STABILITY CRITERIA FOR LPV TIME-DELAYED SYSTEMS: THE DELAY-INDEPENDENT CASE [1]

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Abstract: This paper analyzes the stability of Linear Parameter Varying (LPV) time-delayed systems. Several delay-independent stability conditions are presented, which are derived using appropriately selected Lyapunov-Krasovskii functionals. Depending on the system parameter dependence, these functionals can be selected to obtain increasingly non-conservative results. Using relaxation methods and gridding techniques these stability tests can be cast as Linear Matrix Inequalities (LMI's). These LMI's can be solved efficiently using available software.

Keywords: Time-delay, stability analysis, linear systems, parameters.

1. LPV TIME-DELAY SYSTEMS

Several linear time-delayed systems depend on parameters whose values are not known a priori, but they can be measured or estimated on-line. Assuming that the parameters enter the system dynamics *without delay*, an LPV time-delayed system has the form

$$\dot{x}(t) = A(\gamma(t))\,x(t) + A_d(\gamma(t))\,x(t-\tau) \qquad (1)$$

In (1) τ is a constant, unknown delay with $\tau \in [0,\bar{\tau}]$ and γ is a system parameter that it is assumed to belong to a known polytope Γ. Often, it is also assumed that the rate of γ is known to belong to a polytope, Γ_r. The scope of this paper is to derive conditions that will guarantee stability for (1) for all $(\gamma,\dot{\gamma}) \in \Gamma \times \Gamma_r$, and all $\tau \in [0,\bar{\tau}]$. In this paper the discussion is restricted to scalar parameters, thus Γ and Γ_r are closed intervals of the real line. The results can be generalized to the case when γ is a vector, but the computations become more cumbersome.

In this paper stability conditions are derived only for the case when $\tau \in [0,\infty)$. This case is referred to in the literature as *delay-independent* stability, since stability is ensured for any amount of delay. If the stability conditions hold for $\tau \in [0,\bar{\tau}]$ with $\bar{\tau} < \infty$ then the stability is *delay-dependent* (Niculescu *et al.*, 1997*a*). Delay-dependent stability conditions for LPV time-delayed systems are given in (Zhang and Tsiotras, 2000). Other results on LPV time-delayed systems have been presented in (Wu and Grigoriadis, 1997).

In the proofs Lyapunov methods are used to determine stability. It is therefore needed to characterize the positive definite functionals for Linear Time Varying, time-delayed systems.

1.1 *Positive definite functionals*

Linear Parameter Varying systems can be considered as a special class of Linear Time-Varying (LTV) systems. The proofs of stability for LTV (time-delayed) systems hinge on the following well-known facts that are repeated here for completeness. These results can be found, for example, in (Verriest, 1994; Kolmanovskii and Myshkis, 1992).

Let C_τ denote the set of continuous functions defined over the interval $[-\tau,0]$ and let $V : \mathbb{R}_+ \times C_\tau \to \mathbb{R}_+$ be a continuous functional such that $V(t,0) = 0$. Let also Ω denote the class of scalar nondecreasing continuous functions α such that $\alpha(r) > 0$ for $r > 0$ and $\alpha(0) = 0$. The functional $V(t,\psi)$ is called *positive definite (negative definite)* if there exist a function $\alpha \in \Omega$ such that $V(t,\psi) \geq \alpha(|\psi(0)|)$ (respectively, $V(t,\psi) \leq$

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$-\alpha(|\psi(0)|)$ for all $t \in \mathbb{R}$ and $\psi \in C_\tau$. It is said to have an *infinitesimal upper bound* if $|V(t,\psi)| \leq \alpha(\sup_t |\psi(t)|)$. The following fact provides the main tool used to show (global) asymptotic stability in this paper.

Theorem 1.1. ((Verriest, 1994; Niculescu *et al.*, 1997*a*)). Given some $\tau > 0$, assume that there exist a positive definite continuous functional $V : \mathbb{R}_+ \times C_\tau \to \mathbb{R}_+$, with infinitesimal upper bound whose derivative $\dot{V}$ is a negative definite functional. Then the trivial solution of the LTV, time-delayed system $\dot{x}(t) = A(t)x(t) + A_d(t)x(t-\tau)$ is (globally) uniformly asymptotically stable.

The following lemma is useful for recognizing positive definite functionals, as the ones used in this paper. In the following, x_t denotes the function with domain $[-\tau, 0]$ that coincides with x on the interval $[t-\tau, t]$, i.e., $x_t : [-\tau, 0] \to \mathbb{R}^n$ such that $x_t(\theta) = x(t+\theta)$ for $\theta \in [-\tau, 0]$.

Lemma 1.1. Let Γ be a compact interval of the real line. Consider the continuous functional $V : \mathbb{R}_+ \times C_\tau \to \mathbb{R}_+$ defined by

$$V(t,x_t) = x^T(t)P(\gamma(t))x(t) + \int_{-\tau}^{0} x^T(t+\theta)Q(\gamma(t,\theta))x(t+\theta)\,\mathrm{d}\theta$$

where $\gamma(t) \in \Gamma$ for all $t \geq 0$ and $P(\gamma) > 0$ and $Q(\gamma) > 0$ for all $\gamma \in \Gamma$. Then V is a positive definite functional with an infinitesimal upper bound.

Proof. Let $x_t(\theta) = x(t+\theta)$, then $x_t \in C_\tau$ for all $t \geq 0$. For each $\gamma \in \Gamma$ we have that $V(t,x) \geq \lambda_{\min}[P(\gamma)]|x(t)|^2$. Let $c_1 = \min_\gamma \lambda_{\min}[P(\gamma)] > 0$. This always exists since Γ is compact. Therefore, $V(t,x_t) \geq c_1|x_t(0)|^2$ and V is positive definite with $\alpha(r) = c_1 r$. To show that V has an infinitesimal upper bound, notice that

$$\begin{aligned}
V(t,x_t) &= x_t^T(0)P(\gamma(t))x_t(0) + \int_{-\tau}^{0} x_t^T(\theta)Q(\gamma(t,\theta))x_t(\theta)\,\mathrm{d}\theta \\
&\leq \max_\gamma \lambda_{\max}[P(\gamma)]\,|x_t(0)|^2 + \max_\theta |x_t(\theta)|^2 \int_{-\tau}^{0} \lambda_{\max}[Q(\gamma(t,\theta)]\,\mathrm{d}\theta \\
&\leq c_2 \max_\theta |x_t(\theta)|^2
\end{aligned}$$

where $c_2 = \max_\gamma \lambda_{\max}[P(\gamma)] + \tau \max_\gamma \lambda_{\max}[Q(\gamma)] > 0$. ∎

2. DELAY-INDEPENDENT STABILITY

The first four results in the paper deal with systems having a specific polynomial dependence on the parameter γ. In the following, the dependence on t has been suppressed for notational simplicity.

Theorem 2.1. Consider the LPV time-delayed system (1) and assume that $A(\gamma) = A_0 + \gamma A_1 + \gamma^2 A_2$ and $A_d(\gamma) = A_{d0} + \gamma A_{d1}$ where $\gamma \in \Gamma$, with Γ any compact sub-interval of $\mathbb{R}$. If there exist constant, positive-definite matrices P and Q such that

$$M_1 = \begin{bmatrix} A_0^T P + PA_0 + Q & PA_{d0} & PA_1 \\ A_{d0}^T P & -Q & A_{d1}^T P \\ A_1^T P & PA_{d1} & A_2^T P + PA_2 \end{bmatrix} < 0$$

then system (1) is asymptotically stable for any value of parameter $\gamma \in \Gamma$ and for any $\tau \in [0, \infty)$.

Proof. Consider the following Lyapunov-Krasovskii functional

$$V(t,x) = x^T(t)Px(t) + \int_{-\tau}^{0} x^T(t+\theta)Qx(t+\theta)\,\mathrm{d}\theta$$

Clearly, by Lemma 1.1 V is positive definite and has an infinitesimal upper bound. The derivative of V along the trajectories of (1) is

$$\begin{aligned}
\dot{V}(t) &= 2x^T(t)P(A_0 + \gamma A_1 + \gamma^2 A_2)x(t) + x^T(t)Qx(t) \\
&\quad + 2x^T(t)P(A_{d0} + \gamma A_{d1})x(t-\tau) \\
&\quad - x^T(t-\tau)Qx(t-\tau)
\end{aligned}$$

The last equation can be written as

$$\dot{V}(t) = \begin{bmatrix} x(t) \\ x(t-\tau) \\ \gamma x(t) \end{bmatrix}^T M_1 \begin{bmatrix} x(t) \\ x(t-\tau) \\ \gamma x(t) \end{bmatrix} < 0 \quad (2)$$

Since Γ is compact the previous inequality holds uniformly for all $\gamma \in \Gamma$. Hence $\dot{V}$ is negative definite and from Theorem 1.1 the system (1) is asymptotically stable (Kolmanovskii and Myshkis, 1992). ∎

In Theorem 2.1 the set Γ can be arbitrarily large. Hence, the conditions of the theorem guarantee that system (1) is stable for any (bounded) values of the parameter $\gamma \in \mathbb{R}$. It requires, however, that $A_2^T P + PA_2 < 0$, and $A_0^T P + PA_0 + Q < 0$, i.e., the matrices A_0 and A_2 must be Hurwitz. This condition induces unnecessary conservatism. Assuming that the parameter γ is known to belong to a compact interval, the conditions for delay-independent stability for (1) can be relaxed. The following two lemmas will be helpful in the sequel.

Lemma 2.1. Consider the following parameter dependent matrix $F(\gamma) = \gamma^2 F_2 + \gamma F_1 + F_0$ where $\gamma \in [\underline{\gamma}, \overline{\gamma}]$. If $F_2 \geq 0$, then $F(\gamma)$ is a convex, matrix-valued function, that is,

$$\lambda F(\gamma_1) + (1-\lambda)F(\gamma_2) \geq F(\lambda\gamma_1 + (1-\lambda)\gamma_2) \quad (3)$$

for all $\gamma_1, \gamma_2 \in [\underline{\gamma}, \overline{\gamma}]$ and any scalar $0 \leq \lambda \leq 1$. If $F_2 > 0$ then $F(\gamma)$ is a strictly convex, matrix-valued function, i.e., (3) is satisfied with strict inequality for all $0 < \lambda < 1$. Moreover, if $F_2 \geq 0$ and $F(\gamma^\#) < 0$ for $\gamma^\# \in \{\underline{\gamma}, \overline{\gamma}\}$, then $F(\gamma) < 0$ for all $\gamma \in [\underline{\gamma}, \overline{\gamma}]$.

Proof. The proof is straightforward and thus, omitted. ∎

Lemma 2.2. Consider the following parameter dependent matrix

$$F(\gamma_1,\gamma_2) = \gamma_1^2 F_2 + \gamma_1 F_1 + F_0(\gamma_2)$$
$$F_0(\gamma_2) = F_{01} + \gamma_2 F_{02}$$

where $\gamma_i \in [\underline{\gamma}_i, \overline{\gamma}_i] = \Gamma_i$ for $i = 1,2$. Let $\Gamma_i^\# = \{\underline{\gamma}_i, \overline{\gamma}_i\}$ denote the vertices of Γ_i for $i = 1,2$. If $F_2 > 0$ and $F(\gamma_1^\#, \gamma_2^\#) < 0$ for $(\gamma_1^\#, \gamma_2^\#) \in \Gamma_1^\# \times \Gamma_2^\#$ then $F(\gamma_1,\gamma_2) < 0$ for all $(\gamma_1,\gamma_2) \in \Gamma_1 \times \Gamma_2$.

Proof. For any $\gamma_i \in \Gamma_i$ one can find $0 \le \lambda_i \le 1$, such that $\gamma_i = \lambda_i \underline{\gamma}_i + (1-\lambda_i)\overline{\gamma}_i$, for $i = 1,2$. A straightforward computation shows that

$$\begin{aligned} F(\gamma_1,\gamma_2) \le{}& \lambda_1\lambda_2 F(\underline{\gamma}_1, \underline{\gamma}_2) + \lambda_1(1-\lambda_2)F(\underline{\gamma}_1,\overline{\gamma}_2) \\ &+ \lambda_2(1-\lambda_1)F(\overline{\gamma}_1,\underline{\gamma}_2) \\ &+ (1-\lambda_1)(1-\lambda_2)F(\overline{\gamma}_1,\overline{\gamma}_2) \end{aligned}$$

Since $F(\gamma_1^\#,\gamma_2^\#) < 0$ for all $\gamma_i^\# \in \Gamma_1^\#$ it follows immediately that $F(\gamma_1,\gamma_2) < 0$ for all $(\gamma_1,\gamma_2) \in \Gamma_1 \times \Gamma_2$. ■

The next result gives sufficient conditions for delay-independent stability using a parameter dependent Lyapunov function.

Theorem 2.2. Consider the LPV time-delayed system (1) and $\gamma \in \Gamma = [\underline{\gamma}, \overline{\gamma}]$. Consider matrix-valued functions $P : \Gamma \to \mathbb{R}^{n\times n}$ and $Q : \Gamma \to \mathbb{R}^{n\times n}$ such that

$$P(\gamma) > 0, \quad Q(\gamma) > 0, \quad \forall\, \gamma \in \Gamma \tag{4}$$

and

$$M_2(\gamma_1,\gamma_2) = \begin{bmatrix} \begin{pmatrix} P(\gamma_1)A(\gamma_1) \\ +A^T(\gamma_1)P(\gamma_1) \\ +Q(\gamma_1) \end{pmatrix} & P(\gamma_1)A_d(\gamma_1) \\ A_d^T(\gamma_1)P(\gamma_1) & -Q(\gamma_2) \end{bmatrix} < 0 \tag{5}$$

for all $\gamma_i \in \Gamma,\ i = 1,2$. Then the system (1) is asymptotically stable for all $\gamma \in \Gamma$ and $\tau \in [0,\infty)$.

Proof. Consider the following Lyapunov-Krasovskii functional $V : \mathbb{R}_+ \times C_\tau \to \mathbb{R}_+$

$$V(t,x_t) = x^T(t)P(\gamma(t))x(t) + \int_{t-\tau}^{t} x^T(\theta)Q(\gamma(\theta))x(\theta)\, d\theta$$

where $P(\gamma)$ and $Q(\gamma)$ are defined as (4). From (4) and Lemma 1.1, it follows that V is positive definite with an infinitesimal upper bound. The derivative of V along the trajectories of (1) is

$$\begin{aligned} \dot V(t,x_t) ={}& 2x^T(t)P(\gamma(t))A(\gamma(t))x(t) \\ &+2x^T(t)P(\gamma(t))A_d(\gamma(t))x(t-\tau) \\ &+x^T(t)Q(\gamma(t))x(t) \\ &- x^T(t-\tau)Q(\gamma(t-\tau))x(t-\tau) \end{aligned}$$

or

$$\dot V(t,x_t) = \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}^T M_2(\gamma_1,\gamma_2) \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix} \tag{6}$$

where $\gamma_1 = \gamma(t)$ and $\gamma_2 = \gamma(t-\tau)$. Inequality (5) implies that the matrix $M_2(\gamma_1,\gamma_2)$ is negative definite for all $\gamma_1,\gamma_2 \in \Gamma$. Since Γ is compact, then $-\dot V(t,x_t) > -\min_{\gamma_1,\gamma_2} \lambda_{\max}[M_2(\gamma_1,\gamma_2)]\,(|x(t)|^2 + |x(t-\tau)|^2) > c\,|x(t)|^2$ where $c = -\min_{\gamma_1,\gamma_2} \lambda_{\max}[M_2(\gamma_1,\gamma_2)] > 0$ and the system (1) is asymptotically stable. ■

Equations (4) and (5) represent an infinite dimensional system of Linear Matrix Inequalities. A common way to reduce to a finite set of LMI's is to use gridding of the parameter space. According to this approach, one selects a set of basis functions $f_i(\gamma),\ (i = 1,2,...n_1)$ and $g_i(\gamma),\ (i = 1,2,...n_2)$ and expands P and Q in terms of these basis functions as

$$P(\gamma) = \sum_{i=1} P_i f_i(\gamma) \quad \text{and} \quad Q(\gamma) = \sum_{j=1}^{n_2} Q_j g_j(\gamma)$$

One then seeks matrices $P_i,\ (i = 1,2,\ldots,n_1)$ and $Q_j,\ (j = 1,2,\ldots,n_2)$ such that $\sum_{i=1}^{n_1} P_i f_i(\gamma) > 0$ and $\sum_{j=1}^{n_2} Q_j g_j(\gamma) \ge 0$ for all $\gamma \in \Gamma$ and

$$\begin{bmatrix} \begin{pmatrix} \sum_{i=1}^{n_1} P_i f_i(\gamma_1)A(\gamma_1)+ \\ A^T(\gamma_1)\sum_{i=1}^{n_1} P_i f_i(\gamma_1)+ \\ \sum_{j=1}^{n_2} Q_j g_j(\gamma_1) \end{pmatrix} & \sum_{i=1}^{n_1} P_i f_i(\gamma_1)A_d(\gamma_1) \\ A_d^T(\gamma_1)\sum_{i=1}^{n_1} P_i f_i(\gamma_1) & -\sum_{j=1}^{n_2} Q_j g_j(\gamma_2) \end{bmatrix} < 0$$

for all $\gamma_i \in \Gamma,\ i = 1,2$.

Theorems 2.1 and 2.2 did not consider time variations of the parameter γ. In that respect, the tests provided by Theorems 2.1 and 2.2 can be potentially very conservative, since they ensure – in principle – stability for arbitrarily fast variations of γ. On the other hand, if indeed γ varies very fast then it is expected that $\gamma(t)$ and $\gamma(t-\tau)$ can be treated independently, at least for large enough delays. This motivates our choice of the Lyapunov function and the treatment of $\gamma(t)$ and $\gamma(t-\tau)$ in the stability test (5) as independent parameters. It also justifies the use of delay-independent stability tests. If, on the other hand, $\dot\gamma$ is bounded by a (relatively) small, known, upper bound the previous test may be conservative. Generally speaking, for small variation rates, delay-dependent stability should be used. One possible method to approach this problem is to eliminate, say, γ_2 using the fact that $\gamma_1 = \gamma_2 + \dot\gamma(\xi)\tau$, for some $\xi \in [t-\tau, t]$ and then take into account any known bounds for $\dot\gamma$. The resulting stability test is then, clearly, delay-dependent.

Next, stability tests are derived that take explicitly into account the knowledge of the bound of the rate variation of the parameter . Since the bound on $\dot\gamma$ may be arbitrarily small, one may no longer treat $\gamma(t)$ and $\gamma(t-\tau)$ as independent. In fact, for $\dot\gamma = 0$, one has that $\gamma(t) = \gamma(t-\tau)$ for all $t \ge 0$. It should be pointed out, however, that according to the previous discussion these results (since they are delay-independent) may be overly conservative.

2.1 *Tests for Bounded Parameter Variation Rates*

The following theorem uses the knowledge of bounds of $\dot{\gamma}$ to provide less conservative stability tests than the ones in Theorems 2.1 and 2.2.

Theorem 2.3. Consider the LPV time-delayed system (1) with $\gamma \in \Gamma = [\underline{\gamma}, \overline{\gamma}]$ and $\dot{\gamma} \in \Gamma_r = [\underline{\dot{\gamma}}, \overline{\dot{\gamma}}]$. Consider matrix valued functions $P : \Gamma \to \mathbb{R}^{n\times n}$ and $Q : \Gamma \to \mathbb{R}^{n\times n}$ such that $P(\gamma) > 0, \quad Q(\gamma) > 0$ for all $\gamma \in \Gamma$ and

$$M_3(\gamma_1,\gamma_2,\nu) = \begin{bmatrix} \begin{pmatrix} P(\gamma_1)A(\gamma_1) + (\ )^T \\ +Q(\gamma_1) + \dfrac{\partial P}{\partial \gamma}\nu \end{pmatrix} & P(\gamma_1)A_d(\gamma_1) \\ A_d^T(\gamma_1)P(\gamma_1) & -Q(\gamma_2) \end{bmatrix} < 0 \quad (7)$$

for all $\gamma_1,\gamma_2 \in \Gamma$ and $\nu \in \Gamma_r$. Then the system (1) is asymptotically stable for all $(\gamma,\dot{\gamma}) \in \Gamma \times \Gamma_r$ and $\tau \in [0,\infty)$.

Proof. Consider the following Lyapunov-Krasovskii functional

$$V(t,x_t) = x^T(t)P(\gamma(t))x(t) + \int_{t-\tau}^{t} x^T(\theta)Q(\gamma(\theta))x(\theta)\ d\theta$$

Similarly to Theorem 2.2, V is positive definite with infinitesimal upper bound. The derivative of V along the trajectories of (1) is

$$\begin{aligned}\dot{V}(t,x_t) &= 2x^T(t)P(\gamma(t))A(\gamma(t))x(t) + x^T(t)\frac{\partial P}{\partial \gamma}\dot{\gamma}x(t) \\ &+ 2x^T(t)P(\gamma(t))A_d(\gamma(t))x(t-\tau) \\ &+ x^T(t)Q(\gamma(t))x(t) - x^T(t-\tau)Q(\gamma(t-\tau))x(t-\tau)\end{aligned}$$

or

$$\dot{V}(t,x_t) = \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}^T M_3(\gamma(t),\gamma(t-\tau),\dot{\gamma}(t)) \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}$$

Inequality (7) implies that M_3 is negative definite for all $\gamma \in \Gamma$ and $\dot{\gamma} \in \Gamma_r$. Since Γ and Γ_r are compact, $-\dot{V}(t,x_t) > -c(|x(t)|^2 + |x(t_\tau)|^2) > c|x(t)|^2$ where $c = -\min_{\gamma_1,\gamma_2,\nu} \lambda_{\max}[M_3(\gamma_1,\gamma_2,\nu)] > 0$ and thus, the system (1) is asymptotically stable. ■

2.2 *A Relaxation Approach*

The previous result, as well as Theorem 2.2, requires gridding of the parameter space $\Gamma \times \Gamma_r$. This can be cumbersome since for fine gridding, several matrix inequalities have to be solved simultaneously. In certain cases, when the parameter dependence in the matrices A and A_d is relatively simple (low order polynomial) gridding may be avoided using multi-convexity arguments and relaxation methods at the expense of increasing conservatism (Tuan and Apkarian, 1999). Next, several special cases are explored when gridding can be avoided. To derive the following result, it is assumed that $\Gamma = [-1,1]$. In case $\Gamma \neq [-1,1]$, one can choose $\tilde{\gamma} = [2\gamma - (\overline{\gamma} + \underline{\gamma})]/(\overline{\gamma} - \underline{\gamma})$, such that $\tilde{\gamma} \in [-1,1]$. This simplification can always be made without loss of generality and results in more compact formulas.

Theorem 2.4. Consider the system (1) where $A(\gamma) = A_0 + \gamma A_1 + \gamma^2 A_2$ and $A_d(\gamma) = A_{d0} + \gamma A_{d1} + \gamma^2 A_{d2}$ where $\gamma \in [-1,1]$, and $\dot{\gamma} \in [\underline{\dot{\gamma}}, \overline{\dot{\gamma}}]$. Assume that there exist negative-definite matrices Q_4, Q_2, P_2, positive-definite matrices Q_0, P_0 and symmetric matrices Q_1, Q_3, P_1 such that

$$Q_0 \pm Q_1 + 2Q_2 > 0, \quad -Q_2 \pm Q_3 + Q_4 > 0, \quad (8)$$

$$P_0 \pm P_1 + P_2 > 0 \quad (9)$$

for all $\gamma \in [-1,1]$, $\dot{\gamma} \in [\underline{\dot{\gamma}}, \overline{\dot{\gamma}}]$ and

$$\gamma_1^{\#2} N_2 + \gamma_1^{\#} N_1 + \gamma_2^{\#2} N_3 + \gamma_2^{\#} N_4 + N_0(\dot{\gamma}^{\#}) < 0 \quad (10)$$

where $\gamma_i^{\#} \in \{-1,1\}$ and $\dot{\gamma}^{\#} \in \{\underline{\dot{\gamma}}, \overline{\dot{\gamma}}\}$, and where

$$N_2 = \alpha_1 \Theta_1 + \alpha_2 \Theta_2 + \Theta_3 \geq 0 \quad (11)$$

$$N_1 = \frac{1-\alpha_1}{2}\Theta_1 + \frac{3-3\alpha_2}{4}\Theta_2 + \Theta_4 \quad (12)$$

$$N_0(\dot{\gamma}) = \frac{3\alpha_1 - 3}{16}\Theta_1 + \frac{\alpha_2 - 1}{4}\Theta_2 + \Theta_5(\dot{\gamma}) \quad (13)$$

$$N_3 = \begin{bmatrix} 0 & 0 \\ 0 & -Q_4 - (1-\beta)Q_3 - Q_2 \end{bmatrix} \geq 0 \quad (14)$$

$$N_4 = \begin{bmatrix} 0 & 0 \\ 0 & -\dfrac{3}{4}\beta Q_3 - Q_1 \end{bmatrix} \quad (15)$$

where

$$\Theta_1 = \begin{bmatrix} A_2^T P_2 + P_2 A_2 + Q_4 & P_2 A_{d2} \\ A_{d2}^T P_2 & 0 \end{bmatrix}$$

$$\Theta_2 = \begin{bmatrix} \begin{pmatrix} A_1^T P_2 + A_2^T P_1 \\ +P_1 A_2 + P_2 A_1 + Q_3 \end{pmatrix} & P_1 A_{d2} + P_2 A_{d1} \\ A_{d2}^T P_1 + A_{d1}^T P_2 & 0 \end{bmatrix}$$

$$\Theta_3 = \begin{bmatrix} \begin{pmatrix} A_2^T P_0 + P_0 A_2 \\ +A_1^T P_1 + (\ )^T + Q_2 \end{pmatrix} & (\ P_2 A_{d0} + P_1 A_{d1}\ ) \\ A_{d1}^T P_1 + A_{d0}^T P_2 & A_{d2}^T P_0 A_{d2} \end{bmatrix}$$

$$\Theta_4 = \begin{bmatrix} \begin{pmatrix} A_1^T P_0 + P_0 A_1 \\ +A_0^T P_1 + P_1 A_0 + Q_1 \end{pmatrix} & P_0 A_{d1} + P_1 A_{d0} \\ A_{d1}^T P_0 + A_{d0}^T P_1 & 0 \end{bmatrix}$$

$$\Theta_5(\dot{\gamma}) = \begin{bmatrix} \begin{pmatrix} A_0^T P_0 + P_0 A_0 \\ -2\nu_m P_2 + \dot{\gamma} P_1 + Q_0 \end{pmatrix} & P_0 A_{d0} \\ A_{d0}^T P_0 & -Q_0 + \dfrac{1}{4}\beta Q_3 \end{bmatrix}$$

where $\nu_m = \max\{|\underline{\dot{\gamma}}|, |\overline{\dot{\gamma}}|\}$, and where the pair (α_1,α_2) takes any of the four possible combinations $(\alpha_1,\alpha_2) \in \{(0,0),(0,1),(1,0),(1,1)\}$ and $\beta \in \{0,1\}$. Then the system (1) is asymptotically stable for all $\gamma \in [-1,1]$, all $\dot{\gamma} \in [\underline{\dot{\gamma}}, \overline{\dot{\gamma}}]$ and all $\tau \in [0,\infty)$.

Proof. Consider the Lyapunov-Krasovskii functional

$$V(t,x_t) = x^T(t)P(\gamma(t))x(t) + \int_{t-\tau}^{t} x^T(\theta)Q(\gamma(\theta))x(\theta)\ d\theta$$

where $P(\gamma) = P_0 + \gamma P_1 + \gamma^2 P_2$ and $Q(\gamma) = Q_0 + \gamma Q_1 + \gamma^2 Q_2 + \gamma^3 Q_3 + \gamma^4 Q_4$. From $P_2 < 0$, Equation (9) and Lemma 2.1 one has that $P(\gamma) > 0$ uniformly for all $\gamma \in [-1,1]$. Now write $Q(\gamma)$ as follows $Q(\gamma) = [2\gamma^2 Q_2 + \gamma Q_1 + Q_0] + \gamma^2[\gamma^2 Q_4 + \gamma Q_3 - Q_2]$. Then (8) along with Lemma 2.1 imply that $Q(\gamma) > 0$ uniformly for all $\gamma \in [-1,1]$. The derivative of V along the system (1) is

$$\begin{aligned}\dot{V}(t) = {} & 2x^T(t)(P_0 + \gamma_1 P_1 + \gamma_1^2 P_2)(A_0 + \gamma_1 A_1 + \gamma_1^2 A_2)x(t) \\ & + x^T(t)[\dot{\gamma}_1 P_1 + 2\gamma_1\dot{\gamma}_1 P_2]x(t) \\ & + 2x^T(t)(P_0 + \gamma_1 P_1 + \gamma_1^2 P_2)(A_{d0} + \gamma_1 A_{d1} + \gamma_1^2 A_{d2})x(t-\tau) \\ & + x^T(t)[Q_0 + \gamma_1 Q_1 + \gamma_1^2 Q_2 + \gamma_1^3 Q_3 + \gamma_1^4 Q_4]x(t) \\ & - x^T(t-\tau)[Q_0 + \gamma_2 Q_1 + \gamma_2^2 Q_2 + \gamma_2^3 Q_3 + \gamma_2^4 Q_4]x(t-\tau)\end{aligned}$$

where $\gamma_1 = \gamma(t), \gamma_2 = \gamma(t-\tau)$. Since $P_2 < 0$, it follows that $2\gamma_1\dot{\gamma}_1 x^T(t)P_2 x(t) \leq -2\nu_m x^T(t)P_2 x(t)$. One can then rewrite the equation for $\dot{V}$ as

$$\begin{aligned}\dot{V}(t) \leq {} & \gamma_1^4\Big\{2x^T(t)[P_2A_2 + 0.5Q_4]x(t) \\ & + 2x^T(t)P_2A_{d2}x(t-\tau)\Big\} \\ & - \gamma_2^4 x^T(t-\tau)Q_4 x(t-\tau) \\ & + \gamma_1^3\Big\{2x^T(t)[P_1A_2 + P_2A_1 + 0.5Q_3]x(t) \\ & + 2x^T(t)[P_1A_{d2} + P_2A_{d1}]x(t-\tau)\Big\} \\ & - \gamma_2^3 x^T(t-\tau)Q_3 x(t-\tau) \\ & + \gamma_1^2\Big\{2x^T(t)[P_0A_2 + P_1A_1 + P_2A_0 + 0.5Q_2]x(t) \\ & + 2x^T(t)[P_0A_{d2} + P_1A_{d1} + P_2A_{d0}]x(t-\tau)\Big\} \\ & - \gamma_2^2 x^T(t-\tau)Q_2 x(t-\tau) \\ & + \gamma_1\Big\{2x^T(t)[P_0A_1 + P_1A_0 + 0.5Q_1]x(t) \\ & + 2x^T(t)[P_0A_{d1} + P_1A_{d0}]x(t-\tau)\Big\} \\ & - \gamma_2 x^T(t-\tau)Q_1 x(t-\tau) \\ & + \Big\{2x^T(t)[P_0A_0 - \nu_m P_2 + 0.5Q_0 + 0.5\dot{\gamma}P_1]x(t) \\ & + 2x^T(t)P_0A_{d0}x(t-\tau) - x^T(t-\tau)Q_0 x(t-\tau)\Big\}\end{aligned} \tag{16}$$

Notice now that since $P_0 > 0$ it follows that the inequality $2x^T(t)P_0A_{d2}x(t-\tau) \leq x^T(t)P_0x(t) + x^T(t-\tau)A_{d2}^T P_0 A_{d2} x(t-\tau)$ holds. Also, it can be immediately verified that for all $\gamma \in [-1,1]$ [2] the following inequalities hold. Namely, $\gamma^2 \geq \gamma^4 \geq \frac{1}{2}\gamma - \frac{3}{16}$, $\gamma^2 \geq \gamma^3 \geq \frac{3}{4}\gamma - \frac{1}{4}$. Then $\gamma^4 y \leq \gamma^2 y$ if $y \geq 0$ and $\gamma^4 y \leq (\frac{1}{2}\gamma - \frac{3}{16})y$ if $y < 0$. Therefore, $\gamma^4 y \leq \max\left\{\gamma^2 y, (\frac{1}{2}\gamma - \frac{3}{16})y\right\}$ and $\gamma^3 y \leq \max\left\{\gamma^2 y, (\frac{3}{4}\gamma - \frac{1}{4})y\right\}$. These inequalities imply that

$$\begin{aligned}&\gamma_1^4\Big\{2x^T(t)[P_2A_2 + 0.5Q_4]x(t) + 2x^T(t)P_2A_{d2}x(t-\tau)\Big\} \\ &\leq \max\left\{(\frac{\gamma_1}{2} - \frac{3}{16})\begin{bmatrix} x(t) \\ x(t-\tau)\end{bmatrix}^T \Theta_1 \begin{bmatrix} x(t) \\ x(t-\tau)\end{bmatrix},\right. \\ &\left. + \gamma_1^2 \begin{bmatrix} x(t) \\ x(t-\tau)\end{bmatrix}^T \Theta_1 \begin{bmatrix} x(t) \\ x(t-\tau)\end{bmatrix}\right\}\end{aligned} \tag{17}$$

and

$$\begin{aligned}&\gamma_1^3\Big\{2x^T(t)[P_1A_2 + P_2A_1 + 0.5Q_3]x(t) \\ &+ 2x^T(t)[P_1A_{d2} + P_2A_{d1}]x(t-\tau)\Big\} \\ &\leq \max\left\{\gamma_1^2 \begin{bmatrix} x(t) \\ x(t-\tau)\end{bmatrix}^T \Theta_2 \begin{bmatrix} x(t) \\ x(t-\tau)\end{bmatrix},\right. \\ &\left. + (\frac{3\gamma_1}{4} - \frac{1}{4})\begin{bmatrix} x(t) \\ x(t-\tau)\end{bmatrix}^T \Theta_2 \begin{bmatrix} x(t) \\ x(t-\tau)\end{bmatrix}\right\}\end{aligned} \tag{18}$$

Since $Q4 < 0$ and $\gamma_2 \in [-1,1]$, it follows that $-\gamma_2^4 x^T(t-\tau)Q_4x(t-\tau) \leq -\gamma_2^2 x^T(t-\tau)Q_4 x(t-\tau)$. Moreover, if $x^T(t-\tau)Q_3x(t-\tau) > 0$, then $-\gamma_2^3 x^T(t-\tau)Q_3x(t-\tau) < (-\frac{3}{4}\gamma_2 + \frac{1}{4})x^T(t-\tau)Q_3x(t-\tau)$, whereas if $x^T(t-\tau)Q_3x(t-\tau) < 0$ then $-\gamma_2^3 x^T(t-\tau)Q_3x(t-\tau) < -\gamma_2^2 x^T(t-\tau)Q_3x(t-\tau)$. Therefore in either case, $-\gamma_2^3 x^T(t-\tau)Q_3x(t-\tau) \leq \max\{(-\frac{3}{4}\gamma_2 + \frac{1}{4})x^T(t-\tau)Q_3x(t-\tau), -\gamma_2^2 x^T(t-\tau)Q_3x(t-\tau)\}$ or that

$$\begin{aligned}-\gamma_2^3 x^T(t-\tau)Q_3x(t-\tau) &\leq -(1-\beta)\gamma_2^2 x^T(t-\tau)Q_3x(t-\tau) \\ &- \beta(\frac{3}{4}\gamma_2 - \frac{1}{4})x^T(t-\tau)Q_3x(t-\tau)\end{aligned}$$

where $\beta \in \{0,1\}$.

Collecting all the previous results and substituting in (16), one obtains that

$$\dot{V}(x) \leq \begin{bmatrix} x(t) \\ x(t-\tau)\end{bmatrix}^T N(\gamma_1,\gamma_2,\dot{\gamma}) \begin{bmatrix} x(t) \\ x(t-\tau)\end{bmatrix} \tag{19}$$

where $N(\gamma_1,\gamma_2,\dot{\gamma}) = \gamma_1^2 N_2 + \gamma_1 N_1 + +\gamma_2^2 N_3 + \gamma_2 N_4 + N_0(\dot{\gamma})$ and where N_0, N_1, N_2, N_3, N_4 as in (11)-(13). The inequalities $N_3 \geq 0$ and $N_2 \geq 0$ along with inequality (10) imply that $N(\gamma_1,\gamma_2\dot{\gamma}) < 0$ for all $(\gamma_1,\gamma_2) \in [-1,1]^2$ and $\dot{\gamma} \in [\underline{\dot{\gamma}}, \overline{\dot{\gamma}}]$. The asymptotic stability of (1) then follows immediately from (19). ■

Theorem 2.5. Consider the LPV time-delayed system (1) with $A(\gamma) = A_0 + \gamma A_1$ and $A_d(\gamma) = A_{d0} + \gamma A_{d1}$ where $(\gamma,\dot{\gamma}) \in G = [\underline{\gamma},\overline{\gamma}] \times [\underline{\dot{\gamma}},\overline{\dot{\gamma}}]$. Assume that there exist a negative definite matrix Q_2 and symmetric matrices P_0, P_1, Q_0, Q_1 which satisfy the following LMI's

$$Q(\gamma^\#) = Q_0 + \gamma^\# Q_1 + {\gamma^\#}^2 Q_2 > 0 \tag{20a}$$
$$P(\gamma^\#) = P_0 + \gamma^\# P_1 > 0 \tag{20b}$$

for all $\gamma^\# \in \{\underline{\gamma},\overline{\gamma}\}$ and

$$\begin{bmatrix} \begin{pmatrix} P(\gamma_1)A(\gamma_1) + (\ )^T \\ +\nu P_1 + Q(\gamma_1)\end{pmatrix} & P(\gamma_1)A_d(\gamma_1) \\ A_d^T(\gamma_1)P(\gamma_1) & -Q(\gamma_2)\end{bmatrix} < 0 \tag{21}$$

[2] See, for example, (Tuan and Apkarian, 1999).

for all $\nu \in [\underline{\dot{\gamma}}, \overline{\dot{\gamma}}]$, $\gamma_i \in [\underline{\gamma}, \overline{\gamma}]$, $i = 1, 2$. Then the system (1) is delay-independent stable for all $(\gamma, \dot{\gamma}) \in G$.

Proof. Consider the parameter dependent Lyapunov-Krasovskii functional

$$V(t, x_t) = x^T(t)P(\gamma(t))x(t) + \int_{-\tau}^{0} x^T(t+\theta)Q(\gamma(t+\theta))x(t+\theta)\, d\theta$$

where $P(\gamma) = P_0 + \gamma P_1$ and $Q(\gamma) = Q_0 + \gamma Q_1 + \gamma^2 Q_2$. Since $Q_2 < 0$ $Q(\gamma)$ is concave in γ. From (20a) and Lemma 2.1 it follows that $Q(\gamma) > 0$ for all $\gamma \in [\underline{\gamma}, \overline{\gamma}]$. From the LMI (20b), it also follows that $P(\gamma) > 0$ for all $\gamma \in [\underline{\gamma}, \overline{\gamma}]$. Therefore, $V(t, x_t)$ is a positive definite functional with an infinitesimal upper bound. Calculation of the derivative of V yields

$$\dot{V}(t) = \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}^T M_4(\gamma(t), \gamma(t-\tau), \dot{\gamma}(t)) \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}$$

where $M_4(\gamma_1, \gamma_2, \nu)$ is the matrix in (21). Let $\gamma_1 = \gamma(t), \gamma_2 = \gamma(t-\tau)$ and $\nu = \dot{\gamma}(t)$. Then the matrix inequality $M_4(\gamma_1, \gamma_2, \nu) < 0$ for all $\nu \in [\underline{\dot{\gamma}}, \overline{\dot{\gamma}}]$, $\gamma_i \in [\underline{\gamma}, \overline{\gamma}]$, $i = 1, 2$ implies that M_4 is uniformly negative definite for all γ and $\dot{\gamma}$. Thus, the derivative of V is negative definite along the trajectories of (1) and the system (1) is asymptotically stable for all $(\gamma, \dot{\gamma}) \in G$ and $\tau \in [0, \infty)$. ■

The following result differs from the previous developments because it makes use of the Razumikhin stability theorem (Hale and Lunel, 1993).

Theorem 2.6. Consider the time-delayed (1) with $A(\gamma) = A_0 + \gamma A_1$ and $A_d(\gamma) = A_{d0} + \gamma A_{d1}$ where $\gamma \in [\underline{\gamma}, \overline{\gamma}]$. If there exist a positive-definite matrix P and a positive scalar $\alpha > 0$ that satisfy

$$A_d^T(\gamma^\#)PA_d(\gamma^\#) < \alpha P \qquad (22)$$

$$A^T(\gamma^\#)P + PA(\gamma^\#) + P + \alpha P < 0 \qquad (23)$$

where $\gamma^\# \in \{\underline{\gamma}, \overline{\gamma}\}$, then system (1) will be delay-independent stable.

Proof. Since $A_{d1}^T PA_{d1} \geq 0$, from Lemma 2.1 and condition (22) it follows that $A_d^T(\gamma)PA_d(\gamma) < \alpha P$ for all $\gamma \in [\underline{\gamma}, \overline{\gamma}]$. Consider the following positive definite function $V(x) = x^T Px$. Its derivative along the trajectories of (1) is

$$\begin{aligned}
\dot{V}(t) &= 2x^T(t)P[A(\gamma)x(t) + A_d(\gamma)x(t-\tau)] \\
&= x^T(t)[PA(\gamma) + A^T(\gamma)P]x(t) \\
&+ [x^T(t)PA_d(\gamma)x(t-\tau) + x^T(t-\tau)A_d^T(\gamma)Px(t)] \\
&= x^T(t)[PA(\gamma) + A^T(\gamma)P]x(t) \\
&+ [x^T(t)P^{\frac{1}{2}}P^{\frac{1}{2}}A_d(\gamma)x(t-\tau) \\
&+ x^T(t-\tau)A_d^T(\gamma)P^{\frac{1}{2}}P^{\frac{1}{2}}x(t)] \\
&\leq x^T(t)[PA(\gamma) + A^T(\gamma)P]x(t) \\
&+ [x^T(t)Px(t) + x^T(t-\tau)A_d^T(\gamma)PA_d(\gamma)x(t-\tau)]
\end{aligned}$$

Since $A_d^T(\gamma)PA_d(\gamma) < \alpha P$ for all $\gamma \in [\underline{\gamma}, \overline{\gamma}]$ and $[\underline{\gamma}, \overline{\gamma}]$ is compact, there exists a $\delta > 0$, such that $A_d^T(\gamma)PA_d(\gamma) < (\alpha - \delta)P$ for all $\gamma \in [\underline{\gamma}, \overline{\gamma}]$. Let $\alpha' = \frac{\alpha}{\alpha - \delta} > 1$ and $p(s) = \alpha' s$.

Assume that

$$\begin{aligned}
V(x(t+\theta)) &= x^T(t+\theta)Px(t+\theta) \\
&< p(V(x(t))) = \alpha' x^T(t)Px(t), \quad \forall \theta \in [-\tau, 0]
\end{aligned}$$

Thus $x^T(t-\tau)A_d^T(\gamma)PA_d(\gamma)x(t-\tau) < (\alpha - \delta)x^T(t-\tau)Px(t-\tau) < \alpha'(\alpha - \delta)x^T(t)Px(t) = \alpha x^T(t)Px(t)$. Using the last inequality, one obtains

$$\dot{V}(t) < x^T(t)[PA(\gamma) + A^T(\gamma)P + P + \alpha P]x(t), \quad \forall \gamma \in [\underline{\gamma}, \overline{\gamma}]$$

and from (23) $\dot{V}$ is negative definite. From Razumikhin Theorem (Hale and Lunel, 1993; Niculescu *et al.*, 1997*b*) we finally obtain the assertion of the theorem. ■

3. CONCLUSIONS

Several delay-independent stability tests for Linear Parameter Varying (LPV) systems subject to delays have been developed. Bounds on the parameter variation can be incorporated by using parameter-dependent Lyapunov-Krasovskii functionals. Parameter gridding or relaxation methods can be used to case these tests as convex optimization problems (LMIs) which can be solved efficiently using current computer software. Future work will focus on controller synthesis.

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