ASYMPTOTIC STABILITY OF LINEAR SYSTEMS WITH MULTIPLE TIME-INVARIANT STATE-DELAYS

Jianrong Zhang, Carl R. Knospe [*,1] **Panagiotis Tsiotras** [**]

[*] *Department of Mechanical and Aerospace Engineering*
University of Virginia
Charlottesville, VA 22903, USA
Email: `jz9n@virginia.edu, crk4y@virginia.edu`
[**] *School of Aerospace Engineering*
Georgia Institute of Technology
Atlanta, GA 30332, USA
Email: `p.tsiotras@ae.gatech.edu`

Abstract: In this paper, we investigate the asymptotic stability of the linear systems with multiple time-invariant delays. The delay elements are removed from the original system by using a parameter-dependent Padé approximation, resulting in a delay-free comparison system with real parameter uncertainties. We then show that, the robust stability of the comparison system, not only ensures the asymptotic stability of the original time-delay system, but also guarantees an *a priori* upper bound on the degree of conservatism which only depends on the order of the Padé approximation used. Finally, we present a new, delay-dependent condition, formulated in terms of Linear Matrix Inequalities, for the asymptotic stability of the time-delay systems.

Keywords: Time-delay systems; Stability; Padé approximation.

1. INTRODUCTION

The analysis of time-delay systems has attracted much interest over a half century, especially in the last decade. The recent book (Dugard and (Eds), 1997) contains an extensive collection of papers dealing with both delay-dependent and delay-independent stability conditions. Much interest in the literature has focused on searching for sufficient conditions which are numerically tractable but are not too conservative. Many such conditions involve, either explicitly or implicitly, covering the delay elements with some (convex) sets so as to obtain numerically tractable stability conditions (Zhang *et al.*, 1999*a*). Furthermore, the conservatism of the analysis can be reduced by choosing appropriate covering sets, based on delay elements' properties (Zhang *et al.*, 1999*b*). In (Zhang *et al.*, 2000), the parameter-dependent Padé approximation was used to cover the delay element and obtain a simple delay dependent stability condition for the linear system with a *single time-delay*. This condition guarantees an *a priori* degree of conservatism upper bound. Moreover, its delay margin can be calculated explicitly without incurring any additional conservatism. This condition can also be reduced with some (typically small) conservatism to finite-dimensional LMIs.

Many published Lyapunov-based stability analysis results (see (Li and de Souza, 1996; Niculescu *et al.*, 1995; Park, 1999) and the references therein) use a model transformation to transform a system with a single delay into a sys-

[1] This research was supported by the National Science Foundation of the United States under Grant DMI−9713488.

tem with distributed delay. The recent results of (Gu, 1999; Kharitonov and Melchor, 1999) demonstrated that this transformation introduces additional dynamics and hence any stability criteria based on this transformation will be inherently conservative if the additional dynamics have unstable poles. Our Padé approximation based approach, however, does not involve such model transformation, and therefore does not suffer the inherent conservatism incurring in those Lyapunov-based results.

The objective of this paper is to extend the above approach to the linear systems with multiple time-invariant delays. In particular, we will develope a sufficient stability condition which ensures an *a priori* upper bound on the degree of conservatism. This upper bound only depends on the order of the Padé approximation used, and can be reduced to any desired degree by increasing the order of the approximation. This condition can be reduced (with some small conservatism) to LMIs, which can be solved efficiently.

2. PRELIMINARIES

Consider the linear time-delay system

$$\dot{x}(t) = Ax(t) + \sum_{k=1}^{K} A_k x(t-\tau_k) \qquad (1)$$

where the time delays $\tau_k \in [0, \bar{\tau}_k]$, $\bar{\tau}_k > 0$, $k = 1, \cdots, K$, are constant, unknown and independent, and we assume that $\bar{A} := A + \sum_{k=1}^{K} A_k$ is Hurwitz. We denote the delay vector $\tau = [\, \tau_1 \ \cdots \ \tau_K \,]$, and the delay set $\prod_{k=1}^{K}[0, \bar{\tau}_k] := \left\{ [\, \tau_1 \ \cdots \ \tau_K \,] | \tau_k \in [0, \bar{\tau}_k], k = 1, \cdots, K \right\}$.

Definition 1. The system (1) is said to be asymptotically stable on $\prod_{k=1}^{K}[0, \bar{\tau}_k]$ if

$$\Psi(s, \tau_1, \cdots, \tau_K) \neq 0, \ \forall s \in \bar{\mathbf{C}}_+, \tau_k \in [0, \bar{\tau}],$$

where $\Psi(s, \tau_1, \cdots, \tau_K) := \det(sI_n - A - \sum_{k=1}^{K} A_k e^{-\tau_k s})$ is the characteristic function associated with system (1).

Compared with the single-delay case, the analysis of linear systems with multiple delays case is much more complicated. As a matter of fact, in the general non-commensurate delays case, this problem is $\mathcal{NP}$-hard (Toker and Ozbay, 1996). Consequently, it is very unlikely to find efficient algorithms to solve this problem exactly in the general case. Our objective is to find efficient conditions which are numerically tractable yet are not too conservative. Moreover, it turns out that the analysis of the stability regions is rather complex even for the scalar differential equation involving only two delays (Hale and Huang, 1993). In particular, this system may have multiple "maximum" stability margins, for instance, the system may have stability margins which are unbounded in two different directions (Niculescu and Chen, 1999). This phenomenon complicates our analysis, and herein, we introduce the definition of *actual delay margin with a proportionality ratio vector.*

Definition 2. The **actual delay margin $\bar{\tau}^*$ for the system (1) with the proportionality ratio vector** $\nu := [\, l_1 \ \cdots \ l_K \,]$,where $l_k > 0$, $1 \leq k \leq K$,and $\max_{1 \leq k \leq K}(l_k) = 1$, is defined by

$$\bar{\tau}^* := \nu \bar{\tau}_0^* = [\, l_1 \bar{\tau}_0^* \ \cdots \ l_k \bar{\tau}_0^* \,]$$

where

$$\bar{\tau}_0^* := \sup \left\{ \bar{\tau}_0 | (1) \text{ is a. s. on } \prod_{k=1}^{K} [0, l_k \bar{\tau}_0] \right\}.$$

The system (1) is said to be **delay-dependent** if $\bar{\tau}_0^*$ is finite, and **delay-independent** otherwise.

Next, we give the definition of the *degree of conservatism* of a stability criterion. It provides a quantitative measure for the performance of the criterion in the sense of conservatism.

Definition 3. Suppose $\mathcal{P}$ is a condition that ensures that (1) is asymptotically stable on $\prod_{k=1}^{K}[0, \bar{\tau}_k]$. If (1) is delay-dependent with actual delay margin $\bar{\tau}^* = \nu \bar{\tau}_0^*$ with the proportionality ratio vector $\nu := [\, l_1 \ \cdots \ l_K \,]$,where $l_k > 0$, $1 \leq k \leq K$,and $\max_{1 \leq k \leq K}(l_k) = 1$, then the **degree of conservatism (d.o.c.) of** P is defined by

$$d.o.c.(\nu) := \frac{\bar{\tau}_0^* - \bar{\tau}_{\mathcal{P}_0}^*}{\bar{\tau}_0^*}$$

where

$$\bar{\tau}_{\mathcal{P}_0}^* := \sup \left\{ \bar{\tau}_0 | \mathcal{P} \text{ is true on } \prod_{k=1}^{K} [0, l_k \bar{\tau}_0] \right\}.$$

Moreover, $\bar{\tau}_{\mathcal{P}}^* := \nu \bar{\tau}_{\mathcal{P}_0}^* = [\, l_1 \bar{\tau}_{\mathcal{P}_0}^* \ \cdots \ l_K \bar{\tau}_{\mathcal{P}_0}^* \,]$ is said to be the **delay margin provided by P with the same proportionality ratio vector ν.**

Definition 4. Consider a linear, time-invariant (finite-dimensional) system $G(s)$ interconnected with an uncertain block $\Delta \in \underline{\Delta}$ ($\underline{\Delta}$ is a set of linear time-invariant stable systems), as shown in Figure 1, denoted as $\sum[G(s), \Delta(s)]$. Then the system is said to be **robustly stable** if $G(s)$ is internally stable, the interconnection is well-posed and it remains internally stable for all $\Delta \in \underline{\Delta}$.

Now, we decompose $A_k = H_k F_k$ where $H_k \in \Re^{n \times q_k}, F_k \in \Re^{q_k \times n}$ have full rank, and denote $H := [\, H_1 \ \cdots \ H_K \,]$,and $F := [\, F_1^T \ \cdots \ F_K^T \,]^T$.

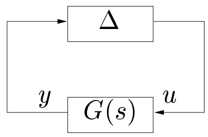

Fig. 1. An interconnection system.

Our principle tool for the stability analysis of (1), is the zero exclusion condition principle, which is based on the value set analysis of the delay elements $e^{-\tau_k s}$.

Lemma 1. The system (1) is asymptotically stable on $\prod_{k=1}^{K}[0,\bar{\tau}_k]$ if and only if

$$\Psi(j\omega,\tau_1,\cdots,\tau_K)\neq 0,\ \forall\omega\geq 0,\tau_k\in[0,\bar{\tau}_k].$$

Proof. This follows from the work of (Datko, 1978). ■

Applying this lemma, we have the following stability condition.

Lemma 2. (**Zero Exclusion Condition**) The system (1) is asymptotically stable on $\prod_{k=1}^{K}[0,\bar{\tau}_k]$ if and only if

$$\det[I_q - G(j\omega)\Phi(j\tau\omega)]\neq 0,\ \forall\omega\geq 0,\tau_k\in[0,\bar{\tau}_k], \qquad (2)$$

where $q=q_1+\cdots+q_K$, $G(s):=F(sI_n-\bar{A})^{-1}H$ and $\Phi(\tau,s) := \mathrm{diag}\{\phi(\tau_1 s)I_{q_1},\cdots,\phi(\tau_K s)I_{q_K}\}$, $\phi(\tau_k s)=e^{-\tau_k s}-1$.

Examining the stability of (1) by checking the condition (2) directly is nontrivial, because (2) implies solving a transcendental equation. An indirect but intuitive approach of examining whether (2) holds, is to cover $\Phi(\tau,j\omega)$ with another set $\underline{\Phi}(\omega)$, that is, to find a value set $\underline{\Phi}(\omega)$ such that

$$\Phi(\tau,j\omega)\in\underline{\Phi}(\omega),\ \forall\omega\geq 0,\tau_k\in[0,\bar{\tau}_k].$$

Then (2) holds if

$$\det[I_q - G(j\omega)\Delta(j\omega)]\neq 0,\ \forall\omega\geq 0,\Delta(j\omega)\in\underline{\Phi}(\omega).$$

which is satisfied if the interconnection $\sum[G(s),\Delta(s)]$ (referred to as the comparison system in the sequel) is robustly stable. The conservatism of this approach mainly arises from the manner in which the covering set $\underline{\Phi}$ is chosen based on the properties of the delay element (Zhang *et al.*, 1999*a*).

In (Zhang *et al.*, 2000), the authors introduced a less conservative covering set for the delay element, by using Padé approximation, and provided sufficient conditions for (1) in the single delay case which guarantee an *a priori* degree of conservatism upper bound. Herein, we will generalize these results to the multiple-delay case.

We consider the *mth* order ($m\geq 3$) *diagonal* Padé approximation to e^{-s} as follows (Perron, 1957; Saff and Varga, 1975)

$$R_m(s)=\frac{N_m(s)}{N_m(-s)}$$

where

$$N_m(s)=\sum_{l=0}^{m}\frac{(2m-l)!(-s)^l}{l!(m-l)!}.$$

Now, we define the following sets:

$$\Omega_A(\omega,\bar{\tau}_k) := \{e^{-j\tau_k\omega}|\tau_k\in[0,\bar{\tau}_k]\},$$
$$\Omega_B(\omega,\bar{\tau}_k) := \{R_m(j\theta_k\alpha_m\omega)|\theta_k\in[0,\bar{\tau}_k]\},$$
$$\Omega_C(\omega,\bar{\tau}_k) := \{R_m(j\theta_k\omega)|\theta_k\in[0,\bar{\tau}_k]\}.$$

where $\alpha_m := \frac{\omega_c}{2\pi}$, and ω_c is the phase crossover frequency of $R_m(j\omega)$ at the -2π line:

$$\omega_c := \min\{\omega>0|R_m(j\omega)=1\}.$$

It can be found that for $m=3,4$ and 5, $\alpha_m\approx 1.2329, 1.0315$, and 1.00363 respectively.

The function $R_m(s)$ and the above sets have several important properties which are summarized as the following lemma.

Lemma 3. (Zhang *et al.*, 2000) For every integer $m\geq 3$, the following statements hold:

(a) All poles of $R_m(s)$ are in the open left half complex plane.

(b) $\Omega_C(\omega,\bar{\tau}_k)\subseteq\Omega_A(\omega,\bar{\tau}_k)\subseteq\Omega_B(\omega,\bar{\tau}_k)$, $\forall\omega\geq 0,\bar{\tau}_k>0$.

(c) $\lim_{\mathbf{m}\to\infty}\alpha_m=1$.

3. MAIN RESULTS

Now, we replace the delay elements $e^{-\tau_k s}$ with the real rational functions $R_m(\theta_k\alpha_m s)$ and $R_m(\theta_k s)$ and denote the resultant finite-dimensional interconnection systems as $\sum_B(\theta) := \sum(G(s),P_d(\alpha_m s))$ and $\sum_C(\theta) := \sum(G(s),P_d(s))$, respectively, where $P_d(s) := \mathrm{diag}\{[R_m(\theta_1 s)-1]I_{q_1},\cdots,[R_m(\theta_K s)-1]I_{q_K}\}$. The following theorem gives a *sufficient* condition for the stability of (1).

Theorem 1. The system (1) is asymptotically stable on $\prod_{k=1}^{K}[0,\bar{\tau}_k]$, if the comparison system $\sum_B(\theta)$ is robustly stable for $\theta\in\prod_{k=1}^{K}[0,\bar{\tau}_k]$.

Proof. This follows directly by using Lemma 2 and Lemma 3. ■

The following theorem presents a *necessary* condition for the stability of (1), which can be used to estimate the d.o.c. of a stability criterion. The proof of this theorem is rather technical and it is omitted due to limited space.

Theorem 2. If (1) is asymptotically stable on $\prod_{k=1}^{K}[0,\bar{\tau}_k]$, then $\sum_C(\theta)$ is robustly stable for $\theta\in\prod_{k=1}^{K}[0,\bar{\tau}_k]$.

The following result shows that the d.o.c. of Theorem 1 is bounded by a function of α_m.

Theorem 3. The d.o.c. of Theorem 1 with any proportionality ratio vector ν satisfies

$$d.o.c.(\nu) \leq \frac{\alpha_m - 1}{\alpha_m}. \qquad (3)$$

Moreover, $d.o.c.(\nu) \to 0$ as $m \to \infty$.

Proof. Let $\bar{\tau}^* = \nu\bar{\tau}_0^*$ be the actual delay margin of (1), and $\bar{\tau}_B^* = \nu\bar{\tau}_{B_0}^*$ be the delay margin provided by Theorem 1, both with the same proportionality ratio vector ν. Let $\bar{\tau}_C^* := \nu\bar{\tau}_{C_0}^*$, where

$$\bar{\tau}_{C_0}^* := \sup\left\{\bar{\tau}_0 | \sum_C(\theta) \text{ is a. s. } \forall\theta \in \prod_{k=1}^{K}[0, l_k\bar{\tau}_{C_0}^*]\right\}.$$

Then, clearly, we have $\bar{\tau}_C^* = \alpha_m\bar{\tau}_B^*$. In addition, from Theorem 2, $\sum_C(\theta)$ is robustly stable for $\theta \in \prod_{k=1}^{K}[0, \bar{\tau}_k]$ whenever (1) is asymptotically stable on $\prod_{k=1}^{K}[0, \bar{\tau}_k]$. Therefore,

$$\bar{\tau}_{C_0}^* \geq \bar{\tau}_0^*$$

which immediately yields (3). Finally, we have $d.o.c.(\nu) \to 0$ as $m \to \infty$, because $\lim_{\mathbf{m}\to\infty} \alpha_m = 1$. ■

Remark 1. For $k = 3, 4$ and 5, $\frac{\alpha_m - 1}{\alpha_m} \approx 18.9\%, 3.05\%$ and 0.361%,respectively. For higher order k, this bound can be further reduced. This bound depends only on the order of Padé approximation used. It is independent of $\bar{\tau}^*$, A and A_k, and hence the d.o.c. of Theorem 1 is guaranteed for any linear system with time-invariant delays.

Remark 2. The traditional manner in using Padé approximations for time-delay systems, such as (Wang and Hu, 1999), is to simply replace delay element $e^{-\tau_k s}$ with the approximations $R_m(\tau_k s)$, by assuming *small delays* and some dynamical properties (such as bandwidth) of the system, because the Padé approximations are accurate only when $\tau_k s$ are sufficiently small. As a matter of fact, Theorem 2 indicates that the resultant condition is only necessary, and hence it does not guarantee, in general, the stability of the original systems. Theorem 1 shows, however, that if we use the α_m-scaled version of the Padé approximations $R_m(\theta_k\alpha_m s)$,the robust stability of the obtained comparison system rigorously guarantees the stability of the time-delay system without assuming that the delays are small.

3.1 *The Singularity Phenomenon*

The comparison system $\sum_B(\theta)$ is free of delays, but it has real uncertainties θ_k. Some care must be taken when examining its robust stability, however. When some θ_k approaches 0, the system dynamics suffers a fundamental and abrupt change. In particular, the dynamics of $R_m(\theta_k\alpha_m s)$ vanish when θ_k becomes 0, causing the system degeneration. The uncertainties θ_k hence can be regarded as singular perturbations. The general singular perturbation theory (Khalil, 1996) considers only one singular perturbation, and hence can not be applied to this comparison system directly.

The singularity of the system obviously complicates our analysis. For the single delay case, the problem is much simpler, and (Zhang *et al.*, 2000) demonstrated that the delay margin can be calculated explicitly without incurring any additional conservatism. The question of whether a similar conclusion can be drawn for the multiple delays case is still open.

The standard μ−framework (Zhou *et al.*, 1996) can be used to examine the robust stability of the comparison system. However, the singularity issue must be carefully taken into account. Notice that the singularity occurs at small values of θ_k, which, according to the necessary condition Theorem 2, imply the effect of small delays in some sense. In a previous paper of the authors (Zhang *et al.*, 1999*b*), we demonstrated that the value sets of delay element $\frac{e^{-\tau_k s} - 1}{\bar{\tau}_k s}$ can be covered with various convex sets, such as a shifted disk, a filter, or both. These covering sets are more conservative than the covering set $\Omega_B(\omega, \bar{\tau}_k)$ we proposed above, but they do not introduce the singularity for small delays. Now, consider the covering using a filter. We suppose that $f(s)$ is real rational, stable, has minimum phase and relative degree 1, and

$$\left|\frac{e^{-j\tau\omega} - 1}{\bar{\tau}j\omega}\right| \leq |f(j\bar{\tau}\omega)|, \quad \forall\omega \geq 0, \tau \in [0, \bar{\tau}], \bar{\tau} > 0.$$

An example of the function $f(s)$ is given by

$$f(s) = \frac{2s + 7.0711}{s^2 + 4.5434s + 7.0711}.$$

Now, choose a small number $\varepsilon > 0$, and divide the set interval $[0, \bar{\tau}_k]$ into $[0, \bar{\tau}_k] = [0, \varepsilon] \cup [\frac{1}{\alpha_m}\varepsilon, \bar{\tau}_k]$. Correspondingly, the set $\prod_{k=1}^{K}[0, \bar{\tau}_k]$ is divided into 2^K subsets S_i. For each subset S_i, every uncertainty θ_k is either in $[\frac{1}{\alpha_m}\varepsilon, \bar{\tau}_k]$, or in $[0, \varepsilon]$. If θ_k is in $[0, \varepsilon]$,we replace the uncertain function $R_m(\theta_k\alpha_m s)$ into

$$s\varepsilon f(s\varepsilon)\delta_k^c(s)$$

where $\delta_k^c(s)$ is a complex uncertain parameter with $\|\delta_k^c\|_\infty \leq 1$. If θ_k is in $[\frac{1}{\alpha_m}\varepsilon, \bar{\tau}_k]$,we define a transformation $\theta_k = \frac{1}{2}(\bar{\tau}_k - \frac{1}{\alpha_m}\varepsilon)\delta_k^r + \frac{1}{2}(\bar{\tau}_k + \frac{1}{\alpha_m}\varepsilon)$, where $-1 \leq \delta_k^r \leq 1$ is a real uncertain parameter. It can be easily shown that $R_m(\theta_k\alpha_m s)I_{q_k}$ can be expressed as an linear fractional transformation (LFT) (Zhou *et al.*, 1996):

$$R_m(\theta_k\alpha_m s)I_{q_k} = \mathcal{F}_l(P_{\bar{\tau}_k}(s), \delta_k^r I_{mq_k}).$$

Now, pulling out all uncertainties δ_k^c or δ_k^r, we establish a standard interconnection $\sum(\bar{G}_i(s), \Delta_i)$. The robust stability on each subset S_i can then be examined by the small μ theorem. This approach involves 2^K frequency sweeping tests, and hence can be very tedious if K is large.

3.2 *A New LMI Delay-Dependent Stability Criterion*

Herein, we present an alternative approach to the previously discussed $\mu-$framework to the stability analysis. Our result is based upon a parameter-dependent Lyapunov matrix. Some additional conservatism may be introduced, but the resultant stability condition is formulated as a finite set of LMIs.

Let $(A_{P_k}, B_{P_k}, C_{P_k}, D_{P_k})$ be the minimal realization of $P_k(s) := [R_m(\alpha_m s) - 1]I_{q_k}$ and denote n_k as the order of A_{P_k}. For convenience, we let $\tilde{A} := \bar{A} + \sum_{k=1}^{K} H_k D_{P_k} F_k, B_k := B_{P_k} F_k$, and $C_k =: H_k C_{P_k}$. Also, we denote $\mathcal{F}_c := \prod_{k=1}^{K}[0, \bar{\tau}_k]$, and $\mathcal{F}_o := \prod_{k=1}^{K}(0, \bar{\tau}_k]$. The following theorem indicates that under an addition condition, the robust stability of $\sum_B(\theta)$ on $\mathcal{F}_o$ implies the robust stability on the closed set $\mathcal{F}_c$.

Theorem 4. $\sum_B(\theta)$ is asymptotically stable on $\mathcal{F}_o$ if and only if for every $\theta \in \mathcal{F}_o$ there exists a positive definite matrix $X(\theta)$ satisfying the Lyapunov inequality

$$A_L(\theta)^T X(\theta) + X(\theta) A_L(\theta) < -I,$$

where

$$A_L(\theta) := \begin{bmatrix} \tilde{A} & \theta_1^{-\frac{1}{2}} C_1 & \cdots & \theta_K^{-\frac{1}{2}} C_K \\ \theta_1^{-\frac{1}{2}} B_1 & \theta_1^{-1} A_{P_1} & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ \theta_K^{-\frac{1}{2}} B_K & 0 & 0 & \theta_K^{-1} A_{P_K} \end{bmatrix}$$

is the kernel of the closed loop system $\sum_B(\theta)$. If, in addition, the matrix $X(\theta)$ is bounded for all $\theta \in \mathcal{F}_o$, then $\sum_B(\theta)$ is asymptotically stable on $\mathcal{F}_c$.

Now, we present a delay-dependent stability condition for system (1) as follows, which is formulated as a (finite) set of LMIs.

Theorem 5. The system (1) is asymptotically stable on $\prod_{k=1}^{K}[0, \bar{\tau}_k]$, if there exist a constant $\varepsilon > 0$, positive definite matrices $Y_i \in \Re^{n \times n}, i = 0, \cdots, K$, and $X_j \in \Re^{n_k \times n_k}, j = 1, \cdots, K$, and matrices $W_k \in \Re^{n \times n_k}, k = 1, \cdots, K$, such that for all vertices $\bar{\tau}_c$ of the polytope $\prod_{k=1}^{K}[0, \bar{\tau}_k]$,

$$\Pi(\bar{\tau}_c) < 0 \tag{4}$$

and

$$X(\bar{\tau}_c) > 0 \tag{5}$$

where

$$\Pi(\theta) := \begin{bmatrix} \Pi_{1,1}(\theta) & \Pi_{1,2}(\theta) & \cdots & \Pi_{1,K+1}(\theta) \\ * & \Pi_{2,2}(\theta) & 0 & 0 \\ * & * & \ddots & 0 \\ * & * & * & \Pi_{K+1,K+1}(\theta) \end{bmatrix},$$

$$X(\theta) := \begin{bmatrix} Y(\theta) & \theta_1^{\frac{1}{2}} W_1 & \cdots & \theta_K^{\frac{1}{2}} W_K \\ * & X_1 & 0 & 0 \\ * & * & \ddots & 0 \\ * & * & * & X_K \end{bmatrix}$$

with $\Pi_{1,1}(\theta) := Y(\theta)\tilde{A} + \tilde{A}^T Y(\theta) + \sum_{k=1}^{K}(W_k B_k + B_k^T W_k^T) + \varepsilon I_n, \Pi_{1,k+1}(\theta) := Y(\theta) C_k + W_k A_{P_k} + \theta_k \tilde{A}^T W_k + B_k^T X_k$, $\Pi_{k+1,k+1}(\theta) := \theta_k W_k^T C_k + \theta_k C_k^T W_k + X_k A_{P_k} + A_{P_k}^T X_k + \varepsilon I_{n_k}, k = 1, \cdots, K$, and $Y(\theta) := Y_0 + \sum_{k=1}^{K} \theta_k Y_k$. $\Theta(\theta) := Y(\theta) - \sum_{k=1}^{K} \theta_k W_k X_k^{-1} W_k^T$.

Proof. First, we notice that $\Pi(\theta)$ is convex in θ because it is affine in θ. Hence (4) implies that

$$\Pi(\theta) < 0, \quad \forall \theta \in \mathcal{F}_c. \tag{6}$$

Using the properties of Shur Complement, Eq. (5) is equivalent to

$$\Theta(\theta_c) > 0,$$

where $\Theta(\theta) := Y(\theta) - \sum_{k=1}^{K} \theta_k W_k X_k^{-1} W_k^T$. Since $\Theta(\theta)$ is convex in θ,the above inequality implies that

$$\Theta(\theta) > 0, \quad \forall \theta \in \mathcal{F}_c,$$

which is equivalent to

$$X(\theta) > 0, \ \forall \theta \in \mathcal{F}_c.$$

For any $\theta \in \mathcal{F}_o$, multiplying (6) on both sides by $E_\theta := \text{diag}\{I_n, \theta_1^{-\frac{1}{2}} I_{n_1}, \cdots, \theta_K^{-\frac{1}{2}} I_{n_K}\}$ yields

$$E_\theta \Pi(\theta) E_\theta < 0$$

which immediately gives

$$A_L(\theta)^T X(\theta) + X(\theta) A_L(\theta) < -\varepsilon I \tag{7}$$

where $A_L(\theta)$ is the kernel of the closed loop system $\sum_B(\theta)$. Since $X(\theta)$ is bounded for all $\theta \in \mathcal{F}_o$, by Theorem 4, $\sum_B(\theta)$ is also asymptotically stable on $\mathcal{F}_c$. Hence, by Theorem 1, (1) is asymptotically stable on $\mathcal{F}_c$. ∎

4. CONCLUSION

The asymptotic stability of the linear system with multiple time-delays is addressed. We established inner and outer inclusion relation for the value sets of the irrational delay element and a parameter-dependent Padé approximation. We then demonstrated that replacing directly the delay elements with the Padé approximation leads to a necessary stability condition for the time-delay system, while replacing the delay elements with an α_m-scaled Padé approximation results in a sufficient condition. This sufficient condition involves examining the robust stability of a delay-free comparison system with real parameter uncertainties. This condition guarantees an *a priori* upper bound on the degree of conservatism which only depends on the order of the Padé approximation used. Finally, we presented a new, LMI-based stability criterion for the time-delay system. The result of this paper generalized the previous result of the authors (Zhang *et al.*, 2000) to the multiple delays case.

5. REFERENCES

Datko, R. (1978). "A procedure for determination of the exponential stability of certain differential-difference equations.". *Quarterly of Applied Mathematics* pp. 279–292.

Dugard, L. and E.I. Verriest (Eds) (1997). *Stability and Control of Time-delay Systems.* Springer-Verlag.

Gu, K. (1999). "Additional dynamics in transformed time-delay systems". In: *Proc. 38th IEEE Conf. Dec. Contr.*. pp. 4673–4677.

Hale, J. K. and W. Huang (1993). "Global geometry of the stable regions for two delay differential equations". *J. Math. Anal. Appl* **178**, 344–362.

Khalil, H. K. (1996). *Nonlinear systems.* Prentice-Hall.

Kharitonov, V. L. and D. A. Melchor (1999). "Some remarks on transformations used for stability and robust stability analysis of time-delay systems". In: *Proc. 38th IEEE Conf. Dec. Contr.*. pp. 1142–1147.

Li, X. and C. E. de Souza (1996). "Robust stabilization and H_∞ control of uncertain linear time-delay systems". *13th IFAC World Congress* pp. 113–118.

Niculescu, S.-I., A. T. Neto, J.-M. Dion and L. Dugard (1995). "Delay-dependent stability of linear systems with delayed state: An LMI approach". In: *Proc. 34th IEEE Conf. Dec. Contr.*. pp. 1495–1497.

Niculescu, S.-I. and J. Chen (1999). "Frequency sweeping tests for asymptotic stability: A model transformation for multiple delays". In: *Proc. 38th IEEE Conf. Dec. Contr.*. pp. 4678–4683.

Park, P. (1999). "A delay-dependent stability criterion for systems with uncertain time-invariant delays". *IEEE Transactions on Automatic Control* **AC-44**(4), 876–877.

Perron, O. (1957). *Die Lehre von den Kettenbruchen.* 3rd ed.. Stuugart: Teubner.

Saff, E. B. and R. S. Varga (1975). "On the zeros and poles of Padé approximants to e^z". *Numer. Math.* **25**, 1–14.

Toker, O. and H. Ozbay (1996). "Complexity issues in robust stability of linear delay-differential systems". *Math., Contr., Signals, Syst.* **9**, 386–400.

Wang, Z. and H. Hu (1999). "Robust stability test for dynamic systems with short delays by using Padé Approximation". *Nonlinear Dynamics* **18**, 275–287.

Zhang, J., C. R. Knospe and P. Tsiotras (1999*a*). "A unified approach to time-delay system stability via scaled small gain". In: *Proc. American Control Conference.* pp. 307–308.

Zhang, J., C. R. Knospe and P. Tsiotras (1999*b*). "Toward less conservative stability analysis of time-delay systems". In: *Proc. 38th IEEE Conf. Dec. Contr.*. pp. 2017–2022.

Zhang, J., C. R. Knospe and P. Tsiotras (2000). "Stability of linear time-delay systems: A delay-dependent criterion with a tight conservatism bound". In: *Proc. 2000 American Control Conference.*

Zhou, K., J. C. Doyle and K. Glover (1996). *Robust and Optimal Control.* Prentice-Hall. Englewood-Cliffs.