.

Level set methods were first introduced by Osher and Sethian in [14], as a general framework for computing the evolution of interfaces using an implicit representation. The key idea is to represent the interfaces $\Gamma(t)$ by the zero level set of a smooth function $\varphi(x,t)$, such that $\Gamma(t) = \{x : \varphi(x,t) = 0\}$. The motion of $\Gamma(t)$ can be formulated by a Hamilton-Jacobi-Bellman equation for the level set function $\varphi(x,t)$, that is,

$$\varphi_t(x,t) + H(x, \nabla\varphi(x,t)) = 0.$$

The most significant advantage of the level set formulation is that it can easily handle the topological changes of the interfaces, such as splitting, merging, appearing and vanishing. Level set methods have been widely used in many applications including computer vision, image processing, material science, fluid dynamics, control and medical science. The two books [15], [16] offer a comprehensive introduction to level set and similar methods, such as fast marching and fast sweeping algorithms [17]. The latter are specially designed methods for solving static Hamilton-Jacobi-Bellman equations.

In this paper, we apply level sets to solve the Hamilton-Jacobi-Bellman equation arising in the minimum-time control of a vehicle moving along a path of prescribed curvature profile, subject to a bounded acceleration envelope. The solution V to the Hamilton-Jacobi-Bellman equation (value function), is given by the zero level set of the function φ, and provides a solution to the *feedback* control problem; once V is computed, the optimal velocity profile can be calculated immediately for any initial condition, by moving along the gradient curves induced by the level sets of V.

The paper is organized as follows. We first describe the problem and then formulate the corresponding optimal control problem. Next, we give a brief introduction of the level set method, followed by its application for solving the optimal control problem at hand. We conclude the paper by giving a numerical example, which validates the proposed approach.

II. Problem Statement

Consider a vehicle modeled as a point of mass m, and travelling through a prescribed path, with given acceleration limits and fixed boundary conditions, that is, fixed initial and final position and velocity. We seek the velocity profile along the path for minimum travel time as a function of the path arc length. The path is assumed to be described by the radius $R(s)$ (equivalently, the curvature $\kappa(s)$) at each point of the path as a function of the path length coordinate s (see Figure 1). The cartesian coordinates at any point on the path

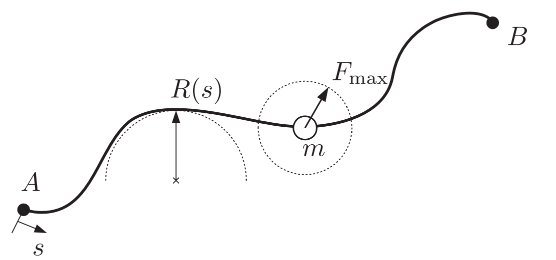

Fig. 1. The vehicle of mass m travels along the prescribed path $R(s)$ in minimum time, given the maximum acceleration limit $F_{\max}/m$.

may be calculated using a standard transformation [4]. The equations of motion are given by

$$m\frac{\mathrm{d}^2 s}{\mathrm{d}\tau^2} = f_t, \tag{1}$$

$$m(\frac{\mathrm{d}s}{\mathrm{d}\tau})^2 = R(s)f_n, \tag{2}$$

where f_t is the tangential component of the force along the path, and f_n is the normal (centripetal) force such that the vehicle tracks the prescribed path. Consider now the following state assignment and change of time scale:

$$t = \beta\tau, \tag{3}$$

$$x_1 = \alpha\beta s, \tag{4}$$

$$x_2 = \alpha\frac{\mathrm{d}s}{\mathrm{d}\tau}, \tag{5}$$

with

$$\alpha = \sqrt{\frac{m}{F_{\max}}}, \tag{6}$$

$$\beta = \alpha\frac{F_{\max}}{m}. \tag{7}$$

The control input in this formulation is f_t, and the maximum overall acceleration limit $F_{\max}/m$ translates to a state-dependent control constraint. Introducing the control variable u, the control constraint may be written as

$$\frac{f_t}{F_{\max}} = u\sqrt{1-\left(\frac{x_2^2}{R(x_1)}\right)^2}, \tag{8}$$

$$|u| \leq 1. \tag{9}$$

The dynamics of the system may then be written as

$$\dot{x}_1 = x_2, \tag{10}$$

$$\dot{x}_2 = u\sqrt{1-\left(\frac{x_2^2}{R(x_1)}\right)^2}, \tag{11}$$

$$|u| \leq 1. \tag{12}$$

Note that for the dynamics to be well defined, the trajectories have to remain inside the region $\mathcal{S}$ of the state space defined by

$$\mathcal{S} = \{(x_1, x_2) : |R(x_1)| \geq x_2^2\}. \tag{13}$$

In the sequel we assume that $(x_1(t), x_2(t)) \in \mathcal{S}$, for all $t \in [0, t_f]$.

III. Optimal Control Formulation

Given fixed boundary conditions,

$$\text{Point } A: \quad (x_1(0), x_2(0)), \tag{14}$$

$$\text{Point } B: \quad (x_1(t_f), x_2(t_f)), \tag{15}$$

the problem is to find the optimal control u that drives the system (10)-(12) from point A to point B in minimum time t_f. We make the natural assumption that the boundary conditions are chosen in such a way that the optimal velocity does not change sign, that is,

$$x_2(t) \geq 0, \qquad \forall\, t \in [0, t_f]. \tag{16}$$

The Hamilton-Jacobi-Bellman (HJB) equation for the corresponding minimum time problem is computed as [7]

$$1 + \frac{\partial V}{\partial x_1}x_2 - \sqrt{1-\left(\frac{x_2^2}{R(x_1)}\right)^2}\left|\frac{\partial V}{\partial x_2}\right| = 0, \tag{17}$$

with boundary condition

$$V(x_1(t_f), x_2(t_f)) = 0. \tag{18}$$

Once the HJB equation is solved and V is known, the optimal control can be found from

$$u = -\mathrm{sgn}\left(\frac{\partial V}{\partial x_2}\right). \tag{19}$$

Before presenting the technique for generating the minimum-time optimal velocity profiles proposed in this paper, we first need to provide a brief introduction of level set methods.

IV. Introduction to Level Set Methods

In [18] Osher and Sethian introduced the concept of a level set formulation to propagate curves and surfaces. The method treats easily self-intersections, topological changes, and kinks in the solution. The problem analyzed in [18] is to move a closed curve

$$\Gamma(t) : [0, t_f] \to \mathbb{R}^2$$

normal to itself with normal velocity ν. The basic idea behind the level set method is to embed the initial position of the front $\Gamma(0)$ as the zero level set of a higher-dimensional function $\varphi : \mathbb{R}^2 \times [0, t_f] \to \mathbb{R}$. The evolution of the function φ is linked to the propagation of the front itself through a time-dependent initial value problem

$$\varphi_t + \nu\sqrt{\varphi_{x_1}^2 + \varphi_{x_2}^2} = 0, \tag{20}$$

$$\varphi(x_1, x_2, 0) = 0, \quad (x_1, x_2) \in \Gamma(0). \tag{21}$$

Equation (20) is referred to in the literature as the *level set equation*. At each instant of time the front is given by the zero level set of the time-dependent level set function φ.

Briefly, one finds a function $\varphi(x_1, x_2, t)$ so that, at $t = 0$, the following conditions hold

$$\varphi(x_1, x_2, 0) = 0 \Leftrightarrow (x_1, x_2) \in \Gamma(0), \tag{22}$$

$$\varphi(x_1, x_2, 0) > 0 \Leftrightarrow (x_1, x_2) \in \Omega, \tag{23}$$

$$\varphi(x_1, x_2, 0) < 0 \Leftrightarrow (x_1, x_2) \in \Omega^c, \tag{24}$$

where $\Omega \subset \mathbb{R}^2$ and $\varphi(x_1, x_2, 0)$ is a uniformly continuous and monotonic strictly decreasing function of the distance to Γ. The interface is to be captured for all later times, by locating the set $\Gamma(t)$ for which φ vanishes. In other words, one requires that $\Gamma(t)$ evolves so that

$$\varphi(x_1, x_2, t) = 0 \Leftrightarrow (x_1, x_2) \in \Gamma(t). \tag{25}$$

A level set formulation can be used to solve the Hamilton-Jacobi-Bellman equation,

$$\varphi_t + H(\varphi_{x_1}, \varphi_{x_2}, x_1, x_2) = 0. \tag{26}$$

It is reminded that solutions to Hamilton-Jacobi-Bellman equations may be nonsmooth even if all data of the problem are smooth. Generalized (i.e., viscosity) solutions are necessary in those cases, as standard derivatives do not exist in the regions where the solution is nonsmooth. Viscosity solutions for Hamilton-Jacobi-Bellman equations were first proposed by Crandall and Lions [19] and the monotone first-order accurate numerical methods were presented by Crandall and Lions in [20]. Later Osher and Sethian [18] used the connection between conservation laws and Hamilton-Jacobi-Bellman equations to construct high order accurate numerical methods. Even though the analogy between conservation laws and Hamilton-Jacobi equations fails in multiple space dimensions, many Hamilton-Jacobi equations can be discretized in a dimension-by-dimension fashion. This culminated in [21], where Osher and Shu proposed a general framework for the numerical solution of Hamilton-Jacobi equations using successful methods from the theory of conservation laws.

A. Discretization

A forward Euler time discretization of (26) can be written as

$$\frac{\varphi^{k+1} - \varphi^k}{\Delta t} + \hat{H}^k(\varphi_{x_1}^-, \varphi_{x_1}^+, \varphi_{x_2}^-, \varphi_{x_2}^+, x_1, x_2) = 0, \tag{27}$$

where $\hat{H}^k(\varphi_{x_1}^-, \varphi_{x_1}^+, \varphi_{x_2}^-, \varphi_{x_2}^+, x_1, x_2)$ is the numerical approximation of $H(\varphi_{x_1}, \varphi_{x_2}, x_1, x_2)$ at time step k. The function $\hat{H}$ is called the *numerical Hamiltonian*, and it is required to be consistent in the sense that

$$\hat{H}(\varphi_{x_1}, \varphi_{x_1}, \varphi_{x_2}, \varphi_{x_2}, x_1, x_2) = H(\varphi_{x_1}, \varphi_{x_2}, x_1, x_2).$$

In order to discretize the derivatives $\varphi_{x_1}^+$, $\varphi_{x_1}^-$, $\varphi_{x_2}^+$, $\varphi_{x_1}^-$, one can use essentially non-oscillatory (ENO) schemes [22], [23], [15]. The basic idea behind the ENO schemes is to find the smoothest possible polynomial interpolant for approximating φ and then to differentiate the interpolant to get φ_{x_1} and φ_{x_2}. The forward Euler time discretization (27) can be extended to higher-order total variation diminishing (TVD) Runge-Kutta (RK) scheme in a straight forward manner. For more details, the reader is referred to [22], [23], [15].

There exist several schemes in the literature [21] for computing the numerical Hamiltonian ($\hat{H}$). For the sake of brevity, here we only briefly describe the Lax-Friedrich's (LF) scheme, which will be used in the sequel to solve the numerical example given in this paper. The Lax-Friedrich's (LF) scheme [20], [15], [21], [23] for discretizing $\hat{H}$ is as follows:

$$\begin{aligned}\hat{H}(\varphi_{x_1}^-, &\varphi_{x_1}^+, \varphi_{x_2}^-, \varphi_{x_2}^+, x_1, x_2) \\ &= H\left(\frac{\varphi_{x_1}^+ + \varphi_{x_1}^-}{2}, \frac{\varphi_{x_2}^+ + \varphi_{x_2}^-}{2}, x_1, x_2\right) \\ &\quad - \frac{1}{2}\alpha^{x_1}(\varphi_{x_1}^+ - \varphi_{x_1}^-) - \frac{1}{2}\alpha^{x_2}(\varphi_{x_2}^+ - \varphi_{x_2}^-),\end{aligned} \tag{28}$$

where,

$$\alpha^{x_1} = \max_{\varphi_{x_1} \in I^{x_1}} |H_1(\varphi_{x_1}, \varphi_{x_2})|, \tag{29}$$

$$\alpha^{x_2} = \max_{\varphi_{x_2} \in I^{x_2}} |H_2(\varphi_{x_1}, \varphi_{x_2})|, \tag{30}$$

where H_1, H_2 are the partial derivatives of H with respect to φ_{x_1} and φ_{x_2} respectively. First, the maximum and minimum values of φ_{x_1} are identified by considering all the values of $\varphi_{x_1}^-$ and $\varphi_{x_1}^+$ on the cartesian mesh. One can then identify the interval

$$I^{x_1} = [\varphi_{x_1}^{\min}, \varphi_{x_1}^{\max}].$$

A similar procedure is used to define

$$I^{x_2} = [\varphi_{x_2}^{\min}, \varphi_{x_2}^{\max}].$$

Next, we apply this theory for the solution of the optimal control problem given in Section III.

V. Optimal Velocity Profile Generation using Level Set Methods

First, we briefly summarize the results of [24], [25] which will be used in this work. To this end, consider a closed target $\mathcal{T}$ for a system evolving according to dynamics,

$$\dot{\mathbf{x}} = f(\mathbf{x}, u), \tag{31}$$

where $\mathbf{x} : [0, t_f] \to \mathbb{R}^2$, $u : [0, t_f] \to \mathbb{R}$, $|u| \le 1$, $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$. The minimum time to reach the target set $\mathcal{T}$ is the viscosity solution of the HJB PDE,

$$H(V_{x_1}, V_{x_2}, x_1, x_2) = 1, \quad (x_1, x_2) \in \mathbb{R}^2 \setminus \mathcal{T}, \tag{32}$$

with

$$V(x_1, x_2) = 0, \quad (x_1, x_2) \in \mathcal{T}, \tag{33}$$

$H(\mathbf{p}, \mathbf{x}) = -\min_{|u|\le 1} \mathbf{p} \cdot f(\mathbf{x}, u)$, $\mathbf{p} = [V_{x_1}, V_{x_2}]^{\mathrm{T}}$, $\mathbf{x} = [x_1, x_2]^{\mathrm{T}}$.

In order to solve (32)-(33), a time dependent Hamilton-Jacobi PDE is found by making the following change of variables,

$$\begin{aligned} V(x_1, x_2) &\leftarrow t, \\ V_{x_1}(x_1, x_2) &\leftarrow -\frac{\varphi_{x_1}(x_1, x_2, t)}{\varphi_t(x_1, x_2, t)}, \\ V_{x_2}(x_1, x_2) &\leftarrow -\frac{\varphi_{x_2}(x_1, x_2, t)}{\varphi_t(x_1, x_2, t)}, \end{aligned}$$

in (32). The algebraic manipulation of the resulting equation leads to the following equation,

$$\varphi_t + H(\varphi_{x_1}, \varphi_{x_1}, x_1, x_2) = 0, \tag{34}$$

where the corresponding initial conditions are

$$\begin{aligned} \varphi(x_1, x_2, 0) &= 0 \quad \text{on } \partial\mathcal{T}, &(35)\\ \varphi(x_1, x_2, 0) &> 0 \quad \text{on } \mathbb{R}^2 \setminus \mathcal{T}, &(36)\\ \varphi(x_1, x_2, 0) &< 0 \quad \text{inside } \mathcal{T}, &(37) \end{aligned}$$

with $\varphi(x_1, x_2, 0)$ a continuous and strictly monotone function of distance to $\partial\mathcal{T}$. Next, we present the key result of [24].

Claim 1: [24] If $\varphi(x_1, x_2, t)$ is the viscosity solution to (34)-(37), then

$$V(x_1, x_2) = \{t : \varphi(x_1, x_2, t) = 0\} \tag{38}$$

is the viscosity solution to (32)-(33).

Hence, by using the level set formulation described above, we find that solving the following equation,

$$\varphi_t - \frac{\partial\varphi}{\partial x_1} x_2 + \sqrt{1 - \left(\frac{x_2^2}{R(x_1)}\right)^2} \left|\frac{\partial\varphi}{\partial x_2}\right| = 0, \tag{39}$$

with the initial condition,

$$\varphi(x_1, x_2, 0) = 0 \Leftrightarrow (x_1, x_2) = (x_1(t_f), x_2(t_f)), \tag{40}$$

$$\varphi(x_1, x_2, 0) > 0 \Leftrightarrow (x_1, x_2) \in \mathbb{R}^2 \setminus (x_1(t_f), x_2(t_f)), \tag{41}$$

is equivalent to solve (17)-(18). Once the solution to (39)-(41) is found, the solution to (17)-(18) can be computed using the relation (38). Subsequently, once the value function $V(x_1, x_2)$ is known, the optimal feedback control can be found using (19) to generate the optimal velocity profiles.

VI. Numerical Example

For a particular example, consider the path having the radius profile shown in Figure 2, taken from [1], [3]. We

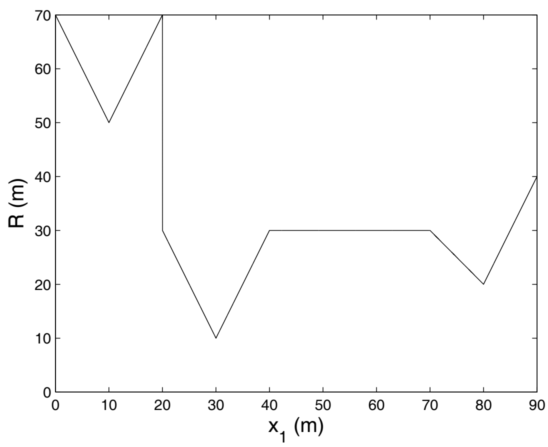

Fig. 2. Radius profile.

consider the same boundary conditions as in [1], that is,

$$\begin{aligned} (x_1(0), x_2(0)) &= (0, 4), &(42)\\ (x_1(t_f), x_2(t_f)) &= (90, 4). &(43) \end{aligned}$$

In order to solve this problem, we used an LF scheme to discretize the Hamiltonian. The derivatives in the LF scheme were computed using a second-order ENO scheme and the temporal integration was performed using a second-order RK scheme. One can choose sufficiently large t_f for computing the optimal cost over the entire feasible domain $\mathcal{S}$ (13). In this work, we pick $t_f = 22$ sec, which is sufficient to find the solution over the whole unconstrained $x_1 - x_2$ space.

The initial-value problem (39)-(41) was solved on a mesh with different grid sizes, namely, 51×51, 101×101, 201×201, 401×401, and 701×701. The zero level sets,

$$\varphi(x_1, x_2, t) = 0,$$

found for all different grids are shown in Figures 3-7, which by equation (38) are the same as the contour plots of the value function $V(x_1, x_2)$. Once the value function V was found, the control u was computed using equation (19). The velocity profile generated by integrating the dynamics of the system (10)-(12) and the computed control (as described in the previous step) for the given initial conditions (42) for all the grids are plotted in Figure 8. For reference of the reader, we have also plotted in the same figure the analytical solution for this problem taken from [1] and the constraint (13). From Figure 8, we see that as the mesh is refined the solution is approaching the analytical solution. Hence, we can solve the initial-value problem for the level set equation (39)-(41) on a sufficiently fine grid offline.

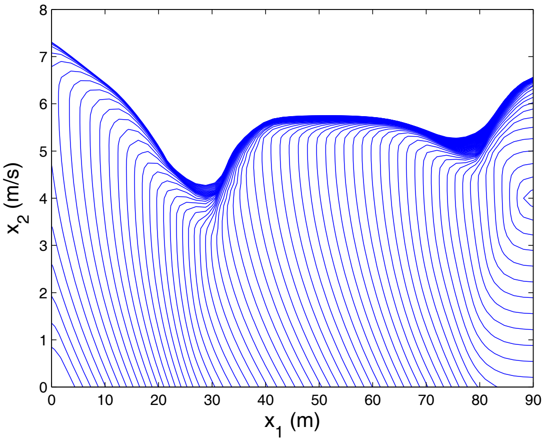

Fig. 3. Contour plots of the cost function V found on a grid of size 51×51.

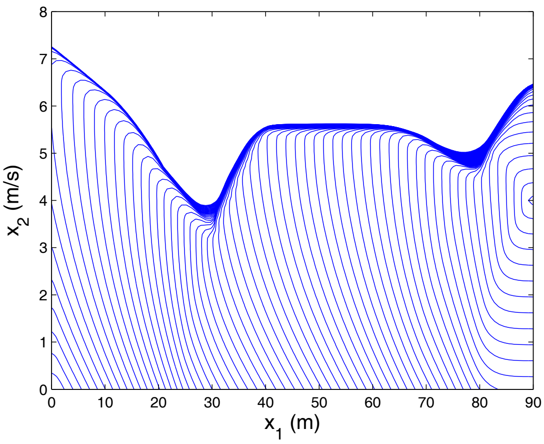

Fig. 4. Contour plots of the cost function V found on a grid of size 101×101.

VII. Conclusions

We have applied level set methods for generating the optimal velocity profile for a vehicle travelling along a prescribed path in minimum time, given a maximum acceleration limit. The application of level set methods allows for the efficient solution of the Hamilton-Jacobi-Bellman equation associated with the optimal control problem at hand. Once the optimal cost is known, the feedback control can be readily computed online to generate the velocity profile quickly. A numerical example demonstrates the proposed approach, and also serves as an independent validation of

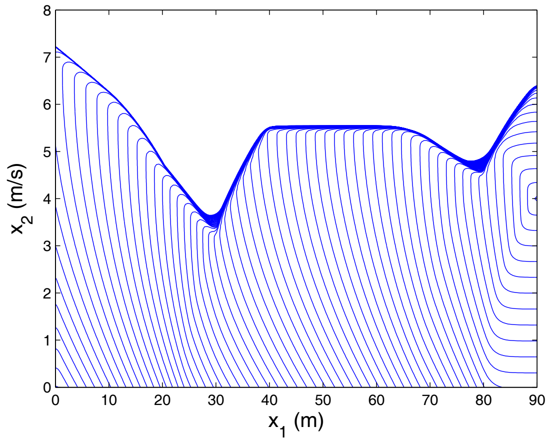

Fig. 5. Contour plots of the cost function V found on a grid of size 201×201.

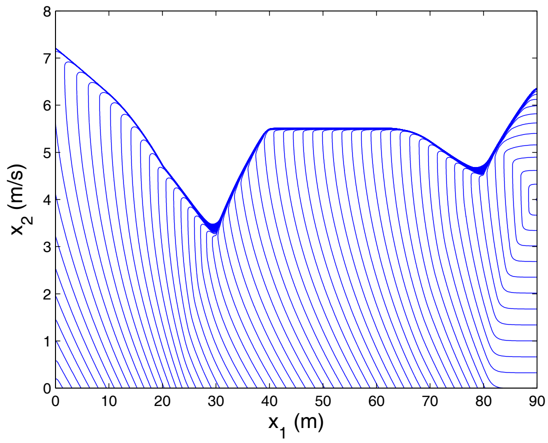

Fig. 6. Contour plots of the cost function V found on a grid of size 401×401.

the optimality of the control law developed earlier in [3] for the same problem.

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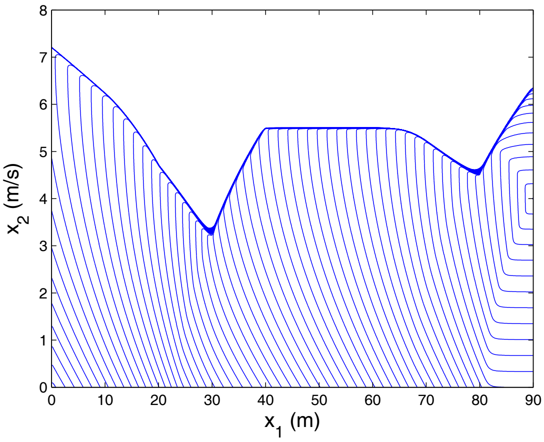

Fig. 7. Contour plots of the cost function V found on a grid of size 701×701.

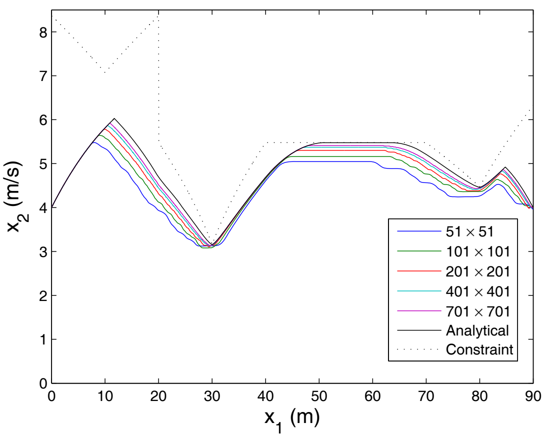

Fig. 8. Optimal velocity profile found for different mesh sizes.

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