Analytic Solution of the Large Angle Problem in Rigid Body Attitude Dynamics

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Abstract

The development of analytic solutions for the forced attitude motion of a rigid body is a formidable task because of the appearance of nonlinear differential kinematic equations when the problem is represented in terms of Eulerian angles. On the other hand, when the problem is cast into the quaternion formulation, the resulting time-varying matrix renders the linear differential equations just as intractable for large ranges of the time variable. In this paper a new kinematic formulation is introduced which characterizes the problem as a single, complex Riccati equation with time-varying coefficients. In this form the highest order nonlinear terms are quadratic, in contrast to the nonlinear trigonometric terms of the Eulerian angles. When the quadratic terms are dropped the new kinematic equations correspond directly to the linearized (small angle) Eulerian formulation. The quadratic kinematic equations are ideal for analytic solutions because an approximate analytic integration of the quadratic terms (which characterize the large angle motion) can be achieved to a high degree of accuracy. The solutions are derived through asymptotic and series expansions and are based entirely on integrals found previously in the small angle theory. The numerical simulations demonstrate that spin axis excursions in excess of 1 radian can be accurately represented by the new analytic solutions.

Introduction

In the last several decades there has been a great deal of activity in the development of analytic solutions for attitude motion of a rigid body in more and more sophisticated problems [1-23]. The earliest works (e.g. Poinsot [24]) address the motion of the *unforced rigid body* and, due to the availability of integrals of the motion (namely energy and momentum), provide exact, compact solutions for both symmetric and asymmetric cases. More recent studies have focused on the *self-excited rigid body*, which Grammel [2, 4] defines as a body free to rotate about a point fixed in the body and space, when it is acted upon by a torque vector arising from internal reactions which do not appreciably change the mass or the mass distribution. The attitude motion of a rigid spacecraft, subject to thruster torques, presents a modern example of a self-excited rigid body, and many of the aforementioned references deal specifically with this problem.

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Nearly all of the analytic solutions for the self-excited rigid body rely on small angle approximations, in which one of the body axes is confined to small angular excursions from its initial orientation in inertial space. This is because most of these developments are based on the Eulerian kinematic equations which are highly nonlinear, in general, but are quite amenable to analysis when small angles are assumed. In fact, much as been achieved through this approach including solutions for the motion of a rigid body subject to time-varying body-fixed torques (Longuski and Tsiotras [22], Tsiotras and Longuski [19, 23]). These solutions, although lengthy, provide important tools for the study of rigid body motion. By examining only the most important terms of an analytic solution (with regard to a specific application), one can gain insight into the behavior of the motion. When more general studies are required, these analytic solutions can provide the basis for efficient computational algorithms, which can in turn be used to study the general behavior through parametric studies (e.g. Longuski and Kia [25]). Such parametric studies would be extremely expensive and time consuming if attemped by numerical integration of the differential equations of motion. A further application of these analytic solutions, is that they can provide for rapid onboard computations of spacecraft maneuvers and hence, can ultimately serve in the missions of autonomous spacecraft.

In this paper we use a new kinematic formulation, (Tsiotras and Longuski [21]), which renders the large angle motion of a rigid body analytically tractable. The approach can be applied to all attitude motion problems. Here we apply it to the self-excited near-symmetric rigid body subject to constant torques about all three axes, as an example.

We mention, in passing, that the new kinematic formulation has also provided breakthroughs in the development of simple and elegant control laws as as shown in [26, 27].

Kinematic Equations in Eulerian Angles

Consider a Type 1: 3-1-2 Euler angle sequence using the nomenclature of Wertz [28]. Type 1 refers to sequences that involve three coordinate axes, whereas Type 2 refers to cases in which the first and third rotations take place about the same axis. For a 3-1-2 sequence the Eulerian angles $(\beta_z, \beta_x, \beta_y)$ are defined by successive rotations about the z, x' and y'' coordinate axes. The resulting kinematic equations are

$$\dot{\beta}_x = \omega_x \cos\beta_y + \omega_z \sin\beta_y \tag{1a}$$

$$\dot{\beta}_y = \omega_y - (\omega_z \cos\beta_y - \omega_x \sin\beta_y)\tan\beta_x \tag{1b}$$

$$\dot{\beta}_z = (\omega_z \cos\beta_y - \omega_x \sin\beta_y)\sec\beta_x \tag{1c}$$

where $(\omega_x, \omega_y, \omega_z)$ are the angular velocity components. The nonlinear nature of these equations coupled with the arbitrary angular velocities make these equations intractable in general. However, we note that β_z is an ignorable variable, so that if we can find expressions for β_x and β_y, then β_z is found by quadrature

$$\beta_z(t) = \int_0^t \{\omega_z(\tau)\cos[\beta_y(\tau)] - \omega_x(\tau)\sin[\beta_y(\tau)]\}\sec[\beta_x(\tau)]\, d\tau \tag{2}$$

Thus, the fundamental problem has two degrees of freedom and the task at hand reduces to the problem of solving for β_x and β_y.

Equations (1) have been used in their linearized form (with β_x and β_y assumed small) to study attitude motion in which the spin axis is the z body axis. The small angle assumption amounts to interpreting the angles β_x and β_y as error angles measured from the desired inertial Z axis orientation. With the further assumption that the product $\beta_y\omega_x$ in (1c) is small compared to ω_z (as is usually the case for spin-stabilized bodies), we have the linearized form of equations (1)

$$\dot{\beta}_x = \omega_x + \beta_y\omega_z \tag{3a}$$

$$\dot{\beta}_y = \omega_y - \beta_x\omega_z \tag{3b}$$

$$\dot{\beta}_z = \omega_z \tag{3c}$$

Again, because of the decoupling of β_z from β_x and β_y, we can concentrate on solving (3a) and (3b). Using the complex notation introduced by Tsiotras and Longuski [20, 21], we can write these two equations in the following single complex equation for the linearized transverse Eulerian angles β_x and β_y

$$\dot{\beta} + i\omega_z\beta = \omega \tag{4}$$

where $\beta \triangleq \beta_x + i\beta_y$ and $\omega \triangleq \omega_x + i\omega_y$.

Equation (4) represents the small angle problem in compact form. For analytic integration of the large angle problem, a different kinematic formulation in the form of a Riccati equation is employed.

Riccati Kinematic Equation

In general for any set of Eulerian angles we have a transformation matrix $\mathbf{A}$ such that

$$\left\{ \begin{array}{c} X \\ Y \\ Z \end{array} \right\} = \mathbf{A} \left\{ \begin{array}{c} x \\ y \\ z \end{array} \right\} \tag{5}$$

where x, y and z are body coordinates and X, Y and Z are inertial coordinates. For the 3-1-2 Euler angle sequence $(\beta_z, \beta_x, \beta_y)$ we have the corresponding transformation matrix $\mathbf{A_{312}}$:

$$\mathbf{A_{312}} = \begin{bmatrix} c\beta_z c\beta_y - s\beta_z s\beta_x s\beta_y & -s\beta_z c\beta_x & c\beta_z s\beta_y + s\beta_z s\beta_x c\beta_y \\ s\beta_z c\beta_y + c\beta_z s\beta_x s\beta_y & c\beta_z c\beta_x & s\beta_z s\beta_y - c\beta_z s\beta_x c\beta_y \\ -c\beta_x s\beta_y & s\beta_x & c\beta_x c\beta_y \end{bmatrix} \tag{6}$$

where c and s denote cosine and sine, respectively.

It is well known that the transformation matrix $\mathbf{A}$ obeys the differential equation

$$\dot{\mathbf{A}} = \mathbf{A}\mathbf{W} \tag{7}$$

where

$$\mathbf{W} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \tag{8}$$

$\mathbf{W}$ is called the *affinor of rotation* by Leimanis [4] or the *angular velocity matrix* by Kane, Likins and Levinson [29]. Integration of equation (7) provides the complete attitude history of the rigid body. Bödewadt [1, 4] proposed a closed-form solution of equation (7) for the self-excited rigid body, but his solution is incorrect in general because the matrix $\mathbf{A}$ does not commute with its integral (see Longuski [17]).

For any row of $\mathbf{A}$ we have

$$[\dot{a} \quad \dot{b} \quad \dot{c}] = [a \quad b \quad c] \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \tag{9}$$

where a, b and c are scalar elements of the row. Noting that

$$a^2 + b^2 + c^2 = 1 \tag{10}$$

we can use stereographic projection of points (a, b, c) on the unit sphere onto the complex plane $w = w_x + i\,w_y$ to get [21]

$$w = w_x + i\,w_y = (b - i\,a)/(1 + c) = (1 - c)/(b + i\,a) \tag{11}$$

The kinematic equation can be derived by differentiating equation (11)

$$\dot{w} = (\dot{b} - i\,\dot{a})/(1 + c) - (b - i\,a)\dot{c}/(1 + c)^2 \tag{12}$$

Using equation (9) and after some algebra we find that

$$\dot{w} + i\,\omega_z w = \frac{\omega}{2} + \frac{\bar{\omega}}{2} w^2 \tag{13}$$

Equation (13) is a differential equation of the Riccati type. An equation of this form first appeared in [30] in connection with some problems in classical differential geometry, however its derivation using the stereographic projection and its use in attitude kinematics was first established in [21].

In order to recover a, b and c from w one can use the following relations

$$a = i\,(w - \bar{w})/(|w|^2 + 1) \tag{14a}$$
$$b = (w + \bar{w})/(|w|^2 + 1) \tag{14b}$$
$$c = -(|w|^2 - 1)/(|w|^2 + 1) \tag{14c}$$

Since we can set the values of a, b and c to those of any row of the transformation matrix $\mathbf{A}$, let us pick the third row of the the matrix $\mathbf{A_{312}}$ in (6):

$$a = -\cos\beta_x \sin\beta_y, \quad b = \sin\beta_x, \quad c = \cos\beta_x \cos\beta_y \tag{15}$$

Then substituting equations (15) into equation(11), we obtain

$$w = (\sin\beta_x + i\cos\beta_x \sin\beta_y)/(1 + \cos\beta_x \cos\beta_y) \tag{16}$$

Here we note that if β_x and β_y are small angles, then

$$w \approx \frac{1}{2}(\beta_x + i\beta_y) = \frac{1}{2}\beta \tag{17}$$

Analytic Integration of the Riccati Equation

Equation (13) is well-suited for analytic investigations of the large angle problem. Let us introduce the small parameter, ϵ, into equation (13):

$$\dot{w} + i\,\omega_z w = \frac{\omega}{2} + \epsilon\,\frac{\bar{\omega}}{2}w^2 \tag{18}$$

When $\epsilon \rightarrow 0$, equation (18) reduces to the small angle problem. In the small angle case we see from equation (17) that $w \approx \frac{1}{2}\beta$ and that equations (4) and (18) are equivalent. When $\epsilon \rightarrow 1$ equation (18) becomes equation (13).

Thus we can use the small parameter, ϵ, as a mathematical device (having no physical interpretation), to control the degree of nonlinearity we wish to model in equation (18). The form of equation (18) is ideally suited to the application of Poincaré's method of small parameters [31] where we assume that the solution is

$$w = w_0 + \epsilon\, w_1 + \epsilon^2 w_2 + ... \tag{19}$$

Substituting equation (19) into equation (18) and collecting terms of like powers in ϵ we obtain

$$\begin{aligned} \dot{w}_0 + i\,\omega_z w_0 &= \omega/2 && (20)\\ \dot{w}_1 + i\,\omega_z w_1 &= \bar{\omega} w_0^2/2 && (21)\\ \dot{w}_2 + i\,\omega_z w_2 &= \bar{\omega} w_0 w_1 && (22) \end{aligned}$$

for the first three differential equations. The solution for equation (20) has already been obtained for the problem of the self-excited near-symmetric rigid body (constant torques about three axes) in [20]. Numerical simulations presented in [21] demonstrate that the inclusion of equation (21) provides a significant improvement in the analytic solution and gives very accurate results for the large angle problem. However we will see that in attempting to integrate the forcing term which appears in equation (21) certain approximations must be made in order to achieve a closed-form solution. Here we present only the first order approximation given in (21) since numerical investigations have demonstrated that this term captures a significant portion of the nonlinearity in (13).

Thus, we will approximate the solution to w as

$$w \approx w_0 + w_1 \tag{23}$$

and the governing differential equation is (from equations (20) and (21)):

$$\dot{w} + i\,\omega_z w \approx \frac{\omega}{2} + \frac{\bar{\omega}}{2}w_0^2 \tag{24}$$

Here we see that equation (24) represents simply one successive approximation to equation (13) where the linear solution, w_0, is substituted into the nonlinear forcing term.

It is convenient to express the solution to equation (24) in terms of the two components w_0 and w_1:

$$\begin{aligned} w_0(t) &= w(0)\exp\left(-i\int_0^t \omega_z(u)\,du\right)\\ &+ \frac{1}{2}\exp\left(-i\int_0^t \omega_z(u)\,du\right)\int_0^t \omega(u)\exp\left(i\int_0^u \omega_z(v)\,dv\right)\,du \end{aligned} \tag{25}$$

$$w_1(t) = \frac{1}{2}\exp\left(-i\int_0^t \omega_z(u)\,du\right)\int_0^t \bar{\omega}(u)w_0^2(u)\exp\left(i\int_0^u \omega_z(v)\,dv\right)\,du \tag{26}$$

Since an explicit form for equation (25) is reported in [20] for our problem, we will concentrate our current effort on equation (26). Before going into details it is first necessary to briefly review the assumptions and results of [20].

The Self-Excited Rigid Body

For the self-excited rigid body, Euler's equations of motion apply

$$M_x = I_x\dot{\omega}_x + (I_z - I_y)\,\omega_y\omega_z \tag{27a}$$
$$M_y = I_y\dot{\omega}_y + (I_x - I_z)\,\omega_z\omega_x \tag{27b}$$
$$M_z = I_z\dot{\omega}_z + (I_y - I_x)\,\omega_x\omega_y \tag{27c}$$

where the principal moments of inertia, I_x, I_y and I_z, are assumed to be constants. The body-fixed moments, M_x, M_y and M_z, may be constants (as in [20]) or time-varying functions (as in [19, 22, 23]).

For our purposes we will assume *constant moments* and *near-symmetry* ($I_y \approx I_x$) about the z spin axis so that the last term of equation (27c) can be dropped. The spin rate can then be integrated to give

$$\omega_z(t) \approx (M_z/I_z)\,t + \omega_{z0}, \qquad \omega_{z0} \triangleq \omega_z(0) \tag{28}$$

We will demonstrate that with these assumptions a very accurate analytic solution can be obtained for the large angle motion. However, the method is not limited to these particular assumptions and extensions to include time-varying body-fixed torques [19, 22, 23] or asymmetric bodies [32, 33] are straightforward.

Without loss of generality we assume that the principal moments of inertia are ordered according to the inequalities $I_z > I_x > I_y$. By defining a new independent variable

$$\tau \triangleq \omega_z(t) \tag{29}$$

and the new complex dependent variable

$$\Omega = \Omega_x + i\,\Omega_y \triangleq \omega_x\sqrt{k_y} + i\,\omega_y\sqrt{k_x} \tag{30}$$

where

$$k_x \triangleq (I_z - I_y)/I_x, \qquad k_y \triangleq (I_z - I_x)/I_y, \qquad k \triangleq \sqrt{k_x k_y} \tag{31}$$

we can rewrite Euler's equations of motion (27a)-(27b) as the complex differential equation

$$\Omega' - i\,\rho\tau\Omega = F \tag{32}$$

where prime denotes differentiation with respect to the new independent variable τ. In equation (32) we have also made use of the definitions

$$\rho \triangleq k(I_z/M_z) \tag{33a}$$
$$F = F_x + i\,F_y \triangleq (M_x/M_z)(I_z/I_x)\sqrt{k_y} + i\,(M_y/M_z)(I_z/I_y)\sqrt{k_x} \tag{33b}$$

For constant F, the solution to equation (32) is

$$\Omega(\tau) = \Omega_0 \exp(\tfrac{1}{2} i\rho\tau^2) + \exp(\tfrac{1}{2} i\rho\tau^2) F I_0(\tau_0, \tau; \rho) \tag{34}$$

where $\Omega_0 = \Omega(\tau_0)\exp(-\frac{1}{2} i\rho\tau_0^2)$ and

$$I_0(\tau_0, \tau; \rho) \triangleq \int_{\tau_0}^{\tau} \exp(-\tfrac{1}{2} i\rho u^2)\, du \tag{35}$$

The integral $I_0(\tau_0, \tau; \rho)$ of equation (35) is discussed in detail in [20, 22], and can be readily calculated in terms of Fresnel integrals:

$$I_0(\tau_0, \tau; \rho) = \sqrt{\pi/|\rho|} \left[sgn(\tau) \tilde{E}\left(\sqrt{|\rho|/\pi}\, \tau\right) - sgn(\tau_0) \tilde{E}\left(\sqrt{|\rho|/\pi}\, \tau_0\right) \right] \tag{36}$$

where

$$\tilde{E}(x) = \begin{cases} E(x) & \text{for} \quad \rho > 0 \\ \bar{E}(x) & \text{for} \quad \rho < 0 \end{cases} \tag{37}$$

and

$$E(x) \triangleq \int_0^x \exp(-\tfrac{1}{2} i\pi u^2) du \tag{38}$$

is the complex Fresnel integral [20, 34]. The term $sgn(\cdot)$ in equation (36) represents the signum function such that $sgn(x) = 1$ for $x > 0$ and $sgn(x) = -1$ for $x < 0$.

Attitude Motion of the Self-Excited Rigid Body

The Small Angle Theory

We now turn our attention back to the kinematic equation (24). By using the new independent variable (29), equation (24) becomes

$$w' + i\lambda\tau w = \lambda(\omega + \bar{\omega} w_0^2)/2 \tag{39}$$

where we have introduced

$$\lambda \triangleq I_z/M_z \tag{40}$$

The zero order equation is

$$w_0' + i\lambda\tau w_0 = \lambda\omega/2 \tag{41}$$

The solution to equation (41) is

$$w_0(\tau) = w_{00} \exp(-\tfrac{1}{2} i\lambda\tau^2) + (\lambda/2) \exp(-\tfrac{1}{2} i\lambda\tau^2) \left[k_1 \Phi_1(\tau_0, \tau; \lambda) + k_2 \Phi_2(\tau_0, \tau; \lambda) \right] \tag{42}$$

where $w_{00} \triangleq w(\tau_0)\exp(\frac{1}{2} i\lambda\tau_0^2)$ and where

$$\Phi_1(\tau_0, \tau; \lambda) \triangleq \int_{\tau_0}^{\tau} \exp(\tfrac{1}{2} i\lambda u^2) \Omega(u)\, du \tag{43a}$$

$$\Phi_2(\tau_0, \tau; \lambda) \triangleq \int_{\tau_0}^{\tau} \exp(\tfrac{1}{2} i\lambda u^2) \bar{\Omega}(u)\, du \tag{43b}$$

and

$$k_1 \triangleq \left(\sqrt{k_x} + \sqrt{k_y} \right) /2k, \qquad k_2 \triangleq \left(\sqrt{k_x} - \sqrt{k_y} \right) /2k \tag{44}$$

Notice that $\Phi_1(\tau_0, \tau; \lambda)$ and $\Phi_2(\tau_0, \tau; \lambda)$ are related by

$$\Phi_2(\tau_0, \tau; \lambda) = \bar{\Phi}_1(\tau_0, \tau; -\lambda) \tag{45}$$

therefore only one of the integrals Φ_1, Φ_2 needs to be evaluated in the expressions (43). Notice that the k_1 and k_2 appear in equation (42) when we write ω of equation (39) in terms of Ω (equation 30)):

$$\omega(\tau) = k_1 \Omega(\tau) + k_2 \bar{\Omega}(\tau) \tag{46}$$

The small angle solution, w_0, of equation (42) can be written most compactly by following the nomenclature of [22], where

$$\begin{aligned} \Phi_1(\tau_0, \tau; \lambda) &= [\Omega_0 - F\, I_0(\tau_0; \rho)] \int_{\tau_0}^{\tau} \exp(\tfrac{1}{2} i \mu u^2)\, du \\ &+ F \int_{\tau_0}^{\tau} \exp(\tfrac{1}{2} i \mu u^2)\, I_0(u; \rho)\, du \end{aligned} \tag{47}$$

where $\mu \triangleq \lambda + \rho = \lambda(1 + k)$. The integral $I_0(\tau_0; \rho)$ is a special case of integrals which have been identified in [22], namely

$$I_n(x; \rho) \triangleq \int_0^x \exp(-\tfrac{1}{2} i \rho u^2) u^n du \tag{48}$$

When $n = 0$ we recognize from equations (48) and (35) that

$$I_0(\tau_0; \rho) \triangleq \int_0^{\tau_0} \exp(-\tfrac{1}{2} i \rho u^2) du = \sqrt{\pi/\rho}\; sgn(\tau_0) \tilde{E}(\sqrt{\rho/\pi}\, \tau_0) \tag{49}$$

The next integral which appears in equation (47) is

$$\int_{\tau_0}^{\tau} \exp(\tfrac{1}{2} i \mu u^2) du = I_0(\tau_0, \tau; -\mu) \tag{50}$$

which we recognize as equation (36). We designate the last integral which appears in equation (47) as

$$J_0(\tau_0, \tau; \mu, \rho) \triangleq \int_{\tau_0}^{\tau} \exp(\tfrac{1}{2} i \mu u^2)\, I_0(u; \rho)\, du \tag{51}$$

Using equation (49) in equation (51), we have

$$J_0(\tau_0, \tau; \mu, \rho) = \sqrt{\pi/|\rho|} \int_{\tau_0}^{\tau} sgn(u) \exp(\tfrac{1}{2} i \mu u^2) \tilde{E}(\sqrt{|\rho|/\pi}\, u)\, du \tag{52}$$

By the change of variable

$$\tilde{u} \triangleq \sqrt{|\rho|/\pi}\, u \tag{53}$$

the integral (52) simplifies to

$$J_0(\tau_0, \tau; \mu, \rho) = (\pi/|\rho|) \int_{\tilde{\tau}_0}^{\tilde{\tau}} sgn(\tilde{u}) \exp(\tfrac{1}{2} i \tilde{\mu} \tilde{u}^2) \tilde{E}(\tilde{u}) d\tilde{u} \triangleq (\pi/|\rho|) \tilde{J}_0(\tilde{\tau}_0, \tilde{\tau}; \tilde{\mu}) \tag{54}$$

where

$$\tilde{\tau}_0 \triangleq \sqrt{|\rho|/\pi}\, \tau_0, \qquad \tilde{\tau} \triangleq \sqrt{|\rho|/\pi}\, \tau, \qquad \tilde{\mu} \triangleq \mu\pi/|\rho| \tag{55}$$

Thus, dropping the tildes for notational convenience, we see that the essential integral involved in the small angle theory is

$$J_0(\tau_0, \tau; \mu) \triangleq \int_{\tau_0}^{\tau} sgn(u) \exp(\tfrac{1}{2} i\mu u^2) \tilde{E}(u)\, du \tag{56}$$

The integral of equation (56) is thoroughly analyzed in [20] where series expansions of the Fresnel integral $\tilde{E}(u)$ were used to integrate termwise.

We can summarize the results for Φ_1 and Φ_2 as follows:

$$\Phi_1(\tau_0, \tau; \lambda) = [\Omega_0 - F\, I_0(\tau_0; \rho)]\, I_0(\tau_0, \tau; -\mu) + F\, J_0(\tau_0, \tau; \mu, \rho) \tag{57a}$$
$$\Phi_2(\tau_0, \tau; \lambda) = [\bar{\Omega}_0 - \bar{F}\, \bar{I}_0(\tau_0; \rho)]\, I_0(\tau_0, \tau; -\kappa) + \bar{F}\, J_0(\tau_0, \tau; -\kappa, \rho) \tag{57b}$$

with $\kappa \triangleq \lambda - \rho$. Thus, the complete solution for the small angle problem is given by equations (42), (57), (49), (50) and (56).

The Large Angle Theory

We now demonstrate the development of the large angle theory based on equation (39). The surprising result of the analysis which follows is that no new integrals appear, so that the extension to the large angle theory is, in principle, no more difficult than the original small angle theory.

For the large angle theory, the solution to equation (39) is

$$w = w_0 + w_1 \tag{58}$$

where w_0 is the small angle (zero order) solution from (42) and w_1 is the first order correction term:

$$w_1(\tau) = \frac{\lambda}{2} \exp(-\tfrac{1}{2} i\lambda\tau^2) \int_{\tau_0}^{\tau} \exp(\tfrac{1}{2} i\lambda u^2)\, \bar{\omega}(u) w_0^2(u)\, du \tag{59}$$

For convenience, let us define the integral in equation (59) as

$$H(\tau_0, \tau) \triangleq \int_{\tau_0}^{\tau} \exp(\tfrac{1}{2} i\lambda u^2) h(u)\, du \tag{60}$$

where

$$h(u) = \bar{\omega}(u) w_0^2(u) \tag{61}$$

It is a very difficult task to compute the integral $H(\tau_0, \tau)$ uniformly for large and small values of the spin rate τ. We can therefore split the problem of computing $H(\tau_0, \tau)$ into two parts, viz. the evaluation of $H(\tau_0, \tau)$ for large (absolute) values of the spin rate and the evaluation of $H(\tau_0, \tau)$ for small values of the spin rate. One expects that with these two cases taken care of, one might be able to extend the solution to the extreme case of spinning through zero spin rate, i.e to cases when the limits of integration in (60) pass from a positive spin rate ($\tau_0 > 0$) to a negative spin rate ($\tau < 0$), or vice versa. To analyze such cases we write equation (60) in the following form

$$H(\tau_0, \tau) = \int_{\tau_0}^{\delta} g(u)\, du + \int_{\delta}^{-\delta} g(u)\, du + \int_{-\delta}^{\tau} g(u)\, du = H_1 + H_2 + H_3 \tag{62}$$

where δ is a (small) constant. By this procedure we can find H_1 and H_3 through asymptotic expansions in terms of τ and H_2 by Taylor series expansion about $\tau = 0$ of the integrand.

The problem of specifying the constant δ to obtain the greatest accuracy becomes a numerical issue dictated by the specific problem being analyzed.

The case of the *asymptotic approximation* of $H(\tau_0, \tau)$, or equivalently of H_1 and H_3 in (62), (i.e., when $\tau > \tau_0 > 0$ or $0 > \tau > \tau_0$) can be treated as follows:

First note that the solutions for ω and w_0 can be written in terms of $I_0(\tau_0, \tau; \rho)$ and $\Phi_1(\tau_0, \tau; \lambda)$ as follows:

$$\omega(\tau) = \omega_0 \exp(\tfrac{1}{2} i\rho\tau^2) + \exp(\tfrac{1}{2} i\rho\tau^2)(F/\sqrt{k}) I_0(\tau_0, \tau; \rho) \tag{63}$$

$$w_0(\tau) = w_{00} \exp(-\tfrac{1}{2} i\lambda\tau^2) + (\lambda/2) \exp(-\tfrac{1}{2} i\lambda\tau^2) \Phi(\tau_0, \tau; \lambda) \tag{64}$$

where $I_0(\tau_0, \tau; \lambda)$ was defined in (35) and $\Phi(\tau_0, \tau; \lambda)$ is now defined as

$$\Phi(\tau_0, \tau; \lambda) \triangleq \int_{\tau_0}^{\tau} \exp(\tfrac{1}{2} i\lambda u^2) \omega(u)\, du \tag{65}$$

Note that, without loss of generality, we assume here only the case of an axisymmetric rigid body since the methodology can be easily generalized to the nonsymmetric case, although the algebra quickly becomes very cumbersome in this case.

The asymptotic expansion of $\bar{I}_0(\tau_0, \tau; \lambda)$ (keeping just the first term) is

$$\bar{I}_0(\tau_0, \tau; \lambda) \sim (i/\rho) \left[\exp(\tfrac{1}{2} i\rho\tau_0^2)/\tau_0 - \exp(\tfrac{1}{2} i\rho\tau^2)/\tau\right] = a_0 + a_1 R_1(\tau; \rho) \tag{66}$$

where

$$R_1(\tau; \rho) \triangleq \exp(\tfrac{1}{2} i\rho\tau^2)/\tau \tag{67}$$

and

$$a_0 \triangleq i \exp(\tfrac{1}{2} i\rho\tau_0^2)/\rho\tau_0, \qquad a_1 \triangleq -i/\rho \tag{68}$$

Similarly, using (66) the asymptotic expansion for $\Phi(\tau_0, \tau; \lambda)$ becomes

$$\Phi(\tau_0, \tau; \lambda) \sim b_0 + b_1 R_1(\tau; \mu) + b_2 R_2(\tau; \lambda) \tag{69}$$

where

$$R_2(\tau; \lambda) \triangleq \exp(\tfrac{1}{2} i\rho\tau^2)/\tau^2 \tag{70}$$

and

$$b_0 \triangleq \frac{F}{\sqrt{k}} \left(\frac{i}{2\rho} \tilde{E}i(\frac{\lambda}{2}\tau_0^2) + \frac{\exp(\frac{1}{2} i\lambda\tau_0^2)}{\rho\mu\tau_0^2} \right) + \frac{\omega_0}{\sqrt{k}} \frac{i}{\mu} \frac{\exp(\frac{1}{2} i\mu\tau_0^2)}{\tau_0} \tag{71a}$$

$$b_1 \triangleq -\frac{F}{\sqrt{k}} \frac{\exp(-\frac{1}{2} i\rho\tau_0^2)}{\rho\mu\tau_0} - \frac{\omega_0}{\sqrt{k}} \frac{i}{\mu} \tag{71b}$$

$$b_2 \triangleq \frac{F}{\rho\lambda\sqrt{k}} \tag{71c}$$

where $\tilde{E}i(\lambda\tau_0^2/2)$ is defined by

$$\tilde{E}i(\frac{\lambda}{2}\tau_0^2) = \begin{cases} Ei(\frac{\lambda}{2}\tau_0^2) & \text{for } \lambda > 0 \\ \bar{E}i(-\frac{\lambda}{2}\tau_0^2) & \text{for } \lambda < 0 \end{cases} \tag{72}$$

and where

$$Ei(x) \triangleq \int_x^{\infty} \frac{e^{iu}}{u}\, du \tag{73}$$

is in essence the *exponential integral* [34].

Substituting (63) and (64) into (61) and after collecting terms, one has for $H(\tau_0, \tau)$ that

$$H(\tau_0, \tau) = \sum_{i=1}^{6} r_i G_i(\tau_0, \tau) \tag{74}$$

where

$$\begin{aligned} r_1 &\triangleq (\bar{F}/\sqrt{k})(\lambda/2)^2, \qquad r_2 \triangleq \bar{\omega}_0(\lambda/2)^2, \qquad r_3 \triangleq \bar{\omega}_0 \lambda w_{00} && (75a) \\ r_4 &\triangleq \bar{F} w_{00}^2/\sqrt{k}, \qquad r_5 \triangleq \bar{\omega}_0 w_{00}^2, \qquad r_6 \triangleq \bar{F}\lambda w_{00}/\sqrt{k} && (75b) \end{aligned}$$

and where

$$\begin{aligned} G_1(\tau_0, \tau) &\triangleq \int_{\tau_0}^{\tau} \exp(-\tfrac{1}{2} i\mu u^2) \bar{I}_0(\tau_0, u; \lambda) \Phi^2(\tau_0, u; \lambda)\, du && (76a) \\ G_2(\tau_0, \tau) &\triangleq \int_{\tau_0}^{\tau} \exp(-\tfrac{1}{2} i\mu u^2) \Phi^2(\tau_0, u; \lambda)\, du && (76b) \\ G_3(\tau_0, \tau) &\triangleq \int_{\tau_0}^{\tau} \exp(-\tfrac{1}{2} i\mu u^2) \Phi(\tau_0, u; \lambda)\, du && (76c) \\ G_4(\tau_0, \tau) &\triangleq \int_{\tau_0}^{\tau} \exp(-\tfrac{1}{2} i\mu u^2) \bar{I}_0(\tau_0, u; \lambda)\, du && (76d) \\ G_5(\tau_0, \tau) &\triangleq \int_{\tau_0}^{\tau} \exp(-\tfrac{1}{2} i\mu u^2)\, du && (76e) \\ G_6(\tau_0, \tau) &\triangleq \int_{\tau_0}^{\tau} \exp(-\tfrac{1}{2} i\mu u^2) \bar{I}_0(\tau_0, u; \lambda) \Phi(\tau_0, u; \lambda)\, du && (76f) \end{aligned}$$

Using the expressions (66) and (69) we first calculate the integral $G_1(\tau_0, \tau)$ in (76) as follows:

$$G_1(\tau_0, \tau) = \sum_{i=1}^{12} p_{1,i} Q_{1,i}(\tau_0, \tau) \tag{77}$$

where

$$\begin{aligned} p_{1,1} &= a_0 b_0^2, \qquad p_{1,2} = a_0 b_1^2, \qquad p_{1,3} = a_0 b_2^2 && (78a) \\ p_{1,4} &= 2a_0 b_0 b_1, \qquad p_{1,5} = 2a_0 b_0 b_2, \qquad p_{1,6} = 2a_0 b_1 b_2 && (78b) \\ p_{1,7} &= a_1 b_0^2, \qquad p_{1,8} = a_1 b_1^2, \qquad p_{1,9} = a_1 b_2^2 && (78c) \\ p_{1,10} &= 2a_1 b_0 b_1, \qquad p_{1,11} = 2a_1 b_0 b_2, \qquad p_{1,12} = 2a_1 b_1 b_2 && (78d) \end{aligned}$$

and where

$$\begin{aligned} Q_{1,1}(\tau_0, \tau) &= \int_{\tau_0}^{\tau} \exp(-\tfrac{1}{2} i\mu u^2)\, du = I_0(\tau_0, \tau; \mu) && (79a) \\ Q_{1,2}(\tau_0, \tau) &= \int_{\tau_0}^{\tau} \exp(\tfrac{1}{2} i\mu u^2)/u^2\, du && (79b) \\ Q_{1,3}(\tau_0, \tau) &= \int_{\tau_0}^{\tau} \exp[\tfrac{1}{2} i(\lambda - \rho) u^2]/u^4\, du && (79c) \\ Q_{1,4}(\tau_0, \tau) &= \ln(|\tau|) - \ln(|\tau_0|) && (79d) \\ Q_{1,5}(\tau_0, \tau) &= \int_{\tau_0}^{\tau} \exp(-\tfrac{1}{2} i\rho u^2)/u^2\, du && (79e) \end{aligned}$$

$$Q_{1,6}(\tau_0,\tau) = \int_{\tau_0}^{\tau} \exp(\tfrac{1}{2}i\lambda u^2)/u^3\,du \tag{79f}$$

$$Q_{1,7}(\tau_0,\tau) = \int_{\tau_0}^{\tau} \exp(-\tfrac{1}{2}i\lambda u^2)/u\,du \tag{79g}$$

$$Q_{1,8}(\tau_0,\tau) = \int_{\tau_0}^{\tau} \exp[\tfrac{1}{2}i(\mu+\rho)u^2]/u^3\,du \tag{79h}$$

$$Q_{1,9}(\tau_0,\tau) = \int_{\tau_0}^{\tau} \exp(\tfrac{1}{2}i\lambda u^2)/u^5\,du \tag{79i}$$

$$Q_{1,10}(\tau_0,\tau) = \int_{\tau_0}^{\tau} \exp(\tfrac{1}{2}i\rho u^2)/u^2\,du \tag{79j}$$

$$Q_{1,11}(\tau_0,\tau) = \left(1/\tau_0^2 - 1/\tau^2\right)/2 \tag{79k}$$

$$Q_{1,12}(\tau_0,\tau) = \int_{\tau_0}^{\tau} \exp(\tfrac{1}{2}i\mu u^2)/u^4\,du \tag{79l}$$

Similarly, for $G_i(\tau_0,\tau)$ $(i = 2,...,6)$ one has that

$$G_2(\tau_0,\tau) = \sum_{i=1}^{6} p_{2,i}Q_{2,i}(\tau_0,\tau) \tag{80}$$

where

$$p_{2,1} = b_0^2, \qquad p_{2,2} = b_1^2, \qquad p_{2,3} = b_2^2 \tag{81a}$$

$$p_{2,4} = 2b_0b_1, \qquad p_{2,5} = 2b_0b_2, \qquad p_{2,6} = 2b_1b_2 \tag{81b}$$

and where

$$Q_{2,1}(\tau_0,\tau) = Q_{1,1}(\tau_0,\tau), \qquad Q_{2,2}(\tau_0,\tau) = Q_{1,2}(\tau_0,\tau) \tag{82a}$$

$$Q_{2,3}(\tau_0,\tau) = Q_{1,3}(\tau_0,\tau), \qquad Q_{2,4}(\tau_0,\tau) = Q_{1,4}(\tau_0,\tau) \tag{82b}$$

$$Q_{2,5}(\tau_0,\tau) = Q_{1,5}(\tau_0,\tau), \qquad Q_{2,6}(\tau_0,\tau) = Q_{1,6}(\tau_0,\tau) \tag{82c}$$

(82d)

For $G_3(\tau_0,\tau)$ one has

$$G_3(\tau_0,\tau) = \sum_{i=1}^{3} p_{3,i}Q_{3,i}(\tau_0,\tau) \tag{83}$$

where

$$p_{3,1} = b_0, \qquad p_{3,2} = b_1, \qquad p_{3,3} = b_2 \tag{84}$$

and where

$$Q_{3,1}(\tau_0,\tau) = Q_{1,1}(\tau_0,\tau), \quad Q_{3,2}(\tau_0,\tau) = Q_{1,4}(\tau_0,\tau), \quad Q_{3,3}(\tau_0,\tau) = Q_{1,5}(\tau_0,\tau) \tag{85}$$

For $G_4(\tau_0,\tau)$ one has

$$G_4(\tau_0,\tau) = \sum_{i=1}^{2} p_{4,i}Q_{4,i}(\tau_0,\tau) \tag{86}$$

where

$$p_{4,1} = a_0, \qquad p_{4,2} = a_1 \tag{87}$$

and where

$$Q_{4,1}(\tau_0,\tau) = Q_{1,1}(\tau_0,\tau), \qquad Q_{4,2}(\tau_0,\tau) = Q_{1,7}(\tau_0,\tau) \tag{88}$$

For $G_5(\tau_0,\tau)$ one has that

$$G_5(\tau_0,\tau) = Q_{1,1}(\tau_0,\tau) \tag{89}$$

and finally for $G_6(\tau_0,\tau)$ one has

$$G_6(\tau_0,\tau) = \sum_{i=1}^{6} p_{6,i} Q_{6,i}(\tau_0,\tau) \tag{90}$$

where

$$p_{6,1} = a_0 b_0, \qquad p_{6,2} = a_0 b_1, \qquad p_{6,3} = a_0 b_2 \tag{91a}$$
$$p_{6,4} = a_1 b_0, \qquad p_{6,5} = a_1 b_1, \qquad p_{6,6} = a_1 b_2 \tag{91b}$$

and where

$$Q_{6,1}(\tau_0,\tau) = Q_{1,1}(\tau_0,\tau), \qquad Q_{6,2}(\tau_0,\tau) = Q_{1,4}(\tau_0,\tau) \tag{92a}$$
$$Q_{6,3}(\tau_0,\tau) = Q_{1,5}(\tau_0,\tau), \qquad Q_{6,4}(\tau_0,\tau) = Q_{1,7}(\tau_0,\tau) \tag{92b}$$
$$Q_{6,5}(\tau_0,\tau) = Q_{1,10}(\tau_0,\tau), \qquad Q_{6,6}(\tau_0,\tau) = Q_{1,11}(\tau_0,\tau) \tag{92c}$$

All the integrals appearing in the equations (79) are of the form

$$\int_{\tau_0}^{\tau} \exp(\tfrac{1}{2} iku^2)/u^n \, du \tag{93}$$

Integrals of this form can be calculated using the recurrence formula [20]

$$\begin{aligned}\int_{\tau_0}^{\tau} \exp(\tfrac{1}{2} iku^2)/u^n \, du &= \left[\exp(\tfrac{1}{2} ik\tau_0^2)/\tau_0^{n-1} - \exp(\tfrac{1}{2} ik\tau^2)/\tau^{n-1}\right]/(n-1) \\ &+ i\frac{k}{n-1}\int_{\tau_0}^{\tau} \exp(\tfrac{1}{2} iku^2)/u^{n-2} \, du, \qquad n = 2, 3, \ldots\end{aligned} \tag{94}$$

The first two terms of this sequence can be computed as follows

$$\int_{\tau_0}^{\tau} \exp(\tfrac{1}{2} iku^2) \, du = I_0(\tau_0,\tau;-k) \tag{95}$$

$$\int_{\tau_0}^{\tau} \exp(\tfrac{1}{2} iku^2)/u \, du = \frac{1}{2}\left[\tilde{E}i(\tfrac{1}{2}k\tau_0^2) - \tilde{E}i(\tfrac{1}{2}k\tau^2)\right] \tag{96}$$

We note in passing that the "naive" direct asymptotic approximation of $H(\tau_0,\tau)$ using integration by parts, that is

$$H(\tau_0,\tau) \sim (i/\lambda)\left[\exp(\tfrac{1}{2}\lambda\tau_0^2)h(\tau_0)/\tau_0 - \exp(\tfrac{1}{2}i\lambda\tau^2)h(\tau)/\tau\right] \tag{97}$$

does not yield accurate results, even for moderately large values of the spin rate. One therefore needs to use the asymptotic expansion method described above.

In order to have an accurate approximation for $H(\tau_0,\tau)$ for small spin rates as required in (62), and in particular for cases such that $\tau > 0 > \tau_0$ we develop a Taylor series expansion for $h(\tau)$ about $\tau = 0$:

$$h(\tau) = h(0) + h'(0)\tau + h''(0)\tau^2/2! + \ldots + h^{(n)}(0)\tau^n/n! + R_{n+1} \tag{98}$$

where the remainder is

$$R_{n+1} = h^{(n+1)}(\xi)\tau^{n+1}/(n+1)! \tag{99}$$

and $0 \leq \xi \leq \tau$. In order to find a general formula for the series, we must be able to compute derivatives to any order. In general if a function is written as a product of two other functions

$$h(\tau) = f(\tau)g(\tau) \tag{100}$$

then the n^{th} order derivative of equation (100) is given by

$$h^{(n)}(\tau) = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} f^{(n-k)}(\tau)g^{(k)}(\tau) \tag{101}$$

In our case, since $h(\tau) = \bar{\omega}(\tau)w_0^2(\tau)$, we have to apply equation (101) twice to obtain

$$h^{(n)}(\tau) = (\bar{\omega}(\tau)w_0^2(\tau))^{(n)} = \sum_{k=0}^{n}\sum_{j=0}^{k} \begin{pmatrix} n \\ k \end{pmatrix} \begin{pmatrix} k \\ j \end{pmatrix} \bar{\omega}^{(n-k)}(\tau)w_0^{(k-j)}(\tau)w_0^{(j)}(\tau) \tag{102}$$

Next we seek general expressions for the derivatives of $\bar{\omega}$ and w_0. We note from equation (46) that

$$\bar{\omega}^{(n)} = k_1\bar{\Omega}^{(n)} + k_2\Omega^{(n)} \tag{103}$$

So we need to find the derivatives of Ω. ¿From equation (32) we have

$$\Omega'(\tau) = i\rho\tau\Omega(\tau) + F \tag{104}$$

The next two derivatives of equation (104) provide

$$\Omega''(\tau) = i\rho\tau\Omega'(\tau) + i\rho\Omega(\tau)\Omega'''(\tau) = i\rho\tau\Omega''(\tau) + 2i\rho\Omega'(\tau) \tag{105}$$

and we deduce the rule

$$\Omega^{(n)}(\tau) = i\rho\tau\Omega^{(n-1)}(\tau) + (n-1)i\rho\Omega^{(n-2)}(\tau), \qquad n = 2, 3, 4, ... \tag{106}$$

which is easily proven by induction. Since in our case the Taylor series expansion is about $\tau = 0$, we have from equations (104)-(106):

$$\Omega'(0) = F \tag{107a}$$

$$\Omega^{(n)}(0) = (n-1)i\,\rho\Omega^{(n-2)}(0), \qquad n = 2, 3, 4, \ldots \tag{107b}$$

To find expressions for $w_0^{(n)}$ we use equation (41):

$$w_0'(\tau) = -i\lambda\tau w_0(\tau) + \frac{\lambda}{2}\omega(\tau) \tag{108}$$

Taking the next two derivatives of equation (108) one obtains

$$w_0''(\tau) = -i\lambda w_0(\tau) - i\lambda\tau w_0'(\tau) + \frac{\lambda}{2}\omega'(\tau) \tag{109}$$

$$w_0'''(\tau) = -2i\lambda w_0'(\tau) - i\lambda\tau w_0''(\tau) + \frac{\lambda}{2}\omega''(\tau) \tag{110}$$

¿From equations (108) and (109) one can easily deduce the recurrence formula

$$w_0^{(n)}(\tau) = -(n-1)i\lambda w_0^{(n-2)}(\tau) - i\lambda\tau w_0^{(n-1)}(\tau) + \frac{\lambda}{2}\omega^{(n-1)}(\tau), \qquad n = 2, 3, 4, \ldots \quad (111)$$

which is also confirmed by induction. For $\tau = 0$ we have

$$w_0'(0) = \frac{\lambda}{2}\omega(0) \quad (112a)$$

$$w_0^{(n)}(0) = -(n-1)i\lambda w_0^{(n-2)}(0) + \frac{\lambda}{2}\omega^{(n-1)}(0), \qquad n = 2, 3, 4, \ldots \quad (112b)$$

Finally, since the Taylor series expansion of $\bar{\omega} w_0^2$ leads to a series of terms in u^n, we need the recurrence relation [22] for the integral (48):

$$I_n(x;\rho) = i\,\frac{x^{n-1}}{\rho}\exp(-\tfrac{1}{2}i\rho x^2) - i\frac{n-1}{\rho} I_{n-2}(x;\rho), \qquad n = 2, 3, 4, \ldots \quad (113)$$

Equation (49) provides $I_0(x;\rho)$ and for $I_1(x;\rho)$ we have [22]

$$I_1(x;\rho) \triangleq \int_0^x \exp(-\tfrac{1}{2}i\rho u^2)\, u\, du = (i/\rho)[\exp(-\tfrac{1}{2}i\rho x^2) - 1] \quad (114)$$

Thus, for small arguments the integral $H(\tau_0, \tau)$ of equation (60) can be computed to any order through equations (98), (102), (103), (107), (49) and (112)-(114). Notice that these expressions are not restricted to the symmetric case (as was assumed for the asymptotic expansions for the sake of simplicity), but include the nonsymmetric case as well.

The large angle solution given by equations (58), (42) and (59) can then be explicitly calculated in terms of asymptotic or Taylor series expansions. We note that the new terms introduced in the large angle theory do not involve any new integrals that were not required in the small angle theory. This fact could have significant implications in the extensions of all previous analytic solutions which were based on small angle assumptions.

Numerical Results

In this section we present a numerical example to demonstrate the previous methodology. We pick a very difficult case in order to exaggerate any weaknesses in the analytic theory. Specifically, we choose a *spin-down maneuver* in which the spin rate is driven from a positive value to a negative one, while constant body-fixed torques act on all three axes. When zero spin rate is reached, the body loses all of its stabilizing momentum and suddenly, under the influence of the transverse torques, moves through large angular excursions of the spin axis. Any analytic theory which provides accurate results for this extreme case would be more than adequate for the vast majority of practical applications.

A Galileo-type spacecraft will serve as an example [19] where representative values for the mass properties are taken as

$$I_x = 2854\ kg\, m^2, \qquad I_y = 2854\ kg\, m^2, \qquad I_z = 4183\ kg\, m^2 \quad (115)$$

For the spin-down maneuver, the body-fixed torques are assumed to be

$$M_x = 2\ Nm, \qquad M_y = 2\ Nm, \qquad M_z = -13.5\ Nm \quad (116)$$

and the initial conditions are

$$\omega_z(0) = 0.3\ r/s, \qquad \omega_x(0) = \omega_y(0) = 0, \qquad \beta_x(0) = \beta_y(0) = \beta_z(0) = 0 \tag{117}$$

For the final condition we assume

$$\omega_z(t_f) = -0.3\ r/s \tag{118}$$

Naturally condition (118) represents a hypothetical case, as no engineer at the Jet Propulsion Laboratory would wish to see the spacecraft spin-down through zero spin rate, since not only would attitude control be lost, but the retro-propulsion module, which depends on centripetal acceleration for operation, would fail. Such a spin-down maneuver represents a catastrophic event.

However, in the interest of theoretical dynamics, we pursue our example. In Figs. 1-5 the solid line represents the exact solution which is found by precise numerical integration of equations (1) and (27). In Figs. 1 and 3 the dotted line represents the analytic solution based the small angle assumption (linear solution). The analytic solution for β_x and β_y is found by analytic integration of equations (32) and (4), namely equation (42). In other words, the analytic solution in Figs. 1 and 3 corresponds to the analytic integration of the linearized system of equations (1) when a small angle assumption for β_x and β_y is imposed. In Fig. 1 we see that after zero spin rate is reached (at approximately 100 seconds) the value of β_x suddenly rises and subsequently exceeds a magnitude of 1 radian (60 deg). The behavior of β_y is similar and is shown in Fig. 3. At such large amplitudes, the nonlinear behavior becomes apparent and we see a significant phase shift between the analytic versus the exact solutions.

The dotted lines in Figs. 2, 4 and 5 represent the results for the large angle solution where only the first correction term (59) is added to the zero order solution, as in equation (58). The approximate analytic solution for β_z is, from (3c) and (28),

$$\beta_z(t) = \int_{t_0}^{t} \omega_z(\tau)\, d\tau = (M_z/I_z)\, t^2/2 + \omega_{z0}\, t + \beta_{z0} \tag{119}$$

Because of the very difficult example problem we have selected, the large variation of the argument τ from large values through zero presents special numerical challenges. For the chosen values, it was found that 10 terms in the Taylor series expansion (98) are required in order to provide an accurate approximation for the integral $H(\tau_0, \tau)$ of equation (60) near $\tau = 0$. The value of δ in (62) is

$$\delta \approx 0.1\ r/s \tag{120}$$

Conclusion

The large angle theory developed in this paper provides significant improvement over the small angle theory, without the requirement of solving any new integrals. The quadratic kinematic equations lead to an important breakthrough for the extension of all previous analytic solutions based on small angle linearization. A numerical simulation for the Riccati kinematic equation shows that it can be very helpful in very accurately describing the motion of a spinning spacecraft during a spin-down maneuver.

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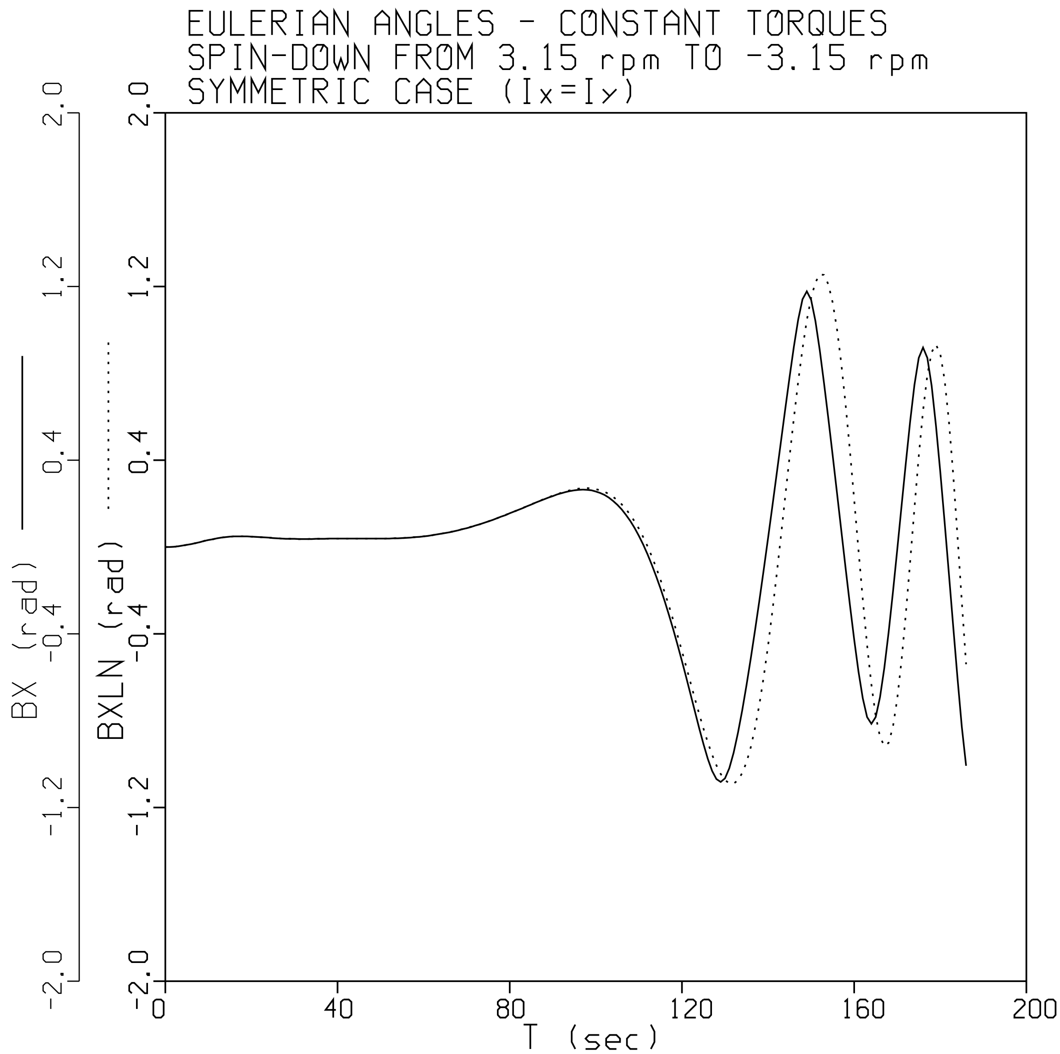

Figure 1: Exact and Analytic Solutions for β_x (small angle theory).

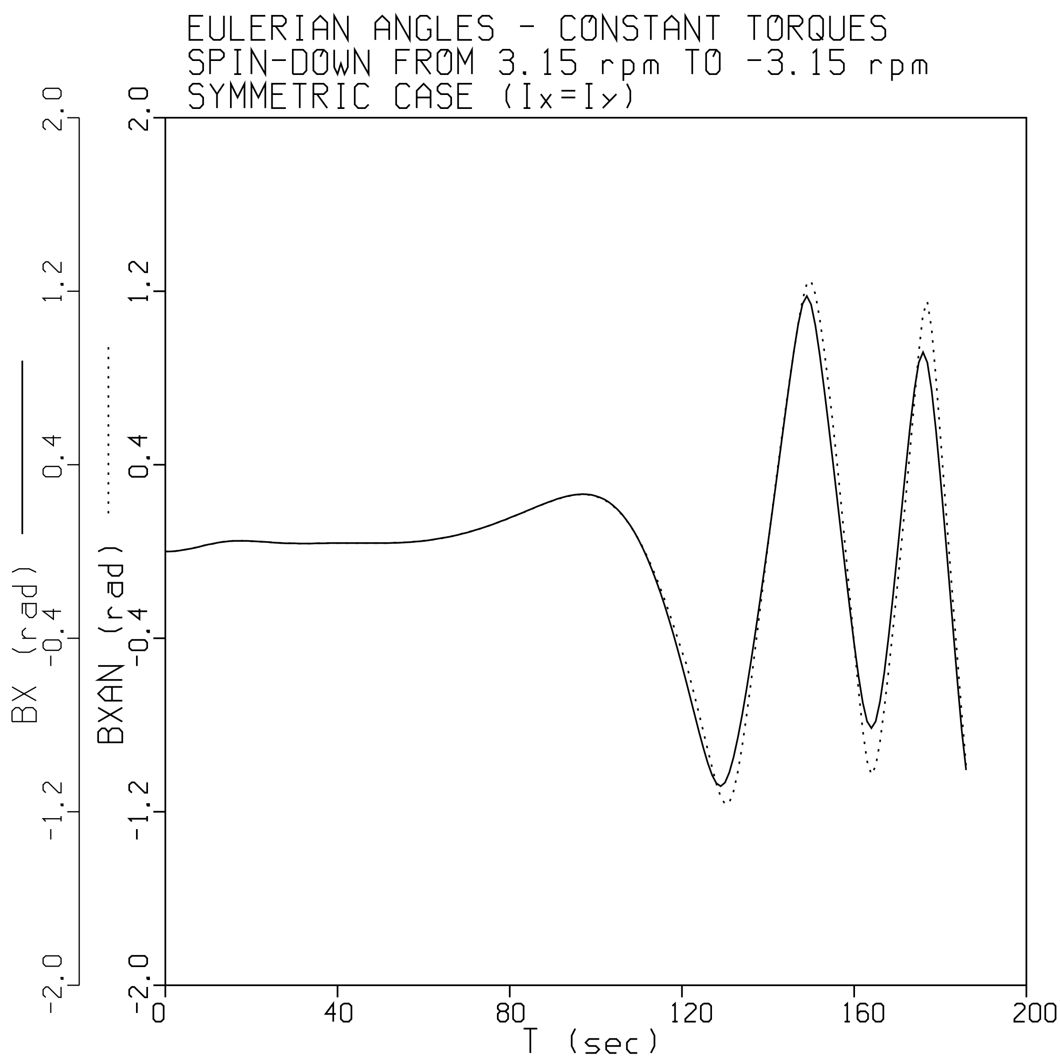

Figure 2: Exact and Analytic Solutions for β_x (large angle theory).

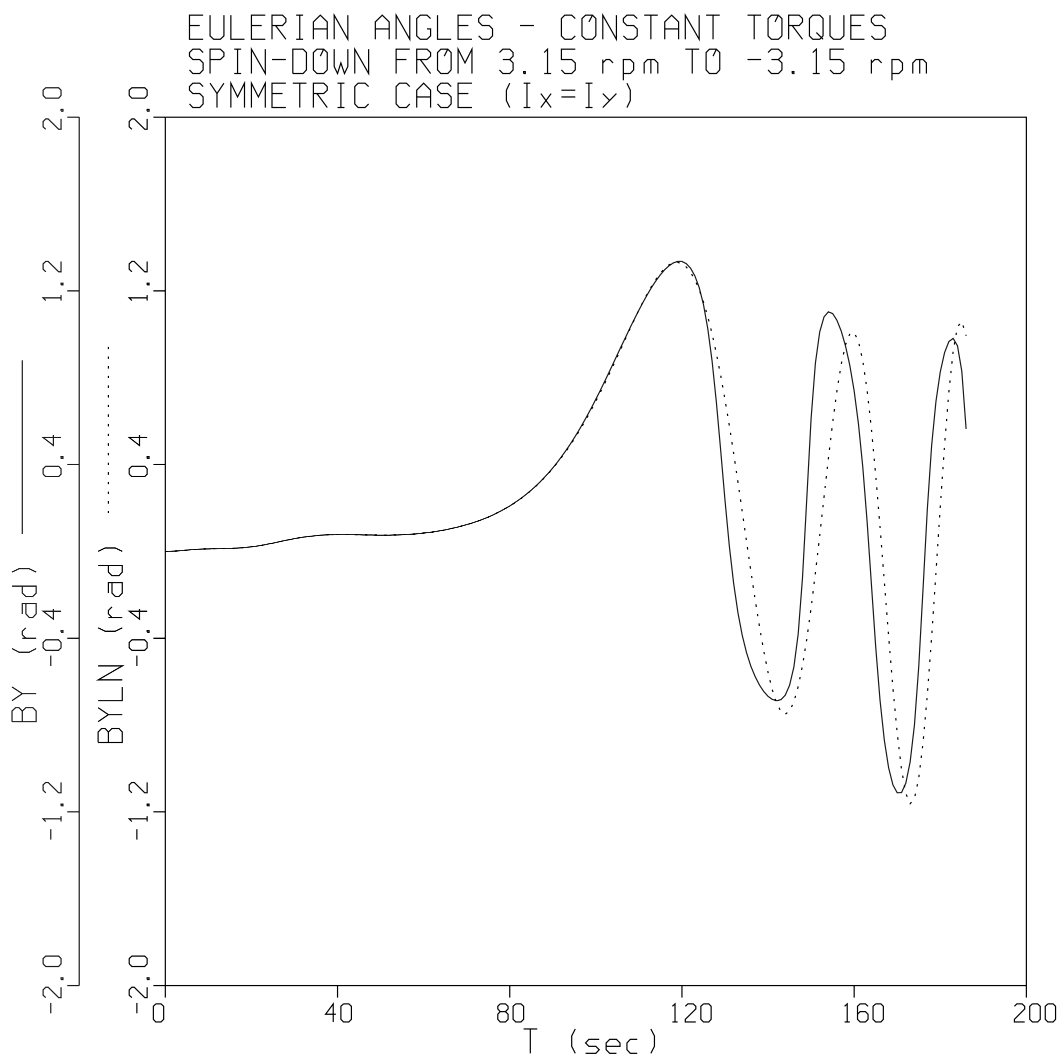

Figure 3: Exact and Analytic Solutions for β_y (small angle theory).

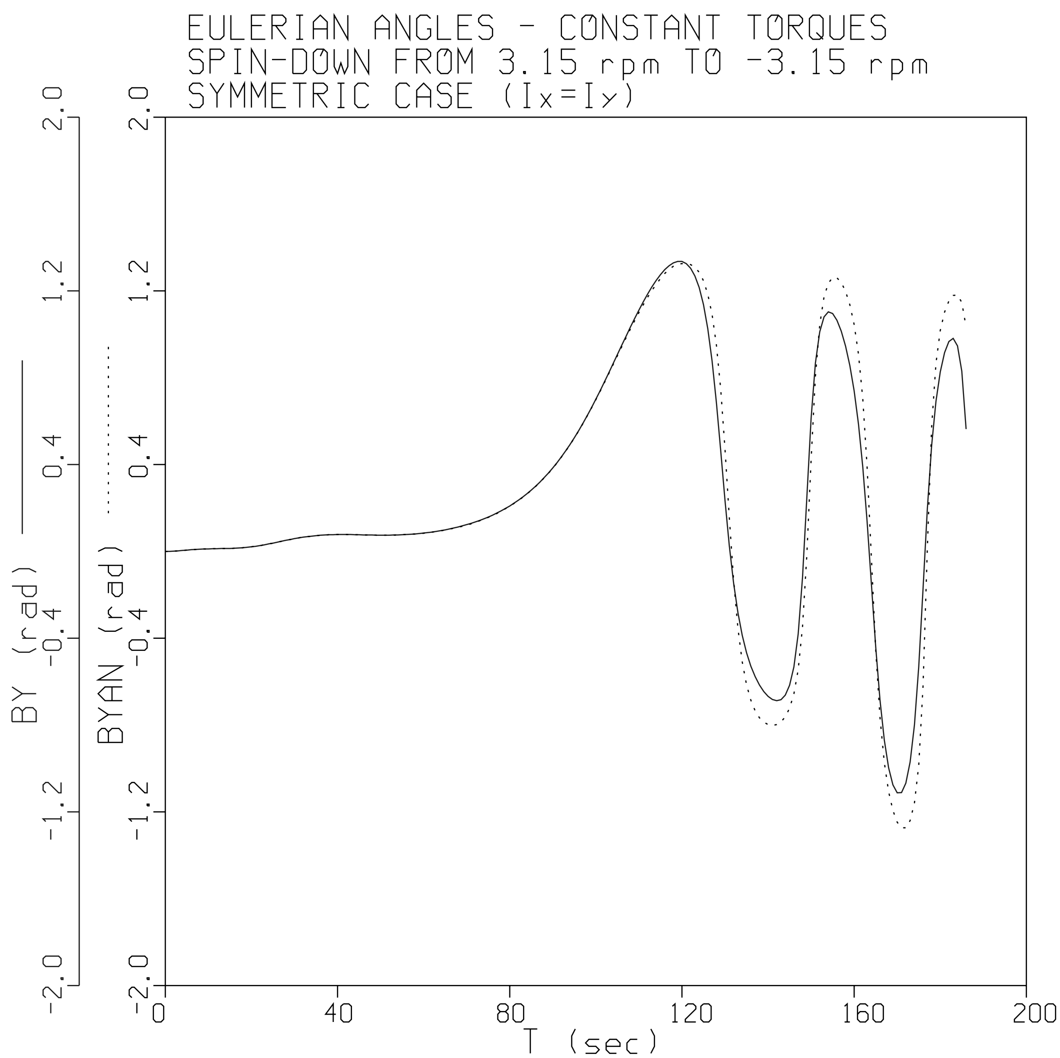

Figure 4: Exact and Analytic Solutions for β_y (large angle theory).

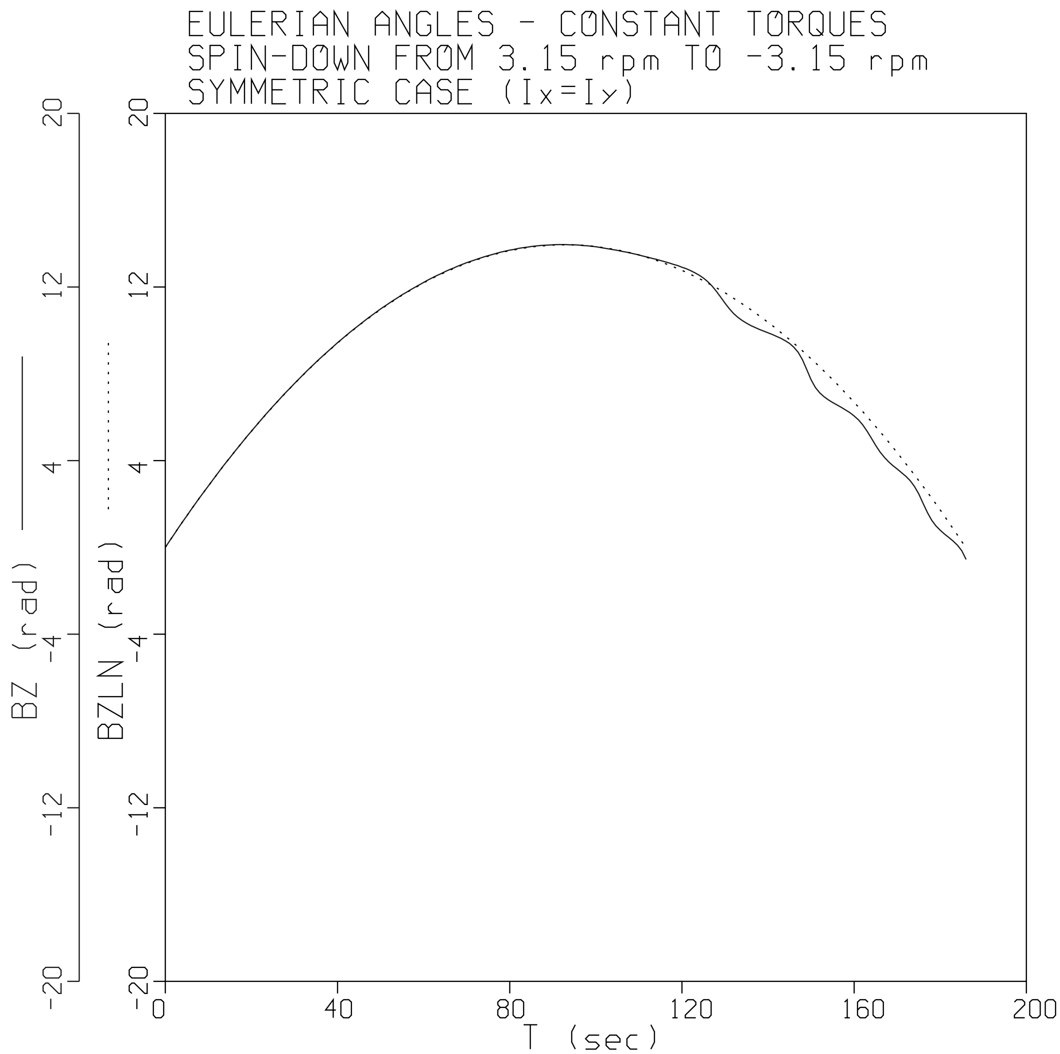

Figure 5: Exact and Analytic Solutions for β_z.