Multiple-Pursuer/One-Evader Pursuit–Evasion Game in Dynamic Flowfields

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In this paper, a reachability-based approach is adopted to deal with the pursuit–evasion differential game between one evader and multiple pursuers in the presence of dynamic environmental disturbances (for example, winds or sea currents). Conditions for the game to be terminated are given in terms of reachable set inclusions. Level set equations are defined and solved to generate the forward reachable sets of the pursuers and the evader. The time-optimal trajectories and the corresponding optimal strategies are subsequently retrieved from these level sets. The pursuers are divided into active pursuers, guards, and redundant pursuers according to their respective roles in the pursuit–evasion game. The proposed scheme is implemented on problems with both simple and realistic time-dependent flowfields, with and without obstacles.

I. Introduction

PURSUIT–EVASION differential games have been studied extensively in the literature. The state of the art focuses on formulating and solving the Hamilton–Jacobi–Isaacs (HJI) partial differential equation, which was introduced by Isaacs in his seminal book *Differential Games: A Mathematical Theory with Applications to Warfare and Pursuit, Control and Optimization* [1]. In the same book, several pursuit–evasion games were introduced and analyzed. The homicidal chauffeur game [1] deals with a pursuit–evasion game between a pursuer having a finite minimum turning radius and an agile evader. A converse version of the homicidal chauffeur game, also known as the suicidal pedestrian game, was studied in [2,3]. The game between two players with curvature constraints is known as the game of two cars and was studied in [4]. A general result for this problem was presented in [5]. Other pursuit–evasion games under specific conditions included the isotropic rocket problem [1] and the lion and man problem [6]. An extension of the game of pursuit with curvature constraints to the three-dimensional space has been addressed in [7]. Stochastic differential games of two players have also been explored, including a stochastic version of the homicidal chauffeur game, which was addressed in [8].

In addition to two-player pursuit–evasion games, multiple-player pursuit–evasion games have also gained attention, mainly owing to the increased current interest in multiagent problems. For example, conditions for target interception with multiple agile pursuers under single integrator dynamics was studied by Pshenichnyi [9]. Conditions for simultaneous capture of k out of n pursuers were derived in [10]. Evasion from many pursuers with integral constraints was discussed in [11]. Finally, in [12], the authors studied the homicidal chauffeur game with multiple cooperative pursuers.

Besides the HJI equation approach, another approach researchers have used when dealing with pursuit–evasion problems is based on reachable set analysis [13–15]. According to this approach, the reachable state space of the pursuers and the evaders is used to find the optimal controls of the pursuer and/or the evader. A reachability set analysis has been applied in performing missile/sensor tradeoffs in homing guidance [16], in obtaining escape strategy under pursuit [17], and in finding pursuer control under control constraints [18].

Despite the plethora of work in this area, few approaches have taken into consideration how dynamic environmental conditions may affect the outcome of the game. For instance, when either the pursuers or the evaders (or both) are small autonomous underwater vehicles or small unmanned aerial vehicles, the presence of time-varying or spatially varying sea currents or winds, respectively, may significantly affect the vehicle's motion. As a result, during pursuit–evasion, the optimal behavior of the players, as is determined by the solution of a differential game, may be greatly affected by the existence of an external dynamically changing flowfield.

Some optimal control problem formulations have taken into account the effect of an external flowfield. For example, in [19], the authors addressed the problem of optimal guidance to a specified position of a Dubins vehicle [20] under the influence of an external flow. The minimum-time guidance problem for an isotropic rocket in the presence of wind has been studied in [21]. The problem of minimizing the expected time to steer a Dubins vehicle to a target set in a stochastic wind field has also been discussed in [22]. However, the same level of attention in the literature has not been devoted to pursuit–evasion games with two (or more) competing agents under the influence of external disturbances (e.g., winds or currents).

In this paper, we consider a multipursuer/one-evader pursuit–evasion game in an external dynamic flowfield that is assumed to be known. Due to the generality of the external flow, Isaacs's approach is not readily applicable [1]. Instead, we follow a different approach and we find the optimal trajectories of the players through a reachable set method. Specifically, we use the level set method [23,24] to generate the reachable sets of both the evader and the pursuers and to retrieve the corresponding optimal control actions at the current location of the agents by backward propagation of their respective reachable sets.

Level set methods have been previously applied by Mitchell et al. [25] and Jang and Tomlin [26] to solve pursuit–evasion games [25,26]. Reference [26] aimed at solving a nonzero sum pursuit–evasion game where the evader and each individual pursuer were assigned their own value functions. The authors of [25] first

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decreased the degrees of freedom of the problem by reformulating it in terms of the relative distance between the pursuer and the evader. The level set method was then applied to solve the corresponding Hamilton–Jacobi–Isaacs equation that governs the backward reachable set from the target set in order to solve the differential game. Our approach differs from that in [25] because we do not attempt to solve the pursuit–evasion game directly by solving the corresponding HJI equation. Instead, we generate the forward reachable sets of the players, and we find the optimal time to capture as the first instance when the reachable set of the evader is fully covered by the reachable set of the pursuer [14]. We then identify the first rendezvous point of the players and retrieve the optimal trajectories and controls of both players through backtracking of their respective trajectories [27,28]. The reason we follow this approach instead of the more direct approach in [25] is due to the dimensionality of the problem. When introducing complex dynamic environmental effects into the system, the pursuit–evasion problem cannot be reduced to a problem described solely in terms of the relative distance between the pursuer and the evader, unless some very restrictive assumptions are imposed on the structure of the external flowfield [29]. In other words, in order to deal with a pursuit–evasion problem between one pursuer and one evader taking place in the presence of a general flowfield, the level set method needs to be implemented on a fourth-dimensional state space. The computational cost of level set methods is very high when the dimension exceeds three or four [30]. On the other hand, the forward reachable set approach used in this paper is quite efficient because the propagated level sets all remain two-dimensional. The approach works even for realistic flows with dynamic currents for which the speed can be much larger than the vehicle speeds [31], and it can treat dynamic obstacles [32]. Finally, because the generation of the reachable sets of each player can be done independently of the other players, the solution can be implemented in a decentralized manner using parallel computing.

II. Problem Formulation

We consider a pursuit–evasion game in the presence of an external flowfield with n pursuers and a single evader. Henceforth, we will refer to the pursuers and the evader collectively as "agents." The dynamics of the pursuers $P_i (i = 1, \ldots, n)$ are given by

$$\dot{X}_P^i = u_P^i + w(X_P^i, t), \qquad X_P^i(t_0) = X_{P_0}^i \tag{1}$$

where $X_P^i := [x_P^i, y_P^i]^T \in \mathcal{D} \subset \mathbb{R}^2$ denotes the position of the ith pursuer. Here, $\mathcal{D}$ denotes a compact subset of $\mathbb{R}^2$ and u_P^i is the control input (i.e., velocity) of the ith pursuer such that $u_P^i \in \mathcal{U}_P^i$ for all $i \in \mathcal{I}$, and $\mathcal{I} = \{1, 2, \ldots, n\}$ stands for the index set of the pursuers. The set $\mathcal{U}_P^i$ consists of all piecewise continuous functions for which the range is included in the set $U_P^i = \{u^i \in \mathbb{R}^2, |u^i| \leq \bar{u}^i\}$, where $|\cdot|$ represents the Euclidean norm and $\bar{u}^i$, $i \in \mathcal{I}$, are constants. If $u_P^i \in \mathcal{U}_P^i$, we say that u_P^i is an admissible control for the ith pursuer. In Eq. (1), $w(X, t) \in \mathbb{R}^2$ represents an exogenous dynamic flow, but it could also represent an endogenous drift owing to the nonlinear dynamics of the agent. It is reasonable to assume that the magnitude of this flow (e.g., winds or currents) is bounded from above by some constant; that is, there exists a constant $\bar{w}$ such that $|w(X, t)| \leq \bar{w}$ for all $(X, t) \in \mathcal{D} \times [t_0, \infty)$.

The objective of the pursuers is to intercept the evader, for which the kinematics is given by

$$\dot{X}_E = u_E + w(X_E, t), \qquad X_E(t_0) = X_{E_0} \tag{2}$$

where $X_E = [x_E, y_E]^T \in \mathcal{D} \subset \mathbb{R}^2$ is the position of the evader; and u_E is its control input (i.e., velocity) such that $u_E \in \mathcal{U}_E$, where $\mathcal{U}_E$ consists of all piecewise continuous functions for which the range is included in the set $U_E = \{v \in \mathbb{R}^2, |v| \leq \bar{v}\}$. When $u_E \in \mathcal{U}_E$, we say that u_E is an admissible control of the evader.

The game begins at time $t = t_0$ with initial positions X_{E_0} and $X_{P_0}^i$, $(i \in \mathcal{I})$, for the evader and the pursuers, respectively, and terminates when the evader coincides with at least one of the pursuers; in which case, capture occurs. That is, capture implies that there exists $i \in \mathcal{I}$ and a terminal time $T \geq t_0$ such that $X_P^i(T) = X_E(T)$. Equivalently, the game terminates if, for any admissible control of the evader $u_E \in \mathcal{U}_E$, there exists a set of admissible controls $(u_P^1, \ldots, u_P^n) \in \mathcal{U}_P^1 \times \ldots \times \mathcal{U}_P^n$ of the pursuers such that $X_P^i(T) = X_E(T)$ for some $i \in \mathcal{I}$ and some time $T \geq t_0$. The pursuers aim to minimize the time to capture if possible, whereas the evader prefers to avoid capture for as long as possible.

Let

$$\overline{X} = [X_E^T, X_P^{1T}, X_P^{2T}, \ldots, X_P^{nT}]^T \in \mathbb{R}^{2(n+1)}$$

denote the state of the game. The game begins at initial time $t_0 = 0$ with initial positions

$$\bar{X}_0 = [X_{E_0}^T, X_{P_0}^{1T}, X_{P_0}^{2T}, \ldots, X_{P_0}^{nT}]^T$$

and terminates when at least one of the pursuers reaches the location of the evader. The terminal time T of the game is defined by

$$T = \inf\{t \in \mathbb{R}_+ : X_P^i(t) = X_E(t), i \in \mathcal{I}\} \tag{3}$$

Let $J(\gamma_P^1, \gamma_P^2, \ldots, \gamma_P^n, \gamma_E) = T$ be the cost function of the game, where $\gamma_P^i, \gamma_E : \mathbb{R}_+ \times \mathbb{R}^4 \mapsto \mathbb{R}^2$ denotes the feedback strategies of the pursuers and the evader, respectively: namely, $\gamma_P^i(t, \overline{X}(t))$ and $\gamma_E(t, \overline{X}(t))$, where $\bar{X}(t)$ is the solution of the system of equations

$$\dot{X}_P^i = \gamma_P^i(t, \overline{X}(t)) + w(X_P^i, t), \qquad i \in \mathcal{I} \tag{4}$$

$$\dot{X}_E = \gamma_E(t, \overline{X}(t)) + w(X_E, t) \tag{5}$$

subject to $\overline{X}(0) = \overline{X}_0$. It is assumed that each player has perfect knowledge of the dynamics of the system represented by Eqs. (1) and (2), the constraint sets U_P^i and U_E, the cost function J, as well as the initial state $\overline{X}_0$. It is also assumed in this paper that the value V of the game [1] exists; that is,

$$V = \min_{\gamma_P^1, \ldots, \gamma_P^n} \max_{\gamma_E} J = \max_{\gamma_E} \min_{\gamma_P^1, \ldots, \gamma_P^n} J \tag{6}$$

Note that pursuit–evasion games of the form addressed in this work are a specific class of differential games for which the Isaacs condition [1] holds owing to the separability of the dynamics and the cost, and hence the value of the game exists [33]. The objective of this paper is to find the open-loop representation of the optimal strategies of the pursuer and the evader. In particular, we use a reachability-based method to obtain optimal controls $u_P^\star(t) = \gamma_P^\star(t, \overline{X}^\star(t))$ and $u_E^\star(t) = \gamma_E^\star(t, \overline{X}^\star(t))$, with $\overline{X}^\star$ denoting the corresponding optimal state trajectory of the system [Eqs. (4) and (5)] with strategies $\gamma_P^{i\star}(t, \overline{X}^\star(t))$ and $\gamma_E^\star(t, \overline{X}^\star(t))$.

III. Problem Analysis

To solve the differential game introduced in the previous section (that is, in order to find the conditions for capture and to derive the corresponding optimal controls and trajectories of both players), we make use of reachable set analysis. Reachable sets provide a quick snapshot of all possible future trajectories of the agent, and thus succinctly encode all possible future positions of the agent under any possible control action. Knowledge of the reachable sets of the pursuer and the evader can then be used to draw conclusions about the potential meeting of the two at some future time (or not). In this paper, we use this intuition behind the information conveyed by the reachable set of each player to solve the pursuit–evasion problem under minimal assumptions about the maximum number of players and the environment they operate in. Because the computation of the reachable sets for each player can be done independently from the other players, the proposed method is decentralized and scales well with the number of players, which is something that is not the case with more traditional approaches that require directly the solution of a HJI partial differential equation (see also the discussion at the end of Sec. I).

We start this section with some basic definitions and facts about reachable sets that will be useful throughout the paper.

Definition III.1 [34]: The reachable set $\mathcal{R}(X_0 t)$ at time $t \geq t_0$ of a system of the form of Eq. (1) or Eq. (2) starting at initial condition $X(t_0) = X_0$ is the set of all the points that can be reached by the agent at time t.

In particular, the reachable set of the ith pursuer at time $\tau \geq t_0$, denoted by $\mathcal{R}_P^i(X_{P_0}^i, \tau)$, is the set of all points $X \in \mathbb{R}^2$, such that there exists a trajectory satisfying Eq. (1) for all $t \in [t_0, \tau]$ with $X_P(t_0) = X_{P_0}^i$ and $X_P(\tau) = X$. Similarly, the reachable set $\mathcal{R}_E(X_{E_0}, \tau)$ of the evader at time $\tau \geq t_0$ is the set of all points $X \in \mathbb{R}^2$ such that there exists a trajectory satisfying Eq. (2) for all $t \in [t_0, \tau]$ with initial condition $X_E(t_0) = X_{E_0}$ and terminal condition $X_E(\tau) = X$.

Definition III.2 [28]: The boundary of the reachable set is the reachability front.

The reachability fronts of the ith pursuer and the evader at time $t \geq t_0$ will be denoted by $\partial\mathcal{R}_P^i(X_{P_0}^i, t)$ and $\partial\mathcal{R}_E(X_{E_0}, t)$, respectively.

Definition III.3: Given the reachable sets of the pursuers, we define the usable reachable set of the evader at time $t \geq t_0$ as

$$\mathcal{R}_E^\star(X_{E_0}, t) = \{X \in \mathcal{D}\colon X = X_E(t) \text{ and } X_E(\tau) \notin \bigcup_{i=1}^n \mathcal{R}_P^i(X_{P_0}^i, \tau), \quad \forall\, \tau \in [t_0, t]\} \tag{7}$$

From this definition, it is clear that $\mathcal{R}_E^\star(X_{E_0}, t) \subseteq \mathcal{R}_E(X_{E_0}, t)$. The definition implies that $\mathcal{R}_E^\star(X_{E_0}, t)$ is the set of all terminal points of the evader at time t, for which the trajectories do not pass through any reachable sets of the pursuers at any time in the interval $[t_0, t]$. In other words, $\mathcal{R}_E^\star(X_{E_0}, t)$ is the set of terminal points of all “safe” evader trajectories.

Suppose now that, at some time $t_c > t_0$ and for some $i \in \mathcal{I}$, we have that $\mathcal{R}_E(X_{E_0}, t_c) \subseteq \mathcal{R}_P^i(X_{P_0}^i, t_c)$. It follows that, for each $u_E \in \mathcal{U}_E$, there exists $u_P^i \in \mathcal{U}_P^i$ such that $X_P^i(t_c) = X_E(t_c)$. In other words, capture of the evader is guaranteed at time t_c by the ith pursuer. Note that $\mathcal{R}_E^\star(X_{E_0}, t_c) = \varnothing$ in this case.

If, on the other hand, for some $t_e > t_0$, we have that $\mathcal{R}_E^\star(X_{E_0}, t_e) \neq \varnothing$, then it follows that there exists $u_E \in \mathcal{U}_E$ such that capture can be avoided in the time interval $[t_0, t_e]$, no matter how the pursuers choose their (admissible) controls. In other words, if $\mathcal{R}_E^\star(X_{E_0}, t_e) \neq \varnothing$, the game will not terminate in the time interval $[t_0, t_e]$.

The previous observations lead to the following theorem, which is the main theoretical result of this paper. It is used later in order to develop an efficient numerical algorithm for solving the pursuit–evasion game with multiple pursuers in the presence of a known dynamic flowfield.

Theorem III.4: Let

$$T = \inf\Big\{t \in [t_0, +\infty)\colon \mathcal{R}_E^\star(X_{E_0}, t) = \varnothing\Big\}$$

If $T < \infty$, then capture is guaranteed for any time greater than T, whereas the evader can always escape within a time smaller than T. Hence, T is the time to capture if both players play optimally. Furthermore, let X_f denote the location where the evader is captured by at least one of the pursuers. Then, we have that

$$X_f \in \mathcal{X} = \Big\{X \in \mathcal{D}\colon X = X_E(T) \text{ and } X_E(\tau) \notin \bigcup_{i=1}^n \mathcal{R}_P^i(X_{P_0}^i, \tau), \quad \forall\, \tau \in [t_0, T)\Big\} \tag{8}$$

Proof: Because U_P^i is compact and convex for all $i \in \mathcal{I}$, it follows that, for each $(t, X) \in [t_0, \infty) \times \mathcal{D}$, the sets $\{u + w(X, t)\colon u \in U_P^i\}$ are compact and convex for all $i \in \mathcal{I}$. Also, because by assumption $u_P^i(t)$ and $w(X, t)$ are bounded for all $X \in \mathcal{D}$ and $t < \infty$, the solution of Eq. (1) exists on $[t_0, t_f]$ for all $t_f < \infty$. Therefore, by Filippov's theorem [35], the reachable sets $\mathcal{R}_P^i(X_{P_0}^i, t)$ are compact for all $t \in [t_0, t_f]$ and $i \in \mathcal{I}$. Similarly, $\mathcal{R}_E(X_{E_0}, t)$ is compact for all $t \in [t_0, t_f]$.

Because $\mathcal{R}_E^\star(X_{E_0}, T) = \varnothing$, it follows from Definition III.3 that, for any trajectory $X_E(\cdot)$ of the evader that satisfies Eq. (2) subject to an admissible evading control $u_E \in \mathcal{U}_E$, there exists $\tau \in [t_0, T]$ such that

$$X_E(\tau) \in \bigcup_{i=1}^n \mathcal{R}_P^i(X_{P_0}^i, \tau)$$

Therefore, $X_E(\tau) \in \mathcal{R}_P^k(X_{P_0}^k, \tau)$ for some $k \in \mathcal{I}$. In other words, there exists at least one admissible control $u_P^k \in \mathcal{U}_P^k$ for the kth pursuer, such that $X_P^k(\tau) = X_E(\tau)$. Therefore, regardless of the strategy of the evader, it can be captured by the kth pursuer at some time $\tau \in [t_0, T]$. This implies that capture is guaranteed for any time greater than or equal to T.

On the other hand, because T is the first time such that $\mathcal{R}_E^\star(X_{E_0}, T) = \varnothing$, it follows that $\mathcal{R}_E^\star(X_{E_0}, t) \neq \varnothing$ for all $t_0 \leq t < T$. Hence, for any $t \in [t_0, T)$, there exists $X_t \in \mathcal{R}_E^\star(X_{E_0}, t)$. That is,

$$X_t = X_E(t) \text{ and } X_E(\tau) \notin \bigcup_{i=1}^n \mathcal{R}_P^i(X_{P_0}^i, \tau)$$

for all $\tau \in [t_0, t]$, for some trajectory $X_E(\cdot)$ of the evader, defined over the interval $[t_0, t]$. This means that $X_E(\cdot)$ does not pass through the reachable set of any pursuer. Hence, for any $t \in [t_0, T)$, there exist $u_E \in \mathcal{U}_E$ such that $X_E(t) = X_t$; for all $\tau \in [t_0, t]$ and $i \in \mathcal{I}$, there exist no $u_P^i \in \mathcal{U}_P$. It follows that the evader can always avoid capture before time T. From the two previous statements, we conclude that T is the optimal time to capture.

To complete the proof, we just need to show that $X_f \in \mathcal{X}$. For any point $X \in \mathcal{X}$, it is clear that $X \in \mathcal{R}_E(X_{E_0}, T)$, and no pursuer can capture the evader at X before time T. This implies that X should be the destination of the evader if the latter aims to maximize its time to capture. Furthermore, at least one of the pursuers needs to reach X in order to capture the evader. Hence, the point $X = X_f$, where X_f is defined as the location where the evader is captured by at least one of the pursuers. This completes the proof. □

The previous theorem gives us a criterion for capture of the evader; that is, capture is guaranteed when $\mathcal{R}_E^\star(X_{E_0}, t) = \varnothing$ for some $t \in [t_0, \infty)$. Also notice from Eq. (7) that, in general, $\mathcal{R}_E^\star(X_{E_0}, t)$ can be generated by keeping track of the reachable sets of the pursuers and the evader at all time before the capture time. However, when we add some constraints with respect to the speeds of the players, we can replace this criterion with an instantaneous condition that is easier to check and implement. Before we state and prove this result, the following lemma is needed.

Lemma III.5: Let $\bar{u} = \bar{u}_i$ for all $i = 1, \ldots, n$ denote the maximum speed of a pursuer; let $\bar{v}$ denote the maximum speed of the evader, respectively; and assume that $\bar{v} < \bar{u}$. If there exists some time $t_s \geq t_0$ such that $X_E(t_s) \in \mathcal{R}_P(X_{P_0}, t_s)$, then $X_E(t) \in \mathcal{R}_P(X_{P_0}, t)$ for all $t \geq t_s$.

Proof: Because $X_E(t_s) \in \mathcal{R}_P(X_{P_0}, t_s)$ for some time $t_s \geq t_0$, it follows that there exists $u_P \in \mathcal{U}_P$ such that $X_P(t_s) = X_E(t_s)$. By assumption, we have $\bar{v} < \bar{u}$. Therefore, for any $u_E \in \mathcal{U}_E$ that starts from $X_E(t_s)$ at time t_s, the pursuer, by choosing $u_P = u_E$, which is admissible because $\bar{v} < \bar{u}$, can ensure that $X_P(t) = X_E(t)$ for all $t \geq t_s$. Hence, $X_E(t) \in \mathcal{R}_P(X_{P_0}, t)$ for all $t \geq t_s$. □

This lemma essentially states that, when the maximum speed of the evader is smaller than the maximum speed of each pursuer, then once the evader enters the reachable set of a pursuer, it can never leave this reachable set. We are now ready to present the simplified condition on the capture of the evader as follows.

Proposition III.6: When

$$\bar{v} \leq \min_{i \in \mathcal{I}} \bar{u}_i$$

the set $\mathcal{R}_E^\star(X_{E_0}, t)$ satisfies

$$\mathcal{R}_E^\star(X_{E_0}, t) = \mathcal{R}_E(X_{E_0}, t) \setminus \bigcup_{i=1}^{n} \mathcal{R}_P^i(X_{P_0}^i, t) \tag{9}$$

for all $t \geq t_0$. In such cases, the condition $\mathcal{R}_E^\star(X_{E_0}, t) = \varnothing$ is equivalent to the condition

$$\mathcal{R}_E(X_{E_0}, t) \subseteq \bigcup_{i=1}^{n} \mathcal{R}_P^i(X_{P_0}^i, t) \tag{10}$$

Proof: By Definition III.3, for any $X \in \mathcal{R}_E^\star(X_{E_0}, t)$, we have that $X \in \mathcal{R}_E(X_{E_0}, t)$ and

$$X \notin \bigcup_{i=1}^{n} \mathcal{R}_P^i(X_{P_0}^i, t)$$

Therefore,

$$\mathcal{R}_E^\star(X_{E_0}, t) \subseteq \mathcal{R}_E(X_{E_0}, t) \setminus \bigcup_{i=1}^{n} \mathcal{R}_P^i(X_{P_0}^i, t) \tag{11}$$

Now, let

$$X \in \mathcal{R}_E(X_{E_0}, t) \setminus \bigcup_{i=1}^{n} \mathcal{R}_P^i(X_{P_0}^i, t) \tag{12}$$

It follows that there exists a trajectory $X_E(\cdot)$ of the evader, defined over the interval $[t_0, t]$ such that $X = X_E(t)$. Furthermore,

$$X_E(t) \notin \bigcup_{i=1}^{n} \mathcal{R}_P^i(X_{P_0}^i, t)$$

We claim that

$$X_E(\tau) \in \mathcal{R}_E(X_{E_0}, \tau) \setminus \bigcup_{i=1}^{n} \mathcal{R}_P^i(X_{P_0}^i, \tau)$$

for all $\tau \in [t_0, t]$. Because, trivially, $X_E(\tau) \in \mathcal{R}_E(X_{E_0}, \tau)$, it only suffices to show that

$$X_E(\tau) \notin \bigcup_{i=1}^{n} \mathcal{R}_P^i(X_{P_0}^i, \tau)$$

for all $\tau \in [t_0, t]$. Suppose, on the contrary, that there exist $\tau \in [t_0, t]$ and $i \in \mathcal{I}$ such that $X_E(\tau) \in \mathcal{R}_P^i(X_{P_0}^i, \tau)$. Because

$$\bar{v} \leq \min_{i \in \mathcal{I}} \bar{u}_i$$

it follows that, once the evader enters the reachable set of a pursuer, it can never leave the reachable set of this pursuer. Hence, $X_E(\sigma) \in \mathcal{R}_P^i(X_{P_0}^i, \sigma)$ for all $\sigma \geq \tau$, contradicting Eq. (12). It follows that $X \in \mathcal{R}_E^\star(X_{E_0}, t)$, and thus

$$\mathcal{R}_E(X_{E_0}, t) \setminus \bigcup_{i=1}^{n} \mathcal{R}_P^i(X_{P_0}^i, t) \subseteq \mathcal{R}_E^\star(X_{E_0}, t) \tag{13}$$

From Eqs. (11) and (13), it follows that

$$\mathcal{R}_E^\star(X_{E_0}, t) = \mathcal{R}_E(X_{E_0}, t) \setminus \bigcup_{i=1}^{n} \mathcal{R}_P^i(X_{P_0}^i, t)$$

The equivalence of condition $\mathcal{R}_E^\star(X_{E_0}, t) = \varnothing$ with Eq. (10) follows immediately. □

In the case where

$$\bar{v} \leq \min_{i \in \mathcal{I}} \bar{u}_i$$

the optimal time to capture is the first time instant when the union of the reachable sets of the pursuers

$$\bigcup_{i=1}^{n} \mathcal{R}_P^i(X_{P_0}^i, \tau)$$

completely covers the reachable set of the evader $\mathcal{R}_E(X_{E_0}, t)$. If $\bar{v} > \bar{u}_i$, for some $i \in \mathcal{I}$ (the relative maximum speed of the evader is larger than that of the ith pursuer), relation (9) may not always hold. Some admissible evader trajectories may temporarily enter the reachable set of the ith pursuer and exit later on. This is not allowable. To eliminate this possibility in such cases, $\mathcal{R}_E^\star(X_{E_0}, t)$ can be determined by treating the reachable set of the ith pursuer as a dynamic "forbidden" region for the evader [27,32]. That is, whenever the reachable set of the evader intersects the reachable set of any of the pursuers, we can either stop the evolution of the intersected part of the evader's reachable set or let it evolve at the same speed as the reachable set of the pursuer. This way, we can ensure that the terminal points of the admissible trajectories of the evader that temporarily enter the reachable set of the pursuer and exit later on are not included in the usable reachable set of the evader.

IV. Numerical Construction

A. Level Set Method

To construct the reachable sets of the pursuers and the evader, we apply the level set method [23,24]. The level set method is a convenient mathematical tool to track the evolution of the reachability front. It evolves the reachability front by embedding it as a hypersurface in a higher dimension, where time is the additional dimension. Automatic handling of merging and splitting of the fronts and other topological changes are made possible by this higher-dimensional embedding. The level set formulation provides an implicit representation of the front, which offers several advantages over an explicit representation [23,24]. For example, implicit function representations are widely used for describing closed and multivalued curves, for point classification (such as determining whether a point is inside or outside an interface), and for finding intersection points and offsets.

The choice of implicit function is not unique in order to represent a curve. The signed distance function is one of the most common choices and will be used in this paper. Its definition is given as follows:

Definition IV.1: The signed distance function $\varphi(X)$ with respect to a set $\mathcal{R}$ is defined as

$$\varphi(X) = \begin{cases} \min\limits_{Y \in \partial\mathcal{R}} |X - Y|, & \text{if } X \notin \mathcal{R}, \\ -\min\limits_{Y \in \partial\mathcal{R}} |X - Y|, & \text{if } X \in \mathcal{R} \end{cases} \tag{14}$$

Recall that, for any $c \in \mathbb{R}$, the c-level set of a φ is the set $\{X{:}\varphi(X) = c\}$. We hereby use the signed distance function from the reachable set to track the evolution of the fronts of the reachable sets of all agents. This is achieved by expressing the reachable front at time t as the zero-level set of the corresponding signed distance function. Assuming that the signed distance function with respect to the ith pursuer reachable set $\mathcal{R}_P^i(X_{P_0}^i, t)$ at time t is $\phi_P^i(X, t)$, then the evolution of the reachable front $\partial\mathcal{R}_P^i(X_{P_0}^i, t)$ is governed by the viscosity solution of the Hamilton–Jacobi equation [28,36]:

$$\frac{\partial \phi_P^i(X, t)}{\partial t} + \bar{u}|\nabla \phi_P^i(X, t)| + \nabla \phi_P^i(X, t) w(X, t) = 0 \tag{15}$$

with initial condition $\phi_P^i(X, t_0) = |X - X_{P_0}^i|$. Note that

$$\mathcal{R}_P^i(X_{P_0}^i, t) = \{X \in \mathcal{D}{:}\phi_P^i(X, t) \leq 0\}$$

and

$$\partial\mathcal{R}_P^i(X_{P_0}^i, t) = \{X \in \mathcal{D}{:}\phi_P^i(X, t) = 0\}$$

Similarly, the reachable front $\partial\mathcal{R}_E(X_{E_0}, t)$ of the evader is computed by solving the Hamilton–Jacobi equation

$$\frac{\partial \phi_E(X, t)}{\partial t} + \bar{v}|\nabla \phi_E(X, t)| + \nabla \phi_E(X, t) w(X, t) = 0 \tag{16}$$

with initial condition $\phi_E(X, t_0) = |X - X_{E_0}|$, where $\phi_E(X, t)$ is the signed distance function with respect to the reachable set $\mathcal{R}_E(X_{E_0}, t)$ of the evader at time t.

In the case where the condition in Proposition III.6 is not satisfied, we need to track $\partial\mathcal{R}_E^\star(X_{E_0}, t)$ in order to determine the optimal time to capture. Instead of propagating $\partial\mathcal{R}_E^\star(X_{E_0}, t)$ directly, we propagate an intermediate reachable front $\partial\tilde{\mathcal{R}}_E(X_{E_0}, t)$, which can be computed by solving the following modified version of the Hamilton–Jacobi equation:

$$\frac{\partial\tilde{\phi}_E(X,t)}{\partial t} + \tilde{v}(t)|\nabla\tilde{\phi}_E(X,t)| + \nabla\tilde{\phi}_E(X,t)w(X,t) = 0 \tag{17}$$

where

$$\tilde{v}(t) = \begin{cases} \min_{i\in\mathcal{I}} \bar{u}_i, & \text{if } \bigcup_{i=1}^{n} \phi_P^i(X,t) < 0, \\ \bar{v}, & \text{otherwise} \end{cases} \tag{18}$$

and initial condition $\tilde{\phi}_E(X, t_0) = |X - X_{E_0}|$.

The main idea here is to treat the reachable sets of the pursuers as moving obstacles, propagate $\tilde{\mathcal{R}}_E(X_{E_0}, t)$ with the maximum speed of the evader $\bar{v}$ for the parts that fall outside the union of the reachable sets of the pursuers, and to keep pace with the propagation of the reachable set of the slowest pursuer when the front of the evader enters any reachable set of the pursuers. By doing this, we can make sure that the front of the evader does not grow out of the union of the reachable sets of the pursuers. The parts of the reachable front of the evader that do not encounter the reachable sets of the pursuers remain unaffected by the change of speed from $\bar{v}$ to $\tilde{v}$ because these changes are only performed for points inside the reachable sets of the pursuers.

Let

$$\tilde{\mathcal{R}}_E(X_{E_0}, t) = \{X \in \mathcal{D}: \tilde{\phi}_E(X,t) \le 0\}$$

At every time instant t, by construction, $\tilde{\mathcal{R}}_E(X_{E_0}, t)$ excludes all the points X such that $X = X_E(t)$ and

$$X \notin \bigcup_{i=1}^{n} \mathcal{R}_P^i(X_{P_0}^i, t)$$

whereas

$$X \in \bigcup_{i=1}^{n} \mathcal{R}_P^i(X_{P_0}^i, \tau)$$

for some $\tau \in [t_0, t)$. It follows that

$$\mathcal{R}_E^\star(X_{E_0}, t) = \tilde{\mathcal{R}}_E(X_{E_0}, t) \setminus \bigcup_{i=1}^{n} \mathcal{R}_P^i(X_{P_0}^i, t)$$

Moreover, because

$$\mathcal{R}_P^i(X_{P_0}^i, t) = \{X \in \mathcal{D}: \phi_P^i(X,t) \le 0\}$$

the usable reachable set of the evader can also be represented in a form that is more suitable for numerical calculations; that is,

$$\mathcal{R}_E^\star(X_{E_0}, t) = \{X \in \mathcal{D}: \tilde{\phi}_E(X,t) \le 0 \text{ and } \bigcup_{i=1}^{n} \phi_P^i(X,t) \ge 0\}$$

B. Classification of Pursuers

For problems with a large number of pursuers, it is quite possible that optimal capture may not involve all pursuers. That is, not all pursuers need to go after the target at the same time. In [37], for instance, a sequential pursuit strategy was introduced, according to which only a single pursuer participated in the pursuit of the target/evader, although the specific pursuer might change dynamically as the game evolves. In certain applications, such as when the pursuers are subject to energy or fuel limitations, or when they play a dual role as pursuers and guards of a certain region of responsibility, it may be beneficial that some of the pursuers remain inactive. In group pursuit problems involving several pursuers, we may therefore classify the pursuers according to their level of involvement as either redundant, active, or guards. In the following, we elaborate on the motivation of this classification.

1. *Redundant Pursuers*

When we formulate a multiple-pursuers/one-evader pursuit–evasion game, depending on the initial positions of the pursuers and the evader, there may be some pursuers that do not affect the outcome of the pursuit.

Definition IV.2: A pursuer P_k is redundant if, given the pursuer set $\{P_1, \ldots, P_n\}$, the optimal time to capture T is the same as the optimal time to capture $\tilde{T}$ given the pursuer set $\{P_1, \ldots, P_n\} \setminus P_k$.

From the point of view of the pursuers, it is important to identify any redundant pursuers, because fuel or energy savings may result by placing these redundant pursuers on standby, and deploy them only if the evader shows up in their vicinity, or when it is absolutely necessary to ensure capture.

Through the reachable set approach, we can find the minimum number of pursuers needed to capture an evader under optimal time-to-capture pursuit. One way to identify any redundant pursuers is, for each pursuer, to compare the two optimal values of time to capture with and without this pursuer. If these two values turn out to be equal to each other, then this pursuer is redundant.

When the condition

$$\bar{v} \le \min_{i\in\mathcal{I}} \bar{u}_i$$

is satisfied, the following method to determine the redundant pursuers is more practical. Specifically, the jth pursuer is redundant if

$$\mathcal{R}_E(X_{E_0}, T) \subseteq \bigcup_{i=1, i\neq j}^{n} \mathcal{R}_P^i(X_{P_0}^i, T)$$

where T is the optimal time to capture given the pursuer set $\{P_1, \ldots, P_n\}$. For instance, Figs. 1 and 2 show two pursuit–evasion problems restricted in the domain $\mathcal{D} = [0, 25] \times [0, 25]$. The initial positions of the two pursuers and the evader are depicted by the dots. The maximum speeds of the pursuers and the evader are given by

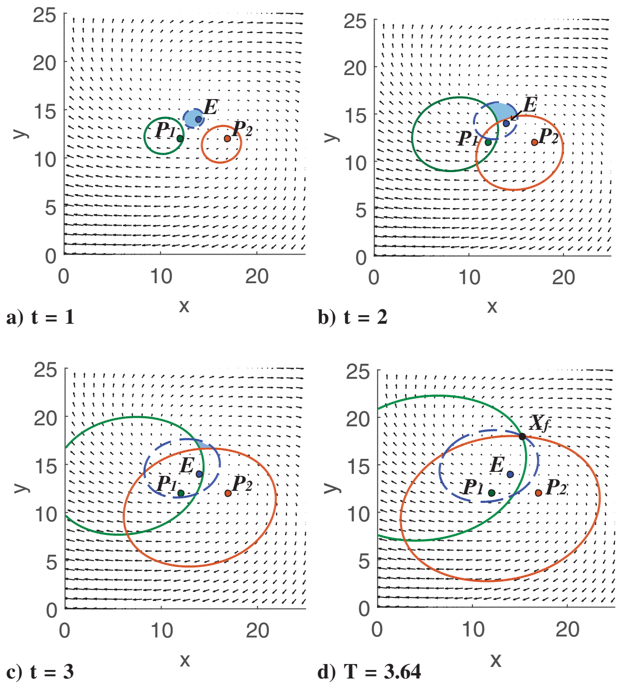

Fig. 1 Evolution of the reachability fronts between two pursuers and one evader.

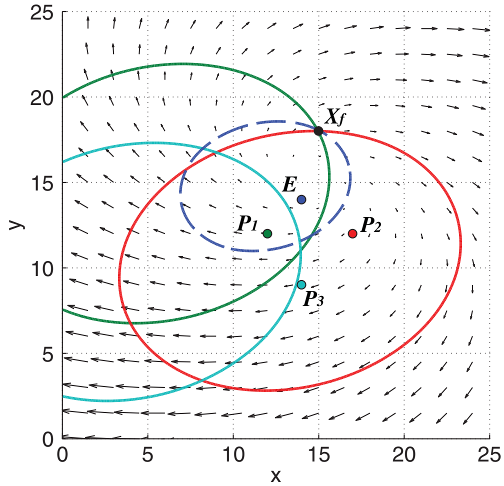

Fig. 2 Level sets of three pursuers and one evader at time *T*. Here, X_f denotes the capture point.

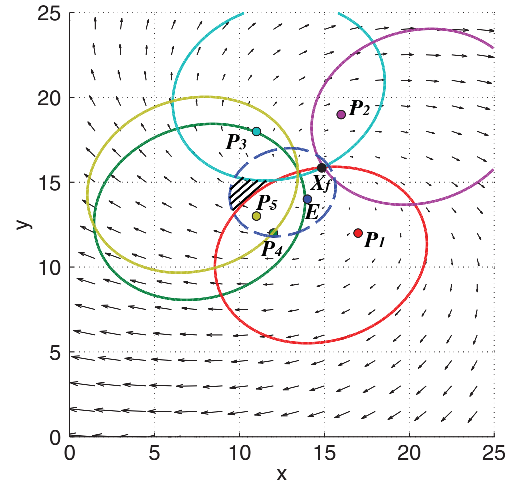

Fig. 3 Level sets of five pursuers and one evader at time *T*. Pursuers P_4 and P_5 are each redundant pursuers by definition, but they cannot be removed together.

$\bar{u}^1 = \bar{u}^2 = 2, \bar{v} = 1$. The vector field of the flowfield is shown in the background. In the first problem, there are two pursuers (P_1 and P_2) against one evader, whereas in the second problem, an additional pursuer (P_3) is added to the pursuer team.

In this example, and all subsequent ones (unless stated otherwise), it will be assumed that the pursuers have a larger maximum speed than the evader. In the absence of an external field, and under simple motion by all players, it is known that the evader can always avoid capture if it has a speed advantage over the pursuers [38]. In the presence of an external field, however, this may not always be the case. Later on, we provide an example where the evader is captured even when all pursuers have a maximum speed that is smaller than the speed of the evader. Please also note that, similarly, an evader may be able to avoid capture from a team of pursuers that has a speed advantage in the presence of an external flowfield.

All examples in this section are subject to a linear flowfield approximated by an affine function $w(X) = A(X - X_s)$, where

$$A = \begin{bmatrix} 0.2 & 0.3 \\ -0.15 & 0.1 \end{bmatrix}, \qquad X_s = \begin{bmatrix} 15 \\ 15 \end{bmatrix} \tag{19}$$

This wind field can be seen as a flow generated from a single singularity point located at X_s, with its characteristics captured by A. Also, the front of the reachable set of the evader is depicted by a dashed elliptical in each of the following examples so that it can be easily distinguished from the front of the reachable sets of the pursuers.

The evolution of the reachability fronts between two pursuers and one evader is depicted in Fig. 1. The usable reachable set of the evader is illustrated in each of the subfigures. Notice that it is known from Theorem III.4 and shown in this example that the capture point X_f is the point in the reachability set of the evader that is not covered by the union of the reachability sets of the pursuers until the capture time T.

For the example shown in Fig. 2, pursuer P_3 turns out to be a redundant pursuer because the optimal time to capture of the evader is the same, regardless of whether P_3 exists or not. If we remove P_3 and its corresponding reachable set at time T, we can recover the case presented in Fig. 1; that is, the outcome of the game is not changed by the presence of pursuer P_3.

Once a redundant pursuer is identified, it can be removed from the set of the active pursuers. It is important to note, however, that we cannot remove two or more redundant pursuers at the same time. Instead, we have to reidentify the redundant pursuers after one redundant pursuer is removed. The reason is that a redundant pursuer may not remain redundant after another redundant pursuer is removed. One such example is shown in Fig. 3, where pursuers P_4 and P_5 guard the same shaded subset of the evader's reachable set, which is the subset of the reachable set of the evader not covered by the union of the reachable sets of pursuers P_1, P_2, and P_3. In this scenario, pursuers P_4 and P_5 are both redundant. However, if we remove both at the same time, then the reachable set of the evader cannot be fully covered by the reachable sets of the remaining pursuers, resulting in the extension of the time to capture. On the other hand, if we reidentify the redundant pursuers after we have removed one of these two pursuers, the other pursuer will not be a redundant pursuer in the updated, reduced set of pursuers.

2. *Active Pursuers and Guards*

Henceforth, we assume that all pursuers are not redundant; otherwise, we can identify and remove any redundant pursuers one by one until no redundant pursuers are left, as explained previously. Under this assumption, we can further divide the pursuer set into two distinct subsets. One subset consists of all the active pursuers, whereas the second subset contains pursuers that do not chase the evader; but, without their presence, there would be no guarantee of capture. The pursuers in the latter subset are called the guards. Once the capture point X_f is found, the active pursuers can be identified as the pursuers for which the boundary of the reachable sets at time T intersects X_f, whereas the rest of the pursuers are guards.

The classification of the pursuer set into active pursuers and guards can be demonstrated by the situation depicted in Fig. 4. For this problem, $\bar{u}^1 = \bar{u}^2 = \bar{u}^3 = \bar{u}^4 = \bar{u}^5 = 2, \bar{v} = 1$. As is shown in this figure, the reachable sets of pursuers P_1, P_2, and P_3 at time T coincide at X_f. These three pursuers need to reach X_f at time T to ensure

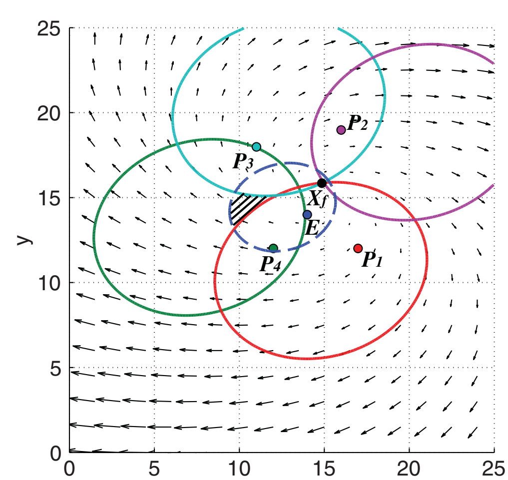

Fig. 4 Level sets of four pursuers and one evader at time *T*. Pursuers P_1, P_2, and P_3 are active pursuers; and P_4 is a guard. Capture occurs at point X_f.

capture of the evader. Hence, these are the active pursuers. On the other hand, pursuer P_4 cannot reach X_f within time T, but its reachable set covers a portion of the reachable set of the evader. Therefore, P_4 acts as a guard of the shaded region depicted in Fig. 4 so that the evader cannot use a control to escape from that area.

It is also worth noting that, if one would like to account for the possibility that the evader may not maneuver optimally, then the process of classifying active pursuers and guards should be repeated at each time step. Otherwise, a pursuer that has been initially classified as a guard might remain inactive even if the evader moves toward it and away from the active pursuers (e.g., the shaded area in Fig. 4).

C. Time-Optimal Paths

In this section, we present a method to retrieve the optimal controls of the evaders and the active pursuers, as well as their corresponding optimal trajectories.

When we deal with multiple pursuers ($n > 1$), the first goal is to find the optimal trajectories of the active pursuers along with their corresponding optimal controls. Because the active pursuers can reach X_f at time T, it is clear that X_f resides on the boundary of their reachable sets; otherwise, capture would have occurred earlier. Therefore, when ϕ_P^i are differentiable, the optimal trajectory for each active pursuer satisfies [27]

$$\frac{dX_P^{i*}}{dt} = \bar{u}^i \frac{\nabla \phi_P^i}{|\nabla \phi_P^i|} + w(X_P^{i*}, t), \qquad i \in \mathcal{I}_A \tag{20}$$

where $\mathcal{I}_A \subseteq \mathcal{I}$ denotes the index set of the active pursuers. The corresponding optimal controls of the active pursuers are thus

$$u_P^{i*} = \bar{u}^i \frac{\nabla \phi_P^i}{|\nabla \phi_P^i|}, \qquad i \in \mathcal{I}_A \tag{21}$$

As for the evader, there are two possible outcomes after the termination of the evolution of the reachable sets of the pursuers and the evader. One possibility is that, at the terminal time T, X_f resides on

$$\partial \tilde{\mathcal{R}}_E(X_{E_0}, T) \ \left(\text{or } \partial \mathcal{R}_E(X_{E_0}, T) \text{ when } \bar{v} \le \min_{i \in \mathcal{I}} \bar{u}^i\right)$$

In this case, it follows that the boundary of the reachable set of the evader is not fully covered for all $t < T$. When differentiable, the optimal trajectory of the evader is then unique and it satisfies the differential equation

$$\frac{dX_E^*}{dt} = \bar{v} \frac{\nabla \phi_E}{|\nabla \phi_E|} + w(X_E^*, t) \tag{22}$$

The corresponding optimal control for the evader is given by

$$u_E^* = \bar{v} \frac{\nabla \phi_E}{|\nabla \phi_E|} \tag{23}$$

It may also happen that X_f lies in the interior of $\tilde{\mathcal{R}}_E(X_{E_0}, T)$, or the interior of $\mathcal{R}_E(X_{E_0}, T)$ when

$$\bar{v} \le \min_{i \in \mathcal{I}} \bar{u}_i$$

This situation occurs when there exists $t_c \in (t_0, T)$ such that

$$\partial \mathcal{R}_E(X_{E_0}, t) \subset \bigcup_{i=1}^{n} \mathcal{R}_P^i(X_{P_0}^i, t)$$

for all $t \in [t_c, T]$. However, some part of the interior of $\mathcal{R}_E(X_{E_0}, t)$ may not be covered until time T. In this case, the optimal control of the evader is not necessarily unique. In particular, the control of the evader can be chosen from the set

$$\mathcal{U}_E^* = \left\{u_E \in \mathcal{U}_E\colon X \text{ satisfies (2) and } X(\tau) \notin \bigcup_{i=1}^{n} \mathcal{R}_P^i(X_{P_0}^i, \tau), \ \forall\, \tau \in [t_0, T]\right\} \tag{24}$$

It follows that an optimal control for the evader is valid, as long as it can bring the evader to X_f at time T without getting captured by any of the pursuers before time T.

V. Simulation Results

We present simulation results for the multiple-pursuer/one-evader pursuit–evasion problem under a realistic flowfield, and for different initial conditions for the pursuers and the evader.

We first consider a state-dependent wind field approximation generalized from the Rankine model of a vortex [39]:

$$w(X) = w_0 + \sum_{i=1}^{n_s} \omega_i A_i (X - X_{s_i}) \tag{25}$$

where

$$\omega_i = \frac{1}{\max\{r_{s_i}^2, |X - X_{s_i}|^2\}} \tag{26}$$

In Eq. (25), n_s is the number of flow singularities; X_{s_i} is the location of the ith flow singularity; r_{s_i} denotes the singularity radius; and A_i is a 2×2 matrix, for which the structure captures the local characteristics of the ith flow singularity. The model approximates the velocity field of a vortex with a linear vector field inside a disk, and the velocity outside of the disk changes with a rate inversely proportional to the squared distance to the center of the disk.

We set the number of flow singularities to $n_s = 3$. The locations of the flow singularities are $X_{s_1} = [18, 18]^T$, $X_{s_2} = [12, 19]^T$, and $X_{s_3} = [14, 12]^T$; and the corresponding radii are $r_{s_1} = 3$, $r_{s_2} = 2$, and $r_{s_3} = 3$, respectively. We also let $w_0 = [0.2, -0.3]^T$. The local wind field matrices are given by

$$A_1 = \begin{bmatrix} 0 & 3 \\ -1.5 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 4 & 2 \\ 0 & -2 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 2 & 1 \\ -2 & 2 \end{bmatrix}$$

In the first example, we formulate a three-pursuer/one-evader problem. The three pursuers are initially located at $X_{P_0}^1 = [13, 13]^T$, $X_{P_0}^2 = [16, 14]^T$, and $X_{P_0}^3 = [14, 17]^T$, respectively. Their corresponding maximum speeds are given by $\bar{u}_1 = 1.5$, $\bar{u}_2 = 1.2$, and $\bar{u}_3 = 0.5$. The initial location of the evader is given by $X_{E_0} = [14, 15]^T$, and its maximum speed is $\bar{v} = 1$. Note that, in this example, the maximum speed of the evader is larger than the speed of one of the pursuers. Therefore, we need to propagate the intermediate reachability front of the evader in order to recover $\mathcal{R}_E^\star(X_{E_0}, t)$.

The optimal time to capture is $T = 4.25$. P_1 is the only active pursuer in this case, whereas P_2 and P_3 are guards. Notice that the optimal time to capture in the case of only P_1 and P_2 against E is $T_{12} = 5.33$. Similarly, the optimal time to capture in the case of P_1 and P_3 against E is $T_{13} = 5.04$. Therefore, P_2 and P_3 are not redundant. Also, the optimal time to capture between P_1 and E is $T_1 = 5.38$. It can be observed from this example, and in accordance with Definition IV.2, that the optimal time to capture is reduced as more (nonredundant) pursuers join the pursuit. The reachable fronts of the pursuers and the evader at time T, as well as the corresponding optimal trajectories of the active pursuer and the evader, are shown in Fig. 5. The curves represent the reachable fronts of the pursuers at the terminal time. As before, arrows on the background represent the external flowfield. In the figure, X_f denotes the capture point.

For the next example, we keep all the initial conditions unchanged, but we modify the maximum speed of the third pursuer from $\bar{u}_3 = 0.5$ to $\bar{u}_3 = 2$. After this change, the condition

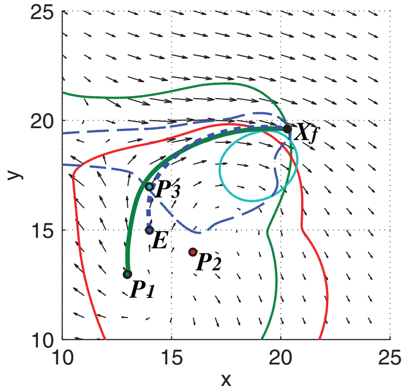

Fig. 5 Optimal trajectory of the evader and the set $\partial\tilde{\mathcal{R}}_E(X_{E_0},t)$, as well as the optimal trajectory of the active pursuer.

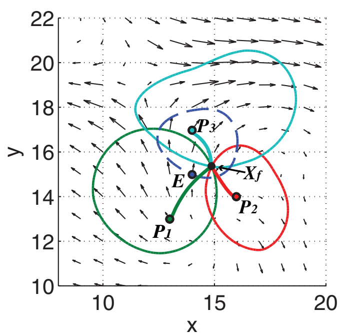

Fig. 6 Optimal trajectories of the three active pursuers. Closed curves represent the reachable fronts of the pursuers at the terminal time.

$$\bar{v} \le \min_{i\in\mathcal{I}} \bar{u}_i$$

is satisfied. By Proposition III.6, it suffices to update $\mathcal{R}_E(X_{E_0},t)$ instead of $\mathcal{R}_E^{\star}(X_{E_0},t)$ and show condition (10). As illustrated in Fig. 6, the simultaneous capture of the three pursuers is required at the optimal time to capture of $T = 1.31$. Optimal trajectories of the three pursuers are depicted to demonstrate the capture of the evader. The terminal position of the evader X_f is denoted by the centermost dot. The optimal trajectory of the evader is not shown because it is not unique and can be picked from Eq. (24).

Next, we consider the case where

$$\max_{i\in\mathcal{I}} \bar{u}_i < \bar{v}$$

In particular, we consider four pursuers with maximum speed $\bar{u}^1 = \bar{u}^2 = \bar{u}^3 = \bar{u}^4 = 0.9$ and an evader with maximum speed $\bar{v} = 1$. The initial positions of the pursuers are given as $X_{P_0}^1 = [13,13]^T$, $X_{P_0}^2 = [15,13]^T$, $X_{P_0}^3 = [15,15]^T$, and $X_{P_0}^4 = [13,15]^T$; whereas the evader is initially located at $X_{E_0} = [14,14]^T$. The flowfield is the same one as in the previous example. Capture in this case occurs at $T = 1.99$, and the corresponding optimal trajectories of the active pursuers are shown in Fig. 7.

The next example intends to demonstrate the effect of the flowfield in the game outcome. To this end, consider a pursuit–evasion game between four pursuers and one faster evader. The initial positions of the pursuers are given as $X_{P_0}^1 = [13,13]^T$, $X_{P_0}^2 = [15,13]^T$, $X_{P_0}^3 = [15,15]^T$, and $X_{P_0}^4 = [13,15]^T$; whereas the evader is initially located at $X_{E_0} = [14,14]^T$. The maximum speeds of the pursuers are set as $\bar{u}^1 = \bar{u}^2 = 0.65$ and $\bar{u}^3 = \bar{u}^4 = 0.95$, and the evader's maximum speed is set to $\bar{v} = 1$. The flowfield is given by $w(X) = A(X - x_s)$, where

$$A = \begin{bmatrix} -0.2 & 0.3 \\ -0.15 & -0.1 \end{bmatrix}, \qquad X_s = \begin{bmatrix} 17 \\ 17 \end{bmatrix} \tag{27}$$

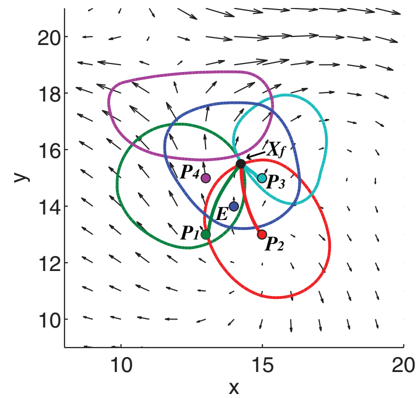

Fig. 7 Optimal trajectories of the three active pursuers and the reachable fronts of the pursuers at the terminal time.

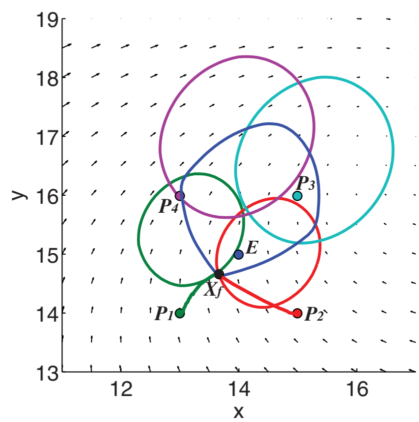

Fig. 8 Reachable fronts of the pursuers and usable reachable front of the evader at the terminal time, and optimal trajectories of the two active pursuers.

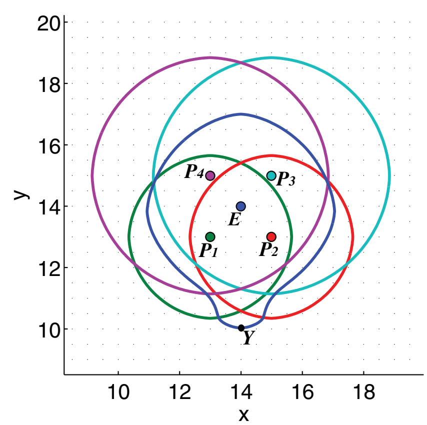

Fig. 9 Reachable fronts of the pursuers and usable reachable front of the evader at $t = 4.41$. Case of faster evader without flowfield.

For this case, capture occurs at $T = 1.71$, as illustrated in Fig. 8. In contrast, the evader escapes in the absence of an external flowfield. This is demonstrated in Fig. 9, where a snapshot of the level sets at $t = 4.11$ is shown. Notice that, at that time instant, the evader reaches the point Y without being captured. The evader can keep avoiding capture after that time because it is faster and Y is outside the convex hull of the pursuers. This example shows that the presence of the flowfield can change the outcome of the game; hence, it is imperative that its effect is quantified and included, if needed, in the game formulation.

We finally apply our algorithm to a pursuit–evasion problem taking place inside a smooth water channel with a circular island obstacle. The external flow enters the rectangle region shown in

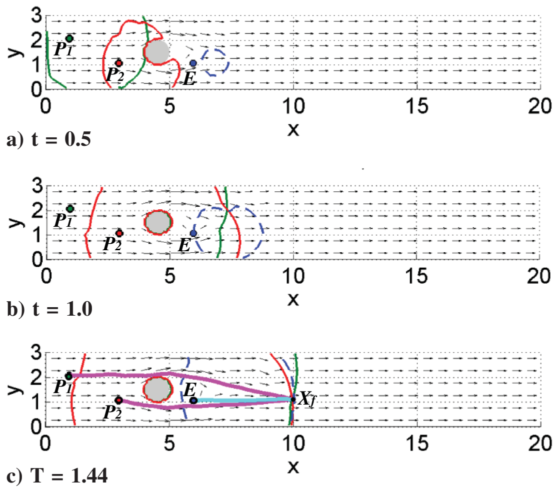

Fig. 10 Evolution of the reachability fronts and optimal trajectories at the optimal time to capture. Arrows on the background represent the time-varying external flowfield.

Fig. 10a from the left edge and drifts past a circular island, which induces vortices downstream. The island is centered at [4.5,1.5] and has a radius of 0.5. More details regarding the simulation of this flowfield can be found in [28]. Through this example, we demonstrate that the proposed algorithm can handle scenarios with arbitrary spatiotemporal flowfields and that the algorithm can be naturally extended to deal with obstacles in a flowfield.

The initial positions of the pursuers and the evader are set as $X^1_{P_0} = [1, 2]^T$, $X^2_{P_0} = [3, 1]^T$, and $X_{E_0} = [6, 1]^T$, respectively. Their corresponding maximum speeds are given by $\bar{u}_1 = 4.5$, $\bar{u}_2 = 3$, and $\bar{v} = 1$. Because $\min\{\bar{u}_1, \bar{u}_2\} > \bar{v}$, we simply evolve the reachability sets of both players. When the reachability front of any one of the players encounters the obstacle, we stop the evolution of the parts on the reachability front that would otherwise go through the obstacle to ensure that the optimal path we find later on is guaranteed to be a collision-free path. Capture occurs at time $T = 1.44$, and the optimal paths of the pursuers and the evader are depicted in Fig. 10c. Their corresponding reachability fronts are also illustrated. Snapshots of the reachability fronts of the pursuer and the evader at times $t = 0.5$ and $t = 1$ are included in Figs. 10a and 10b to demonstrate the evolution of the reachability fronts.

To demonstrate the feedback nature of the proposed strategies, in Fig. 11, we show the result of a game with just two players (one pursuer and one evader), in which the evader employs a suboptimal strategy. Specifically, while the pursuer determines its control action at each instant of time using the reachability set analysis outlined in Sec. III, in Fig. 11 (left), the evader implements a constant bearing strategy given by $u_E = \bar{v}[\cos(\pi/2), \sin(\pi/2)]^T$. Capture occurs at $T = 0.93$, whereas if the evader had acted optimally, capture would have occurred at $T = 1.08$, which is the value of this game. Figure 11 (right) shows another similar scenario where the evader uses the (also suboptimal) strategy $u_E = \bar{v}[\cos(\pi/4), \sin(\pi/4)]^T$. In this case, capture occurs at $T = 1.04$, which is somewhat better than before but still less than the optimal value of $T = 1.08$. For both of these examples, the flowfield is affine, given by $w(X) = A(X - X_s)$, where

$$A = \begin{bmatrix} 0 & 0.3 \\ -0.15 & 0 \end{bmatrix}, \qquad X_s = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \tag{28}$$

For this example, the maximum speeds are $\bar{u} = 4$ and $\bar{v} = 1$; and the initial conditions are $X_{P_0} = [2, 2]^T$ and $X_{E_0} = [4, 4]^T$ for the pursuer and the evader, respectively.

VI. Conclusions

In this paper, differential games between an evader and multiple pursuers in an external dynamic flowfield are considered. Under the assumptions that each player has perfect knowledge of the dynamics of the system, the constraint control sets, the cost function, and the initial state, as well as under the assumption that the value of the game exists, it is shown that the game terminates when the usable reachable set of the evader becomes the empty set for the first time. A simplified condition for capture of the evader can be derived when the maximum speed of the evader is less than the maximum speed of each pursuer. The level set method is adopted to compute and propagate the reachable sets of all the players. Depending on whether a pursuer contributes to the outcome of the game, whether it chases the evader directly, or whether it guards some part of the reachable set of the evader so that the evader does not detour from its optimal trajectory, the pursuers can be, respectively, classified into redundant pursuers, active pursuers, or guards. The redundant pursuers are those that can be removed from the set of pursuers without impacting the outcome of the game. Pursuers that actively chase the evader are the active pursuers. Guards represent pursuers for which their mere presence affects the optimal time to capture, but they do not pursue the evader as long as the latter follows an optimal path. The optimal trajectories and controls of the pursuers and the evader are retrieved by backward propagation along the corresponding levels of the reachable sets. The proposed solution scheme is demonstrated by applying it to multiplayer pursuit–evasion games taking place in realistic strong and time-dependent external flowfields, including a case with an obstacle.

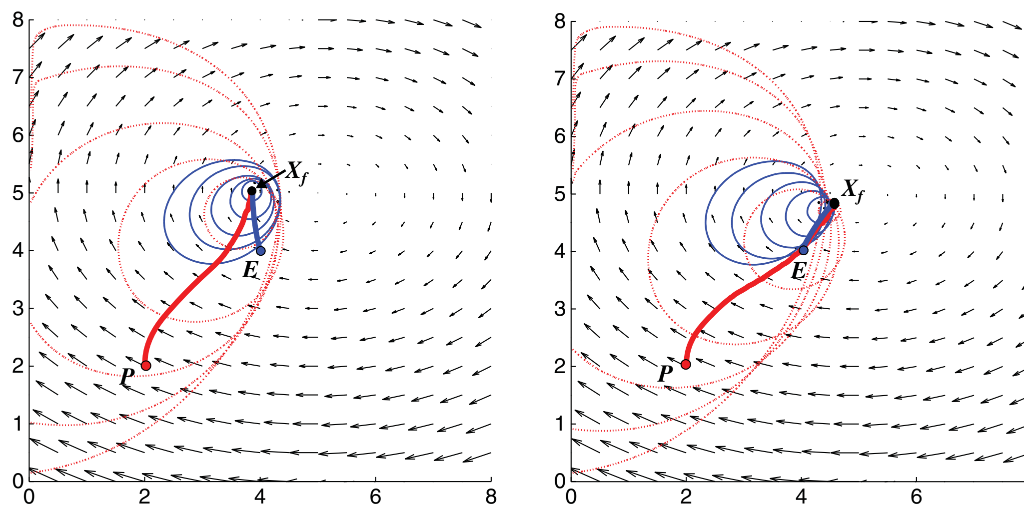

Fig. 11 Evolution of the reachability fronts and optimal trajectories at the optimal time to capture. In this case, the evader plays suboptimally.

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