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# Time-Optimal Control of Axi-Symmetric Rigid Spacecraft Using Two Controls

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**In this paper, we consider the minimum-time reorientation problem of an axi-symmetric rigid spacecraft with two independent control torques mounted perpendicular to the spacecraft symmetry axis. The objective is to reorient the spacecraft from an initial attitude, with some angular velocity, to a final attitude with a certain angular velocity in minimum time. All possible control structures, including both singular and nonsingular arcs, are studied completely by deriving the corresponding formulae and the necessary optimality conditions. It is shown that a second order singular control can be part of the optimal trajectory. It is also shown that for an inertially symmetric and a non-spinning axi-symmetric rigid body, it is possible for infinite-order singular controls to be part of or the whole optimal trajectory. In particular, for a non-spinning axi-symmetric rigid body, the second-order singular trajectory is shown to be an eigenaxis rotation. An efficient method for numerically solving the optimal control problem, based on a cascaded computational scheme which uses both a direct method and an indirect method, is also presented. Numerical examples demonstrate optimal reorientation maneuvers with both nonsingular and singular subarcs, and comparisons are made between eigenaxis rotations and the true time-optimal rotations.**

## Introduction

In recent years, the time-optimal reorientation problem of a rigid spacecraft has been extensively studied by many researchers. In Ref. 1, the minimum-time attitude slewing of a rigid spacecraft is considered. Quasi-linearization is used to solve the Two-Point Boundary-Value Problem (TPBVP) arising from Pontryagin's Minimum Principle. An integral of a quadratic function of the control inputs is used as the performance index instead of the slewing time. The minimum slewing time is determined by sequentially shortening the final time. The corresponding fixed final time problem is solved until the solution can no longer be obtained, or until all the resulting controls are bang-bang. The Euler (eigenaxis) rotation maneuver is used as the initial guess for the numerical computation. Besides the fact that the bang-bang solutions show that the minimum-time trajectories are far from an eigenaxis rotation, numerical results also make the authors suspect that singular controls may appear for a single principal axis rotation of a symmetric body. If this is the case, the singular trajectory is a rotation about a principal axis, i.e., an eigenaxis rotation. In their following work,[2] the authors of Ref. 1 implemented their method to the Naval Research Laboratory's Reconfigurable Spacecraft Host for Attitude and Pointing Experiment (RESHAPE) three-axis maneuver facility. The results of these experiments were presented in Ref. 2.

In Ref. 3, Bilimoria and Wie studied the time-optimal, rest-to-rest, large-angle, principal-axis rotation of an inertially symmetric rigid body. By solving the TPBVP using a shooting method, they obtained a variety of bang-bang controls which showed that the eigenaxis rotation is not time-optimal, in general. Singular controls are considered only in the sense that it is shown that all three controls cannot be singular simultaneously. Later, they extended their work to an axi-symmetric rigid body (with three control torques) and they also studied the principal axis rotation.[4] The emphasis in this latter work is on the effect of the gyroscopic terms in Euler's equations on minimum time. Comparing with the minimum final time obtained for a system with the gyroscopic terms dropped, they showed that the gyroscopic effect increases the final time for a rod-like body and decreases the final time for a disk-like body.

Seywald and Kumar[5] extended the work in this area by analyzing all the possible controls for a general minimum-time reorientation problem of an inertially symmetric rigid body. An elegant derivation of all the possible controls, including bang-bang control subarcs, finite-order singular control subarcs and infinite-order singular control subarcs was developed. It was shown that for rest-to-rest maneuvers, the eigenaxis rotation can appear as a finite-order singular arc, but it is not optimal. It follows that an eigenaxis rotation can, in fact, appear as an optimal infinite-order singular arc.

Scrivener and Thompson[6] explored the minimum-time reorientation of a rigid spacecraft numerically, using a direct method via collocation and nonlinear programming. This method was first introduced by Hargraves and Paris.[7] Instead of dealing with the necessary conditions from Pontryagin's Minimum Principle, the trajectory is first discretized and the optimal trajectory is found in the finite dimensional space of the states and controls at each node using nonlinear programming. The method is shown to be robust in the sense that it does not require accurate initial guesses. Scrivener and Thompson applied this method to the time-optimal, rest-to-rest maneuver of a rigid spacecraft. Comparison was made between their results and the ones in Ref. 4. The results were consistent except the case when the maneuver has a reorientation angle of less than 10 deg. It turns out that in this case – although the maneuver time is the same – the switching structure is different, indicating, possibly, a multiple local minimum of the discretized problem.

Jahangir and Howe[8] considered the problem of controlling a spinning missile in minimum time. The missile was modeled as an axi-symmetric rigid body which is spinning about its symmetry axis. The task was to control the missile from some initial attitude and transverse angular velocity to some final attitude and zero transverse angular velocity, with only a single reaction jet. Instead of solving the TPBVP, the authors integrated the state and co-state equations backwards in time to generate all possible trajectories in a reachable set. The control was assumed to be on-off for the reaction jet. A data storage scheme and a look-up strategy were used to implement this control scheme.

A few researchers have worked with a method called Switching Time Optimization (STO). STO was used by Meier and Bryson[9] to solve the time-optimal control of a two-link manipulator. Byers[10,11] used STO to solve the time-optimal rigid body reorientation problem. More recently, Liu and Singh[12] addressed the weighted time/fuel optimal control of an inertially symmetric spacecraft performing a rest-to-rest maneuver. The authors modified the

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STO method to determine the switching times and total maneuver time of the bang-off-bang control profiles. The results were compared with those of Bilimoria et al. in Ref. 3. The effect of the fuel penalty in the cost on the number of switching times was discussed and an interesting result was presented; namely, as the fuel penalty is beyond a specific value, the eigenaxis control with two switches was shown to be optimal. An apparent drawback of STO is that the switching structure, i.e., the number of switches and how the controls switch, has to be guessed or known in advance. An approach similar to STO has also been used by Ben-Asher *et al.*[13] and Singh *et al.*[14] to compute time-optimal solutions of slewing maneuvers of flexible spacecraft.

In relation to the work in this paper, two articles are of particular interest. First, Chowdhry and Cliff[15] considered the time-optimal reorientation of a rigid body with two control torques. The existence of singular subarcs was also studied. However, the authors only considered the time-optimal control of rigid body angular rates, i.e., they considered only the dynamics of the motion. Hermes and Hogenson[16] studied the same system as the one in the present work. They applied feedback linearization to transform the system to two uncoupled linear double-integrators. The time-optimal controls can then be calculated explicitly.[17] The result is then transformed back to the original space to obtain the explicit feedback control for the original nonlinear system. However, due to the complex relationship between the original controls and the transformed ones, as well as the corresponding control bounds, the controls for the original system are not necessarily time optimal, as pointed out by the authors in their conclusions. In addition, since a double-integrator system has at most one switch and no singular subarcs (and the feedback linearization transformation between the original and the resulting systems is continuous), this method leads to time-optimal controls for the original nonlinear system with at most two switches and no singular subarcs. This is shown *not* to be true in this paper.

In this paper, we address the time-optimal reorientation problem for an axi-symmetric rigid body. The purpose of the control is to drive the symmetry axis from some initial orientation, with some specified angular velocity, to another final orientation, with specified angular velocity. We assume that the relative orientation of the body about the symmetry axis is irrelevant and only the location of the symmetry axis is of interest. This could be the case when the symmetry axis coincides with the boresight or line-of-sight of a camera, an optical telescope, or a gun barrel, for example. Clearly, the relative rotation of the camera or the barrel has no influence on the clarity of the photograph or the accuracy of the projectile. Spin-stabilized spacecraft also fall into this category.

For the axi-symmetric case it turns out that the objective of optimal reorientation of the symmetry axis can be achieved using only two torques about axes that span the plane perpendicular to the symmetry axis. For simplicity, we consider the time-optimal reorientation of an axi-symmetric rigid spacecraft with two control torques acting perpendicular to the symmetry axis and to each other. It can be shown that in this case, the torque axes are also principal directions of the moment of inertia matrix. The spacecraft may be spinning about its symmetry axis with a constant angular velocity. The main effort in this paper is devoted on analyzing the formulae for the possible bang-bang and singular control subarcs and the corresponding necessary conditions.

The paper is organized in the following manner. First, the system model is given and the problem formulation is presented. The optimality conditions are derived from Pontryagin's Minimum Principle, and the singular arcs are analyzed completely. We then apply the analysis to the special case of an inertially symmetric rigid body with two controls and the case of a non-spinning axi-symmetric rigid body. A cascaded computational scheme is discussed in the sequel for numerically solving for the optimal trajectories. This scheme avoids the often intractable task of finding "good" initial guesses for the states, co-states and the optimal switching structure. The numerical results at the end of the paper show the effectiveness of the proposed computational scheme applied to the time-optimal reorientation problem of an axi-symmetric spacecraft with two controls.

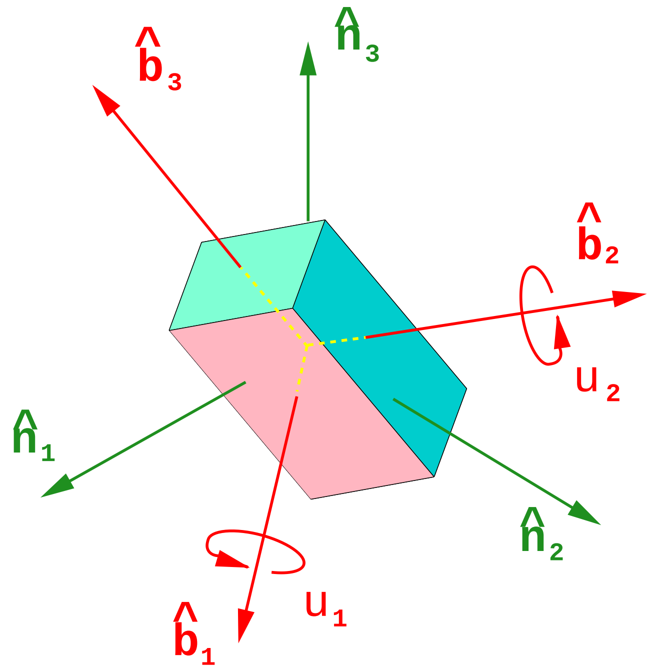


**Fig. 1 Axi-symmetric rigid body with two controls.**

## Problem Formulation

Consider an axi-symmetric rigid body with two control torques as shown in Fig. 1. A body-fixed reference frame $\hat{\mathbf{b}} = (\hat{b}_1, \hat{b}_2, \hat{b}_3)$ is defined with the unit vector $\hat{b}_3$ pointing along the symmetry axis. The control system generates two control torques $T_1$ and $T_2$ along the $\hat{b}_1$ and $\hat{b}_2$ axis, respectively, as shown in Fig. 1. Let $\omega = (\omega_1, \omega_2, \omega_3)^T \in R^3$ denote the angular velocity vector in the $\hat{\mathbf{b}}$ frame, and $I_1$, $I_2$, $I_3$ be the moments of inertia with respect to the three axes just defined. Then Euler's equations[18] with respect to this frame take the form

$$I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3 + T_1 \tag{1a}$$
$$I_2\dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1 + T_2 \tag{1b}$$
$$I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2 \tag{1c}$$

Since $I_1 = I_2$, if we let the initial condition $\omega_3(0) = \omega_{30}$, $\omega_3$ will remain constant throughout the maneuver, and the equations reduce to

$$\dot{\omega}_1 = a\omega_{30}\omega_2 + u_1 \tag{2a}$$
$$\dot{\omega}_2 = -a\omega_{30}\omega_1 + u_2 \tag{2b}$$

where $a = (I_2 - I_3)/I_1$ and $u_1$ and $u_2$ are the new control inputs given by $u_i = T_i/I_i$, $i = 1, 2$. Note that for physical systems we always have that $-1 < a < 1$.

As shown by Tsiotras and Longuski,[19] if $\hat{\mathbf{n}} = (\hat{n}_1, \hat{n}_2, \hat{n}_3)$ denotes the inertial reference frame, then the position of the $\hat{n}_3$ inertial axis in the body fixed $\hat{\mathbf{b}}$ frame can be uniquely described by two variables $w_1$ and $w_2$ which are defined as

$$w_1 = \frac{\beta}{1+\gamma}, \qquad w_2 = \frac{-\alpha}{1+\gamma} \tag{3}$$

where $\alpha$, $\beta$, and $\gamma$ denote the direction cosines of axis $\hat{n}_3$ with respect to $\hat{\mathbf{b}}$ frame, i.e., $\hat{n}_3 = \alpha\hat{b}_1 + \beta\hat{b}_2 + \gamma\hat{b}_3$. It can be shown[19] that $w_1$ and $w_2$ can be obtained by stereographically projecting the unit vector $\hat{n}_3$ onto the $\hat{b}_1$-$\hat{b}_2$ plane.[20] Moreover, $w_1$ and $w_2$ obey the following differential equations[19, 20]

$$\dot{w}_1 = \omega_{30} w_2 + \omega_2 w_1 w_2 + \frac{\omega_1}{2}(1 + w_1^2 - w_2^2) \quad (4a)$$
$$\dot{w}_2 = -\omega_{30} w_1 + \omega_1 w_1 w_2 + \frac{\omega_2}{2}(1 + w_2^2 - w_1^2) \quad (4b)$$

Given the system of Eqs. (2) and (4) we seek to minimize the performance index

$$\min_{u\in U} J = \min \int_{t_0}^{t_f} 1 \, dt \quad (5)$$

subject to the initial conditions

$$\omega(0) = [\omega_1(0), \omega_2(0)]^T \quad \text{given}$$
$$w(0) = [w_1(0), w_2(0)]^T \quad \text{given}$$

and final conditions

$$\Psi\left[\omega(t_f), w(t_f)\right] = 0 \quad (6)$$

where $\Psi : R^4 \mapsto R^k$, $k \le 4$ is a given smooth vector function, and the control constraint set is given by

$$U = \{u \ : \ |u_i| \le u_{i\max}, \quad i = 1, 2\} \quad (7)$$

with $u_{i\max} > 0$, $i = 1, 2$.

This problem has the following physical interpretation. For an observer in the $\hat{\mathbf{b}}$ frame, the location of the inertial $\hat{n}_3$ axis is given by $w_1$ and $w_2$. The time-optimal control problem then consists of reorienting the spacecraft between given relative locations of the $\hat{n}_3$ (expressed in the body frame). We point out that using only Eqs. (2) and (4) it is not possible to specify the absolute orientation of the spacecraft in the inertial frame. In particular, it is not possible to determine the relative orientation of the spacecraft about the $\hat{n}_3$ axis. That would require, of course, a third attitude parameter to complement $w_1$ and $w_2$.[19] Such “reduced” attitude information may be sufficient, for example, in case of reorientation of the symmetry axis of an axi-symmetric spacecraft along a given direction (e.g., the line-of-sight of an optical telescope).

## Optimality Conditions

For simplicity, let the state vector $[\omega_1, \omega_2, w_1, w_2]^T \in R^4$ be denoted by the vector $[x_1, x_2, x_3, x_4]^T \in R^4$ and let $m = \omega_{30}$. Then Eqs. (2) and (4) are re-written as

$$\dot{x}_1 = amx_2 + u_1 \quad (8a)$$
$$\dot{x}_2 = -amx_1 + u_2 \quad (8b)$$
$$\dot{x}_3 = mx_4 + x_2 x_3 x_4 + \frac{x_1}{2}(1 + x_3^2 - x_4^2) \quad (8c)$$
$$\dot{x}_4 = -mx_3 + x_1 x_3 x_4 + \frac{x_2}{2}(1 + x_4^2 - x_3^2) \quad (8d)$$

The Hamiltonian $H$ for this problem is defined by

$$\begin{aligned} H &= 1 + \lambda_1 \dot{x}_1 + \lambda_2 \dot{x}_2 + \lambda_3 \dot{x}_3 + \lambda_4 \dot{x}_4 \\ &= 1 + am\lambda_1 x_2 + \lambda_1 u_1 - am\lambda_2 x_1 + \lambda_2 u_2 \\ &\quad + m\lambda_3 x_4 + \lambda_3 x_2 x_3 x_4 + \lambda_3 x_1 (1 + x_3^2 - x_4^2)/2 \\ &\quad - m\lambda_4 x_3 + \lambda_4 x_1 x_3 x_4 + \lambda_4 x_2 (1 + x_4^2 - x_3^2)/2 \end{aligned} \quad (9)$$

The co-state equations, defined by $\dot{\lambda} = -(\partial H/\partial x)^T$, are

$$\dot{\lambda}_1 = am\lambda_2 - \lambda_3(1 + x_3^2 - x_4^2)/2 - \lambda_4 x_3 x_4 \quad (10a)$$
$$\dot{\lambda}_2 = -am\lambda_1 - \lambda_4(1 + x_4^2 - x_3^2)/2 - \lambda_3 x_3 x_4 \quad (10b)$$
$$\dot{\lambda}_3 = -\lambda_3 x_2 x_4 - \lambda_3 x_1 x_3 + m\lambda_4 - \lambda_4 x_1 x_4 + \lambda_4 x_2 x_3 \quad (10c)$$
$$\dot{\lambda}_4 = -\lambda_3 x_2 x_3 + \lambda_3 x_1 x_4 - m\lambda_3 - \lambda_4 x_1 x_3 - \lambda_4 x_2 x_4 \quad (10d)$$

with $\lambda(t_f)$ given by the transversality condition

$$\lambda^T(t_f) = \nu^T \frac{\partial \Psi}{\partial x(t_f)} \quad (11)$$

where $\nu \in R^k$ is a constant multiplier vector.

From Pontryagin’s Minimum Principle,[17] the optimal control $u^*$ is chosen such that the Hamiltonian $H$ is minimized, i.e.,

$$u^*(t) = \arg \min_{u\in U} H(x(t), \lambda(t), u), \qquad \forall t \ge 0 \quad (12)$$

Since both controls $u_1$ and $u_2$ appear only linearly in the Hamiltonian $H$, the optimal control $u_i^*$ is given by

$$u_i^* = \begin{cases} +u_{i\max} & \text{if } \lambda_i < 0 \\ -u_{i\max} & \text{if } \lambda_i > 0 \\ \text{singular} & \text{if } \lambda_i \equiv 0 \end{cases} \qquad i = 1, 2 \quad (13)$$

The transversality condition associated with the final time $t_f$ is given by

$$H(t_f) = 0 \quad (14)$$

Equation (9) shows that the Hamiltonian $H$ is not an explicit function of time $t$, hence $H(t) \equiv 0$, for $t \in [t_0, \ t_f]$.

## Singular Control Analysis

Let $S_i$, $i = 1, \ 2$ be the switching functions, defined by

$$S_i = \frac{\partial H}{\partial u_i}, \qquad i = 1, 2 \quad (15)$$

Here $S_i = \lambda_i$, $i = 1, \ 2$. From Eq. (13), $u_i^*$ is singular whenever $S_i \equiv 0$ during a nonzero interval $[t_1, \ t_2] \subset [t_0, \ t_f]$. In this case the control component $u_i^*$ is determined implicitly by the condition $S_i \equiv 0$. Indeed, $u_i^*$ can be obtained by differentiating $S_i \equiv 0$ until the control component $u_i$ appears explicitly.[21] For $u_i^*$ to be optimal, it is required that $S_i$ be differentiated an *even* number of times[5, 22, 23] Therefore, $u_i^*$ can be determined by solving

$$u_i^* = \arg \left\{ \left( \frac{d^{2k_i} S_i}{dt^{2k_i}} \right) = 0 \right\}, \qquad i = 1, 2 \quad (16)$$

where $2k_i$ is the least number of differentiations of $S_i$ that are required until the corresponding $u_i$ appears. It is also evident that the switching functions and their time derivatives up to $(2k_i - 1)$th order are zero along the singular subarc $[t_1, \ t_2]$, i.e., $S_i = \dot{S}_i = \ldots = S_i^{(2k_i-1)} = 0$, $\ t \in [t_1, \ t_2]$. In addition, Kelley’s optimality condition[23, 24] (also known as the Generalized Legendre-Clebsch condition)

$$(-1)^{k_i} \frac{\partial}{\partial u_i} \left[ \frac{d^{2k_i} S_i}{dt^{2k_i}} \right] \ge 0 \quad (17)$$

has to be satisfied along an optimal singular subarc.

A complete analysis of all possible singular control cases, i.e., with two and only one control being singular, is presented in the following two subsections. Before proceeding with this analysis, note that $\dot{S}_1$ and $\dot{S}_2$ are given by Eqs. (10a) and (10b), respectively, and that

$$\ddot{S}_1 = am^2\lambda_1 + m(1 + a)\dot{\lambda}_2 + (\lambda_3 x_4 - \lambda_4 x_3) x_2 \quad (18a)$$
$$\ddot{S}_2 = am^2\lambda_2 - m(1 + a)\dot{\lambda}_1 - (\lambda_3 x_4 - \lambda_4 x_3) x_1 \quad (18b)$$

Another important equation which we will frequently use in the derivation is

$$\frac{d}{dt}(\lambda_3 x_4 - \lambda_4 x_3) = (am\lambda_1 + \dot{\lambda}_2) x_1 + (am\lambda_2 - \dot{\lambda}_1) x_2 \quad (19)$$

which can be readily derived from Eq. (8) and Eq. (10). Henceforth, it will be assumed that $m \ne 0$ and $a \ne 0$; namely, the rigid body is *not inertially* symmetric and it is spinning about its symmetry axis with a nonzero angular velocity $\omega_3 = m$. The cases $a = 0$ and/or $m = 0$ are treated separately in Section .

**Case I: both $u_1$ and $u_2$ are singular**

From Eq. (13), when both $u_1$ and $u_2$ are singular during $t \in [t_1,\ t_2] \subset [t_0,\ t_f]$, we have

$$S_1 = \lambda_1 \equiv 0, \qquad S_2 = \lambda_2 \equiv 0 \tag{20}$$

which imply that

$$\dot{S}_1 = \dot{\lambda}_1 \equiv 0, \qquad \dot{S}_2 = \dot{\lambda}_2 \equiv 0 \tag{21}$$

Substitution of Eqs. (20) and (21) into Eq. (9) yields

$$H = 1 + m(\lambda_3 x_4 - x_3 \lambda_4) \tag{22}$$

and substitution of Eqs. (20) and (21) into Eq. (18) yields

$$\ddot{S}_1 = (\lambda_3 x_4 - \lambda_4 x_3) x_2, \quad \ddot{S}_2 = -(\lambda_3 x_4 - \lambda_4 x_3) x_1 \tag{23}$$

From Eqs. (14) and (22), we have that $H \equiv 0$ implies that $(\lambda_3 x_4 - x_3 \lambda_4) \neq 0$. Thus, during the singular arc $[t_1,\ t_2]$, Eq. (23) imply $x_2 = 0$ and $x_1 = 0$. Taking the third time derivative of $S_1$ and $S_2$ and using Eqs. (8a) and (8b), we get explicit expressions for controls $u_1$ and $u_2$. In particular, $S_1^{(3)} = 0$ implies that $u_2 = 0$ and $S_2^{(3)} = 0$ implies that $u_1 = 0$. The control $u_2$ appears in the third time derivative of $S_1$ and $u_1$ appears in the third time derivative of $S_2$. Since the controls appear after differentiating the switching functions an odd number of times, the controls are not optimal.[22]

The previous analysis has shown that the case when both $u_1$ and $u_2$ are singular is ruled out. Nevertheless, a more careful look reveals that this case may be very close to optimal for large values of the spin-rate $m$. To this end, recall that along a singular arc, the conditions $\lambda_1 = \lambda_2 = \dot{\lambda}_1 = \dot{\lambda}_2 = 0$ should be satisfied. Thus, Eqs. (10a) and (10b) imply that

$$\begin{bmatrix} -(1 + x_3^2 - x_4^2)/2 & -x_3 x_4 \\ -x_3 x_4 & -(1 + x_4^2 - x_3^2)/2 \end{bmatrix} \begin{bmatrix} \lambda_3 \\ \lambda_4 \end{bmatrix} = 0 \tag{24}$$

Since $(\lambda_3, \lambda_4) \neq (0,0)$ the previous coefficient matrix must be singular. Calculating the determinant of this matrix, one obtains that a singular arc with both $u_1 = u_2 = 0$ can occur only if $x_3^2 + x_4^2 = 1$.

Using the fact that $x_1 = x_2 = 0$ we have that along a singular arc the system equations are given by

$$\dot{x}_3 = m x_4 \tag{25a}$$
$$\dot{x}_4 = -m x_3 \tag{25b}$$

The solution of the previous system is given by

$$\begin{bmatrix} x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} \cos mt & \sin mt \\ -\sin mt & \cos mt \end{bmatrix} \begin{bmatrix} x_3(0) \\ x_4(0) \end{bmatrix} \tag{26}$$

Note that if $x_3^2(0) + x_4^2(0) = 1$ then $x_3^2(t) + x_4^2(t) = 1$ for all $t \geq 0$. This implies that singular subarcs indeed exist for specific boundary conditions. From the definition of the state vector, the condition $x_3^2 + x_4^2 = 1$ corresponds to the case when the inertial $\hat{n}_3$ and the body $\hat{b}_3$ axes are perpendicular to each other. For an observer in the spacecraft frame, the inertial $\hat{n}_3$ axis rotates about $\hat{b}_3$ axis at a constant rate $-m$ rad/sec in the $\hat{b}_1 - \hat{b}_2$ plane. Both initial and final conditions correspond to different locations of the $\hat{n}_3$ axis in the $\hat{b}_1 - \hat{b}_2$ plane. The singular solution suggests letting the body coast from the initial to the final position by a pure rotation about the $\hat{b}_3$ axis. This situation is shown in Fig. 2.

The time to complete the maneuver can be calculated explicitly from Eq. (26). For example, in case $(x_3(0), x_4(0)) = (1,0)$ and $(x_3(t_f), x_4(t_f)) = (0,-1)$ one obtains that

$$t_f = \frac{\pi}{2m} \tag{27}$$

The optimality conditions in this section have shown that this maneuver is not optimal. Indeed, simulations using

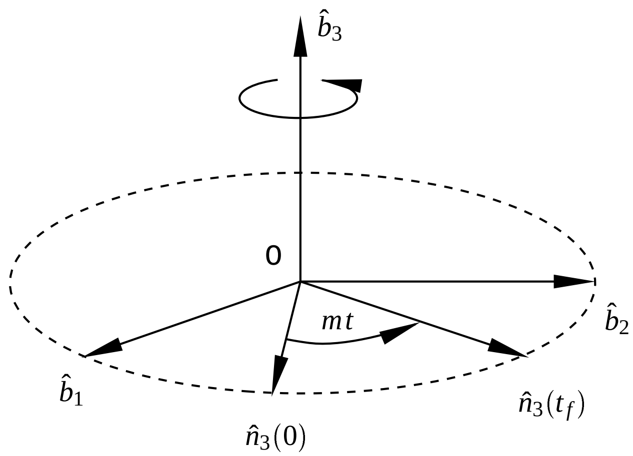


**Fig. 2 Singular control maneuver ($u_1 = u_2 = 0$).**

the numerical scheme described in Section  showed that a bang-bang solution consisting of one switch for each $u_1$ and $u_2$ gives a better (smaller) final time, i.e., an additional nutation decreases the maneuver time. Table 1 gives the numerical results for $u_{i\max} = 1,\ i = 1,2$ and for initial and final conditions as above. Note that these results are valid for any $u_{i\max}$. As the upper bound of the control $u_{i\max}$ is decreased, the bang-bang (optimal) solution increases and approaches the coasting maneuver as $u_{i\max} \to 0$.

**Table 1 Comparison between singular and bang-bang solutions.**

| $m$ (rad/sec) | $t_f$(singular) | $t_f$(bang-bang) |
|---|---|---|
| 0.5 | 3.141592 | 2.328614 |
| 1.0 | 1.570796 | 1.507866 |
| 1.5 | 1.047197 | 1.038082 |
| 2.0 | 0.785398 | 0.783575 |
| 4.0 | 0.392699 | 0.392674 |
| 6.0 | 0.261799 | 0.261798 |
| 10.0 | 0.157080 | 0.157080 |
| $\infty$ | 0 | 0 |

The results in Table 1 show that the difference of the final time between the coasting maneuver in Eq. (26) and the bang-bang solution decreases as the spin-rate $m$ increases. This agrees with our intuition. For this example, the two solutions give essentially the same value of $t_f$ for $m = 10$ rad/sec. For values above $m = 10$ rad/sec numerical issues prevent accurate calculation of the optimal trajectory using EZopt (see Section  for a discussion on the numerical scheme used in this paper to calculate the optimal trajectories). The results in Table 1 correspond to an inertia parameter $a = 0.5$, but similar results were obtained for other values of $a$. Therefore, the results are generic regardless of whether the body is prolate or oblate. We can conclude that the singular maneuver is not optimal in this case. In practice, however, when the spinning rate is high, this coasting maneuver can be used as a suboptimal minimum-time maneuver.

**Case II: only $u_1$ is singular**

In the previous subsection, we have ruled out the possibility that both controls $u_1$ and $u_2$ become singular at the same time. This observation is in accordance with similar results for the inertially symmetric case with three controls.[5] In this subsection we will assume that only $u_1$ is singular during some interval $[t_1,\ t_2] \subset [t_0,\ t_f]$, while $u_2$ is bang-bang. From Eq. (13) we therefore have $S_1 = \lambda_1 = 0$ which implies that along the singular arc, $\dot{S}_1 = \dot{\lambda}_1 = 0$. Substitution of these two equations into Eq. (18a) yields $\ddot{S}_1 = m(1+a)\dot{\lambda}_2 + (\lambda_3 x_4 - \lambda_4 x_3) x_2$. Because the control $u_1$ does not appear in the equation of $\ddot{S}_1$, we have to take the third derivative of $S_1$,

$$S_1^{(3)} = am^3(1+a)\lambda_2 - m(1+2a)(\lambda_3 x_4 - \lambda_4 x_3) x_1 +$$

$$x_1 x_2 \dot{\lambda}_2 + amx_2^2 \lambda_2 + u_2(\lambda_3 x_4 - \lambda_4 x_3) \tag{28}$$

where we have made use of the fact that $\lambda_1 = \dot{\lambda}_1 = 0$.

Again, the control $u_1$ does not appear in the equation of $S_1^{(3)}$, so we shall take the fourth derivative of $S_1$. The control $u_1$ appears in $S_1^{(4)}$ explicitly; i.e.,

$$S_1^{(4)} = A_1 + B_1 u_1 \tag{29}$$

where $A_1$ and $B_1$ are the coefficients which are given by

$$\begin{aligned} A_1 &= [2a(a+1)^2 m^3 - 2amx_1^2 + 2amx_2^2 + 2x_1 u_2]\dot{\lambda}_2 - \\ &\quad (4a^2 m^2 x_1 - 3amu_2) x_2 \lambda_2 && (30a) \\ B_1 &= x_2 \dot{\lambda}_2 - m(1+2a)(\lambda_3 x_4 - \lambda_4 x_3) && (30b) \end{aligned}$$

and $\dot{\lambda}_2$ is given by Eq. (10b). From the discussion in Section , the optimal singular control $u_1$ is of second order and is given by

$$u_1 = -\frac{A_1}{B_1} \tag{31}$$

Kelley's necessary condition for optimality requires that

$$B_1 \geq 0 \tag{32}$$

**Case III: only $u_2$ is singular**

The same analysis as for the case when only $u_1$ is singular can be repeated for $u_2$ if only $u_2$ is singular while $u_1$ is bang-bang. In general, the optimal singular control $u_2$ is of second order and is given by

$$u_2 = -\frac{A_2}{B_2} \tag{33}$$

where

$$\begin{aligned} A_2 &= [-2a(a+1)^2 m^3 - 2amx_1^2 + 2amx_2^2 + 2x_2 u_1]\dot{\lambda}_1 - \\ &\quad (4a^2 m^2 x_2 + 3amu_1) x_1 \lambda_1 && (34a) \\ B_2 &= x_1 \dot{\lambda}_1 - m(1+2a)(\lambda_3 x_4 - \lambda_4 x_3) && (34b) \end{aligned}$$

Kelly's necessary condition for optimality requires

$$B_2 \geq 0 \tag{35}$$

## Special Cases

In the discussion in the previous section, it was assumed that $m \neq 0$ and $a \neq 0$. In this section, we will consider two special cases when $a = 0$ and $m = 0$, respectively. These two cases correspond to an *inertially* symmetric rigid body and a *non-spinning axi-symmetric* rigid body, respectively. For these cases, the equations are simplified significantly and a better insight is gained about the optimal solutions.

**Inertially Symmetric Rigid Body** ($a = 0$)

For an inertially symmetric rigid body, it is $a = 0$, and the dynamics are simply

$$\begin{aligned} \dot{x}_1 &= u_1 && (36a) \\ \dot{x}_2 &= u_2 && (36b) \end{aligned}$$

while the kinematics remain the same as given by Eqs. (8c) and (8d). In this section, we assume that $m \neq 0$, i.e., the rigid body has a nonzero angular velocity component about the $\hat{b}_3$ axis. As before, we examine the three different cases separately.

*Case* I*: both $u_1$ and $u_2$ are singular*

In this case, as in Section , since both $u_1$ and $u_2$ are singular, we have $\lambda_1 = \dot{\lambda}_1 = \lambda_2 = \dot{\lambda}_2 = 0$. The Hamiltonian then becomes $H = 1 + m(\lambda_3 x_4 - x_3 \lambda_4)$. From Eq. (18) the second derivative of $S_1$ and $S_2$ yields $\ddot{S}_1 = (\lambda_3 x_4 - \lambda_4 x_3) x_2$ and $\ddot{S}_2 = -(\lambda_3 x_4 - \lambda_4 x_3) x_1$. Since the Hamiltonian has to be zero along the whole trajectory, and since $m \neq 0$, one obtains $(\lambda_3 x_4 - x_3 \lambda_4) \neq 0$. Now letting the second derivative of $S_1$ and $S_2$ be zero yields $x_2 = 0$ and $x_1 = 0$. Taking the third time derivative of $S_1$ and $S_2$ and letting it be zero, we get $u_2 = 0$ and $u_1 = 0$. As in Section , since the controls $u_1$ and $u_2$ appear in the third time derivative of $S_1$ and $S_2$, these controls are not time optimal.[22]

*Case* II*: only $u_1$ is singular ($u_2$ is bang-bang).*

In this case, since $u_2$ is bang-bang, we have $x_2 \neq 0$, except possibly at some isolated points. The control $u_1$ is assumed singular, so we have $\lambda_1 = \dot{\lambda}_1 = 0$, and following the same approach as in Section  one obtains that the singular control is of second order and is given by

$$u_1 = -\frac{2x_1 u_2 \dot{\lambda}_2}{x_2 \dot{\lambda}_2 - m(\lambda_3 x_4 - \lambda_4 x_3)} \tag{37}$$

where

$$\dot{\lambda}_2 = -\lambda_4 \frac{1 + x_4^2 - x_3^2}{2} - \lambda_3 x_3 x_4 \tag{38}$$

Kelley's optimality condition requires

$$x_2 \dot{\lambda}_2 - m(\lambda_3 x_4 - \lambda_4 x_3) \geq 0 \tag{39}$$

Since in this case the Hamiltonian takes the simple form

$$H = 1 + \lambda_2 u_2 - \dot{\lambda}_2 x_2 + m(\lambda_3 x_4 - \lambda_4 x_3) \tag{40}$$

and since $u_2 = -u_{2\max}\mathrm{sgn}(\lambda_2)$, we have

$$\dot{\lambda}_2 x_2 - m(\lambda_3 x_4 - \lambda_4 x_3) = 1 + \lambda_2 u_2 = 1 - u_{2\max}|\lambda_2| \tag{41}$$

and Kelley's optimality condition is equivalent to $u_{2\max}|\lambda_2| \leq 1$.

From Eqs. (37) and (41), we can see that the optimal control $u_1$ is only defined when $|\lambda_2| \neq 1/u_{2\max}$ except at some isolated points. In practice, it is possible that $|\lambda_2| = 1/u_{2\max}$ along the singular arc and $u_1$ is no longer defined in Eq. (37). In this case, since $\lambda_2$ is continuous,[17] either $\lambda_2 = 1/u_{2\max}$ or $\lambda_2 = -1/u_{2\max}$ holds. Therefore, $\dot{\lambda}_2 = 0$, and $\ddot{S}_1 = 0$ implies $\lambda_3 x_4 - \lambda_4 x_3 = 0$ since $x_2 \neq 0$. From Eq. (18a), we can see that $S_1^{(3)}$ automatically equals to zero, and all higher order derivatives of $S_1$ will be zero identically as well. Therefore, the optimal control $u_1$ is an infinite-order singular control and it can be chosen arbitrarily as long as the boundary conditions are satisfied.[5] In this case, substitution of $\lambda_3 x_4 - \lambda_4 x_3 = 0$ into $\dot{\lambda}_1 = 0$ and $\dot{\lambda}_2 = 0$ yields $\lambda_3 = \lambda_4 = 0$ along the infinite-order singular arc.

*Case* III*: only $u_2$ is singular ($u_1$ is bang-bang).*

This case is similar to Case II above. If $|\lambda_1| \neq 1/u_{1\max}$ except possible at some isolated points, the optimal singular control $u_2$ is of second order and is given by

$$u_2 = -\frac{2x_2 u_1 \dot{\lambda}_1}{x_1 \dot{\lambda}_1 - m(\lambda_3 x_4 - \lambda_4 x_3)} \tag{42}$$

where

$$\dot{\lambda}_1 = -\lambda_3 \frac{1 + x_3^2 - x_4^2}{2} - \lambda_4 x_3 x_4 \tag{43}$$

Kelley's optimality condition requires $|\lambda_1| \leq 1/u_{1\max}$. If $|\lambda_1| = 1/u_{1\max}$ along the singular trajectory, the optimal singular control $u_2$ is of infinite-order and can be chosen arbitrarily as long as all the boundary conditions are satisfied. In this case necessarily $\lambda_3 = \lambda_4 = 0$ along the singular arc.

**Non-spinning Axi-Symmetric Rigid Body** ($m = 0$)

If the body is not spinning, i.e., $m = 0$, then the system equations are simplified as

$$\dot{x}_1 = u_1 \tag{44a}$$
$$\dot{x}_2 = u_2 \tag{44b}$$
$$\dot{x}_3 = x_2 x_3 x_4 + x_1(1 + x_3^2 - x_4^2)/2 \tag{44c}$$
$$\dot{x}_4 = x_1 x_3 x_4 + x_2(1 + x_4^2 - x_3^2)/2 \tag{44d}$$

These equations describe the dynamics and kinematics of both an axi-symmetric and an inertially symmetric body. Again in the analysis we will discuss the possibility of both controls being singular and only one control being singular.

*Case* I*: both* $u_1$ *and* $u_2$ *are singular*

In this case, as in Section , since both $u_1$ and $u_2$ are singular during $[t_1, t_2] \subset [t_0, t_f]$, we get $\lambda_1 = \dot{\lambda}_1 = \lambda_2 = \dot{\lambda}_2 = 0$. Substituting these equations into Eq. (9), it follows immediately that $H = 1$. This contradicts the necessary condition which states that the Hamiltonian has to be zero along the whole trajectory. Therefore both controls being singular is impossible for a non-spinning symmetric body.

*Case* II*: only* $u_1$ *is singular (*$u_2$ *is bang-bang)*

In this case, since $u_2 = -u_{2\max}\,\mathrm{sgn}(\lambda_2)$, we have $x_2 \neq 0$ except possibly at some isolated points. The control $u_1$ is assumed to be singular, so by taking successive derivatives of $\lambda_1$ one obtains $\dot{\lambda}_1 = -\lambda_3(1 + x_3^2 - x_4^2)/2 - \lambda_4 x_3 x_4 = 0$ and $\ddot{\lambda}_1 = (\lambda_3 x_4 - \lambda_4 x_3)x_2 = 0$. Since $x_2 \neq 0$, necessarily $\lambda_3 x_4 - \lambda_4 x_3 = 0$. From Eq. (28) we have for the third derivative of $S_1$,

$$S_1^{(3)} = x_1 x_2 \dot{\lambda}_2 \tag{45}$$

If $\dot{\lambda}_2 \neq 0$, then letting $S_1^{(3)} = 0$, we have $x_1 = 0$. Taking the fourth derivative of $S_1$, we have that $u_1$ explicitly appears in $S_1^{(4)}$. Letting $S_1^{(4)} = 0$, we get the explicit expression for the second order optimal singular control $u_1$ as

$$u_1 = 0 \tag{46}$$

Kelley's optimality condition requires that $x_2\dot{\lambda}_2 \geq 0$. Since in this case the Hamiltonian takes the simple form $H = 1 + \lambda_2 u_2 - \dot{\lambda}_2 x_2$ we have

$$\dot{\lambda}_2 x_2 = 1 + \lambda_2 u_2 = 1 - u_{2\max}|\lambda_2| \tag{47}$$

and Kelley's optimality condition is equivalent to $u_{2\max}|\lambda_2| \leq 1$. Substituting $\lambda_3 x_4 - \lambda_4 x_3 = 0$ into $\dot{\lambda}_1 = 0$, we get $\lambda_3 = 0$ necessarily along the singular arc, so $\dot{\lambda}_2 \neq 0$ implies $\lambda_4 \neq 0$. Therefore, $\lambda_3 x_4 - \lambda_4 x_3 = 0$ implies that $x_3 = 0$. Thus along the singular subarc, $x_1 = x_3 = 0$.

If $\dot{\lambda}_2 = 0$ along the singular arc, from Eq. (45), we can see that $S_1^{(3)}$ automatically equals to zero, and all higher order derivatives of $S_1$ will be zero identically as well. Therefore, the optimal control $u_1$ is an infinite-order singular control and it can be chosen arbitrarily as long as the boundary conditions are satisfied.[5] From Eq. (47), $\dot{\lambda}_2 = 0$ implies $\lambda_2 = \pm 1/u_{2\max}$. Substitution of $\lambda_3 x_4 - \lambda_4 x_3 = 0$ into $\dot{\lambda}_1 = 0$ and $\dot{\lambda}_2 = 0$ yields that $\lambda_3 = \lambda_4 = 0$ along an infinite-order singular arc.

*Case* III*: only* $u_2$ *is singular (*$u_1$ *is bang-bang).*

Consider the case when only $u_2$ is singular and $u_1$ is bang-bang. Then the same analysis as in Case II yields the following results. If $\dot{\lambda}_1 \neq 0$, except possible at some isolated points, the optimal singular control $u_2$ is of second order and given by

$$u_2 = 0 \tag{48}$$

Kelley's optimality condition requires that $u_{1\max}|\lambda_1| \leq 1$. Along the singular subarc, $x_2 = x_4 = 0$.

If $\dot{\lambda}_1 = 0$ along the singular arc, the optimal singular control $u_2$ is of infinite order and can be chosen arbitrarily as long as all the boundary conditions are satisfied. In this case, $\lambda_1 = \pm 1/u_{1\max}$ and $\lambda_3 = \lambda_4 = 0$.

We note that from Eqs. (46) and (48), it can be seen that the second order singular arc for a non-spinning body is an eigenaxis rotation.

## A Numerical Approach for Computing Optimal Solutions

The optimal solutions are obtained numerically using a cascaded computational scheme. Both a direct method and an indirect method are used in this scheme. A direct method is applied first to get initial guesses for the indirect method, which is then solved to obtain accurate optimal solutions. The idea of combining direct and indirect methods for solving optimal control problems was introduced by Stryk and Bulirsch[25] and later by Seywald and Kumar[26] to take advantage of both the good convergence properties of the direct methods and the accuracy of the indirect methods.

This scheme is illustrated in Fig. 3. Three programs are used in this numerical approach: EZopt, COSCAL and BNDSCO. BNDSCO solves the problem using an indirect method, i.e., the optimal solutions are determined by solving the Multipoint Boundary Value Problem arising from Pontryagin's Minimum Principle using a multiple shooting method.[27] It converges quickly to the optimal solution, and the solution obtained is of very high accuracy. However, the radius of convergence of this method is rather small since it requires very good initial guesses for the states, controls, co-states, Lagrange multipliers and switching structure.[25, 26] A major difficulty lies in the fact that in most cases we do not know the optimal switching structure in advance. Also, initial guesses for the co-states and the Lagrange multipliers are nontrivial because these do not, in general, have any intuitive physical interpretation.

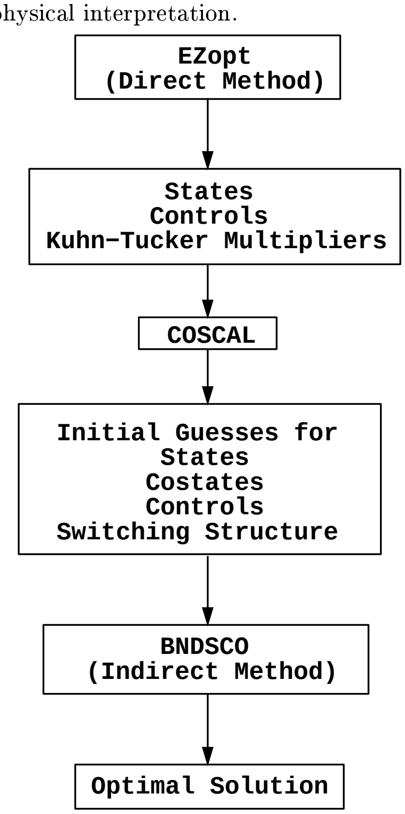


**Fig. 3 Cascaded computational scheme for optimal solution computation.**

EZopt[28] solves the problem using a direct method, namely, the optimal solution is determined by directly minimizing the cost criterion through collocation and nonlinear programming. The radius of convergence of the direct method is usually much larger than that of the indirect method.[25,26] The program converges upon much less accurate initial guesses. The speed of convergence is, however, much slower compared with BNDSCO. Since this method does not involve co-states, one needs to provide only initial guesses for the states and controls. In addition, the switching structure does not have to be known in advance. A disadvantage of this method is that the solutions obtained may not be as accurate as those obtained from an indirect method.[6,25] This is especially true around switching points and when singular subarcs appear as part of the overall optimal solution. The accuracy of the solution depends on the discretization scheme and the number of the discrete nodes. However, these solutions are good enough to roughly determine the trajectories, states, controls, switching structure and, if they exist, singular subarcs. Thus, they provide good initial guesses for a direct optimization software package such as BNDSCO.

Based upon the foregoing discussion, we have developed a software package that combines the two programs (EZopt and BNDSCO) together to overcome the drawbacks of each method. That is, we use the results from EZopt as an initial guess for BNDSCO. With this initial guess, BNDSCO typically converges very fast and gives accurate and reliable results. In addition, the optimality of the solution can be readily checked from the time history of the corresponding switching functions. One major obstacle with this approach is that BNDSCO needs the initial guesses for the co-states (in addition to the states and the correct switching structure) which EZopt does not provide. Thus, the program COSCAL was developed by the authors to calculate the co-states at each node from the Kuhn-Tucker multipliers associated with the nonlinear programming, provided by EZopt. The methodology is based on the work of Seywald and Kumar.[26]

## Numerical Results

EZopt, COSCAL and BNDSCO together form a cascaded computational scheme which is very effective in carrying out the optimal control computations. It has been used extensively by the authors to solve several optimal control problems. In this section we present three numerical examples for the minimum-time reorientation problem, demonstrating the trajectories with bang-bang control subarcs, finite-order control subarcs, and infinite-order control subarcs. Finally, we will give a comparison between the eigenaxis rotation and the true minimum-time rotation for an axi-symmetric rigid body.

### Bang-bang control example

For the problem at hand, bang-bang control is obtained in most situations, including both rest-to-rest and non rest-to-rest maneuvers. The optimal control is given in Eq. (13). As an example of a bang-bang maneuver, consider the following initial and final conditions: $x(0) = [0, 0, 1.5, -0.5]$ and $x(t_f) = [0, 0, 0, 0]$. The parameters $a$ and $m$ are chosen to be $a = 0.5$ and $m = -0.5$ rad/sec. The control inputs are assumed to be bounded by $u_{1\max} = u_{2\max} = 1.0$.

This example represents a "rest-to-rest" maneuver with respect to the two control axes since the body is spinning only about its symmetry axis at a constant rate. The initial boundary condition corresponds to a relative attitude such that the initial angle between the $\hat{b}_3$ axis and $\hat{n}_3$ axis is 115.38 deg. An optimal control is to be found to reorient the body until $\hat{b}_3$ and $\hat{n}_3$ axes are aligned in a rest situation. The optimal control for this example was found to be bang-bang with the first control having one switch and the second control having two switches. The minimum time to complete this maneuver is 2.61 sec.

Figures 4 and 5 show the control inputs and the corresponding switching functions. Figure 6 shows the history of the angular velocities $\omega_1$ and $\omega_2$, and Fig. 7 shows the time history of $w_1$ and $w_2$. Recall that $w_1$ and $w_2$ represent the relative position of the inertial axis $\hat{n}_3$ with respect to the body fixed frame $\hat{\mathbf{b}}$. In these figures, the solid lines stand for the optimal results obtained from BNDSCO (states, controls and co-states), while the circles show the initial guesses obtained from EZopt (states and controls) or COSCAL (co-states).

From these plots, one can see that the solution obtained from EZopt almost captures the properties of the optimal

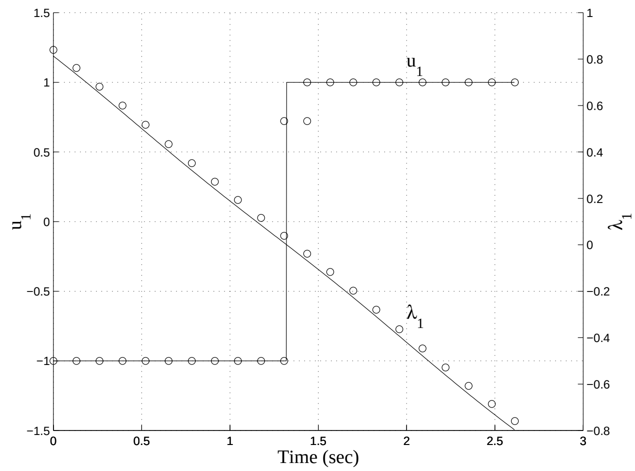


**Fig. 4 Control $u_1$ and co-state $\lambda_1$ for bang-bang maneuver.**

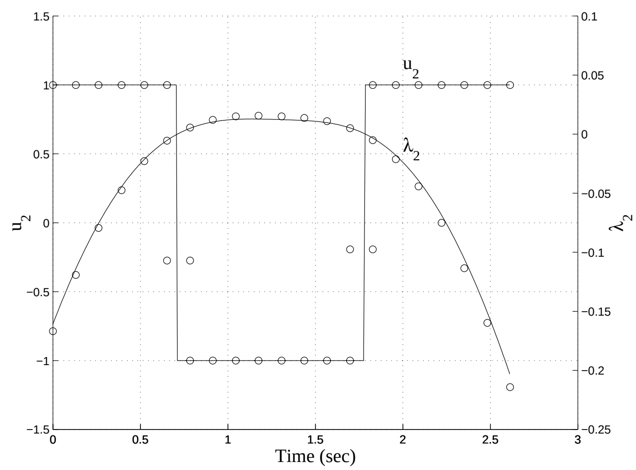


**Fig. 5 Control $u_2$ and co-state $\lambda_2$ for bang-bang maneuver.**

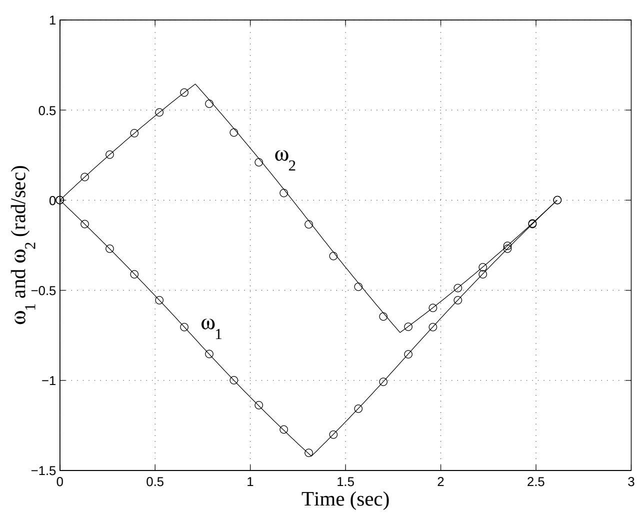


**Fig. 6 Angular velocities $\omega_1$ and $\omega_2$ for bang-bang maneuver.**

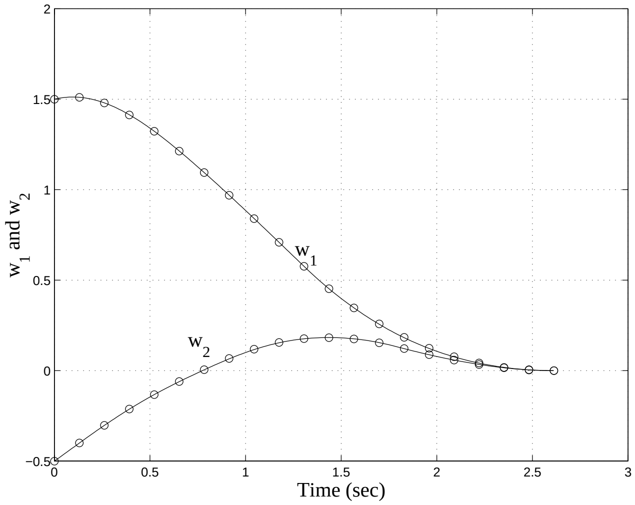


**Fig. 7 Time histories of $w_1$ and $w_2$ for bang-bang maneuver.**

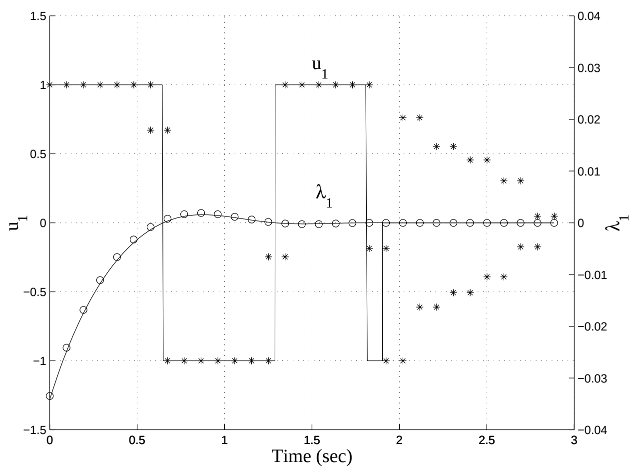


**Fig. 8 Control $u_1$ and co-state $\lambda_1$ for singular subarc maneuver.**

solution, although some discrepancy exists at the switching points. The plots also show that COSCAL provides very accurate guesses for the co-states.

For the calculations shown in Figs. 4-7, 21 nodes were used for EZopt. The initial guesses for EZopt are trivial (all the initial guess values are zero). EZopt converged in 2 minutes on a SPARCstation 5. 11 nodes are used in BNDSCO, and with the initial guesses provided by EZopt and COSCAL, BNDSCO converged in less than 2 seconds.

### Finite-order singular control example

Finite-order singular subarcs can be part of an optimal solution only in some particular situations. As an example, a second order singular control was observed for the boundary conditions $x(0) = [-0.45, -1.1, 0.1, -0.1]$ and $x(t_f) = [0, 0, 0, 0]$. The parameters for this case are given by $a = 0.5$ and $m = 0$, and the control inputs are bounded by $u_{1\max} = u_{2\max} = 1.0$. This example corresponds to a *non-spinning* axi-symmetric body. For these values of $w_1$ and $w_2$ the angle between $\hat{b}_3$ axis and $\hat{n}_3$ axis is 16.1deg. The control is required to reorient the body until the symmetry axis $\hat{b}_3$ is aligned with the inertial axis $\hat{n}_3$ in a rest situation. It turns out that the first control has a singular subarc and the second control is bang-bang. The expression for the optimal singular control is given by Eq. (46). The minimum time is 2.88 sec. The results of the numerical simulations are shown in Figs. 8-12.

It should be noted here that although the analysis in Section does not preclude the existence of singular subarcs in the case when $m \neq 0$, we were unable to find optimal trajectories with singular subarcs for the spinning case. Fig-

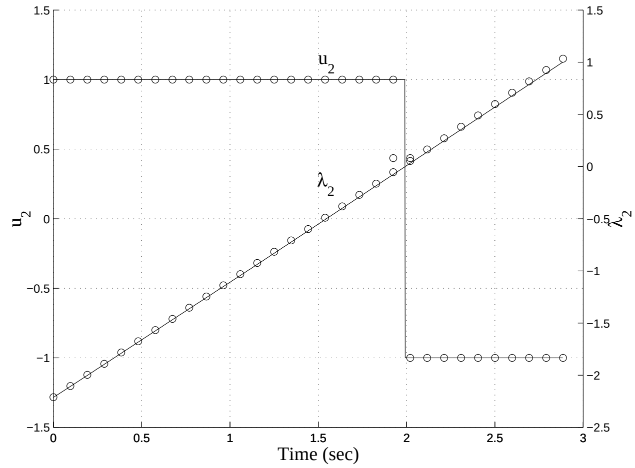


**Fig. 9 Control $u_2$ and co-state $\lambda_2$ for singular subarc maneuver.**

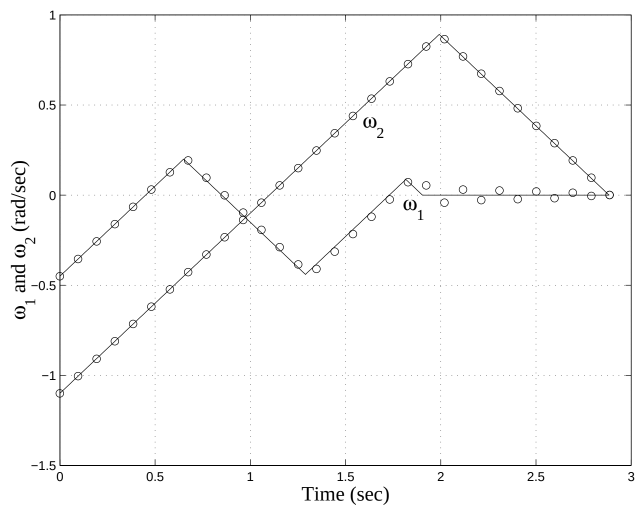


**Fig. 10 Angular velocities $\omega_1$ and $\omega_2$ for singular subarc maneuver.**

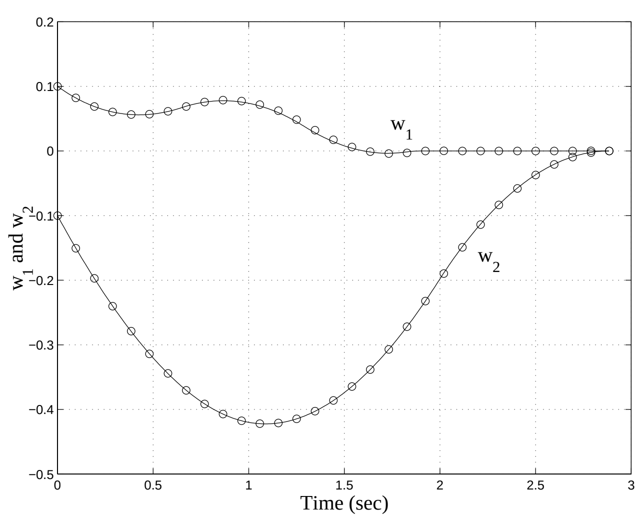


**Fig. 11 Time histories of $w_1$ and $w_2$ for singular subarc maneuver.**

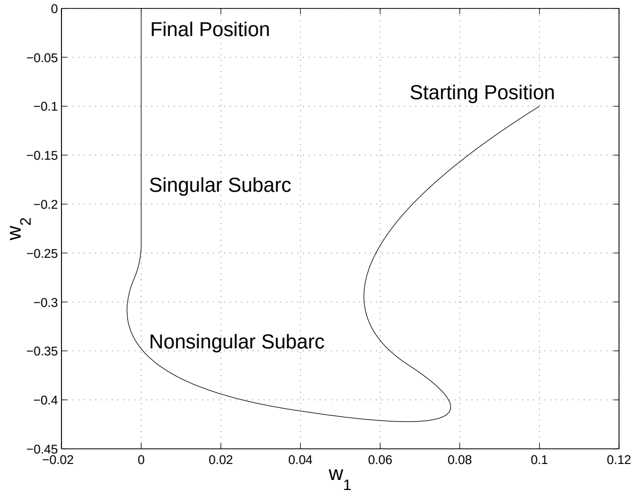


**Fig. 12 Optimal trajectory with singular subarc in $w_1$-$w_2$ plane.**

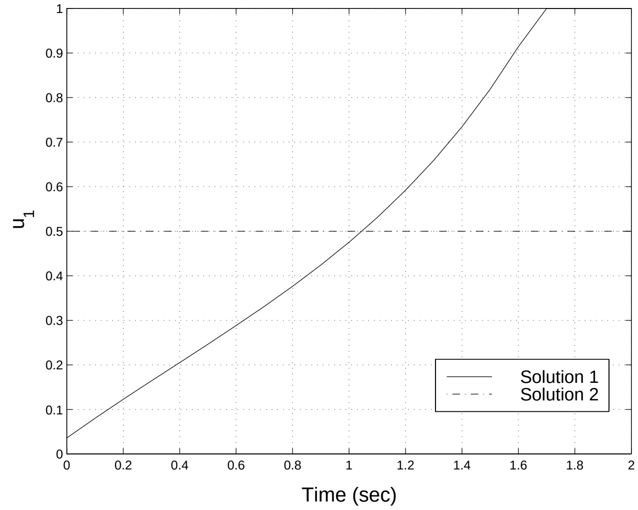


**Fig. 13 Two possible solutions for $u_1$ for an infinite-order singular arc maneuver.**

ures 8 and 9 show the control inputs and the corresponding switching functions. Figure 10 shows the time history of the angular velocities $\omega_1$ and $\omega_2$ and Fig. 11 shows the time history of $w_1$ and $w_2$. Figure 12 shows the same trajectory on the $w_1$-$w_2$ plane. From Fig. 8 we can see that the control $u_1$ is singular after $t = 1.904$ sec. In the $w_1$-$w_2$ plane, if the body is not spinning about its symmetry axis, then an eigenaxis rotation is represented by a straight line. Figure 12 indicates that along the bang-bang subarc, the time-optimal trajectory is not an eigenaxis rotation and along the singular subarc the time optimal trajectory is an eigenaxis rotation.

In these figures, the solid lines show the optimal solution obtained from the solution of the TPBVP and the stars indicate the solution of the direct method, which was used as an initial guess for the TPBVP solver. From these plots it is seen that the bang-bang subarc obtained from EZopt captures the optimal bang-bang subarc very well. The singular control subarc obtained from EZopt is not as accurate, but the output from EZopt gives a good understanding about the existence and location of a singular subarc. Again, COSCAL captures the time history of the co-states very well.

In this example, the calculations in EZopt were performed using 31 nodes, which required EZopt about 5 minutes to converge from all-zero initial guesses. On the other hand, using the output from EZopt/COSCAL, BNDSCO converged in about 2 seconds.

The appearance of a singular subarc in the optimal trajectory deserves special mention. Kelley's necessary condition alone (which was found to be satisfied for this example) does *not* guarantee that the singular subarc will indeed be part of the composite optimal trajectory. The boundary conditions will determine if this is true or not. Even in the case when a trajectory composed of bang-bang and singular subarcs satisfies the boundary conditions, the first order necessary conditions, and Kelley's condition on the singular subarc, it is still not guaranteed that this solution is optimal. In particular, the joining between bang-bang and singular subarcs has to satisfy certain junction conditions. Such optimality conditions are given in Refs. 23 and 29. For a second order singular arc (as the one in Fig. 8) the main condition in Ref. 29 states that the optimal control should be continuous at each junction, i.e., a jump discontinuity when joining a non-singular and a singular control is not allowed. At first glance, this seems to contradict the result in Fig. 8. However, the results in Ref. 29 assume that the control is piecewise analytic. This is shown not to be the case for singular arcs of even order (and also for arcs of odd order greater than one) in Ref. 30; see also Ref. 31. Thus, for even-order singular subarcs (and odd-order singular arcs of order greater than one) the junction between singular and non-singular arcs is not analytic, i.e., the control consists of a sequence of an infinite number of switchings between $u = u_{\min}$ and $u = u_{\max}$ with the time between switchings rapidly decreasing. More relevant to our case is the fact that singular controls may manifest themselves as the cumulative effect of the infinite number of bang-bang control actions (chattering). If this is the case, the solution of the differential equations have to be interpreted in the Filippov sense,[32] and the singular control is then the "equivalent" control action associated with the chattering control.[31]

The previous discussion reinforces our observations for the singular control in Fig. 8. The solution from EZopt shows that the optimal control switches rapidly after $t = 1.904$. The subarc after that point is identified as a singular subarc, and the solid line stands for the optimal solution (given from BNDSCO) which uses the "equivalent" singular control $u_1 = 0$, obtained using the necessary conditions. This singular control has the equivalent effect of a bang-bang control with infinite number of switchings. It must be pointed out that the substitution of a chattering bang-bang control with its "equivalent" singular form is more than a mathematical convenience. In practice, it is often preferable to use the "equivalent" singular control action instead of switching between the upper and lower bounds infinitely fast. At any rate, in both cases, the optimal state trajectory is the same.

### Infinite-order singular control example

As a demonstration of the infinite-order singular control, the boundary conditions $x(0) = [0,0,0,0]$ and $x(t_f) = [1.0, 2.0, \text{free}, \text{free}]$ are considered. In this example $a = 0$, $m = -0.3$ and $u_{1\max} = u_{2\max} = 1$ are also assumed. The infinite-order singular control corresponding to these parameters is discussed in Section . From the boundary conditions we can see that the purpose of this maneuver is to accelerate the angular velocity components $\omega_1$ and $\omega_2$ from zero to 1.0 rad/sec and 2.0 rad/sec, respectively. The final position is not important in this case. Since $\dot{x}_1 = u_1$ and $\dot{x}_2 = u_2$, the minimum time that $x_2$ reaches 2.0 rad/sec is 2 seconds, during which, $x_1$ can obtain its final value $x_1(t_f) = 1$ rad/sec in many ways. Therefore, an infinite-order singular control is a possible solution for this problem. Two possible solutions for $u_1$ are presented. Figures 13 and 14 show the two possible solutions for the control $u_1$ and the angular velocity $\omega_1$. Similar infinite-order results can be obtained when $m = 0$, with this case discussed in Section .

### Comparison Between Eigenaxis Rotation and Time-Optimal Rotation

It has been shown in the previous sections that the eigenaxis rotations, in general, are not time optimal. A comparison between eigenaxis rotations and time-optimal rotations

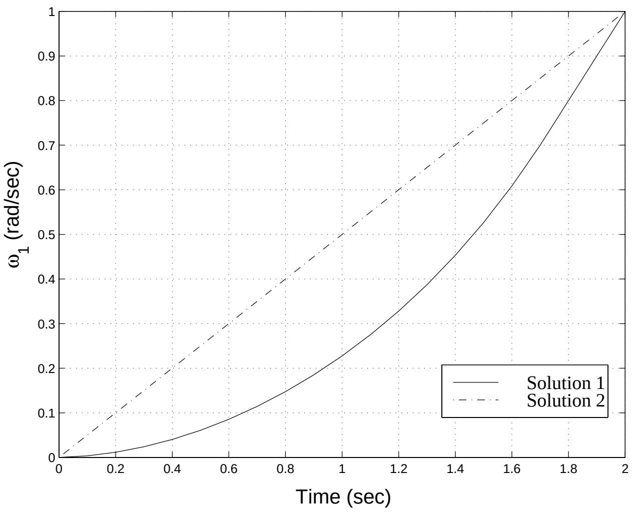


**Fig. 14 Two possible solutions for $\omega_1$ for an infinite-order singular arc maneuver.**

of an *inertially symmetric* rigid body with three controls can be found in Bilimoria and Wie .[3] In this section, we compare eigenaxis rotations and time-optimal rotations for an axi-symmetric rigid body with two controls which are bounded by $u_{\max} = 1$.

Figure 15 shows the configuration of the desired maneuver and eigenaxis rotation. Initially the $\hat{\mathbf{b}}_3$ and $\hat{\mathbf{n}}_3$ axes are perpendicular to each other, in a rest situation. The $\hat{\mathbf{n}}_3$ axis lies in the $\hat{\mathbf{b}}_1$-$\hat{\mathbf{b}}_2$ plane and the angle from the $\hat{\mathbf{b}}_1$ axis to the $\hat{\mathbf{n}}_3$ axis is denoted by $\theta$. The desired maneuver is such that the $\hat{\mathbf{b}}_3$ axis and $\hat{\mathbf{n}}_3$ axis are aligned in a rest situation.

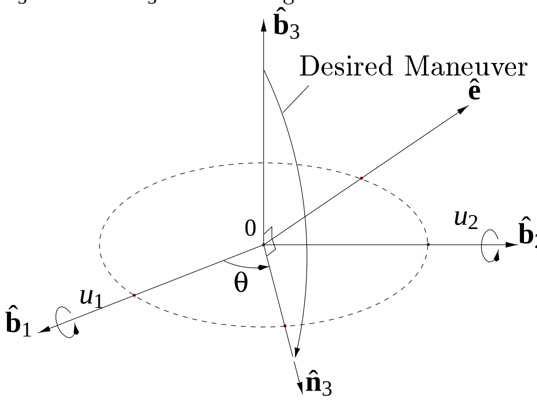


**Fig. 15 Desired maneuver for the comparison between eigenaxis rotation and optimal rotation.**

The minimum-time solution for this rest-to-rest maneuver can be obtained for any value of $\theta$ using the software EZopt/COSCAL/BNDSCO. It turns out that for $\theta = n\pi/4$, $n = 0, 1, \cdots, 7$ the time-optimal maneuver is an eigenaxis rotation, and for other values of $\theta$, the time-optimal maneuver is not an eigenaxis rotation. In addition, the time-optimal maneuvers in each interval $[i\pi/4, (i+1)\pi/4]$, $i = 0, 1, \cdots, 7$ have similar switching structure.

The previous reorientation can also be obtained by an eigenaxis rotation about an eigenaxis $\hat{\mathbf{e}}$ which is fixed in the inertial frame, lies in the $\hat{\mathbf{b}}_1$-$\hat{\mathbf{b}}_2$ plane and is perpendicular to the $\hat{\mathbf{n}}_3$ axis. Suppose $0 \le \theta \le \pi/4$. Then the available control input along the $\hat{\mathbf{e}}$ axis is bounded by $1/\cos\theta$. Hence the minimum time $t_e$ for the time-optimal rest-to-rest eigenaxis rotation can be calculated analytically; i.e.,

$$t_e = \sqrt{2\pi \cos\theta} \tag{49}$$

Table 2 shows the comparison between the eigenaxis rotation and the minimum-time rotation. Due to symmetry, only the maneuver times for $\theta \in [0, \pi/4]$ are shown, since the rotation times for other values of $\theta$ can be calculated similarly. Shown in the table are the switching sequences for $u_1$ and $u_2$, the final time $t_f$ of the eigenaxis rotations and the minimum-time rotation, and the time savings using the min-time rotation.

**Table 2 Comparison between eigenaxis and time-optimal rotations.**

| | Eigenaxis Rotation | | |
|---|---|---|---|
| $\theta$ | $u_1$ | $u_2$ | $t_f$ |
| 0 | $[0]$ | $[1, -1]$ | 2.5066 |
| $\pi/24$ | $[-\tan\theta, \tan\theta]$ | $[1, -1]$ | 2.4959 |
| $\pi/12$ | $[-\tan\theta, \tan\theta]$ | $[1, -1]$ | 2.4639 |
| $\pi/8$ | $[-\tan\theta, \tan\theta]$ | $[1, -1]$ | 2.4093 |
| $\pi/6$ | $[-\tan\theta, \tan\theta]$ | $[1, -1]$ | 2.3327 |
| $5\pi/24$ | $[-\tan\theta, \tan\theta]$ | $[1, -1]$ | 2.2327 |
| $\pi/4$ | $[-1, 1]$ | $[1, -1]$ | 2.1078 |

| | Minimum-Time Rotation | | | |
|---|---|---|---|---|
| $\theta$ | $u_1$ | $u_2$ | $t_f$ | Savings |
| 0 | $[0]$ | $[1, -1]$ | 2.5066 | 0% |
| $\pi/24$ | $[-1, 1, -1]$ | $[1, -1]$ | 2.4742 | 0.87% |
| $\pi/12$ | $[-1, 1, -1]$ | $[1, -1]$ | 2.4191 | 1.82% |
| $\pi/8$ | $[-1, 1, -1]$ | $[1, -1]$ | 2.3527 | 2.35% |
| $\pi/6$ | $[-1, 1, -1]$ | $[1, -1]$ | 2.2787 | 2.31% |
| $5\pi/24$ | $[-1, 1, -1]$ | $[1, -1]$ | 2.1973 | 1.59% |
| $\pi/4$ | $[-1, 1]$ | $[1, -1]$ | 2.1078 | 0% |

## Conclusions

The time-optimal reorientation control problem of an axi-symmetric rigid spacecraft with two control torques has been studied in detail. It is assumed that no control torque is available along the symmetry axis and the remaining two torques are perpendicular to the symmetry axes and to each other. The spacecraft may be spinning about its symmetric axis. The relative rotation about the symmetry axis is therefore indeterminate. A complete analysis of all the possible time-optimal control structures is presented, including cases with singular and nonsingular subarcs. It is shown that second order singular arcs and infinite-order singular arcs can appear as part of the optimal trajectory for specific boundary conditions. Results are also presented for an inertially symmetric rigid body with two controls. It is shown that for a non-spinning, axi-symmetric body, the second order singular arc is an eigenaxis rotation. A cascaded computational scheme is developed and used for the numerical computation of the optimal trajectories. The method does not require any *a priori* knowledge of the optimal switching structure. Examples show that this is a very effective approach to compute optimal trajectories numerically.

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