Optimal Control of Rigid Body Angular Velocity with Quadratic Cost[1]

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Abstract

In this paper, we consider the problem of obtaining optimal controllers which minimize a quadratic cost function for the rotational motion of a rigid body. We are not concerned with the attitude of the body and consider only the evolution of the angular velocity as described by Euler's equations. We obtain conditions which guarantee the existence of *linear* stabilizing optimal and suboptimal controllers. These controllers have a very simple structure.

Key Words: Rigid Bodies, Hamilton Jacobi equation, Riccati equation, optimal control.

1 Introduction

Optimal control of rigid bodies has a long history stemming from interest in the control of rigid spacecraft and aircraft. Most of this research has been directed toward the time-optimal attitude control problem; see, for example, the survey paper (Ref. 1) and the book (Ref. 2). Dixon *et al.* (Ref. 3) considered the fuel-optimal rest-to-rest maneuver for an axisymmetric rigid body. The earliest results on the optimal regulation of angular velocity or, equivalently, angular momentum seem to be Refs. 4–6. Windeknecht (Ref. 7) also examined the problem of optimal regulation of the angular momentum over a finite interval with a quadratic integral penalty on the control variables and a terminal constraint on the state; the weighting matrices in the cost function were identity matrices.

In this paper we seek optimal and suboptimal solutions to the *non-linear quadratic regulator* (NLQR) problem for a rigid body in the sense that a quadratic cost function is to be minimized. We solve the problem of quadratic regulation for the dynamic (angular velocity) equations of a rotating rigid body, by deriving explicit solutions to the associated Hamilton-Jacobi equation (HJE) and Hamilton-Jacobi inequality (HJI). We give necessary and sufficient conditions for the existence of quadratic functions which satisfy the HJE and HJI. These solutions result in *linear* optimal and suboptimal controllers, respectively.

The paper is organized as follows. In Section 2 we present the equations of motion of a rotating rigid body and state the problem to be addressed. In Section 3 we completely characterize the family of quadratic integrals of the unforced system and we present some preliminary results concerning the conditions under which the system is zero-state detectable and zero-state observable. Sections 4 and 5 contain the main results of the paper. The first theorem of the paper (Theorem 4.1) contains conditions under which the NLQR problem is solvable. Theorem 4.2 gives sufficient conditions for the *suboptimal* NLQR problem, i.e., conditions which guarantee boundedness of a quadratic cost. These theorems introduce the HJE and HJI. By restricting consideration to quadratic solutions to the HJE/HJI we seek *linear* solutions to the HJE/HJI and we show that these solutions can be computed by considering only solutions to the associated algebraic Riccati equation (ARE) and algebraic Riccati inequality (ARI),

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of a specific form. In Section 6 we present some special cases in which the optimal feedback NLQR controller can be shown to be linear. We also show that some of the known results in the literature follow immediately from the results presented here. In Section 7 we present an optimization algorithm for computing a positive definite solution to the ARI. We conclude with a numerical example to illustrate the theory.

2 Problem Formulation

2.1 Equations of Motion

The evolution of the angular velocity, equivalently the angular momentum, of a rigid body is described by

$$J\dot{\omega} = (J\omega) \times \omega + Gu\,, \qquad \omega(0) = \omega_0 \tag{1}$$

where $\omega(t) := col(\omega_1(t), \omega_2(t), \omega_3(t))$ with $\omega_i(t) \in \mathbb{R}$ being the i-th component of the angular velocity of the body relative to an inertial reference frame. These components are taken relative to a body-fixed reference frame. The real positive definite matrix J is the inertia matrix of the rigid body at the mass center and expressed relative to the body fixed frame. At time t, the control input is given by $u(t) \in \mathbb{R}^m$. We assume that G is a constant matrix of appropriate dimensions, having full column rank. If we let

$$f(\omega) := J^{-1}[(J\omega) \times \omega], \qquad B := J^{-1}G \tag{2}$$

the system can be described by

$$\dot{\omega} = f(\omega) + Bu, \qquad \omega(0) = \omega_0 \tag{3}$$

Since J is symmetric and positive definite, it has three positive real eigenvalues I_1, I_2, I_3 with three corresponding mutually orthogonal real eigenvectors v_1, v_2, v_3. These eigenvalues and eigenvectors are called the principal moments of inertia and principal axes of inertia, respectively, of the body about its mass center. If two of the principal moments are equal, say $I_1 = I_2$, the body is said to be *axisymmetric* about the axis corresponding to the third eigenvalue, i.e., the 3-axis in this case. If $I_1 = I_2 = I_3$ the body is said to be *symmetric*. In this case J is a multiple of the identity matrix and the system (3) is linear and is simply described by

$$\dot{\omega} = Bu\,, \qquad \omega(0) = \omega_0 \tag{4}$$

2.2 Problem Definition

Consider the control-affine nonlinear system (3). Introducing a *penalty* or *regulated output* $z(t) \in \mathbb{R}^p$, we obtain the following system description:

$$\dot{\omega} = f(\omega) + Bu, \qquad \omega(0) = \omega_0 \tag{5a}$$

$$z = \begin{bmatrix} H\omega \\ Du \end{bmatrix} \tag{5b}$$

where H and D are constant matrices of appropriate dimensions. The associated *uncontrolled system* or *free system* is given by

$$\dot{\omega} = f(\omega), \qquad \omega(0) = \omega_0 \tag{6a}$$

$$z = H\omega \tag{6b}$$

Associated with system (5) is the following quadratic cost functional

$$\mathcal{J}(\omega_0; u) := \int_0^\infty z'(t)z(t)\,dt = \int_0^\infty \|z(t)\|^2\,dt \tag{7}$$

where "$'$" denotes transpose and $\|z\|$ denotes the Euclidean norm (length) of a vector $z \in \mathbb{R}^p$ and is defined by $\|z\|^2 = \sum_{i=1}^p z_i^2 = z'z$. This can also be written as:

$$\mathcal{J}(\omega_0; u) := \int_0^\infty \omega'(t)Q\omega(t) + u'(t)Ru(t)\;dt \tag{8}$$

where $Q = H'H\,, \qquad R = D'D$.

Consider system (5) subject to a memoryless state feedback controller k, i.e.,

$$u = k(\omega) \tag{9}$$

The resulting closed loop system is described by

$$\dot{\omega} = f(\omega) + Bk(\omega) \tag{10a}$$

$$z = \begin{bmatrix} H\omega \\ Dk(\omega) \end{bmatrix} \tag{10b}$$

We are now ready to state the nonlinear quadratic regulator (NLQR) problem.

Problem (NLQR): Find a memoryless state-feedback controller k^* for system (5) such that

(i) the resulting closed-loop system

$$\dot{\omega} = f(\omega) + Bk^*(\omega) \tag{11a}$$

$$z = \begin{bmatrix} H\omega \\ Dk^*(\omega) \end{bmatrix} \tag{11b}$$

is *globally asymptotically stable* about $\omega = 0$;

(ii) for each initial state ω_0 and for every control history $u(\cdot)$ which results in

$$\lim_{t \to \infty} \omega(t) = 0$$

the control history $u^*(\cdot)$ generated by the controller k^* minimizes the cost functional (7), i.e.,

$$\mathcal{J}(\omega_0; u^*) \leq \mathcal{J}(\omega_0; u)$$

If there exists a controller k^* satisfying (i) and (ii), we call it an *optimal stabilizing state-feedback controller* and we say that (5) is *NLQR-solvable.* If, in addition, k^* is linear we say that (5) is *NLQR-solvable via linear control.*

3 Preliminary Results

3.1 Observability and Detectability

Before we present a solution to the NLQR problem associated with system (5) we need to introduce the following concepts (Refs. 8,9) for a general system described by

$$\dot{x} = F(x) \tag{12a}$$

$$z = H(x) \tag{12b}$$

where $x(t) \in \mathbb{R}^n$ and $z(t) \in \mathbb{R}^p$.

Definition 3.1 (Zero-State Observability) System (12) is *zero-state observable* if $z(t) = 0$ for all $t \geq 0$ implies $x(t) = 0$ for all $t \geq 0$.

Definition 3.2 (Zero-state detectability) System (12) is *zero-state detectable* if $z(t) = 0$ for all $t \geq 0$ implies $\lim_{t \to \infty} x(t) = 0$.

Proposition 3.1 Consider system (5) with D full column rank. Then, for any controller k, the closed loop system (10) is zero-state observable (zero-state detectable) if and only if the uncontrolled system (6) is zero-state observable (zero-state detectable).

Proof. Consider z as given by (10b). Clearly $z(t) \equiv 0$ in (10b) implies $H\omega(t) \equiv 0$ and $Dk(\omega(t)) \equiv 0$; since D is full column rank it follows that $k(\omega(t)) \equiv 0$ and the trajectories of (10) evolve according to (6). Thus (10) is zero-state observable (zero-state detectable) if and only if (6) is zero-state observable (zero-state detectable). □

The previous proposition states, in essence, that if D if full column rank then the test of zero-state observability for system (10) reduces to a test on the uncontrolled system (6), i.e., it is an open-loop property. This is the route also followed in Refs. 8–11. From now on, we will always assume that D is full column rank. Without loss of generality, we can redefine the control input so that the matrix D satisfies the condition $D'D = I$.

It turns out that the zero-state observability property for system (6) (or system (10) for that matter) has a very simple characterization in terms of the matrix pair (H, J). We have the following result.

Lemma 3.1 The following statements are equivalent.

(a) System (6) is zero state observable.

(b) System (6) is zero state detectable.

(c)

$$\operatorname{rank}\begin{bmatrix} H \\ HJ \\ HJ^2 \end{bmatrix} = 3 \tag{13}$$

Proof. Clearly (a) implies (b). We now show that (b) implies (c). Suppose on the contrary that (c) does not hold. Then using standard results from the observability of linear systems (Ref. 12), there is an eigenvector v of J such that $Hv = 0$. Thus $v \neq 0$ and, since J is symmetric, v is real and $Jv = \lambda v$ for some real λ. Thus $(Jv) \times v = \lambda(v \times v) = 0$; hence $\omega(t) \equiv v \neq 0$ is a solution of system (6) and $z(t) \equiv Hv = 0$. Hence system (6) is not zero-state detectable.

If we show that (c) implies (a), we are done. Suppose that (a) does not hold. Then there is a nonzero solution $\omega(\cdot)$ of (6) with $z(t) \equiv 0$. Introduce now the momentum variable $p = J\omega$ to obtain the following description.

$$\begin{aligned} \dot{p} &= p \times (J^{-1}p) \\ z &= HJ^{-1}p \end{aligned}$$

Applying Lemma 10.1 in the Appendix, it follows that there exists an eigenvector v of J^{-1} such that $\lim_{t\to\infty} p(t) = v$. Since $z(t) \equiv 0$, we have $HJ^{-1}v = 0$. Since v is an eigenvector of J^{-1}, it is also an eigenvector of J; it now follows that $Hv = HJv = HJ^2v = 0$. This implies that (c) does not hold. So, if (c) holds, then (a) must hold. □

Definition 3.3 When the rank condition (13) holds, we say that the pair (H, J) is observable.

Remark 3.1 From the previous discussion it should be clear that if (H, J) is observable then the closed loop system (10) is zero-state observable with *any* controller k.

3.2 Integrals of the Uncontrolled System

Here we consider quadratic integrals of the motion of the uncontrolled system (6) Consider any scalar-valued function $V : \mathbb{R}^3 \to \mathbb{R}$. If V is continuously differentiable, we let the subscript ω denote differentiation with respect to ω, i.e., $V_\omega(\omega) := \frac{\partial V(\omega)}{\partial \omega} = \nabla V(\omega) = [\ \frac{\partial V}{\partial \omega_1}\ \ \frac{\partial V}{\partial \omega_2}\ \ \frac{\partial V}{\partial \omega_3}\ ]$. The derivative of V along any trajectory $\omega(\cdot)$ of system (6) is given by

$$\frac{d}{dt}V(\omega(t)) = V_\omega(\omega(t))f(\omega(t)) \tag{14}$$

and we have the following definition.

Definition 3.4 A function $V : \mathbb{R}^3 \to \mathbb{R}$ is an *integral of the motion* of (6) if it is constant along every trajectory of (6), i.e., if

$$V_\omega(\omega)f(\omega) = 0 \tag{15}$$

for all $\omega \in \mathbb{R}^3$.

In particular, a quadratic function given by $V(\omega) = \omega' P\omega$, where $P \in \mathbb{R}^{3\times 3}$ is symmetric, is an integral of the uncontrolled system (6) if and only if

$$\omega' Pf(\omega) = 0$$

for all $\omega \in \mathbb{R}^3$.

If J is a multiple of the identity matrix (symmetric body), every quadratic function is an integral of the motion. In the nonsymmetric case, it should be clear from the identities

$$\begin{aligned} (J\omega)' f(\omega) &= \omega'[(J\omega)\times\omega] &= 0 \\ (J^2\omega)' f(\omega) &= (J\omega)'[(J\omega)\times\omega] &= 0 \end{aligned}$$

that the quadratic functions $V_1(\omega) = \omega' J\omega$ and $V_2(\omega) = \omega' J^2\omega = \|J\omega\|^2$ are integrals of the motion of system (6). The first function yields twice the system kinetic energy, whereas the second one is the square of the magnitude of the angular momentum. The following lemma states that every quadratic integral of system (6) (nonsymmetric case) is a linear combination of the above two integrals.

Lemma 3.2 Consider the function $f(\cdot)$ given by (2). A symmetric matrix $P \in \mathrm{I\!R}^{3\times 3}$ satisfies

$$\omega' P f(\omega) = 0 \tag{16}$$

for all $\omega \in \mathrm{I\!R}^3$ if and only if

$$P = \alpha J + \beta J^2 \tag{17}$$

for some real scalars α and β when J is not a multiple of the identity matrix.

Proof. (*Sufficiency*) If (17) holds then,

$$\omega' P f(\omega) = [\alpha\omega + \beta(J\omega)]'[(J\omega)\times\omega]$$

Since the cross product of two vectors is perpendicular to the plane defined by the two vectors, (16) must hold.

(*Necessity*) Suppose (16) holds for all $\omega \in \mathrm{I\!R}^3$. Letting $\tilde{p} := J\omega$, $\tilde{Q} := J^{-1}PJ^{-1}$ and $\tilde{N} := J^{-1}$, condition (16) holds for all $\omega \in \mathrm{I\!R}^3$ if and only if

$$(\tilde{Q}\tilde{p})'[\tilde{p}\times(\tilde{N}\tilde{p})] = 0 \tag{18}$$

for all $\tilde{p} \in \mathrm{I\!R}^3$. Also Equation (17) is equivalent to

$$\tilde{Q} = \alpha\tilde{N} + \beta I \tag{19}$$

Relationship (18) is invariant under the coordinate transformation $\tilde{p} = Rp$ where R is a 3×3 rotation matrix, i.e., $R'R = I$ and $\det R = 1$. In other words, (18) holds for all $\tilde{p}$ if and only if the following relationship holds for all p

$$(Qp)'[p\times(Np)] = 0 \tag{20}$$

where $Q = R'\tilde{Q}R$ and $N = R'\tilde{N}R$. Also, relationship (19) is equivalent to

$$Q = \alpha N + \beta I \tag{21}$$

Since $\tilde{N}$ is symmetric, R can be chosen so that N is diagonal, i.e.,

$$N = \begin{bmatrix} n_1 & 0 & 0 \\ 0 & n_2 & 0 \\ 0 & 0 & n_3 \end{bmatrix}$$

Letting q_{ij} denote the ij-th element of Q and noting that Q is symmetric, relationship (20) results in

$$g(p) = 0 \qquad \text{for all} \quad p \in \mathrm{I\!R}^3 \tag{22}$$

where

$$\begin{aligned} g(p) &= (q_{11}m_1 + q_{22}m_2 + q_{33}m_3)p_1p_2p_3 \\ &+ (q_{12}m_1)p_2^2p_3 + (q_{12}m_2)p_1^2p_3 \\ &+ (q_{13}m_1)p_2p_3^2 + (q_{13}m_3)p_1^2p_2 \\ &+ (q_{23}m_2)p_1p_3^2 + (q_{23}m_3)p_1p_2^2 \end{aligned}$$

with $m_1 = n_3 - n_2$, $m_2 = n_1 - n_3$ and $m_3 = n_2 - n_1$. Relationship (22) holds if and only if

$$q_{11}m_1 + q_{22}m_2 + q_{33}m_3 = 0 \tag{23a}$$
$$q_{12}m_1 = q_{12}m_2 = 0 \tag{23b}$$
$$q_{13}m_1 = q_{13}m_3 = 0 \tag{23c}$$
$$q_{23}m_2 = q_{23}m_3 = 0 \tag{23d}$$

Consider (23b). Since N is not a multiple of the identity matrix, we cannot have $(m_1, m_2) = (0,0)$; hence we must have $q_{12} = 0$. In a similar fashion, (23c) and (23d) yield $q_{13} = 0$ and $q_{23} = 0$ respectively. Hence Q is a diagonal matrix whose diagonal elements satisfy (23a); this latter relationship can be written as

$$\begin{bmatrix} q_{11} \\ q_{22} \\ q_{33} \end{bmatrix}' \left[\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right] = 0 \tag{24}$$

Since the matrix N is not a multiple of the identity matrix, the second and third vectors involved in the above cross product are non-zero and non-parallel; hence their cross product is non-zero and perpendicular to the plane containing them. Relationship (24) tells us that the first vector is perpendicular to the cross product; hence, the first vector must lie in the plane defined by the second and third vectors; i.e.,

$$\begin{bmatrix} q_{11} \\ q_{22} \\ q_{33} \end{bmatrix} = \alpha \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

for some scalars α and β. This is equivalent to (21) and, hence, (17). □

3.3 Stabilizability via Linear Control

Before addressing the problem of obtaining optimal stabilizing controllers, we first address the question of the existence of stabilizing controllers. The following lemma yields a sufficient condition which guarantees the existence of *linear* stabilizing controllers.

Lemma 3.3 Consider system (1) and suppose that

$$\text{rank} \begin{bmatrix} G & JG & J^2G \end{bmatrix} = 3 \tag{25}$$

Then the linear controller

$$k(\omega) = -G'\omega \tag{26}$$

yields a closed loop system which is globally asymptotically stable.

Proof. The closed loop system is described by

$$\dot{\omega} = f(\omega) - J^{-1}GG'\omega$$

Considering the function $V(\omega) = \frac{1}{2}\omega' J\omega$ as a Lyapunov function candidate, its time derivative along any solution of the closed loop system is given by

$$\dot{V}(\omega) = \omega' J f(\omega) - \omega' GG'\omega = -\|G'\omega\|^2 \tag{27}$$

Hence, $\dot{V}(\omega(t)) \leq 0$. Since V is radially unbounded, this implies that all solutions are bounded.

We now note that $\dot{V}(\omega(t)) \equiv 0$ implies that $G'\omega(t) \equiv 0$. Rank condition (25) implies that the pair (G', J) is observable; hence, using Lemma 3.1, we obtain that $G'\omega(t) \equiv 0$ implies $\omega(t) \equiv 0$; hence the only solution for which $\dot{V}(\omega(t)) \equiv 0$ is the zero solution. Global asymptotic stability now follows directly from LaSalle's theorem. □

Definition 3.5 When the rank condition (25) holds, we say that the pair (J, G) is controllable.

4 Sufficient Conditions for Optimality and Suboptimality

4.1 Sufficient Conditions for Solution of the NLQR Problem

The following theorem yields sufficient conditions under which the NLQR problem for system (5) has a solution. The main condition requires the existence of a positive definite function satisfying the Hamilton-Jacobi equation associated with the minimization of (7) subject to the nonlinear dynamics (5). Recall that a function $V : \mathbb{R}^3 \to \mathbb{R}_+$ is called *positive definite* if (Ref. 13)

(a) $V(0) = 0$

(b) $V(\omega) > 0$ for $\omega \neq 0$

(c) $\lim_{\omega \to \infty} V(\omega) = \infty$

We are now ready to give sufficient conditions under which the NLQR problem is solvable.

Theorem 4.1 Consider system (5) with (H, J) observable. Suppose that there exist a continuously differentiable positive definite function $V : \mathbb{R}^3 \to \mathbb{R}_+$ which satisfies the Hamilton-Jacobi equation (HJE)

$$V_\omega(\omega) f(\omega) - (1/4) V_\omega(\omega) BB' V_\omega{}'(\omega) + \omega' H' H \omega = 0, \tag{28}$$

for all $\omega \in \mathbb{R}^3$. Then the feedback controller

$$k^*(\omega) = -(1/2) B' V_\omega{}'(\omega) \tag{29}$$

renders the closed-loop system globally asymptotically stable about the origin and minimizes the cost functional (7). Moreover, for each ω_0, the optimal control history is unique and the minimum value of the cost is $V(\omega_0)$.

Proof. First we show that under the hypotheses of the theorem the feedback controller (29) is globally asymptotically stabilizing. With controller (29) the closed loop system behaves according to

$$\dot{\omega} = f(\omega) - (1/2) BB' V_\omega{}'(\omega) \tag{30}$$

Since the solution V to the HJE is a positive definite function it can be used as a Lyapunov function candidate. Taking the time derivative of V along the trajectories of (30), and using the HJE one obtains

$$\begin{aligned} \dot{V}(\omega) &= V_\omega(\omega) f(\omega) - (1/2) V_\omega(\omega) BB' V_\omega'(\omega) \\ &= -||H\omega||^2 - ||k^*(\omega)||^2 = -||z||^2 \end{aligned}$$

Hence, $\dot{V}(\omega(t)) \leq 0$ and $\dot{V}(\omega(t)) \equiv 0$ if and only if $z(t) \equiv 0$. Since the pair (H, J) is observable, it follows from remark 3.1 that $\omega(t) \equiv 0$. Since V is radially unbounded, all solutions are bounded. Asymptotic stability now follows directly from LaSalle's theorem.

In order to show optimality, consider any initial state ω_0 and any control history $u(\cdot)$ which results in

$$\lim_{t \to \infty} \omega(t) = 0 \tag{31}$$

An easy calculation shows that

$$\frac{d}{dt} V(\omega(t)) = -||z(t)||^2 + ||u(t) - k^*(\omega(t))||^2 \tag{32}$$

where V satisfies (28) and where $z(\cdot)$ is the response of the system subject to the control history $u(\cdot)$. Integrating both sides of (32) we obtain

$$\int_0^T ||z(t)||^2 \, dt = V(\omega(0)) - V(\omega(T)) + \int_0^T ||u(t) - k^*(\omega(t))||^2 \, dt \tag{33}$$

Taking the limit as $T \to \infty$ and using (31) we get $\lim_{T \to \infty} V(\omega(T)) = 0$; hence

$$\int_0^\infty ||z(t)||^2 \, dt = V(\omega_0) + \int_0^\infty ||u(t) - k^*(\omega(t))||^2 \, dt \tag{34}$$

Clearly,

$$\int_0^\infty ||z(t)||^2 \, dt \geq V(\omega_0) \tag{35}$$

and the lower bound $V(\omega_0)$ is achieved if and only if $u(t) \equiv k^*(\omega(t))$. This completes the proof. □

In general, the cornerstone for deriving stabilizing optimal feedback controllers for nonlinear systems is the existence of positive definite solutions to the HJE. For linear time-invariant systems, the above requirement reduces to the well-known requirement for the existence of positive definite (or positive semi-definite) solutions to a matrix Riccati equation, which is the counterpart of the HJE for the linear case. Therefore, the characterization of positive definite solutions to the HJE is fundamental to solving the nonlinear optimal feedback control problem. However, apart from the linear case, to date there does not exist a systematic procedure for obtaining such solutions. One is often compelled to search for solutions of the HJE using series expansions (Refs. 14–17).

4.2 Suboptimal Controllers

In practice it may be very difficult to establish the existence of a positive definite function satisfying (28) for all $\omega \in \mathbb{R}^3$. In such cases, one may restrict oneself to the design of controllers which, although not optimal, are stabilizing and guarantee a bounded value of the cost. This is the result of the next theorem.

Theorem 4.2 Consider system (5) with (H, J) observable. Let $V : \mathbb{R}^3 \to \mathbb{R}_+$ be a positive definite continuously differentiable function which satisfies the Hamilton-Jacobi Inequality (HJI)

$$V_\omega(\omega)f(\omega) - (1/4)V_\omega(\omega)BB'V_\omega'(\omega) + \omega' H' H\omega \leq 0, \tag{36}$$

for all $\omega \in \mathbb{R}^3$. Then the feedback controller

$$k^*(\omega) = -(1/2)B'{V_\omega}'(\omega) \tag{37}$$

globally asymptotically stabilizes (5) about the origin and yields a bounded value for the cost (7). In particular, the cost is bounded above by $V(\omega_0)$.

Proof. The proof follows very closely the proof of Theorem 4.1 and will not be repeated here. □

In most cases it is not an easy task to show that either the Hamilton-Jacobi Equation or Inequality holds for the whole state space. Alternatively, one may dispense altogether with such global results and seek solutions to the HJE/HJI only in a positively invariant subset of the whole state space. This is the route followed in Ref. 18. In the present paper we follow an indirect approach and seek *linear* controllers solving the NLQR problem; equivalently, we seek *quadratic* solutions to the HJE (28) and the HJI (36). In particular, we establish the following fact: The system (5) is NLQR-solvable via linear control if the matrix Riccati equation for the linearized system admits a positive definite solution of a certain structure.

5 Linear Optimal and Suboptimal Controllers

Our motivation here is based on the observation that the nonlinear system (1) has *linear, globally asymptotically* stabilizing controllers. Therefore, we search over this class of controllers for the one that yields the minimum cost. One has to be careful however in following this approach, since — in contrast to linear systems — asymptotic stability of a nonlinear system does not imply, in general, exponential stability, and it may even happen that all linear stabilizing controllers give rise to unbounded cost, i.e., the NLQR problem may not be solvable when one restricts oneself to linear controllers (although it may still be solvable via a nonlinear control).

5.1 Linear Optimal Controllers

In this section we obtain linear optimal controllers by looking for quadratic solutions to the HJE (28). If a quadratic function, given by

$$V(\omega) = \omega' P\omega\,,$$

is a positive definite solution to the HJE, then the corresponding optimal stabilizing controller (recall (29)) is linear and is given by

$$k^*(\omega) = -B'P\omega \tag{38}$$

It can readily be seen that V satisfies the HJE if and only if

$$2\omega' Pf(\omega) + \omega' H' H\omega - \omega' PBB'P\omega = 0, \qquad \forall\ \omega \in \mathbb{R}^3 \tag{39}$$

We are now ready to state the main result for the solution of the NLQR problem via a linear controller.

Theorem 5.1 Consider system (5) with (H, J) observable. A quadratic function, given by $V(\omega) = \omega' P\omega$, satisfies the conditions of Theorem 4.1 if and only if the matrix P is a positive definite solution to the Algebraic Riccati Equation (ARE)

$$H'H - PBB'P = 0 \tag{40}$$

and

$$P = \alpha J + \beta J^2 \tag{41}$$

for some scalars α, β when J not a multiple of the identity matrix.

When these conditions are satisfied, the system is NLQR-solvable via the linear controller (38) and the minimum value of the cost is $\omega_0' P\omega_0$.

Proof. (*Sufficiency*). Since $V(\omega) = \omega' P\omega$ with P a positive definite matrix, the function V is positive definite. Since $P = \alpha J + \beta J^2$ when J is not a multiple of the identity matrix, it follows from Lemma 3.2 that $\omega' P f(\omega) = 0$. ARE (40) now guarantees satisfaction of (39); hence HJE (28) holds. The optimal stabilizing controller is linear and given by equation (38).

(*Necessity*). Suppose that there exists a quadratic function given by $V(\omega) = \omega' P\omega$, which satisfies the conditions of Theorem 4.1. Since V satisfies the HJE, condition (39) must hold. The first term in (39) is a cubic polynomial in ω (recall that $f(\omega)$ is quadratic in ω) and the last two terms are quadratic polynomials in ω. Therefore the left-hand side of equation (39) is the sum of two homogeneous polynomials, one cubic in ω and the other quadratic in ω. Since their sum is zero for all $\omega \in \mathbb{R}^3$, each polynomial must vanish identically for all $\omega \in \mathbb{R}^3$. In other words, (39) holds for all $\omega \in \mathbb{R}^3$ if and only if

$$\omega' P f(\omega) = 0 \quad \text{and} \quad \omega' H' H\omega - \omega PBB'P\omega = 0 \tag{42}$$

for all $\omega \in \mathbb{R}^3$. By Lemma 3.2 the first condition is satisfied if and only if P is of the form $P = \alpha J + \beta J^2$ (when J is not a multiple of the identity). The second condition in (42) is satisfied if and only if P satisfies the algebraic Riccati equation (40). Since V is a positive definite function, the matrix P is positive definite. □

Remark 5.1 Note that (40) is the ARE for the linear quadratic regulator (LQR) problem associated with the linearized system

$$\dot{\omega} = Bu, \qquad \omega(0) = \omega_0 \tag{43a}$$

$$z = \begin{bmatrix} H\omega \\ Du \end{bmatrix} \tag{43b}$$

From standard LQR theory, the linearized problem considered here has a solution if and only if the pair $(0, B)$ is controllable and the pair $(H, 0)$ is observable, i.e., $\operatorname{rank} G = \operatorname{rank} H = 3$. In this case, the optimal controller is given by (38) where P is the positive definite solution to the ARE. Actually, in this case one can show that P is explicitly given by

$$P = W(WBB'W)^{-\frac{1}{2}}W \tag{44}$$

where $W = (H'H)^{\frac{1}{2}}$.

Using the results of Ref. 19, we know that an optimal stabilizing controller for the linearized problem will solve the NLQR problem, at least *locally* for system (5). As a consequence of Theorem 5.1, we see that if there is a solution to the linearized problem, with $P = \alpha J + \beta J^2$ when J is not a multiple of the identity matrix, then the optimal stabilizing controller for the linearized problem is also the (global) optimal stabilizing controller for the nonlinear problem. From Theorem 5.1 we also see that it is not necessary to have a solution to the linearized problem in order to have a solution to the nonlinear problem.

Based on the previous theorem, one can obtain a simple characterization of all state weighting matrices in the cost (7) which guarantee that the sufficient conditions of Theorem 4.1 are satisfied in the nonsymmetric case and (5) is NLQR-solvable via linear control.

Corollary 5.1 Consider system (5) with J not a multiple of the identity matrix and (H, J) observable. Suppose the state weighting matrix $Q = H'H$ in (7) can be written in the form

$$Q = \alpha^2 GG' + \alpha\beta(JGG' + GG'J) + \beta^2 JGG'J \tag{45}$$

for some scalars α, β such that $\alpha I + \beta J > 0$. Then this system is NLQR-solvable via the linear controller

$$k^*(\omega) = -G'(\alpha I + \beta J)\omega \tag{46}$$

For the symmetric case, i.e., when J is a multiple of the identity matrix, P is not required to have any special structure. In this case the system is linear and is given by the linearized system (43). Hence we have the following result.

Lemma 5.1 Consider system (5) with J a multiple of the identity matrix. Then this system is NLQR-solvable via linear control if and only if $\operatorname{rank} H = \operatorname{rank} B = 3$. If this condition is satisfied then the system is NLQR-solvable via the linear controller

$$k^*(x) = -B'P\omega \tag{47}$$

with P given by (44) and the minimum value of the cost is $\omega_0' P\omega_0$.

5.2 Suboptimal Linear Controllers

As in the case of the HJE we restrict our attention here to quadratic solutions of the HJI (36). The following theorem gives necessary and sufficient conditions for the existence of a quadratic V solving the HJI.

Theorem 5.2 Consider system (5) with (H, J) observable. A quadratic function, given by $V(\omega) = \omega' P \omega$, satisfies the conditions of Theorem 4.2 if and only if the matrix P is a positive definite solution to the algebraic Riccati inequality (ARI)

$$H'H - PBB'P \leq 0 \tag{48}$$

and

$$P = \alpha J + \beta J^2 \tag{49}$$

for some scalars α, β when J not a multiple of the identity matrix.

When these conditions are satisfied, the controller given by

$$k^*(\omega) = -B'P\omega \tag{50}$$

renders the closed-loop system globally asymptotically stable about the origin and the cost is bounded above by $\omega_0' P \omega_0$.

Proof. (*Sufficiency*). One can readily show that if a matrix P satisfies the hypotheses of the theorem, then the function $V(\omega) = \omega' P \omega$ satisfies the conditions of Theorem 4.2.

(*Necessity*). Suppose that there exists a quadratic function, given by $V(\omega) = \omega' P \omega$, which satisfies the conditions of Theorem 4.2. Then

$$2\omega' P f(\omega) + \omega' H' H \omega - \omega PBB'P\omega \leq 0, \qquad \forall\, \omega \in \mathbb{R}^3 \tag{51}$$

Let $h_2(\omega) := \omega' H' H \omega - \omega PBB'P\omega$ and $h_3(\omega) := 2\omega' P f(\omega)$. Clearly, h_2 is a homogeneous polynomial of degree 2 and h_3 is a homogeneous polynomial of degree 3. The left-hand side of inequality (51) is the sum of two homogeneous polynomials one of degree 2 and the other of degree 3.

We claim that if $h_3(\omega) + h_2(\omega) \leq 0$ for all $\omega \in \mathbb{R}^3$ then necessarily $h_3(\omega) = 0$ for all $\omega \in \mathbb{R}^3$ and hence, $h_2(\omega) \leq 0$ for all $\omega \in \mathbb{R}^3$. To show this, first notice that by the homogeneity of h_3 and h_2 we have that $h_3(\lambda\omega) = \lambda^3 h_3(\omega)$ and $h_2(\lambda\omega) = \lambda^2 h_2(\omega)$ for all $\omega \in \mathbb{R}^3$ and all $\lambda \in \mathbb{R}$. Suppose now that the claim is false. Then there exist $\bar{\omega} \in \mathbb{R}^3$ such that $h_3(\bar{\omega}) \neq 0$. By the homogeneity of $h_3(\omega)$ and $h_2(\omega)$ we have $\lambda^3 h_3(\bar{\omega}) + \lambda^2 h_2(\bar{\omega}) \leq 0$, for all $\lambda \in \mathbb{R}$, or by dividing by $\lambda^2 \neq 0$ that $\lambda h_3(\bar{\omega}) + h_2(\bar{\omega}) \leq 0$ for all $\lambda \in \mathbb{R}$. Let $\lambda > |h_2(\bar{\omega})|/h_3(\bar{\omega})$ if $h_3(\bar{\omega}) > 0$ or $\lambda < |h_2(\bar{\omega})|/h_3(\bar{\omega})$ if $h_3(\bar{\omega}) < 0$ to arrive to a contradiction.

Using this result, we get that (51) is satisfied for some P if and only if $\omega' P f(\omega) = 0$ for all $\omega \in \mathbb{R}^3$ and $\omega' H' H \omega - \omega PBB'P\omega \leq 0$ for all $\omega \in \mathbb{R}^3$. Equivalently, from Lemma 3.2 equation (51) is satisfied for some P if and only if $P = \alpha J + \beta J^2$ when P is not a multiple of the identity matrix and P satisfies the algebraic Riccati inequality $H'H - PBB'P \leq 0$. Moreover, since $V(\omega) = \omega' P \omega$ is positive definite, P is a positive definite solution of (48). □

Theorem 5.2 along with Theorem 5.1 allow us, without any loss of generality, to dispense with the search of positive definite quadratic solutions to the HJE and HJI and work with the corresponding ARE and ARI instead.

Similarly to Corollary 5.1 we have the following result which characterizes all state weighting matrices for the suboptimal problem.

Corollary 5.2 Consider system (5) with J not a multiple of the identity matrix and (H, J) observable. Suppose the state weighting matrix $Q = H'H$ in (7) satisfies

$$Q \leq \alpha^2 GG' + \alpha\beta(JGG' + GG'J) + \beta^2 JGG'J \tag{52}$$

for some scalars α, β such that $\alpha I + \beta J > 0$. Then the controller

$$k^*(\omega) = -G'(\alpha I + \beta J)\omega \tag{53}$$

renders the closed-loop system globally asymptotically stable about the origin and the cost is bounded above by $\alpha\omega_0' J \omega_0 + \beta {\omega_0}' J^2 \omega_0$.

The structural condition (41) for the solution of (40) is, obviously, very restrictive. It is, nevertheless, as Theorem 5.1 states, a necessary and sufficient condition for the existence of a quadratic V which solves the HJE. If (40)-(41) fails, one may still desire to use a linear controller in attacking the minimization problem (5) subject to (7). Clearly, by restricting optimization to the class of linear controllers the best one can expect is to get an upper bound for the cost (7). Thus from the question of minimizing the cost (7) subject to the dynamics (5) one turns to the question of minimizing an upper bound for this cost using linear controllers via Theorem 5.2.

6 Special Cases

In this section we present some special cases, when the linear controller is indeed the optimal one. In other words, the ARE (40) admits a positive definite solution of the form (41).
Case I: Consider system (1) with

$$G = I$$

and

$$z = \begin{bmatrix} r\omega \\ u \end{bmatrix}, \qquad r > 0$$

Then the problem is that of minimizing

$$\int_0^\infty r^2 \omega'(t)\omega(t) + u'(t)u(t)\, dt \tag{54}$$

subject to the dynamics

$$J\dot{\omega} = (J\omega) \times \omega + u \tag{55}$$

It can be readily shown that in this case $P = rJ$ solves the the ARE (40). Therefore the optimal stabilizing controller in this case is simply given by

$$k^*(\omega) = -r\omega \tag{56}$$

This case has been treated repeatedly in the literature (Refs. 4,6,17). To the authors' knowledge this is the only reported solution to the rigid body NLQR problem thus far.

Case II: Suppose that G and H are *orthogonal matrices*, i.e., $G'G = GG' = I$ and $H'H = HH' = I$. This occurs when the input torques are about mutually perpendicular axes and the components of the output vector $y = H\omega$ are angular velocity components about mutually perpendicular axes. Clearly, in this case $P = J$ is a positive definite solution to ARE (40) and again the optimal controller is linear; it is given by

$$k^*(\omega) = -B'P\omega = -G'\omega \tag{57}$$

Case III: Suppose that $H = G'$. This implies that the actuator input u is *collocated* with the output $y = H\omega$. Again the ARE (40) is solvable with $P = J$. Therefore, the optimal feedback controller is the following simple linear output-feedback controller:

$$k^*(\omega) = -G'\omega = -H\omega = -y \tag{58}$$

7 Optimization Algorithm

A necessary condition for the existence of a positive definite matrix P which solves ARI (48) is that $\operatorname{rank} H \leq \operatorname{rank} G$. In general, the search for a solution to the ARI involves a numerical search algorithm. Since P must have the special structure (49), the search algorithm reduces to a numerical search over the two parameter space (α, β). Unfortunately, the problem is not convex in terms of α, β. On the other hand, the fact that we have only two parameters makes the problem tractable and allows for computation of the global optimizer.

Since an upper bound on the cost is given by $\omega_0' P \omega_0$ for all $\omega_0 \in \mathbb{R}^3$, we consider the following optimization problem:

$$\min_{\alpha,\beta} \ \operatorname{trace}(P) \tag{59}$$

subject to

$$P = \alpha J + \beta J^2, \quad P > 0 \quad \text{and} \quad H'H - PBB'P \leq 0 \tag{60}$$

If (48) is satisfied with equality, then this P solves the (ARE) and yields an optimal controller.

The next example illustrates that although there may not exist a solution of the form (49) which solves the ARE, there may exist a solution of this form solving the ARI.

Example 7.1 Consider system (1) with $J = \mathrm{diag}(2,3,4)$ and

$$B = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \qquad H = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

Since rank $B =$ rank $H = 3$ the unique positive definite solution of the ARE can be computed by (44) to be

$$P = \begin{bmatrix} 0.9268 & -0.0130 & -0.0164 \\ -0.0130 & 0.6766 & -0.1707 \\ -0.0164 & -0.1707 & 2.0374 \end{bmatrix} \tag{61}$$

This matrix is positive definite with eigenvalues 0.6547, 0.9275, 2.0586 but it is not of the form (49) for any $\alpha, \beta \in \mathbb{R}$. However if we choose α, β such as to minimize trace(P) subject to the constraints (60), we obtain the values $\alpha^* = 0.4915$ and $\beta^* = 0.0109$. The positive definite matrix $P = \alpha^* J + \beta^* J^2 = \mathrm{diag}(1.0264, 1.5721, 2.1396)$ satisfies the ARI. The eigenvalues of the matrix $H'H - PBB'P$ for this choice of P are $(0, -0.7067, -26.8513)$.

8 Numerical Example

Consider the system in Example 7.1 and suppose that there is only one torque available and only one rate gyro for angular velocity measurements, both about the axis corresponding to the unit vector $\hat{\mathbf{e}} = [\ 0.5321 \quad 0.2512 \quad 0.6538\ ]'$. Therefore,

$$G = \begin{bmatrix} 0.5321 \\ 0.2512 \\ 0.6538 \end{bmatrix} \tag{62}$$

and the angular velocity measurement is given by $y = G'\omega$. If we choose $H = G'$, the corresponding optimal controller will only require the available measurement y. According to Case III in Section 6, the matrix $P = J = \mathrm{diag}(2,3,4)$ solves the ARE (40) and the optimal stabilizing controller is given by the output feedback controller

$$u = -y \tag{63}$$

Figure 1 contains the results of numerical simulations subject to the initial condition $\omega_0 = (1, -0.5, 1)'$ rad/sec. As expected, the angular velocity tends to zero asymptotically. The running value of the cost is shown in Fig. 2. The minimum value of the cost is $\omega_0' J \omega_0 = 6.75$ which corresponds to the dashed horizontal line in Fig. 2.

9 Conclusions

We have derived conditions for the solution of the optimal and suboptimal quadratic regulation of the nonlinear system which describes the dynamical motion of a rotating rigid body. This is an important example of a nonlinear system whose linearization is, in general, neither detectable nor stabilizable. One thus has to deal with the true nonlinear equations of the system. This system, however, has the property of admitting *globally* asymptotically stabilizing controllers which are *linear*. Motivated by this observation, we search over the class of linear controllers for the one that achieves the best performance.

10 Appendix

Consider a system described by

$$\dot{x} = x \times (Ax) \tag{64a}$$
$$z = Cx \tag{64b}$$

where $x(t) \in \mathbb{R}^3$, $z(t) \in \mathbb{R}^p$ and A, C are constant real matrices of appropriate dimensions. We have the following result.

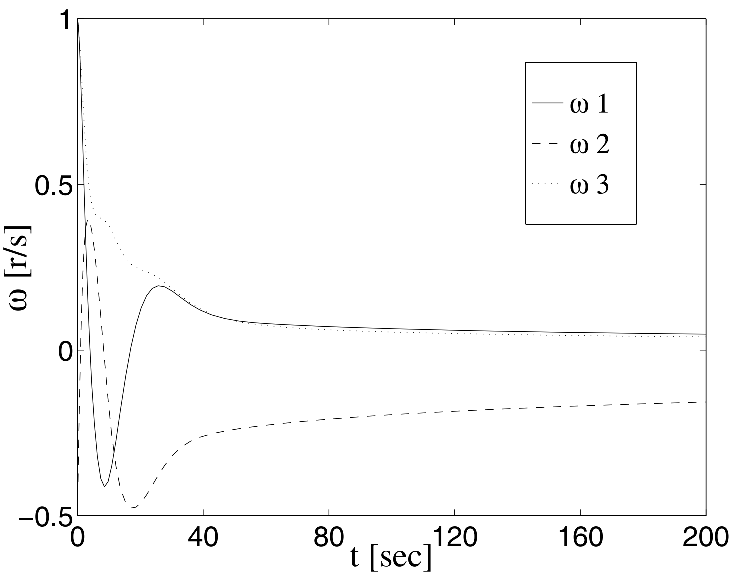

Figure 1: Angular velocity history.

Lemma 10.1 *Consider system (64) with $C \neq 0$. If $x(\cdot)$ is any solution of (64) with $z(t) \equiv 0$, then $x(t) \equiv 0$ or there is an eigenvector v of A such that*

$$\lim_{t\to\infty} x(t) = v \tag{65}$$

Proof. Consider any solution $x(\cdot)$ of system (64) with $z(t) \equiv 0$, i.e. $Cx(t) \equiv 0$. We need to consider different cases depending on the rank of C.

If $\text{rank}(C) = 3$, then the null space of C is trivial and we must have $x(t) \equiv 0$.

If $\text{rank}(C) = 2$, then the null space of C has dimension 1; hence $x(t) = \xi(t)w$ where $Cw = 0$ and $w \neq 0$. Substitution into (64a) and taking the dot product of both sides of the above equation with w yields

$$\dot{\xi}\ ||w||^2 = 0$$

Since $w \neq 0$, we have $\dot{\xi} = 0$; hence $\xi(t) \equiv \xi_0 = \xi(0)$ and $x(\cdot)$ is a constant solution given by $x(t) \equiv v := \xi_0 w$. We also have $v \times (Av) = 0$. If $x(t)$ is not identically zero, then v is nonzero. Since $v \times (Av)$ is zero and v is nonzero, v and Av must be parallel, i.e., there there exist a scalar λ such that

$$Av = \lambda v$$

So, v is an eigenvector of A and (65) holds.

Consider now the case in which $\text{rank}(C) = 1$. First note that the structure of system (64) is invariant under a right-handed orthogonal state transformation; i.e., suppose one introduces a new state $\tilde{x}$ defined by $x = T\tilde{x}$ where $T'T = I$ and $\det T = 1$ then, the resulting description will be of the form

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{x} \times (\tilde{A}\tilde{x}) \\ z &= \tilde{C}\tilde{x} \end{aligned}$$

where $\tilde{A} = T'AT$, $\tilde{C} = CT$. So, without loss of generality we suppose that $Cx = 0$ is equivalent to $x_3 = 0$. Suppose $x(\cdot)$ is a solution for which $z(t) \equiv 0$. Let a_{ij} denote the ij-th element of the matrix A. If we write equation (64a) in scalar form and let $x_3 = 0$, we obtain

$$\begin{aligned} \dot{x}_1 &= (a_{31}x_1 + a_{32}x_2)x_2 \\ \dot{x}_2 &= -(a_{31}x_1 + a_{32}x_2)x_1 \end{aligned}$$

If $a_{31} = a_{32} = 0$, then $\dot{x}_1 \equiv \dot{x}_2 \equiv 0$, hence $x(t) \equiv v := x(0)$. Consider now $(a_{31}, a_{32}) \neq (0, 0)$ and introduce new variables $\eta_1 = a_{32}x_1 - a_{31}x_2$ and $\eta_2 = a_{31}x_1 + a_{32}x_2$. The evolution of these variables is

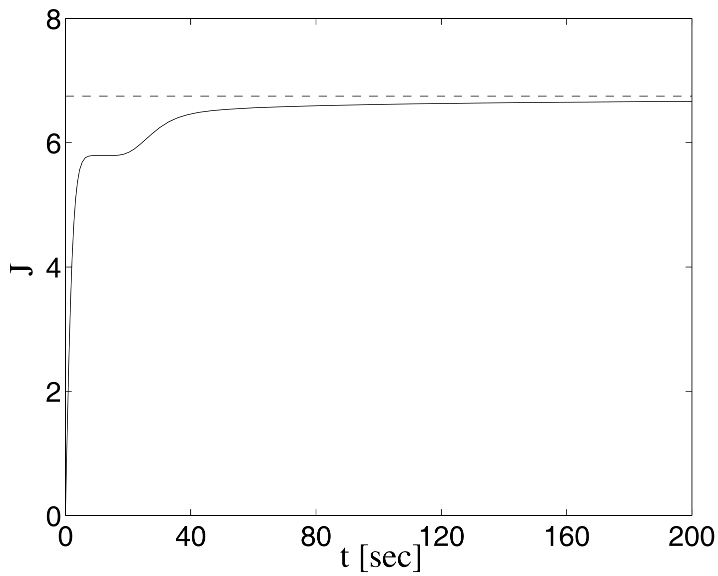

Figure 2: Running cost.

described by

$$\begin{aligned}\dot{\eta}_1 &= \eta_2^2 \\ \dot{\eta}_2 &= -\eta_1\eta_2\end{aligned}$$

If $\eta_{20} = \eta_2(0) = 0$ then $\eta_2(t) \equiv 0$ and $\eta_1(t) \equiv \eta_{10} = \eta_1(0)$ and the solution for η is an equilibrium solution given by $(\eta_1(t), \eta_2(t)) \equiv (\eta_{10}, 0)$; so, (10) holds. Consider now $\eta_{20} \neq 0$. It may readily be verified that along any solution the quantity $\eta_1^2 + \eta_2^2$ is constant. This implies that all non-equilibrium solutions lie on a circle of radius $\sqrt{\eta_{10}^2 + \eta_{20}^2}$; hence all solutions are bounded. The boundedness of η and the first differential equation implies that we must have $\lim_{t\to\infty} \eta_2(t) = 0$. From this we conclude that $\lim_{t\to\infty} \eta_1(t) = (\eta_{10}^2 + \eta_{20}^2)^{\frac{1}{2}}$. So, for this case, we have shown that either $x(t) \equiv 0$ or conclusion (65) holds for some constant vector v. Using differential equation (64a) one can then show that $v \times (Av) = 0$. As we have argued in the previous case, this implies that v is an eigenvector of A. □

References

1. Scrivener, S. L., and Thomson, R. C., *Survey of Time-Optimal Attitude Maneuvers*, Journal of Guidance, Control, and Dynamics, Vol. 17, pp. 225–233, 1994.
2. Junkins, J. L., and Turner, J., *Optimal Spacecraft Rotational Maneuvers.* Elsevier, New York, New York 1986.
3. Dixon, M. V., Edelbaum, T., Potter, J. E., and Vandervelde, W. E., *Fuel Optimal Reorientation of Axisymmetric Spacecraft*, Journal of Spacecraft and Rockets, Vol. 7, pp. 1345–1351, 1970.
4. Athans, M., Falb, P. L., and Lacoss, R. T., *Time-, Fuel-, and Energy-Optimal Control of Nonlinear Norm-Invariant Systems*, IRE Transactions on Automatic Control, Vol. 8, pp. 196–202, 1963.
5. Debs, A. S., and Athans, M., *On the Optimal Angular Velocity Control of Asymmetrical Space Vehicles*, IEEE Transactions on Automatic Control, Vol. 14, pp. 80–83, 1969.
6. Kumar, K. S. P., *On the Optimum Stabilization of a Satellite*, IEEE Transactions on Aerospace Electronic Systems, Vol. 1, pp. 82–83, 1965.
7. Windeknecht, T. G., *Optimal Stabilization of Rigid Body Attitude*, Journal of Mathematical Analysis and Applications, Vol. 6, pp. 325–335, 1963.

8. Lu, W. M., and Doyle, J. C., *$\mathcal{H}_\infty$-Control of Nonlinear Systems via Output Feedback: A Class of Controllers*, Proceedings of the 32nd Conference on Decision and Control, San Antonio, Texas, pp. 166–171, 1993.

9. van der Schaft, A. J., *On a State Space Approach to Nonlinear $\mathcal{H}_\infty$ Control*, Systems and Control Letters, Vol. 16, pp. 1–8, 1991.

10. Byrnes, C. I., Isidori, A., and Willems, J. C., *Passivity, Feedback Equivalence, and the Global Stabilization of Minimum Phase Nonlinear Systems*, IEEE Transactions on Automatic Control, Vol. 36, pp. 1228–1240, 1991.

11. Hill, D., and Moylan, P., *The Stability of Nonlinear Dissipative Systems*, IEEE Transactions on Automatic Control, Vol. 21, pp. 708–711, 1976.

12. Kailath, T., *Linear Systems.* Prentice-Hall, Englewood Cliffs, New Jersey, 1980.

13. Vidyasagar, M., *Nonlinear Systems Analysis.* Prentice-Hall, Englewood Cliffs, New Jersey, 1978.

14. Al'brekht, E. G., *On the Optimal Stabilization of Nonlinear Systems*, Journal of Applied Mathematics and Mechanics, Vol. 25, pp. 1254–1266, 1962.

15. Lukes, D. L., *Optimal Regulation of Nonlinear Dynamical Systems*, SIAM Journal on Control and Optimization, Vol. 7, pp. 75–100, 1969.

16. Willemstein, A. P., *Optimal Regulation of Nonlinear Dynamical Systems on a Finite Interval*, SIAM Journal on Control and Optimization, Vol. 15, pp. 1050–1069, 1977.

17. Yoshida, T., and Loparo, K. A., *Quadratic Regulatory Theory for Analytic Nonlinear Systems with Additive Controls*, Automatica, Vol. 25, pp. 531–544, 1989.

18. Rotea, M., Tsiotras, P., and Corless, M., *Suboptimal Control of Rigid Body Motion with a Quadratic Cost*, 3rd IFAC Nonlinear Symposium on Nonlinear Control Systems Design, Tahoe City, California, 1995.

19. van der Schaft, A. J., *$\mathcal{L}_2$-Gain Analysis of Nonlinear Systems and Nonlinear State Feedback $\mathcal{H}_\infty$ Control*, IEEE Transactions on Automatic Control, Vol. 37, pp. 770–784, 1992.