$\mathcal{H}_\infty$ PERFORMANCE AND ROBUST STABILITY OF LINEAR TIME-DELAY SYSTEMS: A PADÉ COMPARISON SYSTEM APPROACH

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Abstract: This paper presents a comparison system approach to the analysis of $\mathcal{H}_\infty$ performance and robust stability for linear systems with time-invariant delays. Using the properties of the diagonal Padé approximation, we establish a covering set for the delay element via an inner and outer inclusion relation. A comparison system can then be obtained by replacing the delay elements with a frequency-dilated version of a Padé approximation. The resulting conditions can be reduced to finite-dimensional LMIs.

Keywords: Time-delay systems; robust stability; $\mathcal{H}_\infty$ performance; Padé approximation.

1. INTRODUCTION

The analysis of linear time-delay systems (LTDS) has attracted much interest over a half century, especially in the last decade (Dugard and Verriest, 1997). In this paper, we extend the stability analysis results of (Zhang *et al.*, 2000*b*; Zhang *et al.*, 2000*a*) to examine the $\mathcal{H}_\infty$ performance and robustness of LTDS to parametric and/or dynamic uncertainties.

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2. PRELIMINARIES

2.1 *Comparison Systems*

We briefly review the relevant results of (Zhang *et al.*, 2000*a*). Throughout this paper, we will confine our analysis to the case of a single delay so as to simplify the presentation. The results presented can be extended straightforwardly to the case of multiple delay (see (Zhang *et al.*, 2000*a*) for some insights regarding this). Consider a nominal LTDS (denoted as $\sum_A$ in the sequel) subject to exogenous disturbance given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-\tau) + B_1 w(t) \\ z(t) &= C_1 x(t) \end{aligned} \tag{1}$$

where A, $A_d \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times n_w}$, and $C_1 \in \mathbb{R}^{n_w\times n}$ are constant matrices, time-delay $\tau \in [0,\bar{\tau}]$ is constant and unknown, and $w(t) \in \mathcal{L}_2[0,\infty)$ is the exogenous disturbance. Without loss of

We will first consider the stability of this system which may be regarded as homogeneous for such analysis. To begin, we will introduce the following definitions:

Definition 1. The *actual delay margin* $\bar{\tau}^*$ for the system (1) is defined as

$$\bar{\tau}^* := \sup\{\bar{\tau}|(1) \text{ is asymptotcially stable on } [0,\bar{\tau}]\}.$$

The stability of system (1) is said to be *delay-dependent* if $\bar{\tau}^*$ is finite, and *delay-independent* otherwise.

Definition 2. Suppose $\mathcal{P}$ is a condition that ensures that (1) is asymptotically stable on $[0,\bar{\tau}]$. If (1) is delay-dependent with actual delay margin $\bar{\tau}^*$, then the *degree of conservatism (d.o.c.)* of $\mathcal{P}$ is defined by

$$d.o.c. := \frac{\bar{\tau}^* - \bar{\tau}^*_{\mathcal{P}}}{\bar{\tau}^*}$$

where

$$\bar{\tau}^*_{\mathcal{P}} := \sup\{\bar{\tau}|\mathcal{P} \text{ is true on } [0,\bar{\tau}]\}.$$

Moreover, $\bar{\tau}^*_{\mathcal{P}}$ is said to be the *delay margin provided by* $\mathcal{P}$.

In the sequel, we decompose $A_d = HF$ where $H \in \mathbb{R}^{n\times q}$ and $F \in \mathbb{R}^{q\times n}$ have full rank. The following zero exclusion condition lies at the heart of our analysis.

Lemma 1. (*Zero Exclusion Condition*) (Zhang *et al.*, 2000*a*) The system (1) is asymptotically stable on $[0,\bar{\tau}]$ if and only if

$$\det[I_q - G(j\omega)\Phi(j\tau\omega)] \neq 0, \qquad \forall\,\omega \geq 0, \tau \in [0,\bar{\tau}], \tag{2}$$

where

$$G(s) := F(sI_n - \bar{A})^{-1}H = \left[\begin{array}{c|c} \bar{A} & H \\ \hline F & 0 \end{array}\right],$$

and $\Phi(\tau,s) := \phi(\tau s)I_q$, $\phi(\tau s) = e^{-\tau s} - 1$.

An indirect but intuitive approach for examining whether (2) holds, is to cover $\Phi(\tau,j\omega)$ with another set $\underline{\Phi}(\omega)$, that is, to find a value set $\underline{\Phi}(\omega)$ such that

$$\Phi(\tau,j\omega) \in \underline{\Phi}(\omega), \qquad \forall\,\omega \geq 0, \tau \in [0,\bar{\tau}_k].$$

Then (2) holds if

$$\det[I_q - G(j\omega)\Delta(j\omega)] \neq 0, \forall\,\omega \geq 0, \Delta(j\omega) \in \underline{\Phi}(\omega). \tag{3}$$

which is satisfied if the interconnection $\sum[G(s),\Delta(s)]$ (referred to as the comparison system in the sequel) is robustly stable. The conservatism of this approach mainly arises from the manner in which the covering set $\underline{\Phi}$ is chosen, based on the properties of the delay element. In (Zhang *et al.*, 1999*a*), plicitly used in the Lyapunov-based stability criteria of (Verriest and Ivanov, 1994; Niculescu *et al.*, 1995; Li and de Souza, 1996; Park, 1999).

2.2 *Inner and Outer Coverings*

We consider the (m,m)-th ($m \geq 3$) *diagonal* Padé approximation $R_m(s)$ to e^{-s}:

$$R_m(s) = \frac{P_m(s)}{Q_m(s)}$$

where

$$P_m(s) = \sum_{k=0}^{m} \frac{(2m-k)!m!(-s)^k}{(2m)!k!(m-k)!},$$
$$Q_m(s) = P_m(-s).$$

In the sequel, for constant $\bar{\tau} > 0$ and $\omega \geq 0$, we define the following value sets:

$$\Omega_A(\omega,\bar{\tau}) := \{e^{-j\tau\omega}|\tau \in [0,\bar{\tau}]\},$$
$$\mathbf{\Omega}_B(\omega,\bar{\tau}) := \{R_m(j\theta\alpha_m\omega)|\theta \in [0,\bar{\tau}]\},$$
$$\mathbf{\Omega}_C(\omega,\bar{\tau}) := \{R_m(j\theta\omega)|\theta \in [0,\bar{\tau}]\}.$$

where $\alpha_m := \frac{\omega_{cm}}{2\pi}$, and ω_{cm} is the phase crossover frequency of $R_m(j\omega)$ at the -2π line:

$$\omega_{cm} := \min\{\omega > 0|R_m(j\omega) = 1\}.$$

Since $|R_m(j\omega)| = 1$, for every $\omega > 0$, $\Omega_A(\omega,\bar{\tau})$, $\mathbf{\Omega}_B(\omega,\bar{\tau})$ and $\mathbf{\Omega}_C(\omega,\bar{\tau})$ are arcs on the unit circle.

Lemma 2. (Zhang *et al.*, 2000*a*) For every integer $m \geq 3$, the following statements hold:

(a) All poles of $R_m(s)$ are in the open left half complex plane.

(b) $\mathbf{\Omega}_C(\omega,\bar{\tau}) \subseteq \Omega_A(\omega,\bar{\tau}) \subseteq \mathbf{\Omega}_B(\omega,\bar{\tau}), \forall\,\omega \geq 0.$

(c) $\alpha_m \to 1$ as $m \to \infty$.

2.3 *Nominal Stability Analysis*

Next, we replace the delay element $e^{-\tau s}$ with the real rational function $R_m(\theta\alpha_m s)$ and $R_m(\theta s)$ and denote the resulting finite-dimensional interconnection systems as $\sum_B(\theta)$ and $\sum_C(\theta)$, respectively. Therefore, we have

$$\sum\nolimits_A : \quad \begin{aligned} \dot{x}(t) &= \bar{A}x(t) + Hv(t) + B_1w(t) \\ v(t) &= [(e^{-s\tau} - 1)F]x(t) \\ z(t) &= C_1x(t) \end{aligned}$$

$$\sum\nolimits_B : \quad \begin{aligned} \dot{x}(t) &= \bar{A}x(t) + Hv(t) + B_1w(t) \\ v(t) &= [R_m(\theta\alpha_m s)I_q - I_q]Fx(t) \\ z(t) &= C_1x(t) \end{aligned}$$

$$\sum\nolimits_C : \quad \begin{aligned} \dot{x}(t) &= \bar{A}x(t) + Hv(t) + B_1w(t) \\ v(t) &= [R_m(\theta s)I_q - I_q]Fx(t) \\ z(t) &= C_1x(t) \end{aligned}$$

The following theorems gives a *sufficient* condition and a *necessary* condition for the stability of (1).

is asymptotically stable on $[0,\bar{\tau}]$, if the comparison system $\sum_B(\theta)$ is robustly stable for $\theta \in [0,\bar{\tau}]$.

Theorem 2. (Zhang *et al.*, 2000*a*) If (1) is asymptotically stable on $[0,\bar{\tau}]$, then $\sum_C(\theta)$ is robustly stable for $\theta \in [0,\bar{\tau}]$.

Next, we note that the *d.o.c.* of Theorem 1 is bounded by a function that depends only on the order of Padé approximation used, can be reduced to any desired degree, and is independent of $\bar{\tau}^*$, A and A_d.

Theorem 3. (Zhang *et al.*, 2000*a*) For $m \geq 5$, the d.o.c. of Theorem 1 satisfies

$$d.o.c. < 0.16\left(\frac{4.286}{m}\right)^{2m+1}. \tag{4}$$

Thus, *d.o.c.* $\to 0$ as $m \to \infty$. Furthermore, for $m = 3, 4$ and 5, we have *d.o.c.* $\leq 18.9\%, 3.05\%$ and 0.361%, respectively.

The comparison system $\sum_B(\theta)$ is free of delays, but it has parametric (real) uncertainties θ. Some care must be taken when examining its robust stability. Let the minimal realization of $P(s) := [R_m(\alpha_m s) - 1]I_q$ be $P(s) = \left[\begin{array}{c|c} A_P & B_P \\ \hline C_P & D_P \end{array}\right]$. Then the subsystem $P(\theta s) = [R_m(\theta\alpha_m s) - 1]I_q$ can be realized in the state space as

$$\begin{aligned} \theta\dot{x}_p &= A_P x_p + B_P u_p, \\ y_p &= C_P x_p + D_P u_p. \end{aligned}$$

Hence the dynamics of $P(\theta s)$ vanish when θ becomes 0, causing the system degeneration (degree dropping). This singularity of the system at $\theta = 0$ obviously complicates the employment of Theorem 1 for analysis. For the single delay case, the delay margin $\bar{\tau}_B^*$ provided by Theorem 1 can be *explicitly calculated without incurring any additional conservatism.*

Theorem 4. (Zhang *et al.*, 2000*b*) Suppose that the system (1) is asymptotically stable for all $\tau \in [0, \bar{\tau}_a]$, where $\bar{\tau}_a > 0$. Then the delay margin provided by Theorem 1 is given by

$$\bar{\tau}_B^* = \frac{\bar{\tau}_a}{\alpha_m} + \frac{1}{\lambda^+_{\max}(-(M_0 \oplus M_0)^{-1}(M_1 \oplus M_1))}, \tag{5}$$

where $M_0 := \begin{bmatrix} \frac{\bar{\tau}_a}{\alpha_m} A_s & C_s \\ \frac{\bar{\tau}_a}{\alpha_m} B_s & A_P \end{bmatrix}$, $M_1 := \begin{bmatrix} A_s & 0 \\ B_s & 0 \end{bmatrix}$, $A_s := \bar{A} + HD_PF, B_s := B_PF$, and $C_s := HC_P$. .

3.1 *Problem Description*

The $\mathcal{H}_\infty$ performance problem is to examine if the system is asymptotically stable for all $\tau \in [0,\bar{\tau}]$ and satisfies

$$\|T_{zw}\|_\infty \leq \gamma \tag{6}$$

where $T_{zw}(s)$ is the transfer function from the disturbance vector w to performance vector z, and $\gamma > 0$ is the performance measure. To proceed with our analysis, we first provide the following definition.

Definition 3. Suppose that a system $\sum$ has an uncertain constant parameter $\theta \in [0,\bar{\tau}]$, and that $T_{zw}(s,\theta)$ is the transfer function from the disturbance $w \in \mathcal{L}_2[0,\infty)$ to performance vector z. If $\sum$ is asymptotically stable for all $\theta \in [0,\bar{\tau}]$, then the **worst case $\mathcal{H}_\infty$ performance** γ^* of $\sum$ is defined as

$$\gamma^* := \max_{\theta\in[0,\bar{\tau}]} \|T_{zw}(\cdot,\theta)\|_\infty. \tag{7}$$

We have the following theorem regarding the relation among the worst case $\mathcal{H}_\infty$ performances of the systems $\sum_A$, $\sum_B$ and $\sum_C$.

Theorem 5. Suppose $\sum_B$ is asymptotically stable for all $\theta \in [0,\bar{\tau}]$. Then the worst case $\mathcal{H}_\infty$ performances of the systems $\sum_A$, $\sum_B$ and $\sum_C$ satisfy

$$\gamma_C^* \leq \gamma_A^* \leq \gamma_B^*.$$

Proof. *(Only outlined for brevity.)* Since $\sum_B$ is asymptotically stable for all $\theta \in [0,\bar{\tau}]$, from Theorem 1 and Theorem 2, we know that $\sum_A$ is asymptotically stable for all $\tau \in [0,\bar{\tau}]$, and $\sum_C$ is asymptotically stable for all $\theta \in [0,\bar{\tau}]$. Then the transfer functions of these uncertain systems are analytic in s and bounded in the open right half complex plane. From the maximum modulus theorem (Boyd and Desoer, 1985), the $\mathcal{H}_\infty$ norms of each of the uncertain transfer function equal the supremum of their singular values over the imaginary axis. The result then follows from the value set inclusion $\mathbf{\Omega}_C(\omega,\bar{\tau}) \subseteq \Omega_A(\omega,\bar{\tau}) \subseteq \mathbf{\Omega}_B(\omega,\bar{\tau}), \forall\, \omega \geq 0$. ■

Next, we present a sufficient condition to compute the $\mathcal{H}_\infty$ performance of $\sum_A$.

Theorem 6. The system $\sum_A$ is asymptotically stable for any constant time-delay $\tau \in [0,\bar{\tau}]$, and satisfies the following $\mathcal{H}_\infty$ performance bound

$$\gamma_A^* < \gamma,$$

if there exist matrices $X_0 > 0$, $X_0 \in \mathbb{R}^{n\times n}$, $X_1 \in \mathbb{R}^{n\times n}$, $X_{22} > 0, X_{22} \in \mathbb{R}^{n_P\times n_P}$ and $X_{12} \in \mathbb{R}^{n\times n_P}$ such that

$$\Pi(0) < 0,\ \Pi(\bar{\tau}) < 0 \tag{8}$$

$$\begin{bmatrix} X_0 + \bar{\tau} X_1 & \bar{\tau} X_{12} \\ \bar{\tau} X_{12}^T & \bar{\tau} X_{22} \end{bmatrix} > 0 \tag{9}$$

where

$$\Pi(\theta) := \begin{bmatrix} \Pi_{11}(\theta) & \Pi_{12}(\theta) & (X_0 + \theta X_1) B_1 & C_1^T \\ * & \Pi_{22}(\theta) & \theta X_{12}^T B_1 & 0 \\ * & * & -\gamma I_{n_w} & 0 \\ * & 0 & 0 & -\gamma I_{n_w} \end{bmatrix},$$

$\Pi_{11}(\theta) := (X_0 + \theta X_1) A_s + X_{12} B_s + A_s^T (X_0 + \theta X_1) + B_s^T X_{12}^T$, $\Pi_{12}(\theta) := (X_0 + \theta X_1) C_s + X_{12} A_P + \theta A_s^T X_{12} + B_s^T X_{22}$, and $\Pi_{22}(\theta) := \theta X_{12}^T C_s + \theta C_s^T X_{12} + X_{22} A_P + A_P^T X_{22}$.

Proof. Omitted for brevity; similar to (Zhang *et al.*, 2000*a*) in handling singularity when $\theta = 0$. ■

4. ROBUST STABILITY OF LTDS WITH UNCERTAIN DYNAMICS

4.1 *Preliminaries*

In this section, we focus on the robust stability of the LTDS with finite dimensional linear time-invariant (FDLTI) dynamical uncertainties. Since the structure of this type of uncertainties can be well captured by using the linear fractional transformation (LFT), we adopt the following system model for our analysis:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-\tau) + B_2 w(t) \\ y(t) &= C_2 x(t) + D_2 w(t) \\ w(t) &= \frac{1}{\gamma} \Delta_u [y](t) \end{aligned} \tag{10}$$

where A, $A_d \in \mathbb{R}^{n \times n}$, $B_2 \in \mathbb{R}^{n \times n_u}$, and $C_1 \in \mathbb{R}^{n_u \times n}$ are constant matrices, time-delay $\tau \in [0, \bar{\tau}]$ is constant and unknown, $\gamma > 0$ is a constant, and Δ_u is an FDLTI *internally stable uncertain dynamical system* in $\mathcal{U}$, where the structured uncertainty set $\mathcal{U}$ is defined by $\mathcal{U} := \{diag\{\Delta_r, \Delta_c\} |\ \Delta_r \in \mathcal{R}_u, \Delta_c \in \mathcal{C}_u\}$, $\mathcal{R}_u := \{diag\{r_1 I_{k_1^r}, \cdots, r_{l_r} I_{k_{l_r}^r}\} \ |r_i \in \mathbb{R}, |r_i| \leq 1\}$, and $\mathcal{C}_u := \{diag\{c_1 I_{k_1^c}, \cdots, c_{l_c} I_{k_{l_c}^c}, \Delta_1, \cdots, \Delta_{l_u}\} \ |c_i \in \mathbb{C}, |c_i| \leq 1, \Delta_i \in \mathbb{C}^{n_i \times n_i}, \bar{\sigma}(\Delta_i) \leq 1\}$.

Again, we assume $\bar{A} := A + A_d$ is Hurwitz and let $A_d = HF$ where $H \in \mathbb{R}^{n \times q}, F \in \mathbb{R}^{q \times n}$ have full rank. (10) can be rewritten as

$$\begin{aligned} \dot{x}(t) &= \bar{A} x(t) + H v(t) + \frac{1}{\gamma} B_2 \tilde{w}(t) \\ v(t) &= [(e^{-s\tau} - 1)F] x(t) \\ y(t) &= C_2 x(t) + \frac{1}{\gamma} D_2 \tilde{w}(t) \\ \tilde{w}(t) &= \Delta_u [y](t) \end{aligned} \tag{11}$$

where $\tilde{w}(t) = \gamma w(t)$, and

$$G(s) = \left[\begin{array}{c|c} \bar{A} & \begin{bmatrix} H & \frac{1}{\gamma} B_2 \end{bmatrix} \\ \hline \begin{bmatrix} F \\ C_2 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{\gamma} D_2 \end{bmatrix} \end{array} \right].$$

Using the Padé Approximation $R_m(\theta \alpha_m s)$ to replace the delay element, the resultant system has the form

$$\begin{aligned} \dot{x}(t) &= \bar{A} x(t) + H v(t) + B_2 w(t) \\ v(t) &= [R_m(\theta \alpha_m s) I_q - I_q] F x(t) \\ y(t) &= C_2 x(t) + D_2 w(t) \\ w(t) &= \frac{1}{\gamma} \Delta_u [y](t) \end{aligned} \tag{12}$$

The following lemma indicates that (12) is a valid comparison system for (10).

Lemma 3. If the comparison system (12) is robustly stable for every FDLTI internally stable uncertain dynamical system $\Delta_u \in \mathcal{U}$ and $\theta \in [0, \bar{\tau}]$, then the uncertain LTDS (10) is also robustly stable for every FDLTI internally stable uncertain dynamical system $\Delta_u \in \mathcal{U}$ and $\tau \in [0, \bar{\tau}]$.

Proof. Omitted for brevity. ■

Now, the system $v(t) = [R_m(\theta \alpha_m s) I_q - I_q] F x(t)$ can be realized by

$$\begin{aligned} \dot{x}_P &= \theta^{-1} A_P x_P + \theta^{-\frac{1}{2}} B_P F x \\ v &= \theta^{-\frac{1}{2}} C_P x_P + D_P F x \end{aligned}$$

Hence the comparison system (12) becomes

$$\begin{aligned} \dot{x}_L(t) &= A_L(\theta) x_L(t) + B_L w(t) \\ y(t) &= C_L x_L(t) + D_2 w(t) \\ w(t) &= \frac{1}{\gamma} \Delta_u [y](t) \end{aligned} \tag{13}$$

where $x_L(t) = \begin{bmatrix} x(t) \\ x_P(t) \end{bmatrix}$, $A_L(\theta) := \begin{bmatrix} A_\tau & \theta^{-\frac{1}{2}} C_\tau \\ \theta^{-\frac{1}{2}} B_\tau & \theta^{-1} A_P \end{bmatrix}$, $B_L := \begin{bmatrix} B_2 \\ 0 \end{bmatrix}$ and $C_L := \begin{bmatrix} C_2 & 0 \end{bmatrix}$, where $A_\tau := \bar{A} + H D_P F, B_\tau := B_P F$, and $C_\tau := H C_P$.

Since $G_\theta(s)$ depends on the parameter θ, if the small μ theorem to directly test the robust stability of (13), a frequency sweep must be performed for many points of θ, resulting in a very tedious computation. One alternative is to incorporate θ into the uncertainty structure, but then a two-step μ test must be employed to avoid the singularity when $\theta = 0$. Herein, we seek an LMI-based condition that avoids the frequency sweep and can be solved efficiently. This condition is based on the recent advancements in the computation of μ upper bound which we review first.

The structured singular value μ is defined for a constant complex matrix along with a specified uncertainty structure. For robustness analysis of a dynamic system, a frequency sweep is typically used. For the case when the uncertainties are real, it has been demonstrated that it is possible for μ to be discontinuous. Hence, strictly speaking, a test with a frequency sweep with finite number of frequency points does not guarantee the robustness of the system. A common approach to avoid this problem has been to develop the μ upper bounds in the state space to test the robust stability of the dynamical system without any frequency sweep. Our result for LTDS builds on the recent result by (Chen and Sugie, 1998) which employs parameter-dependent multipliers, positive real lemma and the S procedure in its development. A further refinement to this was later developed in (Chen *et al.*, 1999), but the resulting condition is much more complex and is not considered herein.

4.3 *An LMI Condition for LTDS Robust Stability*

Recall that the robust stability of the comparison system (13) ensures the robust stability of (10). For every given θ, (13) is a delay-free closed-loop system subject to structured uncertainty. Its robust stability can be determined by using the result of (Chen and Sugie, 1998) along with the small μ theorem, however, the computation is rather tedious due to the parameter sweep over θ. More importantly, as θ approaches 0, the matrix $A_L(\theta)$ becomes ill-conditioned. A similar technique to that used in (Zhang *et al.*, 2000*a*) (i.e., using appropriately chosen basis function so as to render LMIs affine in θ) can be applied to obtain sufficient conditions which are convex in θ and immune from singularity. Then, rather than testing the whole range $[0, \bar{\tau}]$, we are only required to test the two ends of this interval. This result is stated as the following theorem.

Theorem 7. The system (10) is robustly stable, if there exist following appropriate size real matrices, $C_s(\Delta_r)$, $D_m(\Delta_r)$, $P_{11}(\Delta_r,\theta) = P_{11}(\Delta_r,\theta)^T, P_{12}, P_{22} = P_{22}^T, U_i = U_i^T \geq 0$, $W_i, V_i = V_i^T \geq 0$, and $Q(\Delta_r) > 0$ such that

$$\Pi(\Delta_r,\theta) := \begin{bmatrix} \Pi_{11}(\Delta_r) & \Pi_{12}(\Delta_r) \\ * & -Q(\Delta_r) \end{bmatrix} < 0, \qquad (14)$$

$$P(\Delta_r,\theta) := \begin{bmatrix} P_{11}(\Delta_r,\theta) & \theta^{\frac{1}{2}} P_{12} \\ * & P_{22} \end{bmatrix} > 0, \qquad (15)$$

$$\forall r_i \in \{-1, 1\}, \theta \in \{0, \bar{\tau}\},$$

$$X_{ci} := -He\left\{ \begin{bmatrix} B_r \\ D_r \end{bmatrix} J_i \begin{bmatrix} C_{sri}^A & D_{mri} \end{bmatrix} \right\} + \tilde{U}_i \geq 0, \qquad (16)$$

$$U_i := \begin{bmatrix} & \\ * & V_i \end{bmatrix} \geq 0, \qquad (17)$$

$$\forall i \in \{1, 2, \cdots, l_r\},$$

where $\Delta_r := \mathrm{diag}\{r_1 I_{k_1^r}, r_2 I_{k_2^r}, \cdots, r_{l_r} I_{k_{l_r}^r}\}$,

$$\Pi_{11}(\Delta_r,\theta) := X_r(\Delta_r,\theta) + \begin{bmatrix} B_c \\ 0 \\ D_c \end{bmatrix} Q(\Delta_r) \begin{bmatrix} B_c \\ 0 \\ D_c \end{bmatrix}^T + \sum_{i=1}^{l_r} \begin{bmatrix} U_i & 0 & W_i \\ 0 & 0 & 0 \\ W_i^T & 0 & V_i \end{bmatrix},$$

$$X_r(\Delta_r,\theta) := He\left\{ [X_{ij}(\Delta_r,\theta)]_{3\times 3} \right\}$$

$$X_{11}(\Delta_r,\theta) := A_\tau P_{11}(\Delta_r,\theta) + C_\tau P_{12}^T - B_r \Delta_r C_{sr}^A(\Delta_r),$$

$$X_{12}(\Delta_r,\theta) := \theta A_\tau P_{12} + C_\tau P_{22} - B_r \Delta_r C_{sr}^A(\Delta_r)$$

$$X_{13}(\Delta_r,\theta) := -B_r \Delta_r D_{mr}(\Delta_r),$$

$$X_{21}(\Delta_r,\theta) := B_\tau P_{11}(\Delta_r,\theta) + A_p P_{12}^T$$

$$X_{22}(\Delta_r,\theta) := \theta B_\tau P_{12} + A_p P_{22},$$

$$X_{23}(\Delta_r,\theta) := 0,$$

$$X_{31}(\Delta_r,\theta) := C_2 P_{11}(\Delta_r,\theta) + \gamma C_s^A(\Delta_r) - D_r \Delta_r C_{sr}^A(\Delta_r),$$

$$X_{32}(\Delta_r,\theta) := C_2 P_{12}(\Delta_r) + \gamma C_s^B(\Delta_r) - D_r \Delta_r C_{sr}^B(\Delta_r)$$

$$X_{33}(\Delta_r,\theta) := \gamma D_m(\Delta_r) - D_r \Delta_r D_{mr}(\Delta_r),$$

$$C_s(\Delta_r) := \begin{bmatrix} C_s^A(\Delta_r) & C_s^B(\Delta_r) \end{bmatrix},$$

$$C_s^A(\Delta_r) = \begin{bmatrix} C_{sr}^A \\ C_{sc}^A \end{bmatrix} := \begin{bmatrix} C_{sr0}^A \\ C_{sc0}^A \end{bmatrix} + \sum_{i=1}^{l_r} r_i \begin{bmatrix} C_{sri}^A \\ C_{sci}^A \end{bmatrix},$$

$$C_s^B(\Delta_r) = \begin{bmatrix} C_{sr}^B \\ C_{sc}^B \end{bmatrix} := \begin{bmatrix} C_{sr0}^B \\ C_{sc0}^B \end{bmatrix} + \sum_{i=1}^{l_r} r_i \begin{bmatrix} C_{sri}^B \\ C_{sci}^B \end{bmatrix},$$

$$D_m(\Delta_r) = \begin{bmatrix} D_{mr}(\Delta_r) \\ D_{mc}(\Delta_r) \end{bmatrix} := \begin{bmatrix} D_{mr0} \\ D_{mc0} \end{bmatrix} + \sum_{i=1}^{l_r} r_i \begin{bmatrix} D_{mri} \\ D_{mci} \end{bmatrix},$$

$$P_{11}(\Delta_r,\theta) := \theta X_0 + P_0 + \sum_{i=1}^{l_r} r_i P_i,$$

$$Q(\Delta_r) := Q_0 + \sum_{i=1}^{l_r} r_i Q_i,$$

$$J_i := \mathrm{diag}\{0_{k_1^r}, \cdots, 0_{k_{i-1}^r} I_{k_{i+1}^r}, 0_{k_{i+1}^r}, \cdots, 0_{k_{l_r}^r}\},$$

and $Q_i \in \{diag\{\Gamma_1, \cdots, \Gamma_{l_c}, d_1 I_{n_1}, \cdots, d_{l_u} I_{n_{l_u}}\}$ $|\Gamma_i = \Gamma_i^T \in \mathbb{R}^{k_i^c \times k_i^c}, d_i \in \mathbb{R}\}$.

Proof. Omitted for brevity. ∎

5. NUMERICAL EXAMPLE

Consider the uncertain LTD system motivated by the dynamics of machining chatter (Zhang *et al.*, 1999*b*)

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-\tau) + B_2 w(t) \\ y(t) &= C_2 x(t) \\ w(t) &= \frac{1}{\gamma} \begin{bmatrix} \delta_k & 0 \\ 0 & \delta_c \end{bmatrix} y(t) \end{aligned} \qquad (18)$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-(K_{10}+K)}{m_1} & \frac{K_{10}}{m_1} & 0 & 0 \\ \frac{K_{10}}{m_2} & \frac{-(K_{10}+K_2)}{m_2} & 0 & \frac{-C_0}{m_2} \end{bmatrix}$$

$$A_d = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{K}{m_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{K_1}{m_1} & 0 \\ -\frac{K_{10}}{m_2} & -\frac{C_0}{m_2} \end{bmatrix}, \; C_2 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

and δ_k and δ_c are real valued with $|\delta_k| \leq 1$ and $|\delta_c| \leq 1$.

In this example, we choose the parameters as follows. $m_1 = 1$, $m_2 = 2$, $K_{10} = 10$, $K_2 = 20$, $C_0 = 0.5$, and $K = 1$. From (5), we know that the system is asymptotically stable for all $\tau \in [0, 1.4196]$. So herein, we chose $\bar{\tau} = 1.0$ for our robust stability analysis. The Lyapunov-based stability results of (Park, 1999; Li and de Souza, 1996; Niculescu *et al.*, 1995; Verriest *et al.*, 1993) all fail for the nominal system, and hence their corresponding robust stability tests, if any, cannot be used to examine this uncertain system.

The minimal value of γ found by using Theorem 7 is 6.557. We caution that the conservatism of this result may be significant. We can have some sense of how large the conservatism may be by using a parameter sweep for both δ_k and δ_c and examining the delay-dependent stability of the comparison system using the formula (5) . Using this approach, it was found that a lower bound on the minimal obtainable γ is 1.853. But since real μ may be discontinuous and only a finite number of (δ_k, δ_c) points were checked, it is unknown whether minimal feasible γ is close to1.853 or is nearer to 6.557. For comparison, we note that a μ test with a frequency sweep and D-G scaling yielded a minimal value of γ of 7.124 (the procedure employed two steps to avoid the singularity at $\theta = 0$).

6. CONCLUSIONS

In this paper, we have further extended the stability analysis results developed in (Zhang *et al.*, 2000*a*) to LTDS under exogenous disturbances and parametric/dynamic uncertainties.Sufficient conditions based on LMIs were obtained.

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