The Method of Reflection for Solving the Time-Optimal Hamilton-Jacobi-Bellman Equation on the Interval Using Wavelets

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***Abstract*— In this paper we use the antiderivatives of wavelets to efficiently represent functions which are defined over a bounded interval and which satisfy a boundary condition in the interior of this interval. The functions we want to approximate are typically non smooth at the origin. Such functions appear as solutions to Hamilton-Jacobi-Bellman (HJB) equations for time-optimal control problems. We first give the degree of approximation of the antiderivatives and then propose a Wavelet Reflection Algorithm (WRA) to solve numerically the time-optimal HJB equation on the interval. Several numerical examples demonstrate the advantages of the technique developed in this paper over polynomial expansions.**

I. INTRODUCTION

Wavelets are basis functions (actually, frames) which allow efficient representations of (otherwise regular) functions with isolated singularities, owing to their nice localization properties both in space and frequency domains. Moreover, the wavelet representation of such functions is sparse, in the sense that most of the wavelet coefficients are very small or zero. This property of wavelets allows for compact functional approximations by ignoring the coefficients that are smaller than a prescribed threshold [1].

Wavelets have been used in the recent past for solving hyperbolic, elliptic and parabolic partial differential equations [2], [3]. As a matter of fact, the advantages of wavelets for solving pdes have been noticed early on [2], [4], [5]. Some of the most recent results in this context have appeared in [6]. Glowinski et al, for example, formulated a Galerkin-wavelet method for various boundary value problems [2]. They applied their method to the heat equation and to Burger's equation. However, their methodology may encounter some difficulties. First, Daubechies' wavelets of low order cannot be used due to their lack of sufficient regularity. Second, Dirichlet boundary conditions cannot be applied directly without further modifications. Related results have appeared in [7]. In order to overcome the previous difficulties, Xu and Shann [4] constructed a set of basis functions using the antiderivatives of wavelets. They applied this set of bases to two-point boundary value problems and obtained numerical results of high consistency. The work reported in this paper is a continuation of our recent work on the use of wavelets for the solution of optimal control problems [8], [9], [10].

Standard numerical methods for solving pde's may not be used directly for solving the Hamilton-Jacobi-Bellman pde which arises in the solution of optimal feedback control problems. For instance, the solution (value function) to the time-optimal HJB equation may have discontinuous derivatives. In addition, the value function needs to satisfy a boundary condition in the *interior* of the domain. To address these two problems, in [10] we constructed a frame for the Sobolev Space $W^{1,2}(\Omega)$, (where Ω denotes an open interval) using the antiderivatives of wavelets. Working with the antiderivatives we automatically ensure that the interior boundary condition is satisfied, while at the same time we keep some of the nice properties of wavelets (e.g., a multiresolution decomposition of the solution space). In this paper, we provide the degree of approximation of the frame proposed in [10] and propose yet another method for solving the time-optimal HJB equation.

The paper is organized as follows. First, we offer a brief summary of the wavelets and their properties. We then introduce the antiderivatives of wavelets and investigate their multi-resolution and approximation properties in the Sobolev space $W^{1,2}(\Omega)$. In the second part of the paper we apply this frame in order to find solutions to the time-optimal HJB equation. The method of reflection is introduced to efficiently solve problems for which it is known that the solution is even. Finally, we offer several numerical examples to demonstrate the proposed method. It is shown that the proposed approach offers major advantages in terms of the approximation error and compactness of the representation when compared to polynomial approximations.

II. PRELIMINARIES

A. Nomenclature

The following notation is used in this paper.

1) If $G \subset \mathbb{R}$, then by $\mathrm{cl}(G)$ we denote the closure of G in $\mathbb{R}$.
2) Let $\Omega \subset \mathbb{R}$ be an open set. For any nonnegative integer s, $C^s(\Omega)$ is the vector space consisting of all functions f which, together with all their derivatives of order less than or equal to s, are continuous on Ω.
3) By $\mathbb{R}_+$ we denote the set $\mathbb{R}_+ := \{x \in \mathbb{R} : x > 0\}$.
4) By $x \lesssim y$ we mean $x \leq Cy$, where, C is some positive constant.

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5) The inner product in $L^2(\Omega)$ will be denoted by $\langle f, g\rangle_{L^2(\Omega)} := \int_\Omega f(x)g(x)\,\mathrm{d}x$.

Definition 1 (Sobolev Spaces): Let $\Omega \subset \mathbb{R}$ be an open set. Then the Sobolev space $W^{s,2}(\Omega)$ is defined by

$$W^{s,2}(\Omega) := \left\{ f \in L^2(\Omega) : \ \frac{\mathrm{d}^\alpha f}{\mathrm{d}x^\alpha} \in L^2(\Omega), \quad 0 \le \alpha \le s \right\}$$

with norm $\|f\|_{W^{s,2}(\Omega)} := \sum_{\alpha=0}^{s} \left\| \frac{\mathrm{d}^\alpha f}{\mathrm{d}x^\alpha} \right\|_{L^2(\Omega)}$.

The semi-norm of $f \in W^{s,2}(\Omega)$ is given by $|f|_{W^{s,2}(\Omega)} := \left\| \frac{\mathrm{d}^s f}{\mathrm{d}x^s} \right\|_{L^2(\Omega)}$. For simplicity, we denote the norm $\|\cdot\|_{W^{s,2}(\Omega)}$ by $\|\cdot\|_{s,2,\Omega}$ and the semi-norm $|\cdot|_{W^{s,2}(\Omega)}$ by $|\cdot|_{s,2,\Omega}$. Note that $\|\cdot\|_{0,2,\Omega} = |\cdot|_{0,2,\Omega}$.

In the following we assume that Ω is an open interval of $\mathbb{R}$ with the origin in its interior. The space of interest in this paper is the Sobolev space $W^{1,2}(\Omega)$ that is, the space of functions which are square integrable, having square integrable first derivative over Ω. In addition, we will deal with functions in $W^{1,2}(\Omega)$ which are zero at the origin, that is,

$$W_0^{1,2}(\Omega) := \{f \in W^{1,2}(\Omega) : f(0) = 0\}.$$

It can be readily shown that $W_0^{1,2}(\Omega)$ is a subspace of $W^{1,2}(\Omega)$.

B. Wavelet Fundamentals

In this section, we give a brief overview of the wavelets and their properties. For the details we refer the reader, for example, to [11], [12].

Wavelets are basis functions which induce a multi-resolution decomposition of $L^2(\mathbb{R})$. This is the main property making wavelets attractive in applications. Specifically, wavelets induce the following nested sequence of subspaces

$$\mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \cdots \subset \mathcal{V}_j \subset \mathcal{V}_{j+1} \subset \cdots \subset L^2(\mathbb{R})$$

such that $\bigcup_{j=0}^{\infty} \mathcal{V}_j$ is dense in $L^2(\mathbb{R})$, that is, $\mathrm{cl}\big(\bigcup_{j=0}^{\infty} \mathcal{V}_j\big) = L^2(\mathbb{R})$. The "base" (or coarse-resolution) subspace $\mathcal{V}_0$ is spanned by integer translates of the *scaling function* ϕ: $\mathcal{V}_0 = \mathrm{cl}(\mathrm{span}\{\phi(x-k)\}_{k\in\mathbb{Z}})$. The higher-resolution subspaces $\mathcal{V}_j$ are spanned by dilated versions of the scaling function: $\mathcal{V}_j = \mathrm{cl}(\mathrm{span}\{\phi(2^j x - k)\}_{k\in\mathbb{Z}}), \quad j \ge 0$. The orthogonal complement of $\mathcal{V}_j$ in the larger subspace $\mathcal{V}_{j+1}$ is denoted by $\mathcal{W}_j$ and it is spanned by the *wavelets*: $\mathcal{W}_j = \mathrm{cl}(\mathrm{span}\{\psi(2^j x - k)\}_{k\in\mathbb{Z}}), \quad j \ge 0$, where ψ is the *mother wavelet*, which spans the space $\mathcal{W}_0 = \mathcal{V}_1 \ominus \mathcal{V}_0$.

Let us define the two-parameter family of functions

$$\psi_{j,k}(x) := \begin{cases} \phi(x-k), & \text{for } j = -1, \\ \sqrt{2^j}\,\psi(2^j x - k), & \text{for } j \ge 0, \end{cases}$$

where $k \in \mathbb{Z}$.

The following fact is crucial for the approximating properties of wavelet decompositions.

Theorem 1 (Vanishing Moments [12]): The following are equivalent:

1) The wavelet has m vanishing moments, i.e.,

$$\int_{\mathbb{R}} x^\ell \psi(x)\,\mathrm{d}x = 0, \quad \ell = 0, 1, \cdots, m-1 \qquad (1)$$

2) All polynomials of degree up to $m-1$ can be expressed as a linear combination of shifted scaling functions at any scale. □

A wavelet is of *order* m if it has m vanishing moments.

C. Wavelets as frames for $L^2(\Omega)$

Definition 2 ([4]): Let $\{\varphi_n\}_{n=1}^{\infty}$ be a subset of a Banach Space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$. Let span$\{\varphi_n\}$ be the set of all elements of the form $\sum \alpha_n \varphi_n,\ (\alpha_n \in \mathbb{R})$ which converge (strongly) in $\mathcal{B}$. Then $\{\varphi_n\}_{n=1}^{\infty}$ is said to be a *frame* of $\mathcal{B}$ if span$\{\varphi_n\} = \mathcal{B}$. □

Roughly speaking, frames are redundant (linearly dependent) "bases."

In the following we use the notation $\Omega = (-R, +R)$ and $\Omega_+ = (0, R)$. The proofs of the following results can be found in [10].

Theorem 2 ([10]): Let $\mathrm{supp}\,\psi_{-1,0} = \mathrm{supp}\,\psi_{0,0} = (0, R)$. Then the set $\{\psi_{j,k}|_\Omega : j \ge -1, k \in \mathcal{I}_j\}$ where

$$\mathcal{I}_j := \left\{ k \in \mathbb{Z} \;\middle|\; \begin{array}{ll} 1 - 2R \le k \le R - 1, & j = -1 \\ 1 - (2^j + 1)R \le k \le 2^j R - 1, & j \ge 0 \end{array} \right\}$$

forms a frame for $L^2(\Omega)$. □

Corollary 1 ([10]): Let the space

$$\mathcal{V}_J(\Omega) := \mathrm{span}\{\psi_{j,k}|_\Omega : -1 \le j < J,\ k \in \mathcal{I}_j\}. \qquad (2)$$

Then $\mathcal{V}_J(\Omega) \subset \mathcal{V}_{J+1}(\Omega)$ and $\bigcup_{J=0}^{\infty} \mathcal{V}_J(\Omega)$ is dense in $L^2(\Omega)$. □

Theorem 3 ([10]): The semi-norm $|\cdot|_{1,2,\Omega}$ is equivalent to the norm $\|\cdot\|_{1,2,\Omega}$ in $W_0^{1,2}(\Omega)$. □

Lemma 1 ([10]): Let $\{\psi_{j,k}|_\Omega : j \ge -1,\ k \in \mathcal{I}_j\}$ be a frame for $L^2(\Omega)$. Then, for $j \ge -1$ and $k \in \mathcal{I}_j$, the set

$$\left\{ \Psi_{j,k}(x) := \int_0^x \psi_{j,k}(s)\,\mathrm{d}s,\ \forall x \in \Omega \right\} \subset W_0^{1,2}(\Omega)$$

forms a frame for $W_0^{1,2}(\Omega)$. □

In the next section, we give the degree of approximation when using the antiderivatives of wavelets to represent functions in $W_0^{1,2}(\Omega)$.

III. The Degree of Approximation

Definition 3 ([13]): Let $\Omega \subset \mathbb{R}$ be an open set. An *extension operator* T for $W^{s,2}(\Omega)$ is a bounded linear operator

$$T : W^{s,2}(\Omega) \to W^{s,2}(\mathbb{R})$$

such that $Tz|_\Omega = z$ for every $z \in W^{s,2}(\Omega)$. □

Lemma 2: Let ψ be a wavelet of order at least s. For $j \geq 0$ and $1 \leq s < \infty$, let $f \in W_0^{s,2}(\mathbb{R})$ and $\alpha_{j,k}$ be given by $\alpha_{j,k} = \langle f, \psi_{j,k} \rangle_{L^2(\mathbb{R})}$, where $k \in \mathbb{Z}$. Then

$$|\alpha_{j,k}| \lesssim 2^{-js} |f|_{s,2,S_{j,k}}$$

where, $S_{j,k} = \operatorname{supp} \psi_{j,k}$. □

Proof: Let $f \in W_0^{s,2}(\mathbb{R})$ and $\alpha_{j,k} = \langle f, \psi_{j,k} \rangle_{L^2(\mathbb{R})}$. For any polynomial $q(x)$ of degree less than equal to $s-1$ and a wavelet of order greater than or equal to s, Theorem 1 implies that

$$\int_{\mathbb{R}} q(x) \psi_{j,k}(x) \, \mathrm{d}x = 0$$

Therefore, we can write

$$\alpha_{j,k} = \int_{\mathbb{R}} f \psi_{j,k} \mathrm{d}x - \int_{\mathbb{R}} q \psi_{j,k} \mathrm{d}x = \int_{S_{j,k}} (f-q) \psi_{j,k} \mathrm{d}x$$

or that $\alpha_{j,k} = \langle f - q, \psi_{j,k} \rangle_{L^2(S_{j,k})}$.

$$\begin{aligned}
|\alpha_{j,k}| &= |\langle f - q, \psi_{j,k} \rangle_{L^2(S_{j,k})}| \\
&\leq \|f - q\|_{0,2,S_{j,k}} \|\psi_{j,k}\|_{0,2,S_{j,k}} \\
&= \|f - q\|_{0,2,S_{j,k}} \|\psi_{j,k}\|_{0,2,\mathbb{R}} \\
&= \|f - q\|_{0,2,S_{j,k}}
\end{aligned}$$

where the last equality follows from the fact that $\|\psi_{j,k}\|_{0,2,\mathbb{R}} = 1$. Therefore,

$$|\alpha_{j,k}| \leq \inf_q \|f - q\|_{0,2,S_{j,k}}, \quad j \geq 0, \ k \in \mathbb{Z}. \tag{3}$$

Using the Bramble-Hilbert Lemma [14] we have

$$|f - q|_{0,2,S_{j,k}} \lesssim |S_{j,k}|^s |f|_{s,2,S_{j,k}} \lesssim 2^{-js} |f|_{s,2,S_{j,k}}$$

since $|S_{j,k}| = R/2^j$. Furthermore,

$$\inf_q \|f - q\|_{0,2,S_{j,k}} \leq |f - q|_{0,2,S_{j,k}} \quad \lesssim 2^{-js} |f|_{s,2,S_{j,k}} \tag{4}$$

Combining (3) and (4) one obtains

$$|\alpha_{j,k}| \lesssim 2^{-js} |f|_{s,2,S_{j,k}}$$

This concludes the proof of the lemma. ■

Lemma 3: Let $\Omega = (-R, R)$ and let $\operatorname{supp} \psi_{j,k} = [a, b]$ such that $-R \leq a < R \leq b$ or $a \leq -R < b \leq R$. Then

$$\|\psi_{j,k}(x)\|_{0,2,\Omega} \leq 1. \quad \square$$

Now since $\bigcup_{J=0}^{\infty} \mathcal{V}_J(\Omega)$ is dense in $L^2(\Omega)$ and each $\mathcal{V}_J(\Omega)$ is finite dimensional, we construct finite-dimensional subspaces of $W_0^{1,2}(\Omega)$ from the frame of $\mathcal{V}_J(\Omega)$.

Theorem 4: For $J \geq 0$, let

$$X_J(\Omega) := \operatorname{span}\{\Psi_{j,k} : -1 \leq j < J, \ k \in \mathcal{I}_j\}.$$

Then $X_J(\Omega)$ is a finite-dimensional subspace of $W_0^{1,2}(\Omega)$ and for any $v \in W_0^{1,2}(\Omega) \cap W^{s+1,2}(\Omega)$

$$\inf_{F \in X_J(\Omega)} |v - F|_{1,2,\Omega} \lesssim 2^{-Js} |v|_{s+1,2,\Omega}$$

where, $0 \leq s < \infty$. □

Proof: The proof is rather long but it is similar to the one of Theorem 3.1 in [4], and thus it is omitted. ■

IV. Application to Time-Optimal Control Problems

In this section we provide numerical solutions to the one-dimensional time-optimal HJB equation using the antiderivatives of wavelets as the underlined expansion functions of the solution space.

A. Problem Formulation

Consider an optimal control problem whose dynamics are given by the nonlinear differential equation

$$\dot{x}(t) = f[x(t)] + g[x(t)]\, u \tag{5}$$

with boundary conditions $x(t_0) = x_0$ and $x(t_f) = 0$, where $x(t) \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, and t_f is free. The control is constrained by $|u| \leq 1$. Notice that by writing $g(x)u = |g(x)| \operatorname{sgn}[g(x)]\, u := |g(x)| v$ where $|v| \leq 1$ we can further assume, without loss of generality, that $g(x) \geq 0$ for all $x \in \mathbb{R}$. The cost function to be minimized is

$$\min_{|u| \leq 1} \int_{t_0}^{t_f} \mathrm{d}t. \tag{6}$$

The minimizing control for this problem is obtained from the solution of the following Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{aligned}
1 + \min_{u \in [-1,1]} &\left(\frac{\partial V}{\partial x} \big(f(x) + g(x) u \big) \right) \\
&= 1 + f(x) \frac{\partial V}{\partial x}(x) - g(x) \left| \frac{\partial V}{\partial x}(x) \right| = 0,
\end{aligned} \tag{7}$$

subject to the boundary condition $V(0) = 0$. Once V is known, the optimal control is given in a feedback form as follows (since $g(x) \geq 0$ for all $x \in \mathbb{R}$)

$$u = -\operatorname{sgn} \left(\frac{\partial V}{\partial x} \right). \tag{8}$$

In this paper we deal specifically with symmetric (i.e., even) solutions of (7). The following assumptions will ensure that V is even.

A1: f is an odd function, i.e. $f(x) = -f(-x)$, $\forall\, x \in \mathbb{R}$.
A2: g is an even function, i.e., $g(x) = g(-x)$, $\forall\, x \in \mathbb{R}$.

B. The Wavelet Reflection Algorithm (WRA)

In [15] it is shown that the half space $W^{1,p}(\mathbb{R}_+)$ has the following *extension property*.

Theorem 5 ([15]): Let $v \in W^{1,p}(\mathbb{R}_+)$ $(1 \leq p < \infty)$ and define $\bar{v}$ on $\mathbb{R}$ by

$$\bar{v}(x) = \begin{cases} v(x), & x > 0, \\ v(-x), & x \leq 0. \end{cases} \tag{9}$$

The map $v \mapsto \bar{v}$ defines an extension operator from $W^{1,p}(\mathbb{R}_+)$ onto $W^{1,p}(\mathbb{R})$. □

Corollary 2: Let $\Omega_+ = (0, R)$ and $\Omega = (-R, +R)$. Let $v \in W^{1,2}(\Omega_+)$ and define $\bar{v}$ on Ω as in (9). The map $v \mapsto \bar{v}$ defines an extension operator from $W^{1,2}(\Omega_+)$ onto $W^{1,2}(\Omega)$. □

Let now $V_{\text{sym}} \in W_0^{1,2}(\Omega_+)$ be the value function satisfying the HJB equation (7). From (8) we have that $\frac{\partial V_{\text{sym}}}{\partial x}(x) > 0, \quad \forall\, x \in \Omega_+$. Therefore, the HJB equation (7) for $x \in \Omega_+$ can be written as

$$1 + f(x)\frac{\partial V_{\text{sym}}}{\partial x}(x) - g(x)\frac{\partial V_{\text{sym}}}{\partial x}(x) = 0, \quad x \in \Omega_+,$$

with boundary condition $V_{\text{sym}}(0) = 0$ or, equivalently, as

$$1 + f(|x|)\left|\frac{\partial V_{\text{sym}}}{\partial x}(|x|)\right| - g(|x|)\left|\frac{\partial V_{\text{sym}}}{\partial x}(|x|)\right| = 0, \quad (10)$$

with boundary condition $V_{\text{sym}}(0) = 0$.

We now extend the value function V_{sym} from $W_0^{1,2}(\Omega_+)$ to $W_0^{1,2}(\Omega)$ such that

$$\bar{V}_{\text{sym}}(x) = \begin{cases} V_{\text{sym}}(x), & x \in \Omega_+, \\ V_{\text{sym}}(-x), & x \in \Omega \setminus \Omega_+. \end{cases}$$

Now $\bar{V}_{\text{sym}} \in W_0^{1,2}(\Omega)$ and

$$\frac{\partial \bar{V}_{\text{sym}}}{\partial x}(x) = \begin{cases} \dfrac{\partial V_{\text{sym}}}{\partial x}(x), & x \in \Omega_+, \\ -\dfrac{\partial V_{\text{sym}}}{\partial x}(-x), & x \in \Omega \setminus \Omega_+, \end{cases}$$

or, equivalently, $\left|\frac{\partial \bar{V}_{\text{sym}}}{\partial x}(x)\right| = \left|\frac{\partial V_{\text{sym}}}{\partial x}(|x|)\right|, \quad \forall\, x \in \Omega$.

Therefore, for all $x \in \Omega$ equation (10) can be written as

$$1 + f(|x|)\left|\frac{\partial \bar{V}_{\text{sym}}}{\partial x}(x)\right| - g(|x|)\left|\frac{\partial \bar{V}_{\text{sym}}}{\partial x}(x)\right| = 0, \quad (11)$$

with boundary condition $\bar{V}_{\text{sym}}(0) = 0$.

C. Solution to the HJB equation

We have two cases to consider.

Case 1: Assume that $\frac{\partial \bar{V}_{\text{sym}}}{\partial x} > 0$. In this case equation (11) reduces to

$$1 + f(|x|)\frac{\partial \bar{V}_{\text{sym}}}{\partial x}(x) - g(|x|)\frac{\partial \bar{V}_{\text{sym}}}{\partial x}(x) = 0, \quad x \in \Omega, \quad (12)$$

with boundary condition $\bar{V}_{\text{sym}}(0) = 0$.

Case 2: Assume that $\frac{\partial \bar{V}_{\text{sym}}}{\partial x} < 0$. In this case equation (11) reduces to

$$1 - f(|x|)\frac{\partial \bar{V}_{\text{sym}}}{\partial x}(x) + g(|x|)\frac{\partial \bar{V}_{\text{sym}}}{\partial x}(x) = 0, \quad x \in \Omega, \quad (13)$$

with boundary condition $\bar{V}_{\text{sym}}(0) = 0$.

Let us now denote by $\bar{V}_{\text{sym}_1}$ the solution to equation (12) and by $\bar{V}_{\text{sym}_2}$ the solution to equation (13).

For convenience, in the following we denote $r_1(x) := f(|x|) - g(|x|)$ and $r_2(x) := -f(|x|) + g(|x|)$. Therefore, equations (12) and (13) can be re-written as

$$\text{HJB}_{\text{mod}}(\bar{V}_{\text{sym}_i}, r_i) := 1 + \frac{\partial \bar{V}_{\text{sym}_i}}{\partial x}(x)\, r_i(x) = 0, \quad (14)$$

with boundary condition $\bar{V}_{\text{sym}_i}(0) = 0$, for $i = 1, 2$.

We seek an approximate solution V_{WRA_i} to equation (14) using the method of weighted residuals [16]. To this end, we assume a solution $V_{\text{WRA}_i} \in X_J(\Omega)$, $(i = 1, 2)$ of the form

$$V_{\text{WRA}_i}(x) = \sum_{j=-1}^{J-1} \sum_{k \in \mathcal{I}_j} c_{j,k}^i \Psi_{j,k}(x), \quad i = 1, 2 \quad (15)$$

and $c_{j,k}^i$ are the associated coefficients. Substituting expression (15) into equation $\text{HJB}_{\text{mod}}(\bar{V}_{\text{sym}_i}, r_i) = 0$ results in the error

$$\text{Err}_i := \text{HJB}_{\text{mod}}\left(\sum_{j=-1}^{J-1} \sum_{k \in I_j} c_{j,k}^i \Psi_{j,k}, r_i\right), \quad i = 1, 2. \quad (16)$$

The coefficients $c_{j,k}^i$ are determined by setting the projection of the error (16) on each element that spans the subspace $\mathcal{V}_J(\Omega)$ (namely, $\{\psi_{j,k}\}$ where $-1 \le j < J$ and $k \in \mathcal{I}_j$) to zero. Thus,

$$\langle \text{Err}_i, \psi_{j,k} \rangle_{L^2(\Omega)} := 0, \quad i = 1, 2 \quad (17)$$

This approach will yield two solutions for $\bar{V}_{\text{sym}}$, namely V_{WRA_1} and V_{WRA_2}, such that $\frac{\partial V_{\text{WRA}_1}}{\partial x}(x) > 0$ and $\frac{\partial V_{\text{WRA}_2}}{\partial x}(x) < 0$, $\forall\, x \in \Omega$ respectively. The solution to the HJB equation (7) for all $x \in \Omega$ is then given as follows

$$V_{\text{WRA}}(x) = \begin{cases} V_{\text{WRA}_1}(x), & x \in \Omega_+, \\ V_{\text{WRA}_2}(x), & x \in \Omega \setminus \Omega_+. \end{cases} \quad (18)$$

D. Convergence

We show that the solution (V_{WRA}) obtained using the WRA algorithm converges to $V^\star$ (the unique viscosity solution [17], [18]) as $J \to \infty$.

Theorem 6: Let $V^\star \in C^0(\Omega)$ be the unique viscosity solution defined as

$$V^\star(x) := \begin{cases} V_1^\star(x), & x \in \Omega_+, \\ V_2^\star(x) = V_1^\star(-x), & x \in \Omega \setminus \Omega_+. \end{cases}$$

where, $V_i^\star \in W_0^{1,2}(\Omega) \bigcap W^{s+1,2}(\Omega)$ $(i = 1, 2)$ for some $s \ge 1$, and V_{WRA} be the approximate solution (using WRA) of the time-optimal HJB equation (7). Then $\|V^\star - V_{\text{WRA}}\|_{1,2,\Omega} \to 0$ as $J \to \infty$. □

Proof: The proof is a direct consequence of Theorem 4 in [10] therefore we only give a sketch of the proof here. Let V_{J_i} be the projection of $V_i^\star$ on X_J for $i = 1, 2$. Therefore

TABLE I

NUMBER OF SC USING DAUBECHIES WAVELETS FOR EXAMPLE 1

p	J	M_w	M_{SC}	δ	E_0	E_1
2	3	56	14	0.026	0.0058	0.0458
4	3	136	34	0.013	0.0064	0.0314

TABLE II

NUMBER OF SC USING SYMLETS FOR EXAMPLE 1

p	J	M_w	M_{SC}	δ	E_0	E_1
2	3	56	14	0.026	0.0058	0.0457
4	3	136	35	0.014	0.0089	0.0364

using Theorem 4 we have that $\lim_{J\to\infty} \|V_i^\star - V_{J_i}\|_{1,2,\Omega} = 0$. Then we prove that $\lim_{J\to\infty} \|V_{J_i} - V_{\mathrm{WRA}_i}\|_{1,2,\Omega} = 0$. Now $\lim_{J\to\infty} \|V_i^\star - V_{\mathrm{WRA}_i}\|_{1,2,\Omega} = \lim_{J\to\infty} \|V_i^\star - V_{J_i} + V_{J_i} - V_{\mathrm{WRA}_i}\|_{1,2,\Omega} \leq \lim_{J\to\infty} \|V_i^\star - V_{J_i}\|_{1,2,\Omega} + \lim_{J\to\infty} \|V_{J_i} - V_{\mathrm{WRA}_i}\|_{1,2,\Omega} = 0$ which completes the proof. ■

V. NUMERICAL EXAMPLES

In this section we provide several examples to demonstrate the benefits of the proposed method. In all examples, the solution to the HJB equation is not differentiable at the origin. The problems are also simple enough so that the solution can be obtained analytically. This allows us to make meaningful comparisons on the accuracy of the solution obtained in each case.

Example 1: Consider system (5) with $f(x) = -ax$, $g(x) = 1$, and $a > 0$. The optimal value function is analytically obtained as $V(x) = \frac{1}{a}\ln(a|x| + 1)$.

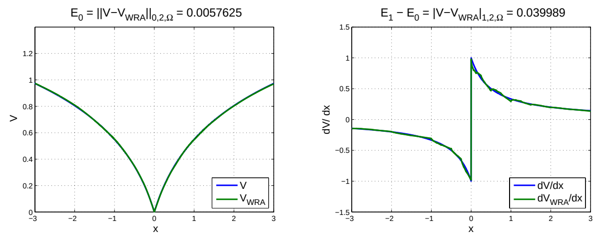
Using the WRA algorithm (for $a = 2$) with the Daubechies wavelets of order $p = 2, 4$ (for scales $J = 3, 3$ respectively) provides the results shown in Fig. 1 and Fig. 2 respectively.

We define the significant coefficients (SC) as the coefficients that are required to calculate the cost function V_{WRA} using the wavelets of order p at the minimum scale J so that the error $E_0 = \|V - V_{\mathrm{WRA}}\|_{0,2,\Omega} \approx 10^{-2}$, where V is the exact solution to the HJB equation. The number of significant coefficients used in the calculation of V_{WRA} with Daubechies wavelets and symlets, and for different values of p, are given in Tables I and II respectively.

In these tables M_w, M_{SC} denote the total number of wavelet coefficients for scale J and the number of SC respectively and δ is the threshold for selecting the SC. For reference, the last column also shows the error $E_1 = \|V - V_{\mathrm{WRA}}\|_{1,2,\Omega}$.

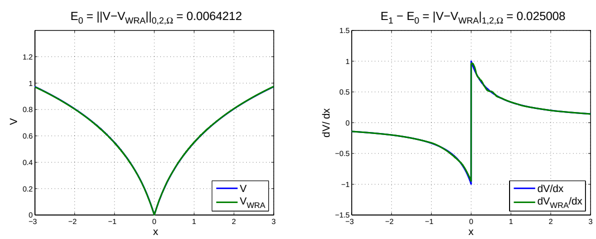
Example 2: Consider system (5) with $f(x) = -\frac{5}{3}x$ and $g(x) = \frac{1+25x^2}{6}$. The optimal value function is analytically obtained as $V(x) = \frac{6|x|}{1+5|x|}$.

Using the WRA algorithm with Daubechies wavelets of order $p = 2, 4$ (for scales $J = 5, 5$ respectively) provides the results shown in Fig. 3 and Fig. 4 respectively. The number of significant coefficients used for calculating V_{WRA} for $p = 2, 4$ using Daubechies wavelets and symlets are given in Tables III and IV, respectively.



(a) Value Function (b) Derivative of Value Function

Fig. 1. Example 1 using Daubechies wavelets ($p = 2, J = 3, M_{SC} = 14$).



(a) Value Function (b) Derivative of Value Function

Fig. 2. Example 1 using Daubechies Wavelets ($p = 4, J = 3, M_{SC} = 34$).

VI. COMPARISON WITH POLYNOMIAL EXPANSIONS

We have also used polynomials to solve the previous examples. The results for both the examples using polynomials are shown in Fig. 5 and Fig. 6. The comparison between the wavelets and polynomials is summarized in Table V which shows the superiority of the use of wavelets when compared to polynomial expansions. (In Table V, M_p are the number of polynomials that were used to solve the problem.)

VII. CONCLUSIONS

In this paper we have proposed a numerical scheme for solving the time-optimal Hamilton-Jacobi-Bellman Equation on the interval. It is well known that the solution is often non-differentiable at the origin. Our approach uses antiderivatives of wavelets as trial functions in order to efficiently capture this behavior. Numerical examples clearly

TABLE III

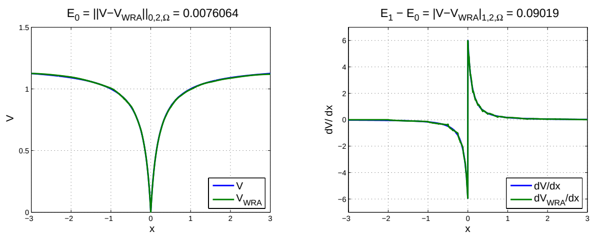
NUMBER OF SC USING DAUBECHIES WAVELETS FOR EXAMPLE 2

p	J	M_w	M_{SC}	δ	E_0	E_1
2	5	204	20	0.035	0.0076	0.0978
4	5	484	39	0.009	0.0058	0.0587

TABLE IV

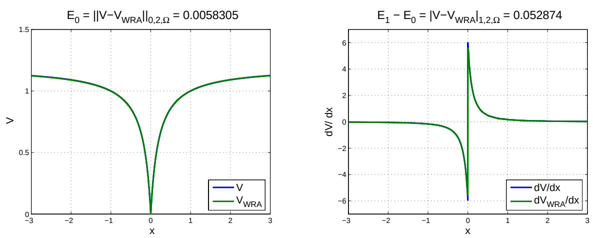
NUMBER OF SC USING SYMLETS FOR EXAMPLE 2

p	J	M_w	M_{SC}	δ	E_0	E_1
2	5	204	20	0.035	0.0076	0.0978
4	5	484	43	0.009	0.0102	0.0444



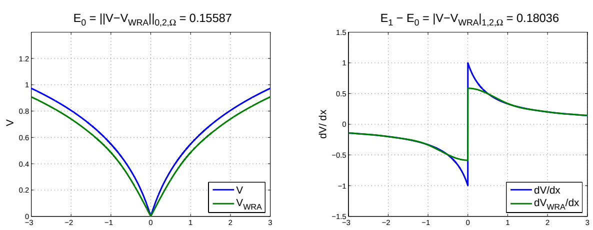
(a) Value Function (b) Its Derivative

Fig. 3. Ex. 2 using Daubechies wavelets ($p = 2, J = 5, M_{SC} = 20$).



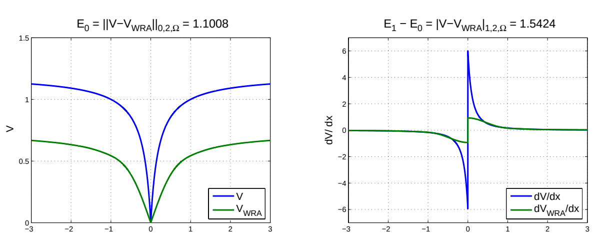
(a) Value Function (b) Its Derivative

Fig. 4. Ex. 2 using Daubechies wavelets ($p = 4, J = 5, M_{SC} = 39$).



(a) Value Function (b) Its Derivative

Fig. 5. Example 1 using polynomials ($M_p = 14$).



(a) Value Function (b) Its Derivative

Fig. 6. Example 2 using polynomials ($M_p = 20$).

TABLE V

WAVELETS VS. POLYNOMIALS

$Ex.$	M_{SC}	E_0	E_1	M_p	E_0	E_1
1	14	0.0058	0.0458	14	0.1559	0.3362
2	20	0.0076	0.0978	20	1.1008	2.5504

demonstrate the advantage of using wavelets in the proposed numerical algorithm as compared to the polynomials.

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