ON ROBUST STABILITY OF LTI PARAMETER DEPENDENT SYSTEMS

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Keywords: Robust stability, guardian map, parameter-dependent LTI systems.

Abstract

In this paper, the complete stability domain for LTI parameter-dependent systems is synthesized by extending existing results in the literature. This domain is calculated through a guardian map which involves the determinant of the Kronecker sum of a matrix with itself. The stability domain is synthesized for both single and multi-parameter dependent LTI systems. The single parameter case is easily computable, whereas the multi-parameter case is more involved. The determinant of the bialternate sum of a matrix with itself is also exploited to reduce the computational complexity of the results.

1 Introduction

In this paper, the stability of linear time invariant parameter-dependent (LTIPD) systems is studied. The need to determine the bounds of the system uncertainty to guarantee stability for the perturbed system has been the subject of intensive research in the past several years. Parameter-dependent Lyapunov functions have been suggested in the literature to find such bounds [1, 2, 3]. However, the use of Lyapunov function methods gives rise to stability conditions that are sufficient but not necessary. References [4] and [5] studied quadratic $\delta-$Hurwitz and $\mathcal{D}-$stability and gave robust stability conditions for static uncertainty. For quadratic stability, Ref. [6] gave necessary and sufficient conditions which are valid even for time-varying linear systems. However, quadratic stability is, in general, more conservative than robust stability [7].Saydy *et al.* [8, 9] defined a particular guardian map and used it to study the stability of LTIPD systems of the form

$$\begin{aligned} &\dot{x} = A(\rho)x\,, \\ &A(\rho) = A_0 + \rho A_1 + \rho^2 A_2 + \ldots + \rho^m A_m \end{aligned} \tag{1}$$

and

$$\begin{aligned} &\dot{x} = A(\rho_1,\rho_2)x\,, \\ &A(\rho_1,\rho_2) = \sum_{i_1,i_2=0}^{i_1+i_2=m} \rho_1^{i_1}\rho_2^{i_2} A_{i_1,i_2}\,. \end{aligned} \tag{2}$$

The guardian map in [8] is the determinant of the Kronecker sum of a matrix with itself. Using this guardian map, Saydy *et al.* gave necessary and sufficient stability conditions wrt a given parameter domain, for the particular LTIPD systems in (1) and (2). This method was latter extended in [10] and [7] to LTI systems with many parameters in the form:

$$\begin{aligned} &\dot{x} = A(\rho_1,\rho_2,\ldots,\rho_m)x\,, \\ &A(\rho_1,\rho_2,\ldots,\rho_m) = A_0 + \sum_{i=1}^{m} \rho_i A_i. \end{aligned} \tag{3}$$

However, the stability conditions in [10] and [7] are only sufficient. Fu and Barmish [11] gave the maximal stability interval around the origin for LTIPD systems of the form (3) with $m = 1$ and A_0 Hurwitz.

Saydy *et al.* [9] and Barmish [10] have also derived stability conditions for a family of $n \times n$ parameter-dependent matrices given by $A(\rho) = \sum_{i=0}^{l} \rho^i A_i$. Their result tests whether the matrix $A(\rho)$ is robustly stable for all ρ in a given compact interval. Reference [7] provides an interval which guarantees robust stability for single and multi-parameter dependent LTI systems. However, this interval is derived from sufficient conditions and hence it is not the maximal robust stability interval. Fu and Barmish [11] presented a method to synthesize the maximal stability interval containing the origin for single parameter-dependent LTI systems.

Most of existing results (e.g., [8, 9, 10]) give necessary and sufficient stability conditions for an a priori given single or multi-parameter interval set. Furthermore, [7] provides a bounded interval set which is only sufficient in guaranteeing the stability of LTIPD systems. A question which arises naturally form this research is how to find the entire stability domain for single or multi parameter dependent systems. The complete stability domain may be composed of one or several pieces of connected sets.

In this paper, we extend existing results to give the entire stability domain for single-parameter dependent LTI systems. We then generalize this result to multi-parameter dependent LTI systems. In order to reduce the computational complexity of the derived stability conditions, the guardian map which involves the determinant of the Kronecker sum of a matrix with itself is replaced by the determinant of the bialternate sum of a matrix with itself. Mustafa [12] studied the robust stability problem of LTIPD systems using the bialternate sum of matrices. Although the determinant of the bialternate sum of a matrix $A \in \mathbb{R}^{n\times n}$ with itself is not a guardian map, it can be used in a similar way as the Kronecker sum to guard Hurwitz matrices with minor changes. The advantage of the bialternate sum is that it involves less calculations than the Kronecker sum. Specifically, the stability test requires the computation of the eigenvalues of the inverse of an $n^2 \times n^2$ matrix if the Kronecker sum is used. This reduces to computing the eigenvalues of the inverse of an $\frac{1}{2}n(n-1) \times \frac{1}{2}n(n-1)$ matrix if the bialternate sum is used.

The notation used in this paper is as follows:

$\otimes \quad \oplus$	Kronecker product and sum
$\star$	Bialternate product
$\lambda_i(A)$	ith eigenvalue of the matrix A
I_n	Identity matrix of dimension $n \times n$ (also denoted I when the dimension is clear from the context)
$\mathrm{int}(\mathcal{D})$	Interior of the set $\mathcal{D}$
$\partial\mathcal{D}$	Boundary of the set $\mathcal{D}$
$\mathcal{A}$	Set of Hurwitz matrices $A \in \mathbb{R}^{n\times n}$
$\bar{A}$	$A \oplus A$, $A \in \mathbb{R}^{n\times n}$
$\tilde{A}$	$A \star I_n + I_n \star A = 2A \star I_n$, $A \in \mathbb{R}^{n\times n}$
$\mathrm{mspec}(A)$	Multispectrum of matrix A, i.e. the set consisting of all the eigenvalues of A, including repeated eigenvalues.
$\mathcal{I}_n$	Index set $\{1, 2, \ldots, n\}$
$\mathcal{I}_n^0$	Index set $\{0, 1, 2, \ldots, n\}$
$\bar{\cup}$	Ordered union of two sets.
$\mathcal{D}^{\#}$	Cardinality of the set $\mathcal{D}$.

2 Preliminaries

2.1 The Guardian Map

Our results rely heavily on the concept of a guardian map for the set of Hurwitz matrices. A guardian map transforms a matrix stability problem to a nonsingularity problem of an associated matrix. The most common guardian map is the one that involves the Kronecker sum of a matrix with itself. The definitions of the Kronecker product and Kronecker sum of two matrices may be found in several standard references (see for example [13]).

Lemma 2.1 ([14]) *Let* $A \in \mathbb{R}^{n\times n}$ *and* $B \in \mathbb{R}^{m\times m}$*. Then* $\mathrm{mspec}(A \oplus B) = \{\lambda_i + \mu_j \ : \lambda_i \in \mathrm{mspec}(A),$ $\mu_j \in \mathrm{mspec}(B),\ i = 1, 2, ..., n$ *and* $j = 1, 2, ..., m\}$.

Lemma 2.2 *Given a matrix* $A \in \mathbb{R}^{n\times n}$*, define* $\bar{A} := A \oplus A$*. Assume that* A *is Hurwitz. Then:*
(i) $\bar{A}$ *is Hurwitz*
(ii) $\det\bar{A} \neq 0$

The following definition is taken from [10].

Definition 1 (Guardian Map) *Let an open set* $\mathcal{S} \subseteq \mathbb{R}^{n\times n}$ *and* ν*:* $\mathbb{R}^{n\times n} \to \mathbb{R}$ *be a given mapping. Then* ν *is said to guard the set* $\mathcal{S}$ *if* $\nu(A) \neq 0$ *for* $A \in \mathcal{S}$

and $\nu(A) = 0$ for $A \in \partial\mathcal{S}$. The map ν is called a guardian map *for $\mathcal{S}$.*

Example 1 The Kronecker sum induces the guardian map $\nu_1 : \mathbb{R}^{n\times n} \to \mathbb{R}$

$$\nu_1(A) := \det(A \oplus A) \tag{4}$$

which guards the set $\mathcal{A}$ of Hurwitz matrices [10].

2.2 Bialternate Sum

For $A, B \in \mathbb{R}^{n\times n}$ with elements a_{ij} and b_{ij}, the bialternate product of A and B is the matrix $F = A \star B$ of dimension $\frac{1}{2}n(n-1) \times \frac{1}{2}n(n-1)$, with elements as follows [15, 12]:

$$f_{\tilde{m}(n,p,q),\tilde{m}(n,r,s)} := \frac{1}{2}\left[\begin{vmatrix} a_{pr} & a_{ps} \\ b_{qr} & b_{qs} \end{vmatrix} + \begin{vmatrix} b_{pr} & b_{ps} \\ a_{qr} & a_{qs} \end{vmatrix}\right],$$

where the index function $\tilde{m}$ is defined as:

$$\tilde{m}(n,i,j) := (j-1)n + i - \frac{1}{2}j(j+1)\,. \tag{5}$$

According to this definition, it is clear that $A \star B = B \star A$. The bialternate sum $\tilde{A}$ of matrix A with itself is defined as [16, 15, 12]

$$\tilde{A} = A \star I_n + I_n \star A = 2A \star I_n\,. \tag{6}$$

If $\tilde{a}_{ij}$ denotes the $i-j\text{th}$ element of $\tilde{A}$ then,

$$\tilde{a}_{\tilde{m}(n,p,q),\tilde{m}(n,r,s)} = \begin{vmatrix} a_{pr} & a_{ps} \\ \delta_{qr} & \delta_{qs} \end{vmatrix} + \begin{vmatrix} \delta_{pr} & \delta_{ps} \\ a_{qr} & a_{qs} \end{vmatrix} \tag{7}$$

where, δ_{ij} is the Kronecker delta $\big(\delta_{ij} = 1, \text{ if } i = j, \ \delta_{ij} = 0, \text{ if } i \neq j\big)$. Clearly, if $A \in \mathbb{R}^{n\times n}$, then $\tilde{A} \in \mathbb{R}^{\frac{1}{2}n(n-1)\times\frac{1}{2}n(n-1)}$. From the definition of the bialternate sum of a matrix with itself, one has immediately that

$$\widetilde{\alpha A} = \alpha\tilde{A}$$
$$\widetilde{A_0 + \rho A_g} = \tilde{A}_0 + \rho\tilde{A}_g$$

where $A, A_0, A_g \in \mathbb{R}^{n\times n}$, and $\alpha, \rho \in \mathbb{R}$.

Theorem 2.1 ([15]) *Let $A \in \mathbb{R}^{n\times n}$. Then* $\text{mspec}(\tilde{A}) = \{\lambda_i(A) + \lambda_j(A)\ i = 2, 3, \ldots, n,\ j = 1, 2, \ldots, i-1\}$.

The following Corollary follows immediately from Theorem 2.1.

Corollary 2.1 *Let $A \in \mathbb{R}^{n\times n}$ be Hurwitz. Then:*
(i) *$\tilde{A}$ is Hurwitz.*
(ii) $\det\tilde{A} \neq 0$.

Remark 1 The determinant of the bialternate sum of a matrix with itself cannot be used as a guardian map of $\mathcal{A}$. To see this, let a matrix $A \in \mathbb{R}^{n\times n}$ with only one eigenvalue zero and all other eigenvalues in the open left half complex plane. In this case, $A \in \partial\mathcal{A}$, but $\det\tilde{A} \neq 0$. However, the map

$$\nu_2(A) = \det A \det\tilde{A} \tag{8}$$

is a guardian map which guards the set $\mathcal{A}$. First, it is easy to see that $\nu_2(A) \neq 0$ if $A \in \mathcal{A}$. Moreover, if $A \in \partial\mathcal{A}$, some eigenvalues of the matrix A are on the $j\omega$-axis and all the others are in the open left half plane of $\mathbb{C}$. Let $\mathcal{F}$ be the set of matrices in $\partial\mathcal{A}$ with at most one eigenvalue at the origin

$$\begin{aligned}\mathcal{F} = \Big\{ & A \in \partial\mathcal{A}\ :\ \lambda_i(A) = 0 \text{ and } \lambda_j(A) \neq 0 \\ & \text{for all } j \neq i,\ \ i, j \in \mathcal{I}_n \Big\}\,.\end{aligned} \tag{9}$$

If $A \in \mathcal{F}$ then $\det A = 0$ and if $A \in \partial\mathcal{A} \setminus \mathcal{F}$ then $\det\tilde{A} = 0$. In either case, $\nu_2(A) = 0$. Hence, $\nu_2(A)$ is a guardian map for the set $\mathcal{A}$ according to the Definition 1. Moreover, $\nu_2(A)$ is easier to compute than $\nu_1(A)$ since the dimension of $\tilde{A}$ is $\frac{1}{2}n(n-1) \times \frac{1}{2}n(n-1)$ whereas that of $\bar{A}$ is $n^2 \times n^2$.

2.3 Some Definitions

Definition 2 *Given $M \in \mathbb{R}^{n\times n}$, let $\tilde{\lambda}_i(M)\,,\ i = 1, \ldots, p$ denote the real, distinct, non-zero eigenvalues of M and define $\tilde{\lambda}_0(M) = 0$. If $p = 0$, let $\mathcal{N}(M) = (-\infty, +\infty)$, otherwise define the open interval $\mathcal{N}(M)$ as follows:*

$$\mathcal{N}(M) := \Big(-\frac{1}{\max\limits_{i\in\mathcal{I}_p^0}\tilde{\lambda}_i(M)}, -\frac{1}{\min\limits_{i\in\mathcal{I}_p^0}\tilde{\lambda}_i(M)}\Big) \tag{10}$$

where,

$$-\frac{1}{\max\limits_{i\in\mathcal{I}_p^0}\tilde{\lambda}_i(M)} = -\infty, \quad \text{if} \quad \max_{i\in\mathcal{I}_p^0}\tilde{\lambda}_i(M) = 0,$$
$$-\frac{1}{\min\limits_{i\in\mathcal{I}_p^0}\tilde{\lambda}_i(M)} = +\infty, \quad \text{if} \quad \min_{i\in\mathcal{I}_p^0}\tilde{\lambda}_i(M) = 0. \tag{11}$$

The following Corollary is a direct consequence of Definition 2.

Corollary 2.2 *For any $M \in \mathbb{R}^{n\times n}$,*

(i) $0 \in \mathcal{N}(M)$

(ii) $\det(I+\rho M) \neq 0$, *for* $\rho \in \mathcal{N}(M)$

Definition 3 *Given $M \in \mathbb{R}^{n\times n}$, let $\tilde{\lambda}_i(M)$, $i = 1,\dots,p$ denote the real, distinct, non-zero eigenvalues of M. Let $r_0 = -\infty$, $r_i = -1/\tilde{\lambda}_i(M)$, $i = 1,2,\dots,p$ and $r_{p+1} = +\infty$ and define the ordered set (after, perhaps, a relabelling of the indices) $\mathcal{B}(M) := \{r_0, r_1, r_2, \dots, r_p, r_{p+1}\}$ such that $r_i < r_{i+1}$.*

Remark 2 From the definition of $\mathcal{B}(M)$, it follows that, for $r \in \mathbb{R}$, $\det(I + rM) = 0$ if and only if $r \in \mathcal{B}(M)$.

3 Maximal Stability Domain of Single Parameter-Dependent LTI Systems

In this section we compute the maximal stability interval containing the origin for a single parameter-dependent LTI system. Theorem 3.1 below is basically a re-statement of the result in [11]. Later this theorem is extended so as to reduce the computations involved through the use of the bialternate sum of matrices.

Theorem 3.1 *Given an open interval Ω in $\mathbb{R}$, and $A_0, A_g \in \mathbb{R}^{n\times n}$, the following two statements are equivalent:*
(i) $0 \in \Omega$, and $A(\rho) := A_0 + \rho A_g$ is Hurwitz for all $\rho \in \Omega$
(ii) A_0 is Hurwitz and $0 \in \Omega \subseteq \mathcal{N}(\bar{A}_0^{-1}\bar{A}_g)$

The proof can be found in [11].

Corollary 3.1 *Given $A_0, A_g \in \mathbb{R}^{n\times n}$ such that A_0 is Hurwitz, let the interval $\mathcal{N}(\bar{A}_0^{-1}\bar{A}_g)$ as in Definition 2. This is the largest interval of $\mathbb{R}$ containing the origin for which the matrix $A_0 + \rho A_g$ is Hurwitz.*

3.1 Improved Stability Condition for Single-Parameter Dependent LTI Systems

The application of the stability condition of Theorem 3.1 is limited owing to the large number of computations required to calculate the inverse of the $n^2 \times n^2$ matrix $\bar{A}_0$, especially when the system is of high order. This limitation can be overcome by using the guardian map of Remark 1 which involves the determinant of the bialternate sum of a matrix with itself. The resulting improved stability condition requires the calculation of the inverses of an $n \times n$ and an $\frac{1}{2}n(n-1) \times \frac{1}{2}n(n-1)$ matrix. Using the map induced by the bialternate sum, one can easily obtain the following robust stability condition, which can also be used to synthesize the maximal continuous robust stability interval that includes the origin.

Theorem 3.2 *Given an open interval Ω in $\mathbb{R}$, and A_0, $A_g \in \mathbb{R}^{n\times n}$, the following two statements are equivalent:*
(i) $0 \in \Omega$, and $A(\rho) := A_0 + \rho A_g$ is Hurwitz for all $\rho \in \Omega$
(ii) A_0 is Hurwitz and $0 \in \Omega \subseteq \mathcal{N}(A_0^{-1}A_g) \cap \mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g)$

Corollary 3.2 *Given $A_0, A_g \in \mathbb{R}^{n\times n}$ such that A_0 is Hurwitz, then $\mathcal{N}(A_0^{-1}A_g) \cap \mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g)$ is the largest continuous interval of $\mathbb{R}$ containing the origin for which the matrix $A_0 + \rho A_g$ is Hurwitz.*

The following result follows immediately from Corollary 3.1 and Corollary 3.2.

Corollary 3.3 *Given $A_0, A_g \in \mathbb{R}^{n\times n}$ suppose that A_0 is Hurwitz. Then,*

$$\mathcal{N}(\bar{A}_0^{-1}\bar{A}_g) = \mathcal{N}(A_0^{-1}A_g) \cap \mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g) \tag{12}$$

4 Complete Stability Domain of Single Parameter-Dependent LTI Systems

4.1 Stability Condition using the Kronecker Sum

Theorems 3.1 and 3.2 give the maximal continuous stability interval in $\mathbb{R}$ which includes the origin. These two theorems nonetheless provide only sufficient conditions for a single-parameter dependent matrix to be Hurwitz, because in many cases the maximal stability interval about the origin is not the complete stability domain. Additionally, the requirement that A_0 is Hurwitz limits the applicability of Theorems 3.1 and 3.2. In this section, our objective is to get the complete stability domain without requiring A_0 to be Hurwitz. This domain turns out to be an open interval or a union of limited number of disjointed open intervals of $\mathbb{R}$ (Theorem 4.1). The complete stability domain is given in Theorem 4.3.

Theorem 4.1 *Let $A_0, A_g \in \mathbb{R}^{n\times n}$ with $\det(A_0 \oplus A_0) \neq 0$. If there exists a stability domain $\Omega \subseteq \mathbb{R}$ such that $A_0+\rho A_g$ is Hurwitz for all $\rho \in \Omega$, then this domain Ω is an open interval or a union of disjointed open intervals of $\mathbb{R}$ and the number of such intervals is finite. Furthermore, this number is no larger than n^2+1.*

Proof. Since the eigenvalues $\lambda_j\big(A_0+\rho A_g\big),\ j = 1,2,\ldots,n$ vary continuously with the parameter ρ if $A_0+\rho_i A_g$ is Hurwitz, for some $\rho_i \in \Omega$, there exists $\delta > 0$ such that $A_0+\rho A_g$ is Hurwitz for $\rho \in (\rho_i-\delta, \rho_i+\delta)$. Therefore, if Ω exists, it must be a union of open intervals. Let Ω be expressed as[1] $\Omega = \bigcup_{i=1}^{m}(\underline{\rho}_i, \bar{\rho}_i)$, where $\underline{\rho}_i, < \bar{\rho}_i$ and m is the (perhaps infinite) number of the disjointed open intervals composing Ω. Since Ω is the exact stability region of ρ, it follows that for every $\underline{\rho}_i \in \mathbb{R}$, $i \in \mathcal{I}_m$, $\mathrm{Re}[\lambda_k(A_0+\underline{\rho}_i A_g)] = 0$ for some $k \in \mathcal{I}_n$. Hence, by Lemma 2.1, $\lambda_{k'}(\bar{A}_0+\underline{\rho}_i\bar{A}_g) = 0$ for some $k' \in \mathcal{I}_{n^2}$ and hence $\det(\bar{A}_0+\underline{\rho}_i\bar{A}_g) = 0$. Since $\det(A_0\oplus A_0) = \det\bar{A}_0 \neq 0$, $\bar{A}_0^{-1}$ exists. Thus,

$$\det(I+\underline{\rho}_i\bar{A}_0^{-1}\bar{A}_g) = 0 \qquad i \in \mathcal{I}_m \tag{13}$$

Since this equation has a finite number of solutions, $m < \infty$. By Definition 3 and equation (13), it follows that $\underline{\rho}_i \in \mathcal{B}(\bar{A}_0^{-1}\bar{A}_g),\ i = 2,3,\ldots,m$. Similarly, one can show that for $\bar{\rho}_i \in \mathbb{R}$, $\bar{\rho}_i \in \mathcal{B}(\bar{A}_0^{-1}\bar{A}_g),\ i = 1,2,\ldots,m-1$. Therefore, $2m \leq (\mathcal{B}(\bar{A}_0^{-1}\bar{A}_g)\cup\{-\infty,+\infty\})^{\#} = \mathcal{B}^{\#}(\bar{A}_0^{-1}\bar{A}_g)$. From the definition of the set $\mathcal{B}(\bar{A}_0^{-1}\bar{A}_g)$ it is clear that $\mathcal{B}^{\#}(\bar{A}_0^{-1}\bar{A}_g) \leq n^2+2$. It follows that $m \leq (n^2+1)$. ∎

[1] With the possibility that $\underline{\rho}_1 = -\infty$ and $\bar{\rho}_m = +\infty$.

Theorem 4.2 *Let $A_0, A_g \in \mathbb{R}^{n\times n}$ with $\det(A_0 \oplus A_0) \neq 0$, and let $p = \mathcal{B}^{\#}(\bar{A}_0^{-1}\bar{A}_g)-2$. Suppose there exists a real number $\rho_i \in \big(r_i, r_{i+1}\big)$, where $r_i, r_{i+1} \in \mathcal{B}(\bar{A}_0^{-1}\bar{A}_g)$, $i \in \mathcal{I}_p^0$ such that $A_0+\rho_i A_g$ is Hurwitz. Then $A_0+\rho A_g$ is Hurwitz for all $\rho \in \big(r_i, r_{i+1}\big)$.*

Proof. The map $\nu_1\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ given by

$$\nu_1(A) = \det(A\oplus A)$$

is a guardian map for the set $\mathcal{A}$ of stable $n\times n$ matrices (see page 303 of [10]). Let $A(\rho) := A_0+\rho A_g$. According to the definition of $\mathcal{B}(\bar{A}_0^{-1}\bar{A}_g)$, if $r_i, r_{i+1} \in \mathcal{B}(\bar{A}_0^{-1}\bar{A}_g)$, $r_i, r_{i+1} \neq \pm\infty$ then $\nu_1(A(r_i)) = 0$ and $\nu_1(A(r_{i+1})) = 0$. Furthermore, $\nu_1(A(\rho)) \neq 0$ if $r_i < \rho < r_{i+1}$. Let now some $\rho_i \in \big(r_i, r_{i+1}\big)$ such that $A_0+\rho_i A_g$ is Hurwitz. But $\nu_1(A(\rho_i)) \neq 0$ since $A(\rho_i)$ is Hurwitz, and since ν_1 is a guardian map, it follows that $A(\rho)$ is Hurwitz for all $\rho \in (r_i,\ r_{i+1})$. ∎

Theorem 4.3 *Given $A_0, A_g \in \mathbb{R}^{n\times n}$ with $\det(A_0 \oplus A_0) \neq 0$ and $\rho \in \mathbb{R}$, let $\bar{A}_0 := A_0 \oplus A_0$, $\bar{A}_g := A_g \oplus A_g$ and let $p = \mathcal{B}^{\#}(\bar{A}_0^{-1}\bar{A}_g)-2$. Define the index set $\mathcal{I} := \big\{i \in \mathcal{I}_p^0 :\ A_0+\rho_i A_g \text{ is Hurwitz for some } \rho_i \in (r_i,\ r_{i+1}),\ \ r_i, r_{i+1} \in \mathcal{B}(\bar{A}_0^{-1}\bar{A}_g)\big\}$ and the open set*

$$\Omega_\epsilon \ := \ \bigcup_{i\in\mathcal{I}}(r_i,\ r_{i+1}) \tag{14}$$

Then, $A_0+\rho A_g$ is Hurwitz if and only if $\rho \in \Omega_\epsilon$.

Proof. To prove sufficiency, assume $\rho \in \Omega_\epsilon$ and let $\rho \in (r_i, r_{i+1})$ for some $i \in \mathcal{I}$. From Theorem 4.2 and the fact that $A_0+\rho_i A_g$ Hurwitz for $\rho_i \in (r_i, r_{i+1})$, it follows that $A_0+\rho A_g$ is Hurwitz. To prove necessity, assume that $A_0+\rho A_g$ is Hurwitz. It follows that $\rho \notin \mathcal{B}(\bar{A}_0^{-1}\bar{A}_g)$. Therefore, there exists $i \in \mathcal{I}_p^0$ such that $r_i < \rho < r_{i+1}$. Since $A_0+\rho A_g$ is Hurwitz, it follows that $i \in \mathcal{I}$. Hence, $\rho \in (r_i, r_{i+1}) \subseteq \Omega_\epsilon$. ∎

Remark 3 Theorem 4.3 can be used to find the *exact* stability domain Ω_ϵ for a parameter-dependent matrix $A(\rho) = A_0+\rho A_g$ where $\rho \in \mathbb{R}$ and $A_0,\ A_g \in \mathbb{R}^{n\times n}$. The procedure involves four steps.

1. Calculate $\bar{A}_0, \bar{A}_g$ and the eigenvalues of the matrix $\bar{A}_0^{-1}\bar{A}_g$.
2. Choose the real, distinct, non-zero eigenvalues of the matrix $\bar{A}_0^{-1}\bar{A}_g$ and construct the set $\mathcal{B}(\bar{A}_0^{-1}\bar{A}_g)$ according to Definition 3.
3. Check whether the matrix $A_0 + \rho_i A_g$ is Hurwitz for *any* $\rho_i \in (r_i, r_{i+1}), i \in \mathcal{I}_p^0, p = \mathcal{B}^{\#}(\bar{A}_0^{-1}\bar{A}_g) - 2$, and construct the index set $\mathcal{I}$.
4. Let Ω_ϵ as in (14).

4.2 Stability Condition using the Bialternate Sum

The need to do intensive numerical calculations in order to calculate the inverse and the eigenvalues of the $n^2 \times n^2$ matrix $\bar{A}_0 = \det A_0 \oplus A_0$ limits the applicability of Theorem 4.3. This limitation can be overcome using a map induced by the bialternate sum of a matrix with itself (see (8) and Remark 1). In this case, it is only needed to calculate the inverse of a matrix of dimension $\frac{1}{2}n(n-1) \times \frac{1}{2}n(n-1)$.

Theorem 4.4 *Let* $A_0, A_g \in \mathbb{R}^{n\times n}$ *with* $\det(A_0 \oplus A_0) \neq 0$. *If there exists a stability domain* $\Omega \subseteq \mathbb{R}$ *such that* $A_0 + \rho A_g$ *is Hurwitz for all* $\rho \in \Omega$, *then this domain* Ω *is an open interval or a union of disjointed open intervals of* $\mathbb{R}$, *and the number of such intervals is finite. Furthermore, this number is no greater than* $\frac{1}{2}(n^2 + n + 2)$.

Proof. The proof is similar to the one for Theorem 4.1 and thus, is omitted. ∎

Remark 4 Since $\frac{1}{2}(n^2+n+2) \leq (n^2+1)$ Theorem 4.4 gives a better estimate for the number of stability intervals than Theorem 4.1.

Theorem 4.5 *Let* $A_0, A_g \in \mathbb{R}^{n\times n}$ *with* $\det(A_0 \oplus A_0) \neq 0$, *and let* $p = (\mathcal{B}(\tilde{A}_0^{-1}\tilde{A}_g)\bar{\cup}\mathcal{B}({A_0}^{-1}A_g))^{\#} - 2$. *Suppose there exists a real number* $\rho_i \in \big(r_i, r_{i+1}\big)$ *where* $r_i, r_{i+1} \in \mathcal{B}(\tilde{A}_0^{-1}\tilde{A}_g)\bar{\cup}\mathcal{B}({A_0}^{-1}A_g)$, $i \in \mathcal{I}_p^0$ *such that* $A_0 + \rho_i A_g$ *is Hurwitz. Then* $A_0 + \rho A_g$ *is Hurwitz for all* $\rho \in \big(r_i, r_{i+1}\big)$.

Proof. The proof is similar to the one of Theorem 4.2 and thus, is omitted. ∎

Theorem 4.6 *Given* $A_0, A_g \in \mathbb{R}^{n\times n}$ *with* $\det(A_0 \oplus A_0) \neq 0$ *and* $\rho \in \mathbb{R}$, *let* $p = \big(\mathcal{B}(A_0^{-1}A_g)\bar{\cup}\mathcal{B}(\tilde{A}_0^{-1}\tilde{A}_g)\big)^{\#} - 2$. *Define the index set* $\mathcal{I} := \big\{i \in \mathcal{I}_p^0 :\ A_0 + \rho_i A_g \text{ is Hurwitz for some } \rho_i \in (r_i, r_{i+1}),\ r_i, r_{i+1} \in \mathcal{B}(A_0^{-1}A_g)\bar{\cup}\mathcal{B}(\tilde{A}_0^{-1}\tilde{A}_g)\big\}$ *and the open set*

$$\Omega_\epsilon := \bigcup_{i\in\mathcal{I}} (r_i, r_{i+1}) \tag{15}$$

Then, $A_0 + \rho A_g$ *is Hurwitz if and only if* $\rho \in \Omega_\epsilon$.

Proof. The proof is similar to the one of Theorem 4.3 and thus,it is omitted. ∎

5 Generalized Stability Condition for Multi-Parameter Dependent LTI Systems

In this section, the robust stability condition for the following multi-parameter dependent LTI system will be studied

$$\dot{x} = (A_0 + \sum_{i}^{k} \rho_i A_{g,i})x\,. \tag{16}$$

Reference [7] gives a stability condition for a system of the form (16), however that condition is only sufficient. Saydy *et al.* [8] used a semi-guardian map[2] to investigate robust stability for the following two-parameter quadratically-dependent matrix over the domain $(r_1, r_2) \in [0,1] \times [0,1]$

$$A(\rho_1, \rho_2) = \sum_{i_1, i_2 = 0}^{i_1+i_2=m} \rho_1^{i_1}\rho_2^{i_2} A_{i_1,i_2}\,. \tag{17}$$

The stability test in [8] requires the parameter domain to be known a priori. Consequently, the test checks whether the matrix is Hurwitz for all values of the parameters in a *given* domain. In this section, we extend the results of Section 4.2.

[2] A map ν: $\mathcal{A} \rightarrow \mathbb{R}$ from the set of $n \times n$ real Hurwitz matrices onto $\mathbb{R}$ is a semi-guardian map if it is continuous, not identically zero and $A \in \partial\mathcal{A} \Rightarrow \nu(A) = 0$.

Lemma 5.1 *Given the vector* $(\rho_1, \rho_2, \ldots, \rho_k)^T \in \mathbb{R}^k$*,* $k \geq 2$*, there exists a real number* r *and* $k-1$ *scalars* $\theta_i \in [0, \pi]$*,* $i = 2, ..., k$ *such that*

$$(\rho_1, \rho_2, \ldots, \rho_k)^T = rv(\theta) \tag{18}$$

where $\theta = (\theta_2, \ldots, \theta_k)^T \in \mathbb{R}^{k-1}$ *and*

$$\begin{aligned} v(\theta) &= (\cos\theta_2,\ \sin\theta_2\cos\theta_3,\ \ldots, \\ &\quad \sin\theta_2\sin\theta_3\cdots\sin\theta_{k-1}\cos\theta_k, \\ &\quad \sin\theta_2\sin\theta_3\cdots\sin\theta_k) \in \mathbb{R}^k \end{aligned} \tag{19}$$

We now use the stability condition of Theorem 3.1, to obtain the following stability condition for the dynamic system in (16).

Theorem 5.1 *Given* $A_0, A_{g,i} \in \mathbb{R}^{n\times n}$*,* $i = 1, \ldots, k$ *with* $\det(A_0 \oplus A_0) \neq 0$*, let* $(\rho_1, \rho_2, \ldots, \rho_k)^T = rv(\theta)$ *as in Lemma 5.1. Let* $p = \mathcal{B}^{\#}(\bar{A}_0^{-1}\bar{A}_g(\theta)) - 2$*,* $A_g(\theta) := \sum_{i=1}^k A_{g,i}v_i(\theta)$*,* $\bar{A}_g(\theta) := A_g(\theta) \oplus A_g(\theta)$*, and define the set* $\Omega_\epsilon(\theta) = \bigcup_{i\in\mathcal{I}(\theta)}(r_i, r_{i+1})$*, where the index set* $\mathcal{I}(\theta)$ *is given by* $\mathcal{I}(\theta) = \big\{i \in \mathcal{I}_p^0 :\ A_0 + r_i' A_g \text{ is Hurwitz for some } r_i' \in (r_i, r_{i+1}), \text{ where } r_i, r_{i+1} \in \mathcal{B}(\bar{A}_0^{-1}\bar{A}_g(\theta))\big\}$*. Let*

$$\Omega'_\epsilon := \bigcup_{\theta\in[0,\pi)^{k-1}} \Big\{y(\theta) \in \mathbb{R}^k :\ y(\theta) = rv(\theta),\ r \in \Omega_\epsilon(\theta)\Big\}$$

Then $A_0 + \sum_i^k \rho_i A_{g,i}$ *is Hurwitz if and only if* $(\rho_1, ..., \rho_k)^T \in \Omega'_\epsilon$*.*

Proof. Applying Lemma 5.1, $(\rho_1, \ldots, \rho_k)^T \in \mathbb{R}^k$ can be expressed as $(\rho_1, \ldots, \rho_k)^T = r\,(v_1(\theta), \ldots, v_k(\theta))^T$. The system matrix in equation (16) can then be rewritten as:

$$\begin{aligned} A_0 + \sum_i^k \rho_i A_{g,i} &= A_0 + r\sum_{i=1}^k A_{g,i}v_i(\theta) \\ &= A_0 + rA_g(\theta) \end{aligned} \tag{20}$$

When the angle vector $\theta \in [0, \pi)^{k-1}$ is given, the system matrix in (20) is a single-parameter matrix which dependents on $r \in \mathbb{R}$. Applying Theorem 4.3, the complete stability domain for r in the direction θ can be calculated from Theorem 4.2 as $\Omega_\epsilon(\theta) = \bigcup_{i\in\mathcal{I}(\theta)}(r_i, r_{i+1})$. The set defined by (20) is the union of the exact stability domains for the parameter r for every $\theta \in \mathbb{R}^{k-1}$. Therefore Ω'_ϵ is the exact stability domain for $(\rho_1, \rho_2, \ldots, \rho_k)^T \in \mathbb{R}^k$. ∎

Theorem 5.2 *Given* $A_0, A_{g,i} \in \mathbb{R}^{n\times n}$*,* $i = 1, \ldots, k$ *with* $\det(A_0 \oplus A_0) \neq 0$*, let* $(\rho_1, \rho_2, \ldots, \rho_k)^T = rv(\theta)$ *as in Lemma 5.1. Let* $p = \big(\mathcal{B}(A_0^{-1}A_g(\theta))\bar{\cup}\mathcal{B}(\tilde{A}_0^{-1}\tilde{A}_g(\theta))\big)^{\#} - 2$*,* $A_g(\theta) := \sum_{i=1}^k A_{g,i}v_i(\theta)$*,* $\tilde{A}_g(\theta) := 2A_g(\theta) \star I_n$*, and define the open set* $\Omega_\epsilon(\theta) = \bigcup_{i\in\mathcal{I}(\theta)}(r_i, r_{i+1})$ *where the index set* $\mathcal{I}(\theta)$ *is given by* $\mathcal{I}(\theta) = \big\{i \in \mathcal{I}_p^0 :\ A_0 + r_i' A_g \text{ is Hurwitz for some } r_i' \in (r_i, r_{i+1}), \text{ where } r_i, r_{i+1} \in \mathcal{B}(A_0^{-1}A_g(\theta))\bar{\cup}\mathcal{B}(\tilde{A}_0^{-1}\tilde{A}_g(\theta))\big\}$*. Let*

$$\Omega'_\epsilon := \bigcup_{\theta\in[0,\pi)^{k-1}} \Big\{y(\theta) \in \mathbb{R}^k :\quad y(\theta) = rv(\theta),\ r \in \Omega_\epsilon(\theta)\Big\} \tag{21}$$

Then $A_0 + \sum_i^k \rho_i A_{g,i}$ *is Hurwitz if and only if* $(\rho_1, ..., \rho_k)^T \in \Omega'_\epsilon$*.*

Proof. The proof is similar to the one of Theorem 5.1 and thus, it is omitted. ∎

Theorems 5.1 and 5.2 give the complete stability domain for multi parameter-dependent matrices. Moreover, these two results do not require that the matrix A_0 is Hurwitz. The drawback of the approach is that the calculation of Ω'_ϵ requires – in general – gridding of the space $[0, \pi)^{k-1}$.

6 Numerical Examples

6.1 Single-Parameter Case

In the following examples, the stability domain for the matrix $A(\rho) = A_0 + \rho A_g$, with $A_0, A_g \in \mathbb{R}^{n\times n}$, $\rho \in \mathbb{R}$, will be calculated by means of Theorems 3.1 and 3.2.

Example 2 Consider the system matrix $A(\rho) = A_0 + \rho A_g$ with

$$A_0 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad A_g = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Since the matrix $A(\rho)$ is upper triangular, the eigenvalues of $A(\rho)$ are always $\{-1, -1\}$. Hence, the largest stability domain for this example is $(-\infty, +\infty)$. From Theorem 3.1, we calculate

$$\bar{A}_0^{-1}\bar{A}_g = \begin{bmatrix} 0 & -0.5 & -0.5 & 0 \\ 0 & 0 & 0 & -0.5 \\ 0 & 0 & 0 & -0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and $\mathrm{mspec}(\bar{A}_0^{-1}\bar{A}_g) = \{0,0,0,0\}$. The largest continuous interval of ρ which includes 0 and guarantees stability for the matrix $A(\rho)$ is $(-\infty,+\infty)$. This agrees with the eigenvalue analysis. Using Theorem 3.2, we have $\tilde{A}_0 = -2$, $\tilde{A}_g = 0$, $\tilde{A}_0^{-1}\tilde{A}_g = 0$ and

$$\begin{aligned}\mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g) &= \mathcal{N}(0) = (-\infty,+\infty)\\ \mathcal{N}(A_0^{-1}A_g) &= \mathcal{N}\Big(\begin{bmatrix}0 & -1\\ 0 & 0\end{bmatrix}\Big) = (-\infty,+\infty).\end{aligned}$$

The stability domain is $\mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g) \cap \mathcal{N}(A_0^{-1}A_g) = (-\infty,+\infty)$, which coincides with the result from Theorem 3.1 and the direct eigenvalue analysis.

Example 3 Consider the matrix $A(\rho) = A_0 + \rho A_g$, where

$$A_0 = \begin{bmatrix}-2 & 0\\ 0 & -1\end{bmatrix}, \quad A_g = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.$$

Direct eigenvalue analysis of $A(\rho)$ gives $\lambda_1 = -2+\rho,\ \lambda_2 = -1-\rho$. Using Theorem 3.1,

$$\bar{A}_0^{-1}\bar{A}_g = \begin{bmatrix}-0.5 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix},$$

and $\mathrm{mspec}(\bar{A}_0^{-1}\bar{A}_g) = \{-0.5,0,0,1\}$. The largest continuous interval of ρ which includes 0 that guarantees stability for $A(\rho)$ is $(-1,2)$ which agrees with the eigenvalue analysis. Using Theorem 3.2, one obtains $\tilde{A}_0 = -3$, $\tilde{A}_g = 0$, $\tilde{A}_0^{-1}\tilde{A}_g = 0$ and

$$\begin{aligned}\mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g) &= \mathcal{N}(0) = (-\infty,+\infty)\\ \mathcal{N}\Big(A_0^{-1}A_g\Big) &= \mathcal{N}\Big(\begin{bmatrix}-\frac{1}{2} & 0\\ 0 & 1\end{bmatrix}\Big) = (-1,+2).\end{aligned}$$

The stability domain is $\mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g) \cap \mathcal{N}(A_0^{-1}A_g) = (-\infty,+\infty) \cap (-1,+2) = (-1,+2)$, which coincides with the result of Theorem 3.1 and the direct eigenvalue analysis.

Example 4 Consider the matrix $A(\rho) = A_0 + \rho A_g$, where

$$A_0 = \begin{pmatrix}-10.64 & 3.395 & 8.841 & 4.558 & -10.25\\ -11.28 & -0.1536 & 14.67 & 9.852 & -13.53\\ 0.7320 & 3.811 & -0.6074 & 2.408 & -10.44\\ -12.14 & 4.938 & 9.649 & 1.152 & -6.297\\ -11.66 & 6.451 & 11.70 & 9.453 & -17.28\end{pmatrix} \quad (22)$$

$$A_g = \begin{pmatrix}-110.9 & -247.0 & 162.4 & -57.61 & 194.2\\ 241.82 & 731.3 & -446.6 & 87.68 & -511.8\\ 366.8 & 987.5 & -617.4 & 181.9 & -777.1\\ 385.3 & 1118.5 & -666.7 & 137.4 & -809.4\\ 100.8 & 237.1 & -142.4 & 57.89 & -234.3\end{pmatrix} \quad (23)$$

Theorems 3.1 and 3.2 give the maximal stability interval around the origin which is $(-0.04632, 0.00241)$. Theorems 4.3 and 4.6 give the exact stability domain, which is $(-0.04632, 0.00241) \cup (4.2279, +\infty)$.

6.2 Multi-Parameter Case

Example 5 This example is from [7]. Consider the matrix $A(\rho) = A_0 + \rho_1 A_1 + \rho_2 A_2$, where

$$A(\rho) = \begin{bmatrix}-2 & 0 & -1\\ 0 & -3 & 0\\ -1 & -1 & -4\end{bmatrix} + \rho_1 \begin{bmatrix}1 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 1\end{bmatrix} + \rho_2 \begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 1 & 0\end{bmatrix} \quad (24)$$

The exact robust stability region for this problem is $(-\infty, 1.75) \times (-\infty, 3)$ (see [7]). Theorem 5.1 or Theorem 5.2, give the exact stability domain as seen in Fig. 1. The same figure shows the exact stability

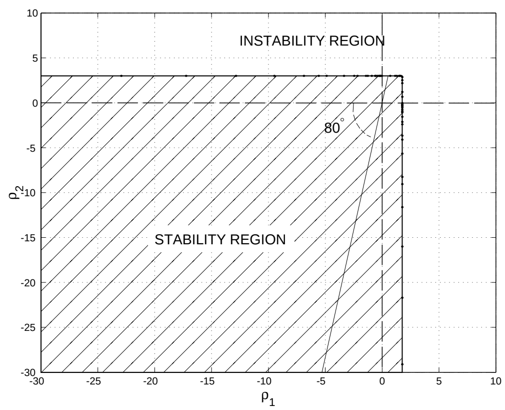


Figure 1: Robust Stability Domain for Example 5.

region along a particular direction ($\theta = 80°$) which, for this case is $(\rho_1 \ \ \rho_2)^T \in \big\{\rho = r\,v(\theta) \ : \ r \in (-\infty, 3.0463),\ v(\theta) = (\cos 80° \ \sin 80°)^T\big\}$.

Example 6 Consider the matrix $A(\rho) = A_0 + \rho_1 A_1 + \rho_2 A_2$, where

$$A_0 = \begin{bmatrix}-2 & 0 & -1\\ 0 & -3 & 0\\ -1 & -1 & -4\end{bmatrix},\ A_1 = \begin{bmatrix}0.9802 & -0.003377 & -0.3599\\ 0.5777 & -0.5721 & 0.9202\\ -0.1227 & 0.2870 & 0.4533\end{bmatrix},$$
$$A_2 = \begin{bmatrix}-0.2641 & -0.1802 & -0.8623\\ 0.7337 & 1.300 & 1.018\\ -0.6962 & 0.5500 & 0.3864\end{bmatrix}.$$

Both Theorem 5.1 and Theorem 5.2 give the same stability domain for the matrix $A(\rho)$, shown in Fig. 2.

Example 7 Consider the matrix $A(\rho) = A_0 + \rho_1 A_1 + \rho_2 A_2$, where

$$A_0 = \begin{bmatrix}-2 & 0 & -1\\ 0 & -3 & 0\\ -1 & -1 & -4\end{bmatrix},\ A_1 = \begin{bmatrix}0.916 & -0.8119 & -0.2168\\ -0.6863 & -0.1001 & -0.4944\\ -0.1673 & 0.7383 & -0.2912\end{bmatrix}$$
$$A_2 = \begin{bmatrix}1.215 & 1.664 & -2.209\\ 0.7542 & -0.1501 & 0.2109\\ 2.199 & 0.6493 & -0.2214\end{bmatrix} \quad (25)$$

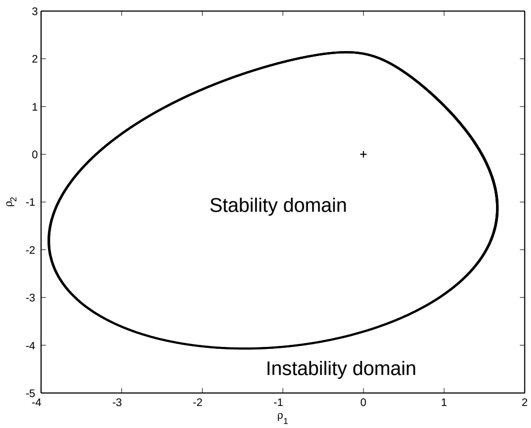


Figure 2: Robust Stability Domain for Example 6.

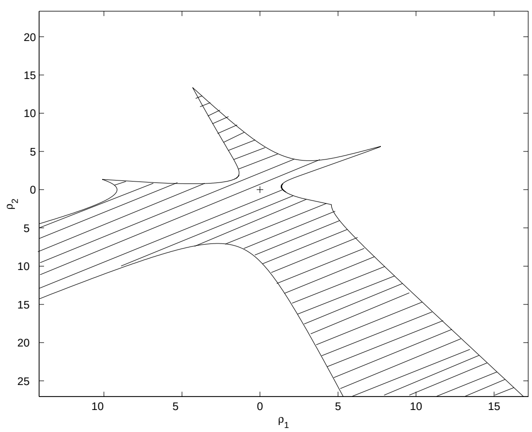


Figure 3: Robust Stability Domain for Example 7.

Both Theorem 5.1 and Theorem 5.2 give the same stability domain for matrix $A(\rho)$, shown in Fig. 3.

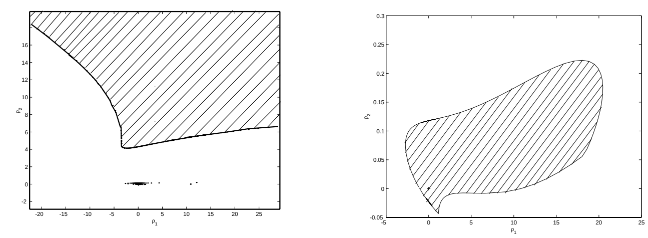
Example 8 Consider the matrix $A(\rho) = A_0 + \rho_1 A_1 + \rho_2 A_2$, where

$$A_0 = \begin{pmatrix} 62.56 & -121.3 & -217.7 & -111.9 & 309.7e+002 \\ -64.81 & 123.1 & 214.78 & 115.4 & -319.4 \\ -7.619 & 19.04 & 25.23 & 21.651 & -52.04 \\ 4.331 & 1.904 & -9.364 & -3.873 & 1.884 \\ -44.28 & 91.39 & 150.5 & 85.74 & -235.0 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} -0.1340 & 0.1139 & 0.2959 & 0.03392 & 0.2288 \\ 0.1747 & -0.2621 & 0.1509 & 0.2436 & 0.2165 \\ 0.1528 & 0.2313 & -0.06069 & 0.2725 & 0.1955 \\ 0.02228 & 0.09418 & 0.2484 & -0.2981 & 0.2262 \\ 0.05797 & 0.1914 & 0.2753 & 0.03664 & -0.1461 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -5.940 & -21.24 & 23.81 & 11.25 & -6.985 \\ -8.853 & -35.44 & 24.58 & 22.03 & 0.9802 \\ -10.05 & -21.45 & 20.03 & 13.64 & -4.311 \\ 0.7771 & -24.14 & 15.17 & 9.370 & 1.589 \\ 2.207 & -14.16 & 13.15 & 3.868 & -8.994 \end{pmatrix}$$

Both Theorems 5.1 and 5.2 give the same stability domain for the matrix $A(\rho)$ shown Fig. 4. In this case, the two-dimensional parameter stability space is composed of two disconnected sets. The area close the origin is zoomed in and is depicted in Fig. 4(b).



(a) (b)

Figure 4: Robust stability domain for Example 8.

7 Conclusions

In this paper we address the problem of stability for Linear Time Invariant Parameter Dependent (LTIPD) systems. We extend previous results in the literature and derive conditions that can be used to compute the exact stability region in the parameters space. Our methodology makes use of the guardian maps induced by the Kronecker and the bialternate sum of a matrix with itself. Although both these maps can be used with the same results, the latter has the benefit of reduced computations.

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