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Stabilization and Optimality Results for the Attitude Control Problem

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Abstract

In this paper we present some recent results on the description and control of the attitude motion of rotating rigid bodies. We derive a new class of globally asymptotically stabilizing feedback control laws for the complete (i.e., dynamics and kinematics) attitude motion. We show that the use of a Lyapunov function which involves the sum of a quadratic term in the angular velocities and a logarithmic term in the kinematic parameters leads to the design of linear controllers. We also show that the feedback control laws for the kinematics minimize a quadratic cost in the state and control variables for all initial conditions. For the complete system we construct a family of exponentially stabilizing control laws and we investigate their optimality characteristics. The proposed control laws are given in terms of the classical Cayley-Rodrigues parameters and the Modified Rodrigues parameters.

Nomenclature

$\hat{e}$	=	Principal axis of rotation
I	=	identity matrix
J	=	inertia matrix
q	=	Euler parameter vector, $(q_0, q_1, q_2, q_3)^T$
$\mathbb{R}$	=	set of real numbers
$\mathbb{R}^n$	=	$n-$dimensional space of real vectors
$\mathbb{R}_+$	=	set of nonnegative real numbers
$S(\cdot)$	=	skew $-$ symmetric 3×3 matrix
$SO(3)$	=	special orthogonal group (rotationgroup)
u	=	torque vector in body $-$ axes, $(u_1, u_2, u_3)^T$
$\partial V/\partial x$	=	gradient of a function $V:\mathbb{R}^n \to \mathbb{R}$, $[\partial V/\partial x_1,\ \partial V/\partial x_2,\ \dots,\partial V/\partial x_n]$
ρ	=	Cayley $-$ Rodrigues parameters, $(\rho_1, \rho_2, \rho_3)^T$
σ	=	modified Rodrigues parameters, $(\sigma_1, \sigma_2, \sigma_3)^T$
Φ	=	Principal angle of rotation
ω	=	angular velocity vector in body axes, $(\omega_1, \omega_2, \omega_3)^T$
$(\cdot)^T$	=	transpose operator
$\|\|\cdot\|\|$	=	$2-$norm of vectors, i.e., $\|x\|^2 = x^Tx$

Introduction

In recent years considerable effort has been devoted to the design of control laws for challenging dynamical systems, such as robot manipulators, high-performance aircraft, and underwater or space vehicles. These systems have similar characteristics. First, they are difficult to control because of the highly nonlinear character of their equations. This is particularly true when these systems are required to perform fast angular maneuvers. In such cases, the cross-coupling terms in the equations become significant, dominating the linear terms. As a result, linear control techniques for such systems are often inadequate[1]. Second, the dynamics of all these systems have the same underlying property: Namely, they describe a rigid (or almost rigid) body in rotational motion. It is of interest to develop a comprehensive theory that will allow a better understanding of the complex dynamic behavior of the motion of rotating bodies. A cornerstone in this effort is the development of alternative descriptions of the kinematics of the rotational motion.

In this paper we are interested in the description and control of the attitude motion using only *minimal*, three-dimensional parameterizations. Two such three-dimensional parameterizations will be used: The classical (Cayley-) Rodrigues and the Modified Rodrigues parameters. Since the Modified Rodrigues kinematic parameterization is neither as widely known nor as frequently used as the classical Rodrigues parameters, we deem it necessary to review some of the advantages of the former. In particular, we show that the Modified Rodrigues parameterization is, in some sense, the "best" three-dimensional representation of the attitude motion.

The scope of this paper is twofold: First, to introduce a kinematic description using the Modified Rodrigues parameters and to briefly discuss its advantages over the other classical three-dimensional parameterizations, in particular the Rodrigues parameters. The second scope of the paper is to demonstrate the potential of the Modified Rodrigues parameters for control applications. In particular, we show how to derive *linear* globally asymptotically stabilizing control laws using these parameters. Since most of the stabilization results also hold for the Rodrigues parameters we derive, in parallel, similar results for these parameters as well. In the second part of the paper we address the problem of the *optimal regulation* of the attitude motion in terms of the previous kinematic parameters. In particular, we investigate the optimality properties of the previously derived stabilizing feedback control laws.

Optimal control theory for a rigid body has a long history[2–5]. Most of these references either address the optimal control problem of the angular velocity equations only[2,4] (i.e., without any reference to the kinematics), or they solve the open-loop optimal control problem which, via Pontryagin's Maximum Principle leads, to a Two-Point-Boundary-Value Problem. This problem can be solved, in general, using numerical techniques[6–8]. At the same time, the *synthesis* problem (i.e., the optimal feedback problem) is much more difficult and has been mainly addressed in the context of *time-optimal* maneuvers (see Ref. 9 and the references therein). Other optimal feedback results include Refs. 10-12. The Reference 13 contains a comprehensive compilation of most of the existing results on the rigid body optimal control. In this paper we seek solutions to the optimal *feedback* regulation problem of a rigid body subject to a quadratic cost where *both* the angular velocity *and* the orientation of the body are regulated. Taking into consideration the cascade interconnection of the system equations, we first state the optimal regulation problem for the kinematics of the attitude motion when the angular velocity acts as a control input. The cost includes a penalty on the orientation parameters and the angular velocity. The actual control input is the acting torque entering the system through Euler's equations (the dynamics). The optimal regulation when the dynam-

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ics is included in the problem, and for general performance indices is not yet solved — as far as the authors knows. However, the optimization problem for the kinematics provides a lower bound on the achievable performance for the whole system for the same cost functional. Actually, we show that if the dynamics is fast (or can be made fast enough through the appropriate choice of the control input) one is able to recover this performance asymptotically. We show how such a controller can be constructed — and thus achieve the optimal performance — under the assumption that there is no penalty on the control effort. This controller will include, in general, a high gain portion. Motivated by the optimal characteristics of this controller we derive an optimal controller which penalizes its high gain portion.

Kinematics Overview

The dynamics of the rotational motion of a rigid body are described by the following set of differential equations

$$J\dot{\omega} = S(\omega)J\omega + u, \qquad \omega(0) = \omega_0 \tag{1}$$

where the symbol $S(\cdot)$ denotes a 3×3 skew-symmetric matrix, that is,

$$S(\omega) := \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}. \tag{2}$$

In addition to Eq. (1), which provides the time history of the angular velocity vector, the orientation of a rigid body is given by a set of kinematic equations. Typically, the Euler parameters (quaternion), the Cayley-Rodrigues parameters, or the Eulerian angles are used to parameterize the attitude kinematics. In the sequel, we briefly review the two main kinematic parameterizations used in this paper, i.e., the Rodrigues parameters and the Modified Rodrigues parameters. Both can be viewed as normalized versions of the Euler parameters.

Let Φ denote the principal angle and let $\hat{\mathbf{e}}$ denote the principal axis associated with the Euler's Theorem[14]. The Euler parameters are defined by

$$q_0 := \cos\frac{\Phi}{2}, \qquad q_i := e_i \sin\frac{\Phi}{2}, \qquad (i = 1,2,3). \tag{3}$$

This is a four-dimensional parameterization, hence redundant. The constraint among the Euler parameters ($q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$) can be eliminated by renormalization. The Rodrigues parameters eliminate the constraint associated with the Euler parameter set – thus reducing the number of coordinates necessary to describe the kinematics from four to three – by introducing the ratio of the Euler parameters as new coordinates

$$\rho_i := \frac{q_i}{q_0}, \qquad (i = 1,2,3). \tag{4}$$

Letting $\rho = (\rho_1, \rho_2, \rho_3)^T \in \mathbb{R}^3$, the associated kinematic equations take the form[14]

$$\dot{\rho} = H(\rho)\omega, \qquad \rho(0) = \rho_0 \tag{5}$$

where

$$H(\rho) := \tfrac{1}{2}(I - S(\rho) + \rho\rho^T) \tag{6}$$

and I denotes the 3×3 identity matrix. It can be immediately shown that the matrix $H(\rho)$ satisfies the following two identities

$$\rho^T H(\rho)\omega = \left(\frac{1+\rho^T\rho}{2}\right)\rho^T\omega, \tag{7}$$

and

$$H^T(\rho)(I+\rho\rho^T)^{-1}H(\rho) = \left(\frac{1+\rho^T\rho}{4}\right) I \tag{8}$$

for all $\omega, \rho \in \mathbb{R}^3$.

The first identity is easily shown by direct calculation. In order to show the second identity, substitute Eq. (6) into the left-hand side of Eq. (8) and expand to get

$$\begin{aligned} & H^T(\rho)(I+\rho\rho^T)^{-1}H(\rho) \\ = \; & \frac{1}{4}(I+S(\rho)+\rho\rho^T)(I+\rho\rho^T)^{-1}(I+S(\rho)+\rho\rho^T) \\ = \; & \frac{1}{4}(I+S(\rho)+\rho\rho^T) - \frac{1}{4}(I+S(\rho)+\rho\rho^T)(I+\rho\rho^T)^{-1}S(\rho) \\ = \; & \frac{1}{4}(I+S(\rho)+\rho\rho^T) - \frac{1}{4}S(\rho) - \frac{1}{4}S(\rho)(I+\rho\rho^T)^{-1}S(\rho) \end{aligned} \tag{9}$$

It can be shown that

$$(I+\rho\rho^T)^{-1} = I - \frac{\rho\rho^T}{1+\rho^T\rho} \tag{10}$$

and

$$S^2(\rho) = -\rho^T\rho I + \rho\rho^T \tag{11}$$

Substituting the last two expressions in Eq. (9) and using the fact that $S(\rho)\rho = 0$ one obtains

$$\begin{aligned} & H^T(\rho)(I+\rho\rho^T)^{-1}H(\rho) \\ = \; & \frac{1}{4}(I+\rho\rho^T) - \frac{S^2(\rho)}{4} + \frac{1}{4}\left(\frac{1}{1+\rho^T\rho}\right)S(\rho)\rho\rho^T S(\rho) \\ = \; & \left(\frac{1+\rho^T\rho}{4}\right) I \end{aligned} \tag{12}$$

as claimed.

The vector ρ of the Rodrigues parameters is related to the principal vector and the principal angle through

$$\rho = \hat{\mathbf{e}}\tan\frac{\Phi}{2} \tag{13}$$

As it is evident from Eq. (13), the classical Cayley-Rodrigues parameters cannot be used to describe eigenaxis rotations of more than 180 deg. This is the reason why the Cayley-Rodrigues parameters were, for the most part, ignored in the literature of attitude dynamics. Revived interest in these parameters stems mainly from their potential advantages in control and stabilization problems[15–17].

If instead of using Eq. (4) one eliminates the Euler parameter constraint by introducing the parameters

$$\sigma_i := \frac{q_i}{1+q_0}, \qquad (i = 1,2,3) \tag{14}$$

one obtains the following set of differential equations in terms of the vector $\sigma = (\sigma_1, \sigma_2, \sigma_3)^T \in \mathbb{R}^3$

$$\dot{\sigma} = G(\sigma)\omega, \qquad \sigma(0) = \sigma_0 \tag{15}$$

where

$$G(\sigma) = \tfrac{1}{2}\left(I - S(\sigma) + \sigma\sigma^T - \frac{1+\sigma^T\sigma}{2} I\right). \tag{16}$$

The proposed kinematic description is derived from the Euler parameters via stereographic projection[18] and due to its similarity to the Rodrigues parameters is referred to in the literature as the *Modified Rodrigues Parameterization*[14,19,20]. The Modified Rodrigues parameters – although closely related to the Cayley-Rodrigues parameters — are superior to them, since they are not limited to eigenaxis rotations of only up to 180 deg. Indeed, from Eqs. (3) and (14) one easily sees that the Modified Rodrigues parameter vector σ is related to the principal axis and the principal angle through

$$\sigma = \hat{\mathbf{e}}\tan\frac{\Phi}{4} \tag{17}$$

which is well-defined for all eigenaxis rotations in the range $0 \leq \Phi < 360$ deg.

It should be apparent that the previous three-dimensional parameterization moves the inherent singularity of the parameterization as far from the equilibrium point (i.e., the origin) as possible; at the same time, only one attitude configuration is eliminated as being singular. This parameterization has the largest domain of validity over all other minimal parameterizations. A parameterization specified in this way allows for a larger set of rigid-body configurations than the other classical three-dimensional parameterizations, such as the Eulerian angles and Rodrigues parameters. In contradistinction, the use of Eulerian angles or Rodrigues parameters eliminates an infinite number of possible orientation configurations.

Similarly to Eqs. (7) and (8), direct calculation shows that the matrix $G(\sigma)$ in Eq. (15) also satisfies the following two identities.

$$\sigma^T G(\sigma)\omega = \left(\frac{1+\sigma^T\sigma}{4}\right)\sigma^T\omega \tag{18}$$

and

$$G^T(\sigma)G(\sigma) = \left(\frac{1+\sigma^T\sigma}{4}\right)^2 I \tag{19}$$

for all $\omega, \sigma \in \mathbb{R}^3$.

Stabilization Results

We are interested in designing feedback controllers that globally asymptotically stabilize the attitude motion. In particular, we are interested in answering the following question: Are there any globally asymptotically stabilizing feedback control laws for the systems of Eqs. (1)-(5) and (1)-(15) which are *linear*? Linear feedback control laws have been previously derived for the case of the four-dimensional parameterization of the Euler parameters[21], but not for the case of non-redundant parameter sets, such as the Rodrigues or the Modified Rodrigues parameters — at least as far as the author knows. In this Section we show that there exist indeed linear globally asymptotically stabilizing controllers in terms of the Rodrigues and the Modified Rodrigues parameters, and thus we answer the previous question in the affirmative.

One word of caution should be mentioned at this point as far as our use of the term "global" stabilization is concerned. Strictly speaking, the attitude motion of a rigid body cannot be globally continuously stabilized since the configuration space of the motion (the rotation group $SO(3)$) is noncontractible. Thus, by "global asymptotic stabilization" we mean here that the system of the corresponding *kinematic parameters* is globally asymptotically stable, i.e., the trajectories in these parameters remain bounded and tend to zero for arbitrary initial conditions. We therefore guarantee asymptotic stability for all initial orientations not corresponding to singular configurations. It is therefore obvious why we insist on parameterizations with the largest possible domain of validity. On the same token, starting from a non-singular orientation, the global asymptotic stability of the closed-loop system *a posteriori* insures the existence of solutions of the differential equations for all $t \geq 0$. If one is worried about the case when the *initial* body orientation corresponds to a singular configuration, one needs only to modify the proposed control laws as follows: because the singular configurations consist a dense set in $SO(3)$ (actually in the case of the Modified Rodrigues parameters is the whole space minus a single, isolated point), *any* control law of arbitrarily short duration will move the body away from this singular configuration; one can then use the stabilizing control laws proposed here. Hence from a practical point of view this slight abuse of the terminology should not cause any concern, since stability is guaranteed for all "generic" initial conditions.

The following assumptions will be used throughout the subsequent discussion. First, we assume that we have complete and accurate knowledge about the state of the system, i.e., the angular velocity vector and the kinematic parameter vector. This assumption allows for the design of state feedback controllers. The question of incomplete information of the state (output feedback) is not addressed in this paper. A special case of output feedback (no angular velocity measurements) can be found in Ref. 22. Second, we will assume that all initial conditions are in the domain of definition of the corresponding differential equations. This assumption is introduced because we are interested only in minimal parameterizations. Well-posedness of the problem limits the valid initial conditions. According to the previous discussion, this is only a mild assumption since one can always trivially modify the proposed control laws in order to incorporate physical orientations corresponding to singular configurations. We will also implicitly assume that the actuators used can generate continuous control profiles. This necessitates either the use of momentum wheels, or gas jet actuators in conjunction with Pulse-Width Pulse-Frequency (PWPF) modulators. Since most current gas jets are of the on-off type, PWPF modulators can be used to produce the continuously varying control profile by generating a pulse command sequence to the thruster valve by adjusting the pulse width and pulse frequency. The average torque thus produced by the thruster equals the demanded torque input[23].

We next derive stabilizing control laws both in terms of the Cayley-Rodrigues parameters, as well as the Modified Rodrigues parameters. Since most of the readers are probably more familiar with the Cayley-Rodrigues parameters we start with the results for this case.

Cayley-Rodrigues Parameters

References 15 and 16 present the main known result concerning asymptotic stabilization using the Rodrigues parameters. According to the results in Refs. 15 and 16, the feedback control

$$u = -k_1\rho(1+\rho^T\rho) - k_2\omega \tag{20}$$

with $k_1 > 0$ and $k_2 > 0$, globally asymptotically stabilizes the system of Eqs. (1) and (5) at the origin.

The proof of the stability of the closed-loop system is based on the use of the following *quadratic* Lyapunov function in terms of the angular velocities and the kinematic parameters[15,16]

$$V = \tfrac{1}{2}\omega^T J\omega + k_1\rho^T\rho \tag{21}$$

Before proceeding with the main results of this section we need to recall an important structural property of the system of Eqs. (1)-(5); namely, it represents a system in cascade form. That is, the control input u drives the angular velocity equations (1) and the angular velocity ω drives the kinematic equations (5). There is no direct connection between the kinematics subsystem and the torque input u. The kinematic equations can be accessed and manipulated only through the angular velocity vector ω. For systems in cascade connection there is an intuitive way to achieve closed-loop stability. The methodology involves a two-step procedure. Namely, one can concentrate first on the stabilization of the second (driven) subsystem (the kinematic equations in our case) treating the driving state as a control-like variable (the angular velocity vector in our case) and then proceed to the stabilization of the complete system.

Recall that a function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called *positive definite* if $V(x) \geq 0$ for all $x \in \mathbb{R}^n$ and $V(x) = 0$ if and only if $x = 0$. It is called *radially unbounded* if $\lim_{||x|| \to \infty} V(x) = \infty$.

Let now the system in Eq. (5) with ω considered as the control variable. Let $U : \mathbb{R}^3 \to \mathbb{R}_+$ be any positive definite, radially unbounded function such that

$$\frac{\partial U}{\partial \rho} = \rho^T h(\rho) \tag{22}$$

with $h : \mathbb{R}^3 \to \mathbb{R}$ a continuous function such that $\rho^T h(\rho) \neq 0$ for all $\rho \neq 0$.

Then we have that the following two conditions hold:

(i) The control law

$$\omega = -k_1 \left(\frac{1+\rho^T\rho}{2}\right) h(\rho)\rho \tag{23}$$

with $k_1 > 0$, globally asymptotically stabilizes the system in Eq. (5) at the origin. Moreover, U is a Lyapunov function for the associated closed-loop system.

(ii) The control law

$$u = -k_1 \left(\frac{1+\rho^T\rho}{2}\right) h(\rho)\rho - k_2\omega \tag{24}$$

with $k_1 > 0$ and $k_2 > 0$, globally asymptotically stabilizes Eqs. (1)-(5) at the origin and

$$V = \tfrac{1}{2}\omega^T J\omega + k_1 U(\rho) \tag{25}$$

is a Lyapunov function for the associated closed-loop system.

In order to show (i) notice that with the control in Eq. (23) the closed-loop system for the kinematics becomes

$$\dot\rho = -k_1 H(\rho) \left(\frac{1+\rho^T\rho}{2}\right) h(\rho)\rho \tag{26}$$

Since U is a positive definite function it can be used as a Lyapunov function candidate. Using Eq. (7) the derivative of U along the trajectories of the closed-loop system is given by

$$\begin{aligned} \dot U &= \frac{\partial U}{\partial \rho}\dot\rho = -k_1\left(\frac{1+\rho^T\rho}{2}\right)\rho^T h(\rho) H(\rho) h(\rho)\rho \\ &= -k_1\left(\frac{1+\rho^T\rho}{2}\right)^2 h^2(\rho)||\rho||^2 \leq 0, \quad \forall \rho \in \mathbb{R}^3 \setminus \{0\} \end{aligned} \tag{27}$$

From Eq. (27) we conclude that the origin $\rho = 0$ is asymptotically stable. Since U is also assumed to be radially unbounded the origin is, in fact, globally asymptotically stable[24].

For the proof of part (ii), consider the complete system of Eqs. (1) and (5) with control law (24) and let the function in Eq. (25) be a Lyapunov function candidate. The derivative of V along the trajectories of the closed-loop system, taking into consideration Eq. (7), is given by

$$\begin{aligned} \dot V &= \omega^T J\dot\omega + k_1 \frac{\partial U}{\partial\rho}\dot\rho \\ &= \omega^T\left(-k_1\left(\frac{1+\rho^T\rho}{2}\right)h(\rho)\rho - k_2\omega\right) + k_1 h(\rho)\rho^T H(\rho)\omega \\ &= -k_2||\omega||^2 \leq 0 \end{aligned} \tag{28}$$

If $\dot V \equiv 0$ then $\omega = \dot\omega \equiv 0$ and from Eqs. (1) and (24) also $\rho \equiv 0$. Thus, the largest invariant set such that $\dot V = 0$ is the origin. By LaSalle's Theorem[24], the system is asymptotically stable at the origin. Since V in Eq. (25) is also radially unbounded, the closed loop system is actually globally asymptotically stable at the origin.

Equations (23) and (24) provide a large family of stabilizing control laws for the system of Eqs. (1)-(5). The main obstruction with the use of these results is, of course, the fact that one does not have a constructive way of generating the positive definite, radially unbounded function U satisfying Eq. (22). On the other hand, *any* such positive definite, radially unbounded function can be used to generate globally asymptotically stabilizing control laws for the system in Eqs. (1)-(5).

The easiest (but by no means the only) choice for U is to pick a positive definite function in terms of the magnitude of ρ alone. The choice

$$U(\rho) = \rho^T\rho \tag{29}$$

for example, satisfies Eq. (22) with $h(\rho) = 2$ and the control law in Eq. (20) follows directly from Eq. (24). If one chooses a different Lyapunov function than the one in Eq. (21) one easily establishes the fact that a *linear* control law suffices to provide global asymptotic stability for the system of Eqs. (1) and (5). To see this, let the system in Eq. (5) with ω considered as a control variable. Observe that the positive definite function

$$U(\rho) = \ln(1+\rho^T\rho) \tag{30}$$

where $\ln(\cdot)$ denotes the natural logarithm, satisfies the conditions of Eq. (22). For U as in Eq. (30) we get that

$$h(\rho) = \frac{2}{1+\rho^T\rho} \tag{31}$$

Therefore the linear feedback control law

$$\omega = -k_1\rho \tag{32}$$

asymptotically stabilizes the system (5) at the origin. Actually, since

$$\begin{aligned} \dot U &= -k_1\left(\frac{2}{1+\rho^T\rho}\right)\rho^T H(\rho)\rho \\ &= -k_1||\rho||^2 \leq -k_1 \ln(1+||\rho||^2) = -k_1 U(\rho) \end{aligned} \tag{33}$$

the control law in Eq. (32) *exponentially* stabilizes the system in Eq. (5) with rate of decay $k_1/2$. (In Eq. (33) we have used the inequality $x \geq \ln(1+x)$ for all $x \geq 0$.)

In addition, the linear control law

$$u = -k_1\rho - k_2\omega \tag{34}$$

with $k_1 > 0$ and $k_2 > 0$, globally asymptotically stabilizes the system in Eqs. (1) and (5) at the origin. This result follows directly from Eq. (24) and the fact that the function

$$V = \tfrac{1}{2}\omega^T J\omega + k_1\ln(1+\rho^T\rho) \tag{35}$$

is a Lyapunov function for the closed-loop system in Eqs. (1)-(5) and (34).

From Eqs. (20) and (21) and Eqs. (34) and (35) it is clear that one has, in essence, traded the complexity of the Lyapunov function with the complexity of the resulting control law. Lyapunov functions of the type (35) (i.e., "quadratic plus logarithmic") have been previously introduced in connection with the attitude stabilization problem of an axisymmetric spacecraft using only two control torques[25].

Modified Rodrigues Parameters

Similar results can also be shown for the case of the Modified Rodrigues parameters. The following results, stated without proof, present the corresponding stabilization results in terms of these parameters.

Let the system in Eq. (15) with ω considered as a control variable. Let $W : \mathbb{R}^3 \to \mathbb{R}_+$ be any positive definite, radially unbounded function such that

$$\frac{\partial W}{\partial\sigma} = \sigma^T g(\sigma) \tag{36}$$

with $g : \mathbb{R}^3 \to \mathbb{R}$ a continuous function, such that $\sigma^T g(\sigma) \neq 0$ for all $\sigma \neq 0$.

Then we have that

(i) The control law

$$\omega = -k_1 \left(\frac{1+\sigma^T\sigma}{4} \right) g(\sigma)\sigma \tag{37}$$

with $k_1 > 0$, globally asymptotically stabilizes (15) at the origin. Moreover, W is a Lyapunov function for the associated closed-loop system.

(ii) The control law

$$u = -k_1 \left(\frac{1+\sigma^T\sigma}{4} \right) g(\sigma)\sigma - k_2\omega \tag{38}$$

with $k_1 > 0$ and $k_2 > 0$, globally asymptotically stabilizes the system (1)-(15) at the origin and

$$V = \tfrac{1}{2}\omega^T J\omega + k_1 W(\sigma) \tag{39}$$

is a Lyapunov function for the associated closed-loop system.

Observing that the positive definite function

$$W(\sigma) = \ln(1+\sigma^T\sigma) \tag{40}$$

satisfies the conditions of Eq. (36) one can also easily show that the linear controller

$$\omega = -k_1\sigma \tag{41}$$

with $k_1 > 0$, globally exponentially stabilizes the system in Eq. (15) at the origin with rate of decay $k_1/2$ (with ω considered as a control variable). Moreover, the linear control law

$$u = -k_1\sigma - k_2\omega \tag{42}$$

with $k_1 > 0$ and $k_2 > 0$, globally asymptotically stabilizes Eqs. (1) and (15) at the origin.

A *nonlinear* feedback control arises if one uses the quadratic function

$$W(\sigma) = 2\sigma^T\sigma \tag{43}$$

instead. Using this choice of W the feedback control

$$u = -k_1\sigma(1+\sigma^T\sigma) - k_2\omega \tag{44}$$

globally asymptotically stabilizes the system of Eqs. (1) and (15) at the origin. The previous control law is, in essence, the counterpart of the control law in Eq. (20) for the case of the Modified Rodrigues parameters.

The results of this section show that there is a similar structure between the Cayley-Rodrigues and the Modified Rodrigues parameters which allows the derivation of stabilizing controllers for the Modified Rodrigues parameters *mutatis mutandis* from the corresponding results for the Cayley-Rodrigues parameters (and vice versa). Despite this apparent similarity, the behavior of the two sets of kinematic parameters can vary greatly in applications. The numerical examples at the end of the paper illustrate the relative merit of the two parameterizations in the description of the attitude motion and in control problems.

As a last remark, we note that all of the proposed control laws have the property that they do not require any information about the body principal moments of inertia and they are therefore *robust* with respect to system parametric uncertainty.

We now turn to the question of *performance* of the previous controllers.

Optimality Results

In this section we show that the linear control laws *for the kinematics* in Eqs. (32) and (41) have certain optimality properties. In particular, we show that the Lyapunov functions of the logarithmic type in Eqs. (30) and (40) solve the Hamilton-Jacobi equation associated with an optimization problem with a performance index which includes a quadratic penalty in the angular velocity and the orientation parameters ρ or σ. We then investigate the optimality properties of a family of nonlinear control laws for the complete system (i.e., dynamics and kinematics).

Recalling the cascade interconnection of the dynamics-kinematics subsystems we first address an optimization problem for the kinematics *only*. To this end, consider the system in Eq. (5) where ω acts as the control variable and let the quadratic performance index

$$\mathcal{J}_1(\rho_0;\omega) := \tfrac{1}{2}\int_0^\infty \{k^2\|\rho(t)\|^2 + \|\omega(t)\|^2\}\, dt \tag{45}$$

where k some positive constant. Notice that this functional is a true performance index in the sense that it penalizes the state (ρ) and the control input (ω).

In the previous section we have shown that the control law

$$\omega^*(\rho) = -k\rho \tag{46}$$

renders the closed-loop system globally exponentially stable at the origin. We will now show that this control law also minimizes (45) and the minimum value of the cost is $\mathcal{J}_1^*(\rho_0) = k\ln(1+\|\rho_0\|^2)$.

According to Hamilton-Jacobi theory the optimal feedback control ω^* for the previous problem is given by

$$0 = \min_{\omega}\{\frac{k^2}{2}\|\rho\|^2 + \frac{1}{2}\|\omega\|^2 + \frac{\partial V}{\partial \rho}H(\rho)\omega\} \tag{47}$$

Therefore, the Hamilton-Jacobi equation associated with the optimal control problem in Eqs. (5)-(45) is given by

$$k^2\|\rho\|^2 - \|H^T(\rho)\frac{\partial^T V}{\partial \rho}\|^2 = 0, \qquad \forall\, \rho \in \mathrm{IR}^3 \tag{48}$$

If V is a positive definite solution to this equation, then a simple calculation shows that the control law

$$\omega^*(\rho) = -H^T(\rho)\frac{\partial^T V}{\partial \rho} \tag{49}$$

minimizes the cost in Eq. (45). Moreover, the optimal cost is $\mathcal{J}^*(\rho_0) = \mathcal{J}(\rho_0;\omega^*) = V(\rho_0)$.

Notice now that the positive definite function defined by

$$V_1(\rho) := k\ln(1+\|\rho\|^2) \tag{50}$$

provides a solution to Eq. (48). The optimal control in Eq. (46) follows then directly from Eq. (50) and Eq. (49) and the optimal cost is given by $V_1(\rho_0) = k\ln(1+\|\rho_0\|^2)$.

A similar result can also be shown for the Modified Rodrigues parameters. We state the result without proof.

Consider the system in Eq. (15) where ω acts as the control variable and let the quadratic performance index

$$\mathcal{J}_1(\sigma_0;\omega) := \tfrac{1}{2}\int_0^\infty \{k^2\|\sigma(t)\|^2 + \|\omega(t)\|^2\}\, dt \tag{51}$$

where k some positive constant.

Then the control law

$$\omega^*(\sigma) = -k\sigma \tag{52}$$

renders the closed-loop system globally exponentially stable at the origin and minimizes (51). Moreover, the minimum value of the cost is $\mathcal{J}_1^*(\sigma_0) = 2k\ln(1+\|\sigma_0\|^2)$.

A similar procedure can also be used to show that the nonlinear control laws

$$\omega^*(\rho) = -k(1+||\rho||^2)\rho \tag{53}$$

and

$$\omega^*(\sigma) = -k(1+||\sigma||^2)\sigma \tag{54}$$

are also optimal with respect to the costs

$$\mathcal{J}_2(\rho_0;\omega) := \tfrac{1}{2}\int_0^\infty \{k^2||\rho(t)||^2(1+||\rho(t)||^2)^2 + ||\omega(t)||^2\}\,dt \tag{55}$$

and

$$\mathcal{J}_2(\sigma_0;\omega) := \tfrac{1}{2}\int_0^\infty \{k^2||\sigma(t)||^2(1+||\sigma(t)||^2)^2 + ||\omega(t)||^2\}\,dt \tag{56}$$

with k some positive constant, for the systems in Eqs. (5) and (15), respectively. Notice that the cost functionals involve quadratic and quartic terms in the state variables in this case. The proof is based on the observation that the positive definite functions $V_2(\rho) = k\|\rho\|^2$ and $V_2(\sigma) = 2k||\sigma||^2$ solve the respective Hamilton-Jacobi equations.

Thus far, we have only considered the kinematics subsystem of the attitude equations, that is, only Eqs. (5), viz. Eqs. (15), with ω acting as a control variable. If the dynamics is sufficiently fast the previous results suffice. In these cases, the optimal angular velocity profile can be implemented through the dynamics without significant degradation in performance.

Consider the feedback control

$$u_{as} = -S(\omega)J\omega - kJH(\rho)\omega - \lambda J(\omega + k\rho) \tag{57}$$

With this control law the closed-loop system in Eqs. (1) and (5) becomes

$$\begin{aligned}\epsilon(\dot\omega + k\dot\rho) &= \omega + k\rho && (58\text{a})\\ \dot\rho &= H(\rho)\omega && (58\text{b})\end{aligned}$$

where $\epsilon = 1/\lambda$. For λ large this is a singularly perturbed system where the dynamics (58a) is the fast subsystem and the kinematics (58b) is the slow subsystem. For $\epsilon = 0$ in Eq. (58a) one obtains $\omega = -k\rho$ as in Eq. (46). Thus, for $\epsilon \to 0$ one obtains that $\omega \to \omega^*$. This implies that

$$\begin{aligned}\mathcal{J}_3(\rho,\omega;u_{as}) &= \textstyle\int_0^\infty \{k^2\|\rho(t)\|^2 + \|\omega(t)\|^2\}\,dt\\ &\to k\ln(1+\|\rho_0\|^2) = \mathcal{J}_1(\rho_0)\end{aligned} \tag{59}$$

and the cost can be made arbitrarily close to $\mathcal{J}_1^*(\rho_0)$ by choosing ϵ sufficiently small. Since, in general, $\epsilon \neq 0$ the optimal cost $\mathcal{J}_1(\rho_0)$ provides only a lower bound on the achievable performance when the actual control input is the body fixed torque.

Similar statements can also be made for the Modified Rodrigues parameters. For the sake of brevity, these results will not be repeated here and are left as an exercise to the reader.

The disadvantage of the control law in Eq. (57) is, of course, that it requires high gain in terms of λ. This may not be acceptable if there are stringent bounds on the available control effort. A more realistic performance index needs to incorporate a penalty on the control u as well. Unfortunately, the optimization problem for a performance index which is quadratic both in the state and the control effort remains formidable. Alternatively, one may investigate the optimality properties of the control law in Eq. (57) and, if possible, modify this control such that its high-gain portion is penalized.

The following procedure is based on the results of Ref. 26, where the authors examined the optimality properties of a class of feedback control laws for relative degree one minimum phase systems. As before, we will present the results for the Rodrigues parameters only. The reader is invited to derive the corresponding control laws and performance functionals for the Modified Rodrigues parameters from the Rodrigues parameters case.

As it is evident from Eq. (57) the last term in this equation is the high-gain portion of the controller. We therefore consider a modified control law of the form

$$u = -S(\omega)J\omega - kJH(\rho)\omega + Jv \tag{60}$$

where v must be kept small. Recalling the desirable properties of the relationship $\omega = -k\rho$, it is natural to introduce the variable

$$z = \omega + k\rho \tag{61}$$

and develop control laws which will make $z \to 0$. The performance index should therefore include a penalty on z as well as a penalty on the control effort v.

Using Eqs. (61) and (60) the system equations can be written as follows

$$\begin{aligned}\dot z &= v && (62\text{a})\\ \dot\rho &= H(\rho)(z - k\rho) && (62\text{b})\end{aligned}$$

It is claimed that the feedback control law

$$v^*(\rho,z) = -\frac{\rho}{\lambda} - \lambda z \tag{63}$$

makes the system in Eq. (62) exponentially stable and minimizes the cost

$$\mathcal{J}_4(\rho,z;v) = \tfrac{1}{2}\int_0^\infty \{\|v + \frac{\rho}{\lambda}\|^2 + 2k\|\rho\|^2 + \lambda^2\|z\|^2\}\,dt \tag{64}$$

The proof is obtained as follows. Notice that the Hamilton-Jacobi equation associated with the previous optimization problem is given by

$$-\tfrac{1}{2}\|\frac{\partial V}{\partial z}\|^2 + k\|\rho\|^2 + \frac{\lambda^2}{2}\|z\|^2 - \frac{\partial V}{\partial z}\frac{\rho}{\lambda} + \frac{\partial V}{\partial \rho}H(\rho)(z - k\rho) = 0 \tag{65}$$

and the optimal control is given by

$$v^*(\rho,z) = -\frac{\rho}{\lambda} - \frac{\partial^T V}{\partial z} \tag{66}$$

Notice now that the positive definite function

$$V_4(\rho,z) = \ln(1+\|\rho\|^2) + \frac{\lambda}{2}\|z\|^2 \tag{67}$$

solves the Hamilton-Jacobi equation (65). The exponential stabilizability of the control law in Eq. (63) then follows easily by using Eq. (67) as the Lyapunov function for the closed-loop system. The minimum value of the cost is given by

$$\begin{aligned}\mathcal{J}_4^*(\rho_0,z_0) &= \mathcal{J}_4(\rho_0,z_0;v^*) = \ln(1+\|\rho_0\|^2)\\ &\quad + \tfrac{\lambda}{2}\|z_0\|^2 = V_4(\rho_0,z_0)\end{aligned} \tag{68}$$

From Eqs. (60) and (63) we have, finally, that the optimal control is

$$u^*(\omega,\rho) = -S(\omega)J\omega - kJH(\rho)\omega - J\frac{\rho}{\lambda} - \lambda J(\omega + k\rho) \tag{69}$$

Moreover,

$$u^* = u_{as} - J\frac{\rho}{\lambda} \tag{70}$$

where u_{as} is the control in Eq. (57). Comparison of Eqs. (63) and (64) shows that the control law

$$\tilde v^*(z) = -\lambda z \tag{71}$$

minimizes the cost

$$\tilde{\mathcal{J}}_4(\rho,z;\tilde v) = \tfrac{1}{2}\int_0^\infty \{\|\tilde v\|^2 + 2k\|\rho\|^2 + \lambda^2\|z\|^2\}\,dt \tag{72}$$

subject to the dynamic constraints

$$\dot{z} = -\frac{\rho}{\lambda} + \tilde{v} \tag{73a}$$

$$\dot{\rho} = H(\rho)(z - k\rho) \tag{73b}$$

The first term in Eq. (72) includes a penalty on the high gain portion of the controller. Moreover, notice that as $\lambda \to \infty$ then $v^* \to -\lambda z$ and $u^* \to u_{as}$ and we recover the results of the control law in Eq. (57).

Numerical Examples

We now demonstrate the previous theoretical results by means of numerical simulations. We first compare the linear control laws in Eqs. (34) and (42) with the nonlinear control laws in Eqs. (20) and (44). We consider a rigid body with inertia parameters $J = diag(10, 6.3, 8.5)\ (kg \cdot m^2)$. The initial angular velocities are zero and the initial orientation corresponds to an eigenaxis/angle representation $\hat{e} = (0.4896, 0.2032, 0.8480)^T$ and $\Phi = 2.5\,rad$. That is, the initial conditions in terms of the Cayley-Rodrigues parameters are given by $\rho(0) = (1.4735, 0.6115, 2.5521)^T$ and in terms of the Modified Rodrigues parameters are given by $\sigma(0) = (0.3532, 0.1466, 0.6118)^T$. The values for the gains were chosen as $k_1 = 2$ and $k_2 = 1$.

The results of the simulations for the Rodrigues parameters are shown in Figures 1-2. The solid lines represent the trajectories with the nonlinear control in Eq. (20) and the dashed lines represent the trajectories with the linear control law in Eq. (34). Figure 1 depicts the behavior of the Rodrigues parameter vector, while Figure 2 shows the time history of the associated control effort. Only the first component of the corresponding vectors are shown here, since the other two components exhibit similar behavior.

These results show that the linear controller seems to perform better than the nonlinear controller. In particular, the trajectories with the linear controller exhibit much smaller overshoot and the trajectories with the nonlinear controller. In addition, the linear controller uses less control effort than the nonlinear controller. This can be explained by observing that the initial condition for the Principal angle gives rise to relatively high initial values for the Rodrigues kinematic parameters. As a result, the additional quadratic term in the control law in Eq. (20) imposes greater control requirements and more drastic control action than the linear controller, especially at the beginning of the maneuver. For smaller initial conditions it is expected that the two controllers — and the corresponding trajectories — will exhibit similar behavior. This is due to the fact that as $\rho \to 0$ the linear and the nonlinear controllers become identical. (Notice that the linearization of the control law in Eq. (20) is the control law in Eq. (34) and hence the linearizations of the corresponding closed-loop systems are the same.)

The corresponding results for the Modified Rodrigues parameters are shown in Figures 3-4. The control effort for the two control laws is comparable although, again, the nonlinear controller initially requires more control energy than the linear controller. In light of the discussion in the previous paragraph it should be clear that the differences between the linear and the nonlinear controller would become more distinct as the initial conditions approach the singularity ($\Phi = 360\,$deg) with the linear controller performing increasingly better as the initial value of $\Phi \to 360\,$deg. Moreover, as $\Phi \to 180\,$deg (the singularity of the Cayley-Rodrigues parameterization) it is expected that more control effort will be required by the control law in Eq. (34) than the one in Eq. (42). This is a consequence of the fact that the Modified Rodrigues parameters behave more "linearly" over a larger domain than the Cayley-Rodrigues parameters[20].

The second numerical example compares the control law in Eq. (69) for different values of the parameter λ. The same initial conditions are chosen as before, and the gain is taken $k = 2$. Figure 5 shows the time history of the Cayley-Rodrigues parameters for $\lambda = 0.1, 1$ and 10. The respective control histories are plotted in Fig. 6. Recall that it is desirable to keep λ small, but one may not make it too small either, since then the fourth term in Eq. (69) may become significant. By varying λ one can shape the acceptable control and states profiles. The cost criterion in Eqs. (64) or (72) can then be used as a guide for the best choice for λ. For this example it appears that a value approximately $\lambda = 1$ provides a reasonable compromise.

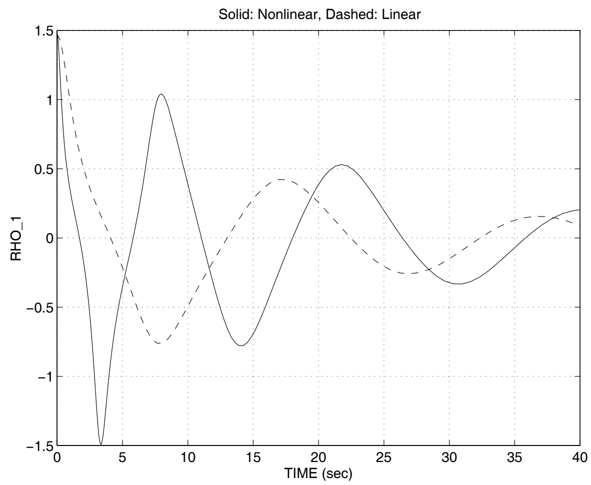

Fig. 1 Controller comparison; Rodrigues parameters history.

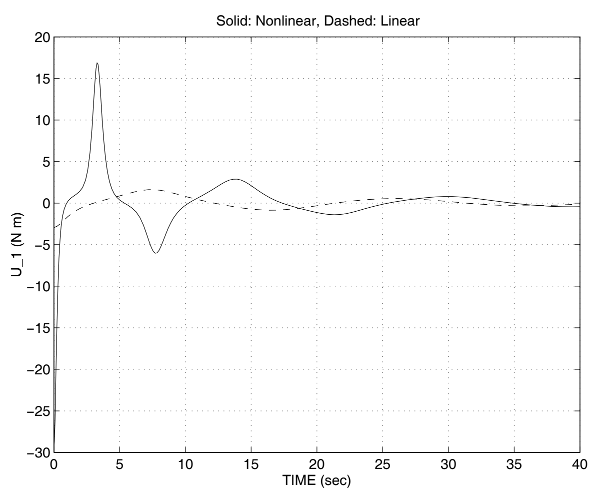

Fig. 2 Controller comparison; Control history for Rodrigues parameters.

Conclusions

We have presented a minimal parameterization for the kinematics of the attitude motion which – although similar to the classical Cayley-Rodrigues parameter – does not have the disadvantage of restricting the principal angle between 0 and 180 deg. In fact, using these parameters, all eigenaxis rotations within $0 \leq \Phi < 360\,$deg are allowed; this implies that the inherent singularity of any three-dimensional parameterization has been moved as far from the equilibrium point as possible. This is clearly a desirable property, especially for attitude control applications. We have also derived a new class of feedback control laws for the attitude stabilization of

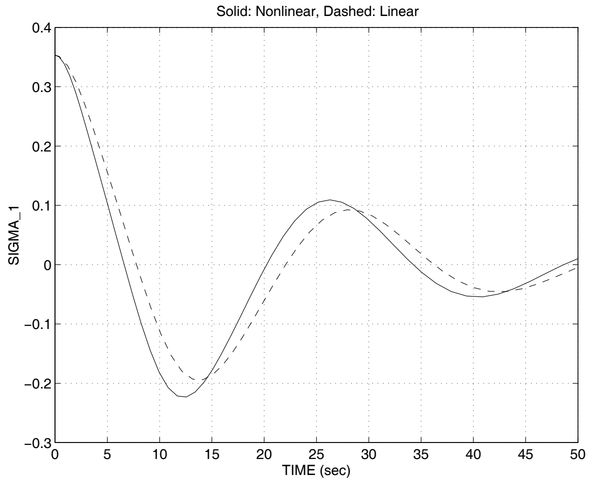

Fig. 3 Controller comparison; Modified Rodrigues parameters history.

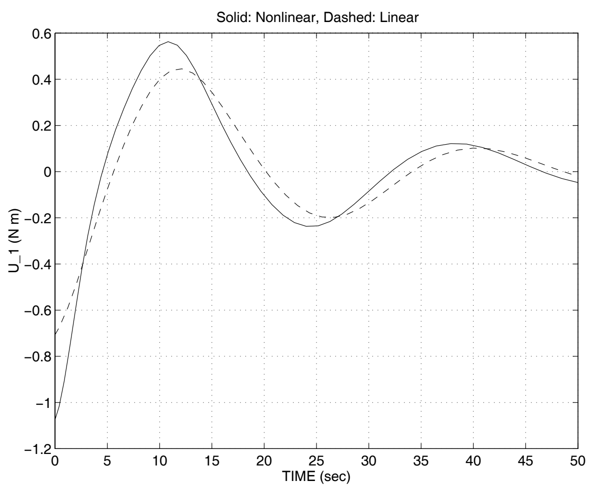

Fig. 4 Controller comparison; Control history for Modified Rodrigues parameters.

a rigid body in terms of minimal, three-dimensional parameterizations. In particular, a new type of Lyapunov function for this class of problems is proposed, which often leads to linear control laws. The proposed Lyapunov functions include a quadratic term in the angular velocities (the kinetic energy) and a logarithmic term in the kinematic parameters. Finally, we have addressed the optimal feedback control problem for the attitude motion subject to a quadratic cost. The optimal controllers are derived by analytically solving the associated Hamilton-Jacobi equations. Although the complete solution to this problem is still unknown, the results reported in this paper may shed some light for possible candidate solutions. The numerical examples at the end of the paper indicate that the Modified Rodrigues parameters are better conditioned for numerical simulations and stabilization purposes than the classical Cayley-Rodrigues parameters. This, along which the fact that they are not limited to eigenaxis rotations of only upto 180 deg, makes the Modified Rodrigues parameters an attractive choice for attitude description and control problems.

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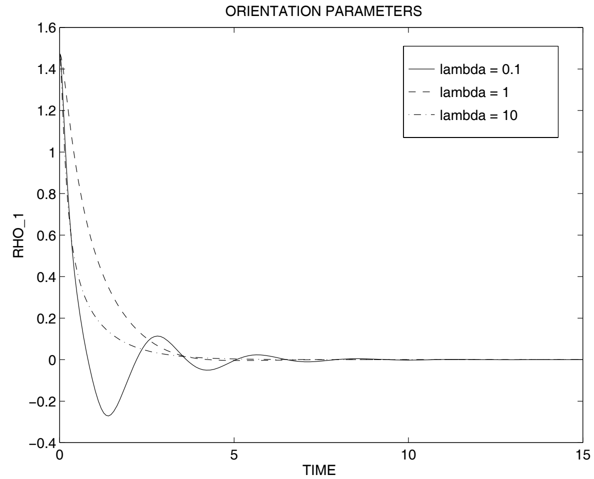

Fig. 5 Orientation parameter response using u^*.

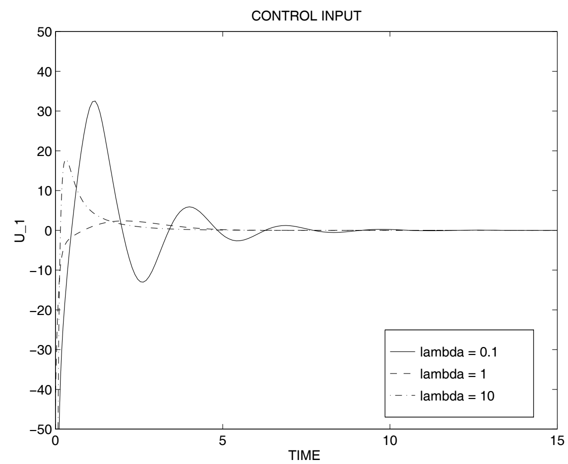

Fig. 6 Control input response using u^*.

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