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# Goddard Problem with Constrained Time of Flight

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**The problem of the vertical flight of a rocket in a resisting medium with a constraint of isoperimetric type is studied. Only the case of maximum altitude for given flight time has been analyzed, but the methodology applies to other types of constraints as well, e.g., the minimum-time problem, due to Mayer reciprocity. Analysis shows that a one-parameter family of singular extremals is generated according to the value of time of flight. Three cases of switching structure have been found, varying with the specified duration of flight, with the most interesting case featuring appearance of a second full-thrust subarc at the departure from the singular subarc, owing to a low value of the upper bound on the thrust.**

## Introduction

THE problem of maximizing the altitude of a rocket in vertical flight, for a given amount of propellant, has been extensively analyzed by many writers since the early days of rocketry. The classical theory of the calculus of variations was employed first and, later, optimal control theory. For strict assumptions on the drag law and the thrust, solutions were found, even in a closed analytical form. The pioneering work was by Goddard[1] and later by Hamel,[2] the latter being the first to point out the existence of a solution by means of the calculus of variations, based on the fact that the mass of the rocket enters linearly in the equation of motion. However, it was not until 1951 that Tsien and Evans,[3] using Hamel's results, treated the problem in detail and carried out computations of the trajectories for two particular cases, namely, one with linear drag dependence on velocity and the other with quadratic drag dependence on velocity.

Leitmann[4-7] later extended their results and derived necessary conditions for the solution. He carried out a similar study for the problem in which the case of the rocket is consumed along with the fuel, changing the area factor in the drag reduction—an assumption used also by Goddard—treating it as a problem of Mayer with two differential equations. Using a totally different approach, Miele[8] proved the sufficiency of the extremal solution established by his predecessors, i.e., that the optimal burning program involves a rapid boost at the beginning of the flight, usually followed by a period of continuous burning (sustain phase) and ending with a zero-thrust period. Faulkner[9] and Leitmann[10] had earlier indicated that for certain types of end conditions, this optimal program may have to be modified to contain a similar second boost at the end of the sustain phase. Miele was the first to extend the previous results to the case in which a time constraint is imposed and the first to suggest the possibility of a more complex sequence of subarcs, also for the case of a general drag model.[11-13] In another early paper, Miele[14] also gave some examples of one-parameter families of singular extremals arising in connection with a related problem, the climbing flight of a jet-propelled aircraft.

In 1964 Ewing and Hazeltine[15] objected to the vagueness that plagued previous solutions based on the calculus of variations and treated the problem in rigorous, mathematical detail. Later on, Munick[16] and Lee and Markus[17] gave the proof of four lemmas that govern the composition of subarcs for an optimal trajectory.

More recently, Ardema[18] obtained closed-form solutions for the vertical rocket flight, using a singular perturbations approach. He showed that singular perturbation methods can be a powerful tool in deriving approximate solutions to optimal control problems that are nonlinear and singular, but the success of those methods is limited by the fact that for such problems it is highly desirable (and sometimes even necessary) to transform the state variables into a system in which the control appears only in the fast equation.

However, although the problem of the vertical rocket flight, in one version or the other, has interested many writers, solutions have been obtained only under the convenient assumption that the thrust has no upper bound or, equivalently, that the upper bound of the thrust is of a sufficiently high magnitude. Furthermore, with the exception of the very brief work by Miele,[19,20] attention was confined to the free-final-time case, assuming that the results extend gracefully to the fixed-final-time case. In such a case, and for sufficiently high upper bound on the thrust, an optimal solution can be obtained that includes, at the most, three subarcs: an initial full-thrust subarc followed by a subarc of variable control effort, or singular subarc, and finally a coasting subarc until maximum altitude has been reached. However, that may not be the case when a constraint of isoperimetric type is added to the problem. Consider, for instance, the problem of extremizing the rise in altitude for a given time or alternatively the problem of extremizing the time of flight for given altitude increase. In the present work, the first case is studied, recognizing the reciprocity of the two cases. Analysis shows that a one-parameter family of singular extremals is generated according to the value of time of flight. It is shown that the optimal control in this case can have a more complex switching structure, mainly due to the possibility of appearance of a second full-thrust subarc following the singular period of burning. The solution is applied to this special case, using the modeling of Zlatskiy and Kiforenko,[21] and numerical results are given.

## Problem Formulation

The following assumptions are made to simplify the physics of the problem. The rocket-powered aircraft is ideally regarded as a particle of variable mass flying over a flat, stationary Earth with Newtonian central gravitational field of inverse square law. The authors considered a single-stage rocket with constant exhaust velocity, stabilized so that its axis is always parallel to the flight path. Moreover, the air density is exponential with altitude, allowing the drag to take the form

$$D(v,h) = D_0(v)\exp(-\beta h) \quad (1)$$

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Usually the restrictions on the function $D_0(v)$ involve the continuity of its first and second derivatives. The assumption that the drag can be written in the form of Eq. (1) is perhaps the most important one because without essential restrictions on the drag there may not even exist an optimal solution. However, one can show[15] that using a drag law of the previous form, an optimal control exists, which is a very significant result since only necessary conditions will be used to test the optimality of the solution.

The vertical path of the rocket obeys the following system of nondimensionalized equations:

$$\dot{h} = v$$
$$\dot{v} = \frac{T - D(v,h)}{m} - h^{-2} \tag{2}$$
$$\dot{m} = -\frac{T}{c}$$

Those three equations correspond to force equilibrium and kinematics along the direction of flight. In those equations, $h$ is the radius distance from the Earth's center, $v$ the velocity of the vehicle, $m$ the mass of the vehicle, $T$ the thrust, $D(v,h)$ the aerodynamic drag, and $c$ the effective exhaust velocity of the gas flow. The previous system of equations has been suitably nondimensionalized using the following quantities:

$$\hat{h} = R_e, \quad \hat{t} = G^{-1/2}R_e^{3/2}, \quad \hat{m} = m_0$$

Here, $R_e$ denotes the radius of the Earth, $G$ the gravitational constant, and $m_0$ the launching mass of the vehicle. Consequently, the forces are nondimensionalized by the initial weight. The aerodynamic drag assumes the following form:

$$D(v,h) = C_D b v \,|v|\, \exp[\beta(1-h)] \tag{3}$$

where the factor $bv\,|v|\,\exp[\beta(1-h)]$ is numerically equal to the product of the velocity head and the characteristic area of the aircraft, $b$ and $\beta$ are constants, and $C_D$ is the zero-lift drag coefficient.

The initial conditions are specified for the three states as $h_0$, $v_0$, and $m_0$. The final value of the mass is also given as $m_f$. The problem is to determine the optimum trajectory of a rocket in vertical flight, from an assigned initial position on the surface of the Earth to the final position where the altitude reaches its maximum value, time of flight being predetermined; i.e., one wants to maximize the altitude at the terminal time:

$$\mathcal{J} = h(t_f) \tag{4}$$

subject to the prescribed boundary conditions, the dynamic equality constraints given by Eq. (1) and the isoperimetric constraint on time:

$$t_c = \int_0^{t_f} \mathrm{d}t \tag{5}$$

Thus, the problem is formulated as a problem of Mayer, i.e., no additional differential equation or integral for the cost $\mathcal{J}$ is required.[22] The guidance of the rocket is achieved by means of the magnitude of the thrust, which is considered a control variable allowed to have jumps, and which is bounded according to the inequality

$$0 \le T \le T_{\max} \tag{6}$$

## Problem Analysis

In the following analysis of the problem, it is tacitly assumed that the experiment has no meaning for $t_f < \bar{t}$, where

$$\bar{t} = (m_0 - m_f)\frac{c}{T_{\max}}$$

is the time that fuel runs out, using full-thrust power. Therefore, only problems of maximizing the altitude for times greater or equal to $\bar{t}$ are considered.

Define the state vector $x = \mathrm{col}(h, v, m)$, and the costate vector $\lambda = \mathrm{col}(\lambda_h, \lambda_v, \lambda_m)$. Then the Hamiltonian takes the form

$$\mathcal{H}(x,\lambda,T) = \lambda_h \dot{h} + \lambda_v \dot{v} + \lambda_m \dot{m} \tag{7}$$

The costate vector satisfies the following differential equation:

$$\dot{\lambda} = -\frac{\partial \mathcal{H}}{\partial x} \tag{8}$$

and its components are given analytically by

$$\dot{\lambda}_h = -\frac{\partial \mathcal{H}}{\partial h} = \frac{\lambda_v}{m}\frac{\partial D}{\partial h} - 2\lambda_v h^{-3}$$
$$\dot{\lambda}_v = -\frac{\partial \mathcal{H}}{\partial v} = \frac{\lambda_v}{m}\frac{\partial D}{\partial v} - \lambda_h$$
$$\dot{\lambda}_m = -\frac{\partial \mathcal{H}}{\partial m} = \frac{\lambda_v}{m^2}(T - D) \tag{9}$$

The transversity condition requires that for the unspecified states at the terminal time, the following relations are to be satisfied by their associated costates:

$$\lambda_h(t_f) = 1$$
$$\lambda_v(t_f) = 0 \tag{10}$$

The time does not appear explicitly in the equation of motion; therefore the Hamiltonian is a first integral, remaining constant throughout the trajectory:

$$\mathcal{H} = C_1 \qquad (0 \le t \le t_f) \tag{11}$$

where $C_1$ is a constant to be determined, satisfying the conditions

$$C_1 > 0 \quad \text{when} \quad t_f < t_{\text{free}}$$
$$C_1 = 0 \quad \text{when} \quad t_f = t_{\text{free}}$$
$$C_1 < 0 \quad \text{when} \quad t_f > t_{\text{free}} \tag{12}$$

and $t_{\text{free}}$ is the final time corresponding to the unconstrained time problem, i.e., when Eq. (5) is absent. Notice that for the minimum-time problem $C_1 = +1$, and for the maximum-time problem $C_1 = -1$.

Using Eqs. (2) and (9), and noting that the control $T$ appears linearly in the equations of motion, one obtains for the Hamiltonian the following form:

$$\mathcal{H} - C_1 = \mathcal{H}_0 + T\mathcal{H}_1 = 0 \tag{13}$$

where $\mathcal{H}_0$ and $\mathcal{H}_1$ are given by

$$\mathcal{H}_0 = \lambda_h v - \lambda_v\left(\frac{D}{m} + h^{-2}\right) - C_1 \tag{14}$$

$$\mathcal{H}_1 = \frac{\lambda_v}{m} - \frac{\lambda_m}{c} \tag{15}$$

where $\mathcal{H}_0$ is the portion of the Hamiltonian independent of the control variable, minus $C_1$, and $\mathcal{H}_1$ is the portion of the Hamiltonian multiplied linearly by the control variable. The $\mathcal{H}_1$ portion is called the *switching function,* and it determines the extremal control history.

## Control Logic

It is known that the maximum of $\mathcal{H}$ with respect to all controls in the admissible set $\mathcal{W} = [0, T_{\max}]$, is a *necessary* condition for optimal control.[23] This maximum principle can be stated as follows:

$$T^* = \underset{T\in\mathcal{W}}{\operatorname{argmax}}\,\mathcal{H}(T) \qquad (0 \le t \le t_f) \qquad \mathcal{H}(T^*) - C_1 \equiv 0 \quad (16)$$

From Eqs. (6) and (16) three possibilities exist for an extremal control, depending on the sign of the switching function:

$$\begin{aligned} T^* &= T_{\max} && \text{when} \quad \mathcal{H}_1 > 0 \\ 0 \le T^* &\le T_{\max} && \text{when} \quad \mathcal{H}_1 = 0 \\ T^* &= 0 && \text{when} \quad \mathcal{H}_1 < 0 \end{aligned} \qquad (17)$$

Isolated points in time $t^*$, where the switching function vanishes, having measure zero, are of no consequence in the composition of an optimal trajectory. However, the second case of Eq. (17) allows the possibility of an interval of singular control, i.e., an interval of control effort with $\mathcal{H}(T)$ stationary. Along such an interval of singular control (singular arc), a positive $\delta$ exists, so that the switching function vanishes for each $t$ in $t^* - \delta \le t \le t^* + \delta$. Hence, the following relationships are fulfilled simultaneously on a singular arc:

$$\mathcal{H}_1 = \dot{\mathcal{H}}_1 = \ddot{\mathcal{H}}_1 = \cdots \equiv 0 \qquad (18)$$

## Evaluation of Singular Control

Along a singular extremal, the graph of the Hamiltonian vs the control is a horizontal line, and the maximum principle gives no information about the possible optimal control because all admissible controls qualify. However, from Eq. (18) the switching function vanishes identically along the singular path, and a singular control that maintains $\mathcal{H}_1 \equiv 0$ along solutions of the canonical equations, Eqs. (2) and (9), can be determined by taking successive time derivatives of the switching function. It can be shown[24] that the control $T$ will always appear explicitly in an even derivative of the switching function, i.e., the following equation must be satisfied:

$$\frac{d^{2q}}{d^{2q}}\left(\frac{\partial \mathcal{H}}{\partial T}\right) = 0 \qquad (19)$$

where $q$ is the smallest integer for which $T$ enters explicitly into the left side of the previous equation. The value of $q$ also denotes the order of the singular arc. Using Eq. (2) and Eq. (9), and enforcing $\mathcal{H}_1 = \dot{\mathcal{H}}_1 = 0$, the first derivative of $\mathcal{H}_1$ takes the following form:

$$\dot{\mathcal{H}}_1 = \frac{\lambda_v}{m^2}\left(\frac{\partial D}{\partial v} + \frac{D}{c}\right) - \frac{\lambda_h}{m} = 0 \qquad (20)$$

Note that this expression is independent of $T$, and therefore the value of the singular control, say $T_0$, is evaluated from the second derivative of $\mathcal{H}_1$, which, it can be shown, has the form

$$\ddot{\mathcal{H}}_1 = A + T_0 B = 0 \qquad (21)$$

resulting in the singular control

$$T_0 = -\frac{A}{B} \qquad (22)$$

Here $q = 1$ and the singular arc is of first order. The equations for $A$ and $B$ are given as follows:

$$A = \sum_{i=1}^{4} Q_i \qquad (23)$$

$$B = \sum_{i=1}^{3} R_i \qquad (24)$$

where $Q_i$, $i = 1,2,3,4$ are given by

$$Q_1 = -\frac{D}{mc}\left(\frac{D}{c} + \frac{\partial D}{\partial v}\right)$$

$$Q_2 = \frac{\partial^2 D}{\partial h \partial v}v - \frac{\partial^2 D}{\partial v^2}\left(\frac{D}{m} + h^{-2}\right)$$

$$Q_3 = \frac{1}{c}\left[\frac{\partial D}{\partial h}v - \frac{\partial D}{\partial v}\left(\frac{D}{m} + h^{-2}\right)\right]$$

$$Q_4 = -\frac{\partial D}{\partial h} + 2mh^{-3}$$

and $R_i$, $i = 1,2,3$ are given by

$$R_1 = \frac{1}{mc}\left(\frac{\partial D}{\partial v} + \frac{D}{c}\right)$$

$$R_2 = \frac{1}{m}\frac{\partial^2 D}{\partial v^2}$$

$$R_3 = \frac{1}{mc}\frac{\partial D}{\partial v}$$

For the drag model of Eq. (3) the partial derivatives required for the evaluation of the above equations are given as follows:

$$\frac{\partial D}{\partial v} = 2C_D b\,|v|\,\exp[\beta(1-h)]$$

$$\frac{\partial D}{\partial h} = -C_D b v\,|v|\,\beta\,\exp[\beta(1-h)]$$

$$\frac{\partial^2 D}{\partial v^2} = 2\operatorname{sgn}(v) C_D b\,\exp[\beta(1-h)]$$

$$\frac{\partial^2 D}{\partial h \partial v} = -2C_D b\,|v|\,\beta\,\exp[\beta(1-h)]$$

where sgn denotes the signum function defined by

$$\operatorname{sgn}(v) = \begin{cases} +1 & \text{if} \quad v > 0 \\ 0 & \text{if} \quad v = 0 \\ -1 & \text{if} \quad v < 0 \end{cases}$$

It is interesting to note in passing that the singular control is in state feedback form.

## Singular Surface

The study of problems in which singular solutions appear is significantly simplified if it is possible to determine a surface that represents the following conditions:

$$\begin{aligned} \mathcal{H}_0 &\equiv \dot{\mathcal{H}}_0 = \cdots \equiv 0 \\ \mathcal{H}_1 &\equiv \dot{\mathcal{H}}_1 = \cdots \equiv 0 \end{aligned} \qquad (25)$$

in the state space of the original state variables. If such an expression

$$S(x,t) = 0 \qquad (26)$$

can be obtained, then the singular control in Eq. (22) is a function only of state variables. This is a very significant result, especially for applications, since in that case, the optimal singular control can be expressed in state feedback form, by taking successive time derivatives of Eq. (26). The surface $S(x,t)$ is then called the *singular control surface,* since the state-variable trajectory corresponding to the singular control $T_0$ must lie on this surface. For this reason, it should be noted that only those regions of $S(x,t)$ corresponding to $T_0 \in \mathcal{W}$ are considered. Note that $S(x,t)$ is also the singular control switching boundary,[25,26] since any point of the state space that does not lie on $S(x,t)$ must feature a bang-bang control.

Obviously, Eq. (25) defines a hypersurface $R(x,\lambda,t) = 0$, in the $2n + 1$ dimensional space of states, costates, and, possibly, time. If a singular surface exists, it will lie on the projection of $R(x,\lambda,t)$ in the $n$-dimensional state subspace. Thus, in principle, it is possible to express the costate variables as a function of the states on the singular portion of the optimal path and find $S(x,t)$, by projecting $R(x,\lambda,t)$ in the $n$-dimensional state subspace. Nevertheless, in practice, one may not be able to solve for the costates. There are cases, however, that it may be possible to determine a relation $\lambda = \lambda(x,t)$ without knowing the general expression of $\lambda(x,t)$, the success of the method depending on the number of state equations and the ability to find a number of functional independent first integrals from Eq. (25). The methodology, often referred to as the *synthesis problem,* proves to be rather simple to apply for the problem under study.

Along the singular surface, the expressions given by Eqs. (14), (15), and (20) vanish. Solving Eq. (20) for $\lambda_h$ and replacing it in Eq. (14) one obtains

$$\lambda_v \left[ \frac{v}{m}\left(\frac{\partial D}{\partial v} + \frac{D}{c}\right) - \left(\frac{D}{m} + h^{-2}\right)\right] = C_1 \tag{27}$$

or equivalently

$$E(v,h,m) = v\left(\frac{\partial D}{\partial v} + \frac{D}{c}\right) - D - mh^{-2} - \frac{C_1}{\lambda_v} m = 0 \tag{28}$$

From Eq. (9) and Eq. (20) also, a simple relationship can be found for the propagation of $\lambda_v$ along the singular arc. If one solves for $\lambda_h$ from Eq. (9) and substitutes in Eq. (20), one obtains

$$\dot{\lambda}_v = -\lambda_v \frac{D}{mc} \tag{29}$$

which, integrated along with Eq. (2), will generate the optimal $\lambda_v$ history on the singular surface. The values of the costates $\lambda_h$ and $\lambda_m$ then can be found by direct substitution of $\lambda_v$ into Eq. (9). Integration of Eq. (29) along the singular subarc will give

$$\lambda_v(t) = C_2 \exp\left[ -\int_{t_{s1}}^{t} \frac{D}{mc}\, dt \right], \qquad t_{s1} \le t \le t_{s2} \tag{30}$$

where $t_{s1}$ and $t_{s2}$ are the times of entry to and exit from the singular subarc, and

$$\lambda_v(t_{s1}) = C_2 \tag{31}$$

which is the initial condition for Eq. (30). Replacing Eq. (30) into Eq. (28), we get

$$E(v,h,m) = v\left(\frac{\partial D}{\partial v} + \frac{D}{c}\right) - D - mh^{-2} - \frac{C_1}{C_2} \exp\left[\int_{t_{s1}}^{t} \frac{D}{mc}\, dt\right] = 0, \qquad t_{s1} \le t \le t_{s2} \tag{32}$$

which is the equation of the singular surface. Note that Eq. (28), or alternatively Eq. (32), generates a one parameter family of singular surfaces according to the value of the constant $C_1/C_2$. This kind of parameterization will be very useful later, at the implementation of the numerical solution.

In the special case of constant gravity—an approximation that has been made in previous works—the costate equations can be integrated in closed form[19] and eliminate the mass from Eq. (28), simplifying considerably the analysis of the problem, since the singular surface takes the very simple form of a single curve in the two-dimensional space of $v,h$. However, such a simplification is not possible in general, and the plot of the singular surface has to be computed in the entire $v,h,m$ state space. A depiction of Eq. (28) for $C_1 = 0$ is given in Fig. 1.

Note that for the particular case of free final time, when $C_1 = 0$, Eq. (28) matches the results obtained by Munick,[16] Tsien and Evans,[3] and others. One also notices from Eq. (28) that for the case when $\lambda_v = 0$ we must also have $C_1 = 0$ or else a singular surface does not exist. However, this remark should not pose difficulties, since at such a case from Eq. (29) $\lambda_v$ is zero everywhere. Then, the costate vector $\lambda$ vanishes as well, which contradicts the maximum principle.[17,23] This also indicates why an optimal candidate cannot include a terminal singular subarc, see Eq. (10).

## Optimality of Singular Subarc

The appearance of singular subarcs in Goddard's problem was confirmed by the research of Tsien and Evans,[3] Hamel,[2] Leitmann,[4-7] and others. However, owing to the particular character of singular controls, the classical necessary conditions, i.e., the Weierstrass condition and its counterpart, the Pontryagin maximum principle, or the Legendre-Clebsch condition, fail to give any information about the maximal or minimal nature of the singular candidate. Moreover, the more stringent Jacobi condition applies only to nonsingular candidates. It was not until 1964 that a new necessary condition for screening singular extremals became available by Kelley.[27] Kelley's result was generalized by Robbins[28,29] and Kelley et al.[24] to give what is now known as the generalized Legendre-Clebsch condition or the Kelley-Contensou test.

This condition can be stated as follows:

$$(-1)^q \frac{\partial}{\partial T}\left[\frac{d^{2q}}{dt^{2q}}\left(\frac{\partial \mathcal{H}}{\partial T}\right)\right] \le 0 \tag{33}$$

where $q$ is the order of the singularity of the arc, as in Eq. (19). For the problem under study, the latter relation using Eq. (22) reduces to the inequality

$$B \ge 0 \tag{34}$$

where $B$ is given by Eq. (24).

Even so, treatments of Jacobi-like conditions and the sufficiency question are fragmentary at present.

## Numerical Solution and Results

The aerodynamic data and vehicle's parameters, with the exception of the value of $T_{max}$, were taken from the work of

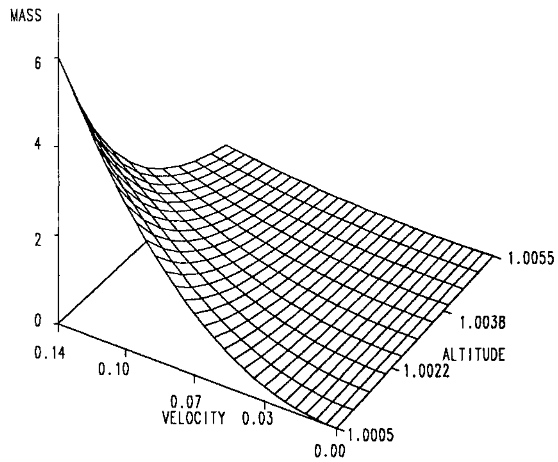


**Fig. 1 Singular surface $E(v,m,h) = 0$.**

Zlatskiy and Kiforenko.[21] Their nondimensionalized values are listed as follows:

$$C_D = 0.05$$
$$b = 6200$$
$$\beta = 500$$
$$T_{\max} = 3.5 \qquad (35)$$

These correspond roughly to the Soviet SA-2 surface-to-air missile, NATO code-named GUIDELINE.[30] The nondimensionalized initial and final values of the state variables are given as follows:

$$h(t_0) = 1$$
$$v(t_0) = 0$$
$$m(t_0) = 1$$
$$m(t_f) = 0.6 \qquad (36)$$

The previous equations for the states, along with the Eq. (10) for the costates, form the boundary conditions for the differential system of Euler equations.

The optimal climbing program must determine the best thrust or velocity history, so the vehicle will move most efficiently along a path and achieve maximal final altitude. The two main deterrents to achieving altitude are the forces of gravity and the aerodynamic drag. Gravity losses are proportional to time of flight, whereas drag losses are proportional to some power of the velocity. Thus, diminishment of gravity losses requires a short flight time, i.e., high velocity, which in its turn tends to increase the drag losses. As a result, arcs of intermediate thrust, i.e., singular arcs, denote the requirement to establish an optimum compromise between gravity and drag losses, at every instant in time. Since both the gravity and the drag forces diminish with increasing altitude, one should expect that such arcs will not appear at flight in vacuum or at very high altitudes. Previous results show that this is indeed the case.[3] A companion paper[31] examines the effect of drag-law variations on the solution of the problem.

Three cases of the switching structure may arise according to the specified value of the final time (see Table 1): 1) full thrust, singular thrust, and zero thrust; 2) full thrust, singular thrust, full thrust, and zero thrust; and 3) full thrust and zero thrust.

The free-terminal-time problem corresponds to the classical first case of switching structure listed earlier, which also yields the absolute maximum final altitude compared to every other value of flight time. This is not a surprising result, since it corresponds to unconstrained final altitude (Table 2).

The switching structure remains unchanged when the final time is increased, with the singular subarc occupying a greater portion of the solution of the whole optimal trajectory, as a result of the requirement to reach the final altitude later. Hence, the rocket consumes its fuel at a slower rate as well.

However, the history of the thrust is considerably different for the case of shortened flight, and the appearance of a second full-thrust subarc, following the variable-thrust subarc, becomes inevitable for sufficiently small values of final time. The reason for this is the fact that for small values of the flight time the singular thrust tends to violate its upper bound. Such a situation is not allowed, from Eq. (6), and therefore the trajectory must depart from singular control, and a bang-bang control must be used. If one allows the control to saturate above, in following the singular subarc, then only a zero-thrust subarc is possible to satisfy the McDanell and Powers necessary condition for joining optimal singular and nonsingular subarcs.[32] In such a case, the mass will not meet its terminal boundary condition (i.e., there is still fuel to be burned). The only appropriate choice is evidently a second full-thrust subarc to make use of the remaining propellant. Departure from the singular surface, however, must occur at a point in time so that the switching function becomes zero at the time when fuel is exhausted. Switching then takes place to a coast that extends to the final time.

The second full-thrust subarc reveals the necessity of burning the fuel at an increasing rate for the case of small values of final time. In fact, the first and second full-thrust subarcs occupy an increasing portion of the whole trajectory as the final time decreases, while the portion allotted for the intermediate variable-thrust subarc decreases (Table 3).

Thus, the third case arises when the two full-thrust subarcs join, and the singular subarc is totally absent from the com-

**Table 1 Variation of switching points with final time**

| Final time | Full thrust subarc | Singular thrust subarc |
|---|---|---|
| 0.058 | 0.99378 | — |
| 0.080 | 0.71428 | — |
| 0.100 | 0.57142 | — |
| 0.120 | 0.47619 | — |
| 0.130 | 0.33827 | 0.36742 |
| 0.140 | 0.25679 | 0.39305 |
| 0.150 | 0.19787 | 0.40441 |
| 0.170 | 0.16153 | 0.38898 |
| 0.198 | 0.11822 | 0.36541 |
| 0.220 | 0.09784 | 0.35164 |
| 0.250 | 0.07760 | 0.33553 |
| 0.280 | 0.06350 | 0.32227 |
| 0.320 | 0.05031 | 0.30788 |
| 0.350 | 0.04313 | 0.29855 |

**Table 2 Variation of final altitude and final velocity with final time**

| Final time | Final altitude | Final velocity |
|---|---|---|
| 0.058 | 1.00405 | 0.14324 |
| 0.080 | 1.00683 | 0.10788 |
| 0.100 | 1.00877 | 0.08612 |
| 0.120 | 1.01028 | 0.06596 |
| 0.130 | 1.01090 | 0.05621 |
| 0.140 | 1.01142 | 0.04789 |
| 0.150 | 1.01173 | 0.03579 |
| 1.170 | 1.01249 | 0.02367 |
| 0.198 | 1.01283 | 0.00000 |
| 0.220 | 1.01265 | −0.01772 |
| 0.250 | 1.01173 | −0.04340 |
| 0.280 | 1.01004 | −0.06932 |
| 0.320 | 1.00659 | −0.10224 |
| 0.350 | 1.00330 | −0.11215 |

**Table 3 Variation of subarc duration with final time**

| $t_f$ | $\Delta t_{FT}/t_f$, full | $\Delta t_{ST}/t_f$, singular | $\Delta t_{FT}/t_f$, full |
|---|---|---|---|
| 0.058 | 0.99378 | — | — |
| 0.080 | 0.71428 | — | — |
| 0.100 | 0.57142 | — | — |
| 0.120 | 0.47619 | — | — |
| 0.130 | 0.33827 | 0.02915 | 0.07372 |
| 0.140 | 0.25679 | 0.13626 | 0.03072 |
| 0.150 | 0.19787 | 0.20654 | — |
| 0.170 | 0.16153 | 0.22745 | — |
| 0.198 | 0.11822 | 0.24719 | — |
| 0.220 | 0.09784 | 0.25380 | — |
| 0.250 | 0.07760 | 0.25793 | — |
| 0.280 | 0.06350 | 0.25877 | — |
| 0.320 | 0.05031 | 0.25757 | — |
| 0.350 | 0.04313 | 0.25542 | — |

posite. A limiting case of this switching structure is the case when no coasting subarc is present, and the trajectory is composed totally of a full-thrust arc. This is the case when $t_f = \bar{t}$.

It should be noted, however, that the possibility of appearance of any of the aforementioned cases in an optimal solution does not depend merely on the final time but on the value of the upper bound of the thrust as well. It is possible, for instance, that a second full-thrust may be required, even for the free-final-time case, if the thrust is bounded above by a small value, whereas, on the other hand, with a very high upper bound on the thrust ($T_{max} \rightarrow \infty$) the optimal trajectory is composed according to the classical sequence: full, singular, and coast. In fact, for the limiting case, when $T_{max} = \infty$, this last combination is the only possible one,[3,4] with an impulsive boost instead of the full-thrust subarc.

Notice that the terminal velocity is positive for $t_f < t_{free}$ and is negative for $t_f > t_{free}$. Obviously, $v = 0$ for $t_f = t_{free}$. This is hardly surprising; from Eq. (10) and Eqs. (13-14) for the coasting subarc ($T = 0$) the following relationship must hold at the terminal time:

$$C_1 = v(t_f) \tag{37}$$

and, using Eq. (12), the proof is complete.

Note that since the boundary conditions, Eqs. (10) and (36), are split, i.e., some of the states and costates are known at the initial time, while others are known at the final time, the problem is a so-called two-point-boundary-value problem (TPBVP). The computational difficulties normally associated with the solution of the stated TPBVP, in connection with the instability of the system of state and costate equations, is aggravated by the inclusion of a singular subarc in the composite optimal path. Numerical solution of TPBVP normally involves the adjustment of the missing initial costates so that along with the specified initial conditions the terminal conditions are met, usually after a trial and error scheme. Thus, it is possible to establish a unique correspondence between the field of extremals and the initial values of the costate functions $\lambda(t_0)$. However, this kind of parameterization does not cover trajectories that contain singular sections. For composite trajectories consisting of singular and nonsingular subarcs, one has to determine the initial costate vector from $\lambda(t_0) \in \Lambda_0$ where the unknown manifold $\Lambda_0$ consists of those values of $\lambda(t_0)$ that will satisfy Eq. (25) simultaneously on the singular subarc. Different points of the manifold $\Lambda_0$ correspond to different initial times for the singular control subarc. In such cases, it is more appropriate to use the moments of time for entering to a singular arc $t_{s1}$ and, departing from it, $t_{s2}$, as parameters of the extremal. This enables us to replace the search of the unknowns $\lambda_h(t_0)$, $\lambda_v(t_0)$, $\lambda_m(t_0)$, associated with the relation $\mathcal{H} = C_1$, by the determination of the optimum values of $t_{s1}$ and $t_{s2}$, to eliminate $\lambda(t_{s1})$ in favor of $x(t_{s1})$, using Eq. (25). To seek the unknowns $h(t_{s1})$, $v(t_{s1})$, and $m(t_{s1})$ associated with Eq. (25)

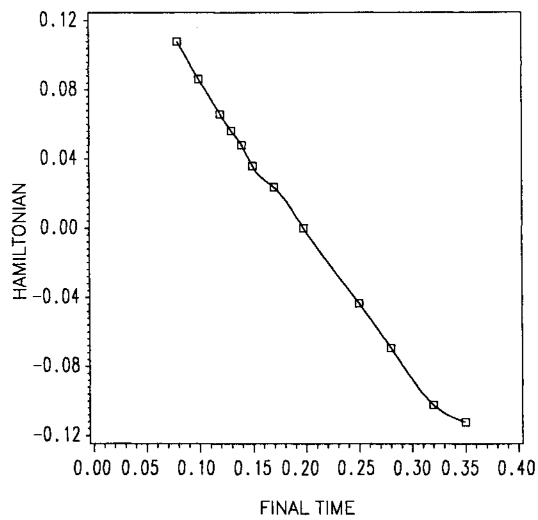


**Fig. 2 Variation of Hamiltonian with final time.**

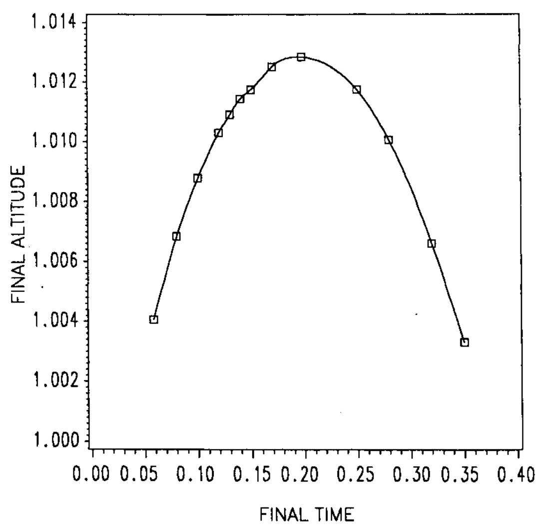


**Fig. 3 Variation of final altitude with final time.**

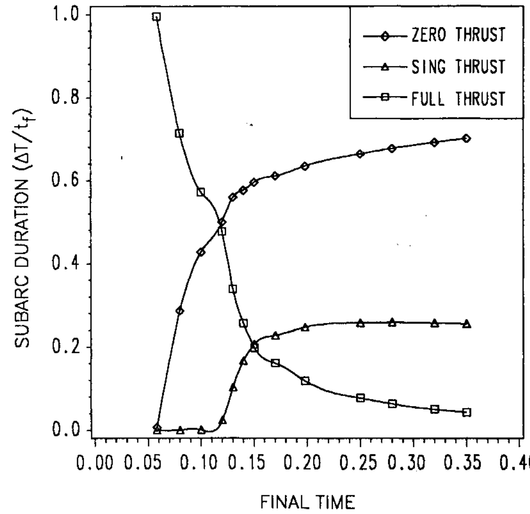


**Fig. 4 Time duration of subarcs vs final time.**

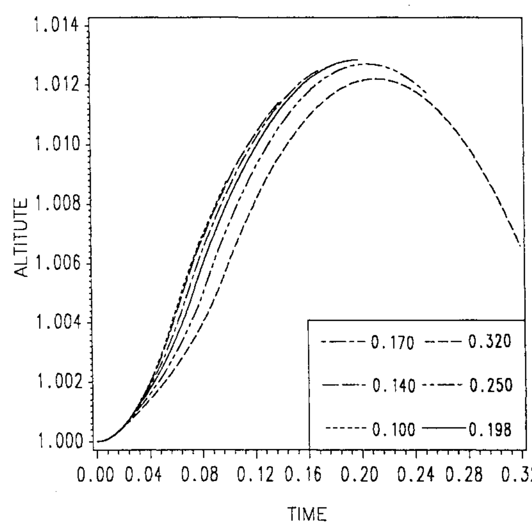


**Fig. 5 Variation of altitude history with final time.**

or—equivalently—with relation, Eq. (26), appears more encouraging than to determine the point $\lambda(t_0)$ on the unknown manifold $\Lambda_0$. This method is especially effective when a priori information is available on the optimum control at the start of the trajectory. The methodology is somewhat analogous to the one used in Zlatskiy and Kiforenko[33]:

Given an initial guess for the ratio

$$\lambda_v' = \frac{\lambda_v}{C_1} \tag{38}$$

one may integrate forward the state Eq. (2) with the assigned initial boundary values of the states until Eq. (27) is satisfied. Solving the system of Eqs. (14) and (15), one finds the values of the other two costates at the entry of the singular arc given by

$$\lambda_h' = \frac{\lambda_v'}{m}\left(\frac{\partial D}{\partial v} + \frac{D}{c}\right) \tag{39}$$

$$\lambda_m' = \lambda_v' \frac{m}{c} \tag{40}$$

These relations assume that the constant $C_1$ has been used as a scaling factor of the costates, due to the homogeneity of Eqs. (9) and (13–15). The values of the costates at the initial time can then be found by backward integration of the state-Euler system, Eq. (2), and Eq. (9) using these values of $\lambda_v'$, $\lambda_h'$, and $\lambda_m'$.

Once on the singular arc, one can integrate forward using the singular thrust from Eq. (22). Integration is continued along the singular arc, checking that the inequality constraint $0 \leq T_0 \leq T_{\max}$ is not violated.

Exit from the singular arc is made at the time when the mass criterion is satisfied. In the case when the singular thrust saturates on its upper bound, the time for the departure is taken so that, at the end of the second full-thrust, both conditions

$$\mathcal{H}_1 = 0 \qquad \text{and} \qquad m_f = 0.6 \tag{41}$$

are satisfied.

A final coasting arc follows until $\lambda_v = 0$. Once the final conditions on the state variables have been satisfied, the final value of the costate variable $\lambda_h$ is used to scale the values of all the components of the costate manifold throughout the trajectory. This ensures the satisfaction of the transversality condition $\lambda_h(t_f) = 1$. Notice that the whole procedure involves two successive scalings of the costates, i.e., first using $C_1$, and the final scaling using $\lambda(t_f)$.

A direct relation between the constant $C_1$ and the final time $t_f$ is evident. Every value of $C_1$ corresponds to a unique value of the final time. The graph of $C_1$ vs $t_f$ appears in Fig. 2.

The numerical solution was obtained using the two-point-boundary-value problem solver, BNDSCO.[34] However, convergence of the solution algorithm is experienced only if the initial guesses for the trajectory are reasonably good. The previously described method gave very accurate first guesses for BNDSCO. With these initial guesses, BNDSCO converged within one or two iterations. With a converged solution in hand, one can generate the whole family of trajectories by parametrically varying the final time in small steps. The results are shown in Figs. 3–7.

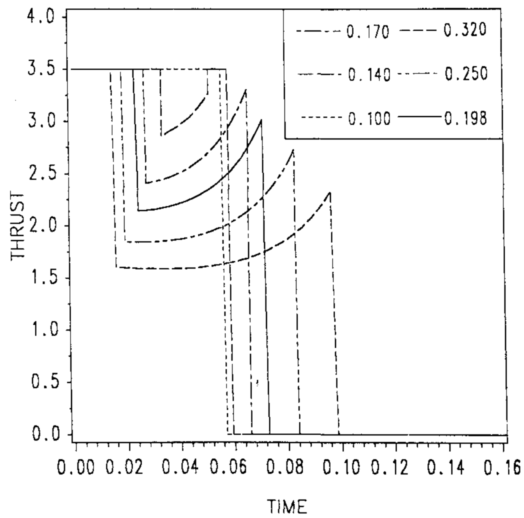


**Fig. 6 Variation of thrust history with final time.**

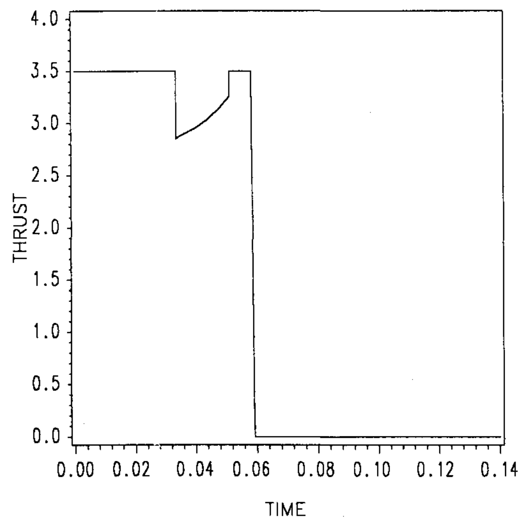


**Fig. 7 Thrust history for full-singular-full-coast sequence.**

## Conclusions

The problem of maximizing the final altitude of a vertically ascending rocket has been analyzed for the case of bounded thrust and quadratic drag law. The case of a constant drag coefficient has been examined, and solutions were obtained for several values of the duration of the flight. The analysis of the problem showed that a one-parameter family of singular extremals is generated, according to the prescribed value of the final time. Moreover, it has been shown that the final value of the time affects the switching structure of the problem, with the most interesting case the appearance of a second full-thrust subarc after the singular subarc, as a result of the boundness of the available thrust. For the vertical flight problem, this is a new result as far as the authors know.

The Kelley necessary condition for singular arcs and the McDanell and Powers condition for joining singular and nonsingular subards were checked and were found to be satisfied.

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