HOHMANN-HOHMANN AND HOHMANN-PHASING COOPERATIVE RENDEZVOUS MANEUVERS

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ABSTRACT

We consider the problem of cooperative rendezvous between two satellites in circular orbits, given a fixed time for the rendezvous to be completed, and assuming a circular rendezvous orbit. We investigate two types of cooperative maneuvers for which analytical solutions can be obtained. One is the case of two Hohmann transfers, henceforth referred to as HHCM, while the other, henceforth referred to as HPCM, is the case of a Hohmann transfer and a phasing maneuver. For the latter case we derive conditions on the phasing angle that makes a HPCM rendezvous cheaper than a cooperative rendezvous on an orbit that is different than either the original orbits of the two participating satellites. It is shown that minimizing the fuel expenditure is equivalent to minimizing a weighted sum of the ΔVs of the two orbital transfers, the weights being determined by the mass and engine characteristics of the satellites. Our results show that, if the time of rendezvous allows for a Hohmann transfer between the orbits of the satellites, the optimal rendezvous is either a non-cooperative Hohmann transfer or a Hohmann-phasing cooperative maneuver. In both these cases, the maneuver costs are determined analytically. A numerical example verifies these observations. Finally, we demonstrate the utility of this study for Peer-to-Peer (P2P) refueling of satellites residing in two different circular orbits.

INTRODUCTION

The problem of fixed-time impulsive orbital transfers has been studied extensively for a long time. Methods developed for determining multi-impulse solutions are primarily based on Lawden's primer vector theory.[1] Lion and Handelsman[2] applied calculus of variations to obtain first-order conditions for the optimal addition of an impulse along the trajectory, and for the inclusion of initial and final coasting periods. Primer vector theory has also been applied to determine multiple-impulse fixed-time solutions to rendezvous between two vehicles in circular orbits.[3] The Clohessy-Wiltshire (C-W) equations[4] have also been used in the literature to obtain minimum-fuel, multiple-impulse orbital trajectories. Primer vector theory has also been applied to the C-W equations.[5] The computation of the minimum-ΔV, two-impulse orbital transfer between coplanar circular orbits is essentially the well-known Lambert's

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problem.[4] The multiple revolution solutions to the Lambert's problem, in which the vehicle can complete several revolutions in the transfer orbit, have been studied in Ref. 6, where it has been shown that if the number of maximum possible revolutions is $N_{\max}$, then the optimal solution is determined by exhaustively investigating a set of $(2N_{\max} + 1)$ candidate minima.[6] Further, studies have led to an algorithm that determines the optimal solution by investigating at most two of the $(2N_{\max} + 1)$ candidate minima.[7]

Although most of the studies in the literature focus on active-passive (non-cooperative) rendezvous, there also exist works that consider active-active (cooperative) rendezvous. The earliest works on cooperative rendezvous considered systems with linear or non-linear dynamics with various performance indices.[8,9] The idea of using differential games to study cooperative rendezvous problems has also been discussed in the literature.[10] The optimal terminal maneuver of two active satellites engaged in a cooperative impulsive rendezvous has been studied in Ref. 11. Determination of the optimal terminal maneuver involves the optimization of the common velocity vector after the rendezvous. Methods to determine optimal time-fixed impulsive cooperative rendezvous using primer vector theory have been developed in Ref. 12. These accommodate cases of fuel-constraints on the satellites, and enable the addition of a mid-course impulse to the trajectory of the vehicle. In Ref. 12, examples show that a non-cooperative solution is cheaper once the time allotted for the rendezvous between the satellites is large enough for a Hohmann transfer to be feasible. Hence, cooperative rendezvous is advantageous when the time allotted for the maneuver is relatively short. However, Ref. 12 did not provide any general characterization of the optimal cooperative solutions.

The minimum-fuel rendezvous of two *power-limited* spacecraft (a spacecraft whose engine power has an upper bound) has also been studied using both non-linear and C-W equations.[13,14] For two power-limited spacecraft of similar mass engaging in a rendezvous, cooperative rendezvous is always found to be cheaper than a non-cooperative rendezvous. Constrained and unconstrained circular terminal orbits have also been analyzed in Ref. 13, where it has been found that the cooperative solution still remains the cheaper option. Analytical solutions using the C-W equations can be used to predict the nature of the terminal orbit of the rendezvous. For instance, for the case of a cooperative rendezvous between two satellites in a circular orbit, the two satellites meet in an orbital slot that is mid-way between the original slots, each satellite essentially removing half of the phase angle.[13]

The determination of optimal cooperative rendezvous maneuvers requires, in general, the solution of a non-linear programming (NLP) problem. Solving an NLP is computationally intensive and there can also be issues with the convergence to a local minimum rather than the global minimum. This may not be an issue if we are interested in a single cooperative rendezvous. However, for problems involving multiple satellites, such as peer-to-peer (P2P) satellite refueling,[15–19] numerous cooperative rendezvous need to be computed as part of the overall combinatorial optimization problem. As a result, the time required to obtain the solution for each rendezvous does become an important factor that hinders the applicability of the approach for the overall P2P problem.

Figure 1 depicts a constellation involving two coplanar circular orbits, in the context of a P2P refueling problem: the fuel-sufficient satellites $s_1, s_2, \ldots, s_8$ in the inner orbit have large amounts of fuel, while the fuel-deficient satellites $s_9, s_{10}, \ldots, s_{16}$ on the outer orbit have small amounts of fuel. The fuel-sufficient and fuel-deficient satellites may rendezvous with each other to exchange fuel, thus redistributing the fuel among all satellites in the constellation. In such a case of P2P refueling,[15] cooperative rendezvous may help in bringing down the fuel expenditure incurred during the ensuing orbital transfers. Efficient determination of the optimal cooperative rendezvous maneuvers is therefore crucial in the overall optimization scheme. We therefore wish to minimize the computations, even at the expense of obtaining suboptimal solutions. Thus, in this work we focus on cooperative rendezvous for which analytic solutions are easy to obtain. Our approach is justified because P2P refueling is a discrete optimization problem; even if the numerical values of the costs associated with the decision variables are not exact, the optimal matching between satellites for refueling most likely will not change.

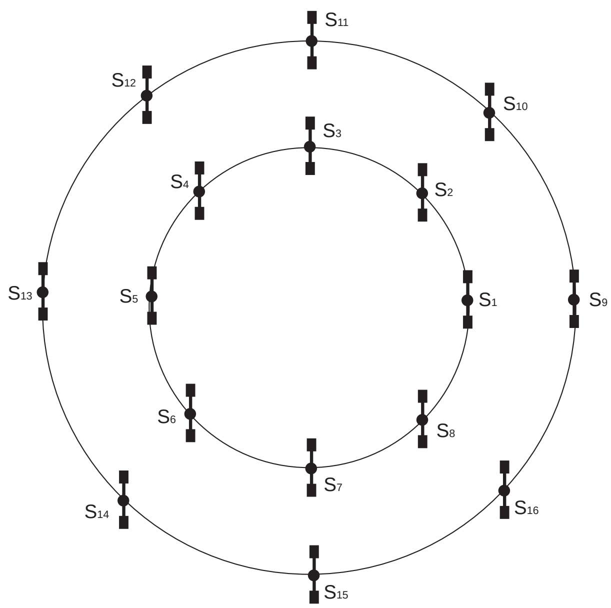


Figure 1. Peer-to-Peer (P2P) refueling scenario.

The previous observations provide us with the main motivation in this paper, namely the study of cooperative rendezvous maneuvers that can be solved analytically or semi-analytically (instead of numerically). In addition, we wish to determine the conditions under which these maneuvers are optimal. For this purpose, we consider a restricted case of the general cooperative rendezvous problem, by constraining the terminal orbit of the rendezvous to be circular. For the problem under consideration, we illustrate the advantages of a cooperative rendezvous (in terms of lower ΔV) over its non-cooperative counterpart, when the time to complete the maneuver is smaller than that required for non-cooperative Hohmann transfers.

The primary contribution of this paper is a characterization of the solutions when the allotted time for each maneuver is not sufficient for non-cooperative Hohmann transfers. In the following sections, we discuss the problem, describe those cooperative maneuvers that can be determined analytically, and investigate the conditions under which they are optimal. We present a numerical study to illustrate these results. Finally, we also demonstrate the benefits of cooperative maneuvers in the context of P2P refueling of a satellite constellation in two different coplanar circular orbits.

PROBLEM DESCRIPTION

We discretize an orbit of radius r into a set of orbital slots Φ_r equally spaced along the orbit. Let $\mathcal{I}$ denote the set of indices for these slots. Let us consider two satellites s_μ and s_ν occupying the orbital slots ϕ_i and ϕ_j in the circular orbits of radius r_i and r_o respectively. Let the initial separation angle between these satellites be θ_0. Now consider an orbital slot $\phi_{k_r} \in \Phi_r$ on the orbit of radius r where a cooperative rendezvous takes place, where $k_r \in \mathcal{I}$. The situation is depicted in Fig. 2. Finally, let the time allotted for a cooperative rendezvous between the two satellites be given by T, and let also the velocity change required for an orbital transfer from slot ϕ_i to slot ϕ_{k_r} be denoted by ΔV_{ik_r}, and the velocity change required for an orbital transfer from slot ϕ_j to slot ϕ_{k_r} be denoted by ΔV_{jk_r}. The total velocity change required for a cooperative rendezvous in which the satellites meet at slot $\phi_{k_r} \in \Phi_r$ is denoted by

$$\Delta V_{ij}^c|_{k_r} = \Delta V_{ik_r} + \Delta V_{jk_r}. \tag{1}$$

This is the total velocity change required for a cooperative rendezvous between the two satellites. Had the satellites

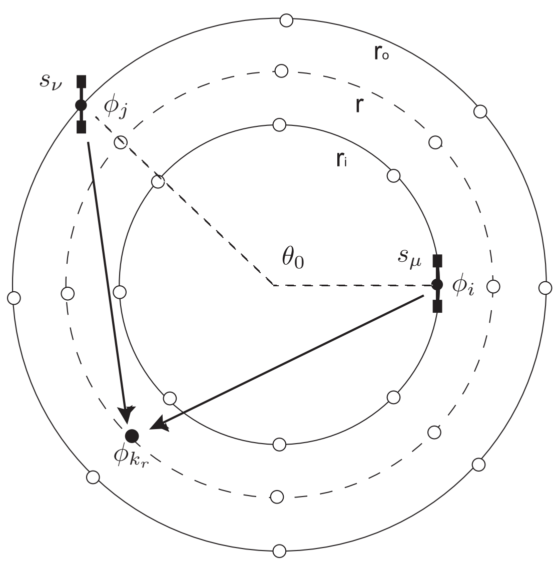


Figure 2. Cooperative rendezvous for the case $r_i \leq r \leq r_o$.

been involved in a non-cooperative rendezvous, then the total velocity change required to complete the rendezvous would be ΔV_{ij} if satellite s_μ were active, and ΔV_{ji} if satellite s_ν were active. Hence, the cases $\phi_{k_{r_i}} = \phi_i$ and $\phi_{k_{r_j}} = \phi_j$ correspond to the two cases of non-cooperative rendezvous. Note that if the cooperative rendezvous takes place on one of the two orbits r_i or r_o, then one of the two orbital maneuvers for the cooperative rendezvous is essentially a phasing maneuver. By phasing maneuver we mean that the first and second impulses occur tangentially to the orbit, and at the same position in the transfer orbit of the maneuvering satellite. The first impulse places the satellite in an higher/lower orbit, while the second impulse places the satellite in the original orbit.

Figure 3 shows the variation of the non-dimensional ΔV with respect to the transfer time when an orbital transfer takes place in the same orbit. With the allowance of coasting, ΔV is a non-increasing function of time, with ΔV-

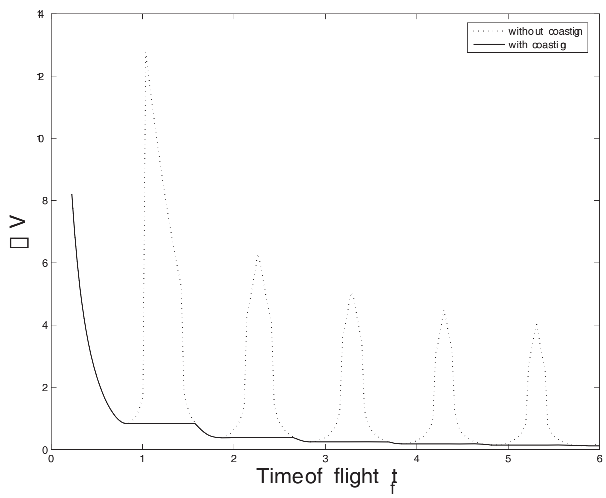


Figure 3. Non-dimensional ΔV ($r_1 = r_2 = 1$, $\theta_0 = 60\,\mathrm{deg}$.

invariant intervals denoting phasing maneuvers. Since the intervals in which ΔV decrease become negligible with increasing time of transfer, the optimal transfers are phasing maneuvers for large times of flight. For such phasing maneuvers, analytical expressions for ΔV are known. Furthermore, if the transfer of the second spacecraft is a Hohmann transfer, then we have an analytic expression for $\Delta V_{ij}^c|_{k_r}$ as well. Similarly, for rendezvous on an intermediate orbit, if both the orbital transfers are Hohmann transfers, then we also have an analytic expression for $\Delta V_{ij}^c|_{k_r}$.

One of the slots in Φ_r results in the cheapest cooperative maneuver between any two satellites meeting on the orbit of radius r. Let us denote this slot by $\phi_c(r)$ and the corresponding total velocity change by $\Delta V_c(r)$. We therefore have

$$\Delta V_c(r) \triangleq \min_{\phi_{k_r} \in \Phi_r} \Delta V_{ij}^c|_{k_r}, \tag{2}$$

and

$$\phi_c(r) \triangleq \arg \min_{\phi_{k_r} \in \Phi_r} \Delta V_{ij}^c|_{k_r}. \tag{3}$$

Assume now that the optimal cooperative rendezvous involving satellites s_μ and s_ν takes place in the orbit of radius $r_{\min}$ and at the orbital slot $\phi_{c,\min}$. This corresponds to the lowest ΔV over all possible orbits and all possible slots.

We also let the corresponding optimal velocity change be $\Delta V_{c,\min}$. We therefore have,

$$\Delta V_{c,\min} \triangleq \min_{r} \Delta V_c(r) \tag{4}$$

and

$$\phi_{c,\min} \triangleq \phi_c(r_{\min}), \quad \text{where} \quad r_{\min} = \arg\min_{r} \Delta V_c(r). \tag{5}$$

HOHMANN-HOHMANN COOPERATIVE MANEUVERS (HHCM)

The first cooperative maneuver involves two Hohmann transfers, that is, both satellites perform Hohmann transfers to complete the cooperative rendezvous. Before we look at the details of the HHCM, let us first consider a single Hohmann transfer.

Hohmann Transfers

For a Hohmann transfer from an orbit of radius r_1 to an orbit of radius r_2 (where, for simplicity, we may assume that $r_1 < r_2$), the semi-major axis of the transfer orbit is given by $a = (r_1 + r_2)/2$, so that the velocity change corresponding to the first impulse is given by

$$\Delta v_1 = \sqrt{2\mu\left(\frac{1}{r_1} - \frac{1}{r_1 + r_2}\right)} - \sqrt{\frac{\mu}{r_1}}, \tag{6}$$

and the velocity change corresponding to the second impulse is given by

$$\Delta v_2 = \sqrt{\frac{\mu}{r_2}} - \sqrt{2\mu\left(\frac{1}{r_2} - \frac{1}{r_1 + r_2}\right)}. \tag{7}$$

Using the above expressions, we have that the total ΔV requirement for the Hohmann transfer is

$$\Delta V^H = \left(\sqrt{\frac{\mu}{r_2}} - \sqrt{\frac{\mu}{r_1}}\right) + \sqrt{2\mu}\left(\sqrt{\frac{1}{r_1} - \frac{1}{r_1 + r_2}} - \sqrt{\frac{1}{r_2} - \frac{1}{r_1 + r_2}}\right). \tag{8}$$

If $r_1 = r_i$ and $r_2 = r_o$, this represents the cost ΔV_{nc}^H of a non-cooperative Hohmann transfer between s_μ and s_ν where s_μ is the active satellite. A similar expression holds when satellite s_ν is active. The available time for the rendezvous and the initial separation angle determines whether a Hohmann transfer is feasible or not. Hence, if a Hohmann transfer from s_μ to s_ν is possible, we have $\Delta V_{ij} = \Delta V_{nc}^H$. By the same token, if a Hohmann transfer from s_ν to s_μ is possible, we have $\Delta V_{ji} = \Delta V_{nc}^H$. Since for Hohmann transfers $\Delta V_{ij} = \Delta V_{ji}$, we finally obtain that $\Delta V_{ij} = \Delta V_{ji} = \Delta V_{nc}^H$.

The phasing angle required for a Hohmann transfer to be feasible is given by[20]

$$\theta_H = \pi\left[1 - \left(\frac{1 + r_1/r_2}{2}\right)^{3/2}\right]. \tag{9}$$

Unless this angle of separation is achieved, the satellite performing the transfer will need to coast for a time τ_H given by[20]

$$\tau_H = \frac{\theta_0 - \theta_H}{2\pi\left(1/T_1 - 1/T_2\right)}, \tag{10}$$

where θ_0 is the initial separation angle, and where $T_s = \left(1/T_1 - 1/T_2\right)^{-1}$ is the synodic period for the orbits concerned, with $T_i = 2\pi\sqrt{r_i^3/\mu}$, for $i = 1, 2$ is the orbital period. Since we are concerned with fixed-time transfers, the maximum time allowed for coasting is given by

$$t_c \leq T - \pi\sqrt{\frac{\left(r_1 + r_2\right)^3}{8\mu}}. \tag{11}$$

Therefore, the separation angle required for a Hohmann transfer should lie between θ_H and $\theta_H + \Delta\theta$, where

$$\Delta\theta = \begin{cases} \dfrac{2\pi}{T_s}\left(T - \pi\sqrt{\dfrac{\left(r_1 + r_2\right)^3}{8\mu}}\right), & \text{if } r_1 < r_2, \\ -\dfrac{2\pi}{T_s}\left(T - \pi\sqrt{\dfrac{\left(r_1 + r_2\right)^3}{8\mu}}\right), & \text{if } r_1 > r_2. \end{cases} \tag{12}$$

A Hohmann transfer is therefore feasible for all separation angles $\theta_0 \in [\theta_H, \theta_H + \Delta\theta_H]$ if $r_1 < r_2$ and $\theta_0 \in [\theta_H + \Delta\theta_H, \theta_H]$ if $r_1 > r_2$. Therefore, all slots on r_i and r_o that satisfy the above condition on the separation angle will allow for a Hohmann transfer to take place within the specified time T.

Optimal HHCM Rendezvous

Let satellites s_μ and s_ν engage in a HHCM rendezvous. We assume that for all orbits of radius r where a cooperative rendezvous can take place, there exists at least one slot $\phi_{k_r} \in \Phi_r$ at which both satellites can perform a Hohmann transfer. Using the expression for the cost of a Hohmann transfer in (8), the total velocity change required for the HHCM rendezvous, when $r_i \leq r \leq r_o$, is given by

$$\begin{aligned} \Delta V_c^H(r) \quad = \quad & \left(\sqrt{\frac{\mu}{r_o}} - \sqrt{\frac{\mu}{r_i}}\right) + \\ & \sqrt{2\mu}\left(\sqrt{\frac{1}{r_i} - \frac{1}{r_i + r}} - \sqrt{\frac{1}{r} - \frac{1}{r_i + r}} + \sqrt{\frac{1}{r} - \frac{1}{r_o + r}} - \sqrt{\frac{1}{r_o} - \frac{1}{r_o + r}}\right). \end{aligned} \tag{13}$$

Taking the derivative of the previous expression with respect to r, we have,

$$\begin{aligned}\sqrt{\frac{2}{\mu}}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta V_c^H\right) &= \left(\frac{1}{r}\right)^2\left[\frac{1}{\sqrt{1/r-1/(r_i+r)}}-\frac{1}{\sqrt{1/r-1/(r_o+r)}}\right]\\ &+ \left(\frac{1}{r_i+r}\right)^2\left[\frac{1}{\sqrt{1/r_i-1/(r_i+r)}}-\frac{1}{\sqrt{1/r-1/(r_i+r)}}\right]\\ &+ \left(\frac{1}{r_o+r}\right)^2\left[\frac{1}{\sqrt{1/r-1/(r_o+r)}}-\frac{1}{\sqrt{1/r_o-1/(r_o+r)}}\right].\end{aligned} \tag{14}$$

By defining the following two parameters

$$\beta_1 = 2\left(r_o+r_i\right)^3, \qquad \beta_2 = r_o\left(r_o+3r_i\right)^2, \tag{15}$$

and by substituting $r = r_i$ in (14), we have

$$\sqrt{\frac{2}{\mu}}\,\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta V_c^H\right)\right]_{r=r_i^+} = \frac{\sqrt{\beta_1}-\sqrt{\beta_2}}{r_i^{3/2}\left(r_i+r_o\right)^{3/2}}. \tag{16}$$

Note that $\beta_1-\beta_2 = \left(r_o-r_i\right)\left[r_o\left(r_o+r_i\right)-2r_i^2\right] > 0$ since $r_o > r_i$. It follows that $0 < \sqrt{\beta_2} < \sqrt{\beta_1}$. We therefore have that

$$\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta V_c^H\right)\right]_{r=r_i^+} > 0. \tag{17}$$

Substituting $r = r_o$ in (14), and by performing similar calculations, we obtain

$$\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta V_c^H\right)\right]_{r=r_o^-} < 0. \tag{18}$$

Similarly, we consider the cost of a HHCM rendezvous for the cases $r < r_i < r_o$ and $r_i < r_o < r$. These two cases yield

$$\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta V_c^H\right)\right]_{r=r_i^-} < 0, \tag{19}$$

and

$$\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta V_c^H\right)\right]_{r=r_o^+} > 0. \tag{20}$$

The results can be summarized as

$$\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta V_c^H\right)\right]_{r=r_i}\begin{cases}<0, & \text{if } r<r_i,\\ >0, & \text{if } r>r_i,\end{cases} \tag{21}$$

and

$$\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta V\right)\right]_{r=r_o} \begin{cases} < 0, & \text{if } r < r_o, \\ > 0, & \text{if } r > r_o. \end{cases} \tag{22}$$

Therefore, we conclude that the ΔV cost for a HHCM rendezvous attains a local minimum when either $r = r_i$ or $r = r_o$. Note that a HHCM rendezvous for $r = r_i$ or $r = r_o$ is actually a non-cooperative Hohmann transfer. It follows that if either Hohmann transfer is possible, then the non-cooperative maneuvers are local minimizers. Figure 4 shows how the cooperative rendezvous cost varies with r for different values of r_o (r_i is fixed at 1). For a cooperative rendezvous at an outer orbit, the cost of the maneuver increases rapidly. As r_o approaches r_i, the cooperative cost for any intermediate orbit r approaches the non-cooperative Hohmann transfer cost and the concave region flattens out. In the limiting case when $r_o \rightarrow r_i$, the minimum is obtained at $r = r_i = r_o$ with the total cost of transfer being zero.

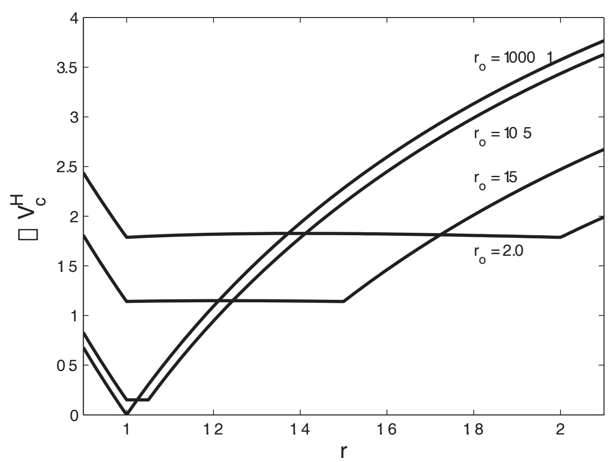


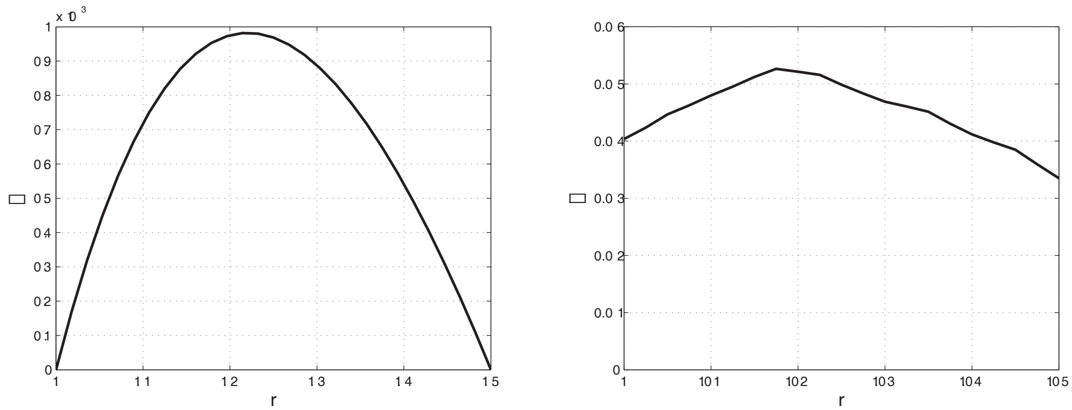
Figure 4. Variation of HHCM cost with r.

For convenience, let the difference of the HHCM and the non-cooperative Hohmann maneuver costs be denoted by the function $\eta(r)$, given by

$$\eta(r) \triangleq \frac{\Delta V_c^H(r) - \Delta V_{nc}^H}{\sqrt{2\mu}}.$$

Clearly, $\eta(r_i) = 0$ and $\eta(r_o) = 0$. The function $\eta(r)$ can be calculated analytically, and an example variation of $\eta(r)$ over r is shown in Fig. 5(a) for the values $r_i = 1$, $r_o = 1.05$, and $T = 1.5$. Note that the distance and time have both been non-dimensionalized, with unit distance equaling the radius r_i, and unit time equaling the time period of the satellite in radius r_i. Fig. 5(a) shows that $\eta(r)$ is marginally sub-optimal for all $r \in (r_i, r_o)$ compared to $r = r_i$ or $r = r_o$. If enough time is available so that Hohmann transfers are possible for the given separation of the satellites, the optimal rendezvous is non-cooperative.

If the optimal cooperative rendezvous is comprised of two Hohmann transfers, we have $\Delta V_c(r) = \Delta V_c^H(r)$.



(a) Variation of $\eta(r)$ (b) Variation of $\zeta(r)$

Figure 5. Auxiliary functions $\eta(r)$ and $\zeta(r)$ ($r_i = 1$, $r_o = 1.5$, $T = 1.5$).

However, both Hohmann transfers may not be possible within the specified time T. In this case $\Delta V_c(r) \neq \Delta V_c^H(r)$. Let us define the following function:

$$\zeta(r) \triangleq \frac{\Delta V_c(r) - \Delta V_c^H(r)}{\sqrt{2\mu}}.$$

Since a Hohmann transfer is the optimal two-impulse transfer between all coplanar circular orbits, a HHCM rendezvous is the optimal cooperative rendezvous at a radius $r \notin \{r_i, r_o\}$. Hence, $\zeta(r)$ measures the sub-optimality of the cooperative rendezvous solution when a HHCM rendezvous is not feasible at any slot on the orbit. The function $\zeta(r)$, for the case of $r_i = 1$, $r_o = 1.05$, and $T = 1.5$, is shown in Fig. 5(b). Note that if a HHCM rendezvous is feasible at any slot on the orbit, we have $\zeta(r) = 0$.

HOHMANN-PHASING COOPERATIVE MANEUVERS (HPCM)

The second cooperative maneuver we investigate is comprised of one Hohmann transfer and one phasing maneuver (HPCM), that is, the cooperative rendezvous occurs at one of the orbits r_i or r_o. Before we look at a HPCM, let us consider first a single phasing maneuver.

Phasing Maneuvers

Optimal two-impulse orbital transfers from one position in a circular orbit to another position on the same orbit in a given time are essentially phasing maneuvers (recall that the cost-invariant intervals in Fig. 3 correspond to phasing maneuvers). Let us consider a phasing maneuver by the satellite located on the orbit of radius $r_\star$. Also, let $T_\star$ denote the time period for the orbit of radius $r_\star$. In our case, either satellite s_μ or satellite s_ν performs a phasing maneuver, that is, $r_\star \in \{r_i, r_o\}$. The satellite can transfer from its original slot (ϕ_i for s_μ and ϕ_j for s_ν) to another orbital slot $\phi_{k_{r_\star}}$ in the same orbit, by performing one of the following two maneuvers:

- A *supersynchronous* maneuver, in which the transfer orbit has a higher apoapsis than $r_\star$,
- A *subsynchronous* maneuver, in which the transfer orbit has a lower periapsis than $r_\star$.

Let us denote the phasing angle by ψ, where $-\pi \leq \psi \leq \pi$. We consider the cases of $\psi < 0$ and $\psi > 0$ separately. For each of these cases, we have one of the two maneuvers (supersynchronous or subsynchronous). We therefore have four cases to consider:[21]

i) Supersynchonous and $\psi > 0$: The velocity change required for this transfer is given by

$$\Delta V^p = 2\sqrt{\frac{\mu}{r_\star}}\left[\sqrt{2-\left(\frac{\ell-1}{\ell-\psi/2\pi}\right)^{2/3}}-1\right], \tag{23}$$

where

$$\ell = \lfloor T/T_\star + \psi/2\pi \rfloor. \tag{24}$$

ii) Supersynchonous and $\psi < 0$: The velocity change required for this transfer is given by

$$\Delta V^p = 2\sqrt{\frac{\mu}{r_\star}}\left[\sqrt{2-\left(\frac{\ell}{\ell-\psi/2\pi}\right)^{2/3}}-1\right]. \tag{25}$$

iii) Subsynchonous and $\psi > 0$: The velocity change required for this transfer is given by

$$\Delta V^p = 2\sqrt{\frac{\mu}{r_\star}}\left[1-\sqrt{2-\left(\frac{\ell}{\ell-\psi/2\pi}\right)^{2/3}}\right]. \tag{26}$$

iv) Subsynchonous and $\psi < 0$: The velocity change required for this transfer is given by

$$\Delta V^p = 2\sqrt{\frac{\mu}{r_\star}}\left[1-\sqrt{2-\left(\frac{\ell+1}{\ell-\psi/2\pi}\right)^{2/3}}\right]. \tag{27}$$

We now consider a cooperative maneuver comprised of a Hohmann transfer and a phasing maneuver. Note that during a HPCM rendezvous the phasing maneuver can occur either on the orbit r_i or on the orbit r_o. Hence there are two possibilities regarding the location of the cooperative rendezvous. In one case, the satellite s_ν performs a Hohmann transfer from r_o to r_i and the satellite s_μ performs a phasing maneuver. In the other case, the satellite s_μ performs a Hohmann transfer from r_i to r_o and the satellite s_ν performs a phasing maneuver. This case is depicted in Fig. 6, in which ψ represents the phasing angle and θ_0 is the initial separation angle between satellites s_μ and s_ν.

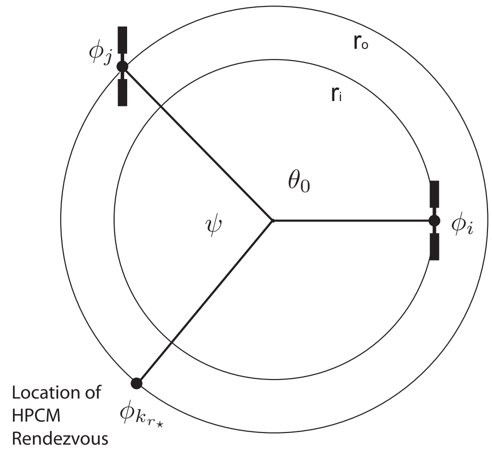


Figure 6. Hohmann-phasing Cooperative Maneuver ($r_\star = r_o$).

Denoting by $\overline{\Delta V}_c(r_\star)$ the cooperative cost, we therefore have

$$\overline{\Delta V}_c(r_\star) = \Delta V^p + \Delta V_{nc}^H, \qquad \star = i, o. \tag{28}$$

Optimality of Hohmann-Phasing Cooperative Maneuvers (HPCM)

The total ΔV for a HPCM rendezvous depends on the phasing angle ψ. The phasing angle ψ determines the location of the cooperative rendezvous on the orbit $r_\star$, where $r_\star = r_i$ or $r_\star = r_o$. In this section, we consider the four cases of phasing maneuvers and find the locations on $r_\star$ for which the corresponding HPCM rendezvous is cheaper (in terms of ΔV) than a cooperative maneuver on an intermediate orbit. According to the previous discussion, these are exactly the locations for which a HPCM rendezvous is feasible.

First, note the following expressions:

$$\lfloor T/T_\star - 1 \rfloor \leq \ell \leq \lfloor T/T_\star \rfloor \quad \text{if} \quad \psi \leq 0, \tag{29}$$

and

$$\lfloor T/T_\star \rfloor \leq \ell \leq \lfloor T/T_\star + 1 \rfloor \quad \text{if} \quad \psi \geq 0. \tag{30}$$

We therefore have,

$$\begin{aligned} \frac{\Delta V_c(r) - \overline{\Delta V}_c(r_\star)}{\sqrt{2\mu}} &= \frac{\Delta V_c(r) - \Delta V_c^H(r)}{\sqrt{2\mu}} + \frac{\Delta V_c^H(r) - \Delta V_{nc}^H}{\sqrt{2\mu}} + \frac{\Delta V_{nc}^H - \overline{\Delta V}_c(r_\star)}{\sqrt{2\mu}} \\ &= \eta(r) + \zeta(r) - \frac{\overline{\Delta V}_c(r_\star) - \Delta V_{nc}^H}{\sqrt{2\mu}} \\ &= \eta(r) + \zeta(r) - \frac{\Delta V_p}{\sqrt{2\mu}} \end{aligned} \tag{31}$$

We are interested in finding the phasing angle such that $\Delta V_c(r) \geq \overline{\Delta V}_c(r_\star)$. It follows that

$$\eta(r) + \zeta(r) \geq \sqrt{\frac{2}{r_\star}} \left[\sqrt{2 - \left(\frac{\ell - 1}{\ell - \psi/2\pi} \right)^{2/3}} - 1 \right], \tag{32}$$

which gives

$$\sqrt{\frac{r_\star}{2}} \left(\eta\left(r\right) + \zeta\left(r\right) \right) \geq \sqrt{2 - \left(\frac{\ell - 1}{\ell - \psi/2\pi} \right)^{2/3}} - 1. \tag{33}$$

Simple calculations lead to

$$\left(\frac{\ell - 1}{\ell - \psi/2\pi} \right) \geq \left[2 - \left(\sqrt{r_\star/2}\left(\eta(r) + \zeta(r)\right) + 1 \right)^2 \right]^{3/2}. \tag{34}$$

This inequality yields

$$\frac{\psi}{2\pi} \geq \left(1 - \frac{1}{\left[2 - \left(\sqrt{r_\star/2}\left(\eta(r) + \zeta(r)\right) + 1 \right)^2 \right]^{3/2}} \right) \ell + \frac{1}{\left[2 - \left(\sqrt{r_\star/2}\left(\eta(r) + \zeta(r)\right) + 1 \right)^2 \right]^{3/2}}. \tag{35}$$

Finally, using (24), the above inequality yields

$$\frac{\psi}{2\pi} \geq \lfloor T/T_\star \rfloor - \frac{\lfloor T/T_\star \rfloor - 1}{\left[2 - \left(\sqrt{r_\star/2}\left(\eta(r) + \zeta(r)\right) + 1 \right)^2 \right]^{3/2}}. \tag{36}$$

Inequality (36) provides a lower bound $\psi_\ell^1(r)$ for the supersynchronous phasing angle. This lower bound is given by

$$\psi_\ell^1(r) = 2\pi \left[1 + \lfloor T/T_\star \rfloor \left(1 - \frac{1}{\left[2 - \left(\sqrt{r_\star/2}\left(\eta(r) + \zeta(r)\right) + 1 \right)^2 \right]^{3/2}} \right) \right], \tag{37}$$

and determines the minimum value of the supersynchronous phasing angle that defines locations on $r_\star$ for which a HPCM rendezvous is feasible. Since for this case we have, by definition, $0 \leq \psi \leq \pi$, the lower bound on the phasing angle is given by $\max\{0, \psi_\ell^1(r)\}$. Naturally, π represents an upper bound for the phasing angle. Similarly, for the case of a supersynchronous phasing maneuver with $\psi < 0$, we can show that

$$\frac{\psi}{2\pi} \geq \left(\lfloor T/T_\star \rfloor - 1 \right) \left(1 - \frac{1}{\left[2 - \left(\sqrt{r_\star/2}\left(\eta(r) + \zeta(r)\right) + 1 \right)^2 \right]^{3/2}} \right) \tag{38}$$

ensures that a HPCM is cheaper than the optimal cooperative rendezvous on any orbit of radius $r \neq r_\star$. The above inequality imposes a lower bound on the supersynchronous phasing angle given by

$$\psi_\ell^2(r) \triangleq 2\pi\left(\lfloor T/T_\star \rfloor - 1\right)\left(1 - \frac{1}{\left[2 - \left(\sqrt{r_\star/2}\left(\eta(r) + \zeta(r)\right) + 1\right)^2\right]^{3/2}}\right). \tag{39}$$

For this case, we have $-\pi \leq \psi \leq 0$ by definition. The lower bound on the phasing angle is therefore given by $\max\{-\pi, \psi_\ell^2(r)\}$. Naturally, it follows by definition that the upper bound on the phasing angle is 0. In summary, for the case of a supersynchronous maneuver, the lower bound on the phasing angle is given by

$$\psi_\ell(r) = \begin{cases} \max\{0, \psi_\ell^1(r)\}, & \text{if } 0 \leq \psi \leq \pi, \\ \max\{-\pi, \psi_\ell^2(r)\}, & \text{if } -\pi \leq \psi \leq 0. \end{cases} \tag{40}$$

Note that the maximum value of $\psi_\ell(r)$ represents a lower bound on the phasing angle that defines the location on $r_\star$ for which a HPCM is optimal. Note also that the maximum for both $\psi_\ell^1(r)$ and $\psi_\ell^2(r)$ occurs when the quantity $\left(2 - \left(1 + \sqrt{r_\star/2}\left(\eta(r) + \zeta(r)\right)\right)^2\right)$ is maximum, equivalently, when $\left(1 + \sqrt{r_\star/2}\left(\eta(r) + \zeta(r)\right)\right)$ is minimum, which occurs when $\eta(r) + \zeta(r)$ is minimum.

For the case of a sub-synchronous maneuver with $\psi > 0$, we have

$$\frac{\psi}{2\pi} \leq \lfloor T/T_\star \rfloor - \frac{\left(\lfloor T/T_\star \rfloor + 1\right)}{\left[2 - \left(1 - \sqrt{r_\star/2}\left(\eta(r) + \zeta(r)\right)\right)^2\right]^{3/2}} \tag{41}$$

as the corresponding condition that makes HPCM cheaper. The above inequality imposes an upper bound $\psi_u^3(r)$ on the sub-synchronous phasing angle in a HPCM

$$\psi_u^3(r) \triangleq 2\pi\left(\lfloor T/T_\star \rfloor - \frac{\left(\lfloor T/T_\star \rfloor + 1\right)}{\left[2 - \left(1 - \sqrt{r_\star/2}\left(\eta(r) + \zeta(r)\right)\right)^2\right]^{3/2}}\right), \tag{42}$$

such that a HPCM rendezvous is feasible. However, for this case, we have, by definition, $0 \leq \psi \leq \pi$. Therefore, $\min\{\psi_u^3(r), \pi\}$ denotes the upper bound on the phasing angle, while the lower bound is zero. Finally, for the case of a sub-synchronous maneuver with $\psi < 0$, we can show that

$$\frac{\psi}{2\pi} \leq 1 + (\lfloor T/T_\star \rfloor - 1)\left(1 - \frac{1}{\left[2 - \left(1 - \sqrt{r_\star/2}\,(\eta(r) + \zeta(r))\right)^2\right]^{3/2}}\right) \tag{43}$$

is required to have a HPCM maneuver to be optimal. This inequality imposes an upper bound $\psi_u^4(r)$ on the subsynchronous phasing angle

$$\psi_u^4(r) \triangleq 2\pi\left[1 + (\lfloor T/T_\star \rfloor - 1)\left(1 - \frac{1}{\left[2 - \left(1 - \sqrt{r_\star/2}\,(\eta(r) + \zeta(r))\right)^2\right]^{3/2}}\right)\right], \tag{44}$$

which is a bound on the location on $r_\star$ for which a HPCM rendezvous is feasible. Since by definition, $-\pi \leq \psi \leq 0$, the upper bound on the phasing angle is given by $\min\{0, \psi_u^4(r)\}$, and the lower bound is given by $-\pi$. Therefore, by combining the two cases of subsynchronous maneuvers, we have the following expression

$$\psi_u(r) = \begin{cases} \min\{\pi, \psi_u^3(r)\}, & \text{if } 0 \leq \psi \leq \pi, \\ \min\{0, \psi_u^4(r)\}, & \text{if } -\pi \leq \psi \leq 0. \end{cases} \tag{45}$$

Note that the minimum of $\psi_u(r)$ over r represents an upper bound on the phasing angle that gives the position on the orbit of radius $r_\star$ for which HPCM is feasible, hence also optimal. Note that the minimum of both $\psi_u^3(r)$ and $\psi_u^4(r)$ occurs when $\left(2 - \left(1 - \sqrt{r_\star/2}\,(\eta(r) + \zeta(r))\right)^2\right)$ is minimum, that is, when $\left(1 - \sqrt{r_\star/2}\,(\eta(r) + \zeta(r))\right)$ is maximum, which occurs when $\eta(r) + \zeta(r)$ is minimum.

In this section, we investigated the HPCM rendezvous, which can occur either on the orbit r_i or the orbit r_o. The location of the rendezvous is given by the phasing angle ψ on the particular orbit in which the phasing maneuver takes place. Hence, the above analysis gives the location of the cooperative rendezvous for HPCM to be the cheapest rendezvous option between the two satellites.

SHORT TIME TO RENDEZVOUS

In the previous sections it has been assumed that a phase-free Hohmann transfer is always possible between the orbits r_i and r_o. However, if the time allowed for rendezvous is sufficiently small, a Hohmann transfer between orbits r_i and r_o and vice versa becomes infeasible. In this section, we consider the case of short-time rendezvous between satellites s_μ and s_ν. We therefore assume that

$$T < \pi\sqrt{\frac{(r_i + r_o)^3}{8\mu}}. \tag{46}$$

Clearly, HPCM maneuvers are not possible in this case. However, HHCM maneuvers may be possible for some orbit

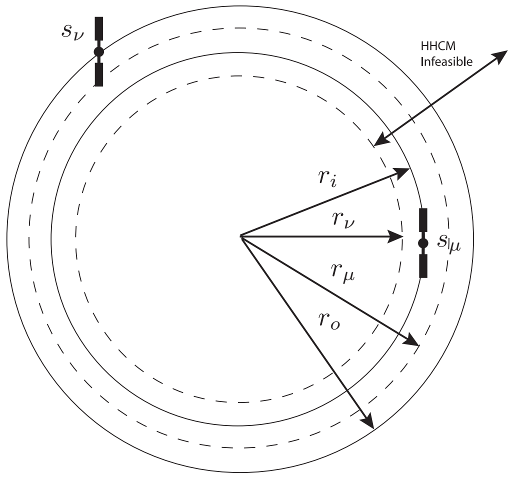


Figure 7. Short time of rendezvous: Feasibility of HHCM.

$r \notin \{r_i, r_o\}$. Let us determine the orbits r for which HHCM maneuvers can occur for short-time rendezvous. To this end, let us investigate if the time T is sufficient for both satellites s_μ and s_ν to perform Hohmann transfers to an orbit of radius r. It follows from inequality (7) that

$$T < \pi\sqrt{\frac{(r + r_o)^3}{8\mu}}, \quad \text{for all} \quad r \geq r_i. \tag{47}$$

Therefore, satellite s_ν cannot perform a Hohmann transfer to an orbit of radius r if $r \geq r_i$. For the given time T, the satellite s_ν can nonetheless perform a Hohmann transfer from orbit r_o to an orbit r, provided $r \leq r_\nu$, where r_ν is defined as

$$r_\nu = \left(\frac{8\mu T^2}{\pi^2}\right)^{1/3} - r_o < r_i. \tag{48}$$

Similarly, for the given time T satellite s_μ can perform a Hohmann transfer from orbit r_i to an orbit of radius r, provided that $r \leq r_\mu$, where r_μ is defined as

$$r_\mu = \left(\frac{8\mu T^2}{\pi^2}\right)^{1/3} - r_i < r_o. \tag{49}$$

Note that $r_\nu < r_\mu < r_o$ and $r_\nu < r_i < r_o$. Hence, a HHCM rendezvous is feasible only for $r \leq r_\nu$. Consequently, if the optimal solution is a HHCM rendezvous, the location of rendezvous is at an orbit of radius $r \leq r_\nu$. Otherwise, the optimal rendezvous takes place on an orbit of radius $r > r_\nu$. Figure 7 shows the two satellites s_μ and s_ν in the orbits r_i and r_o respectively, along with the orbits r_μ and r_ν.

The results of the previous analysis are summarized in Table 1. In the table, T^H_{nc} denotes the time required for a non-cooperative Hohmann transfer between the satellites (which is a function of the initial separation angle θ_0) and T^H_{pf} denotes the time required for a phase-free Hohmann transfer.

Table 1. Summary of results.

Time of Rendezvous	Optimal Solution	Optimal Rendezvous Location
$T \geq T^H_{nc}(\theta_0)$	Non-Cooperative Hohmann Transfer	$r_\star = r_i$ or $r_\star = r_o$
$T^H_{pf} \leq T < T^H_{nc}(\theta_0)$	HPCM	$r_\star = r_i$ or $r_\star = r_o$
$T < T^H_{pf}$	Cooperative Rendezvous	$r_\star \leq r_\nu$ if HHCM

FUEL EXPENDITURE DURING COOPERATIVE RENDEZVOUS

Thus far we have only discussed the minimization of the total velocity change required for a cooperative rendezvous. In this section, we consider the minimization of the true objective, which is the fuel expenditure during the cooperative rendezvous between the satellites s_μ and s_ν. Let $m_{\mathrm{s}\mu}$ and $m_{\mathrm{s}\nu}$ denote the mass of the permanent structure of the satellites s_μ and s_ν respectively, while f^-_μ and f^-_ν denote the initial fuel content of satellites s_μ and s_ν, respectively. For the transfer of s_μ from ϕ_i to ϕ_{k_r}, let ΔV_{ik_r} denote the required velocity change. The fuel expenditure during the transfer is given by

$$p^\mu_{ik_r} = \left(m_{\mathrm{s}\mu} + f^-_\mu\right)\left(1 - e^{-\frac{\Delta V_{ik_r}}{c_{0\mu}}}\right), \tag{50}$$

where $c_{0\mu} = g_0 I_{\mathrm{sp}\mu}$, g_0 is the acceleration due to gravity at the surface of the earth, and $I_{\mathrm{sp}\mu}$ is the specific thrust of the engine. For the transfer of s_ν from ϕ_j to ϕ_ℓ, let $\Delta V_{j\ell}$ denote the required velocity change. The fuel expenditure during this transfer is given by

$$p^\nu_{jk_r} = \left(m_{\mathrm{s}\nu} + f^-_\nu\right)\left(1 - e^{-\frac{\Delta V_{jk_r}}{c_{0\nu}}}\right). \tag{51}$$

The total fuel expenditure during the cooperative rendezvous between satellites s_μ and s_ν is therefore given by

$$p^\mu_{ik_r} + p^\nu_{jk_r} = \left(m_{\mathrm{s}\mu} + f^-_\mu\right)\left(1 - e^{-\frac{\Delta V_{ik_r}}{c_{0\mu}}}\right) + \left(m_{\mathrm{s}\nu} + f^-_\nu\right)\left(1 - e^{-\frac{\Delta V_{jk_r}}{c_{0\nu}}}\right), \tag{52}$$

and is a function of the location (slot ϕ_{k_r} of orbit of radius r) of the cooperative rendezvous. Now, let us assume that the minimum fuel expenditure occurs at the slot $\phi^\star$ of orbit of radius $r^\star$. We will denote all quantities associated with the optimal fuel expenditure by the subscript '$\star$'. In other words, we have

$$p^{\mu}_{ik_{r_\star}} + p^{\nu}_{jk_{r_\star}} \le p^{\mu}_{ik_r} + p^{\nu}_{jk_r} \tag{53}$$

for all possible r and ϕ_{k_r}. Using (52), we have from (53),

$$\left(m_{s\mu} + f^-_\mu\right)\left(e^{-\frac{\Delta V_{ik_r}}{c_{0\mu}}} - e^{-\frac{\Delta V_{ik_{r_\star}}}{c_{0\mu}}}\right) + \left(m_{s\nu} + f^-_\nu\right)\left(e^{-\frac{\Delta V_{jk_r}}{c_{0\nu}}} - e^{-\frac{\Delta V_{jk_{r_\star}}}{c_{0\nu}}}\right) \le 0 \tag{54}$$

Expanding the exponential term, and neglecting higher powers of $\Delta V/c_0 \ll 1^*$, we have

$$\left(m_{s\mu} + f^-_\mu\right)\left(\frac{\Delta V_{ik_{r_\star}}}{c_{0\mu}} - \frac{\Delta V_{ik_r}}{c_{0\mu}}\right) + \left(m_{s\nu} + f^-_\nu\right)\left(\frac{\Delta V_{jk_{r_\star}}}{c_{0\nu}} - \frac{\Delta V_{jk_r}}{c_{0\nu}}\right) \le 0, \tag{55}$$

which reduces to

$$\frac{\left(m_{s\mu} + f^-_\mu\right)}{c_{0\mu}}\Delta V_{ik_{r_\star}} + \frac{\left(m_{s\nu} + f^-_\nu\right)}{c_{0\nu}}\Delta V_{jk_{r_\star}} \le \frac{\left(m_{s\mu} + f^-_\mu\right)}{c_{0\mu}}\Delta V_{ik_r} + \frac{\left(m_{s\nu} + f^-_\nu\right)}{c_{0\nu}}\Delta V_{jk_r} \tag{56}$$

Note that the right-hand side of the above inequality is a function of r and ϕ_{k_r}. The inequality holds for all r and ϕ_{k_r}. Hence, we have

$$\min_{r,\phi_{k_r}}\left[\frac{\left(m_{s\mu} + f^-_\mu\right)}{c_{0\mu}}\Delta V_{ik_r} + \frac{\left(m_{s\nu} + f^-_\nu\right)}{c_{0\nu}}\Delta V_{jk_r}\right] = \frac{\left(m_{s\mu} + f^-_\mu\right)}{c_{0\mu}}\Delta V_{ik_{r_\star}} + \frac{\left(m_{s\nu} + f^-_\nu\right)}{c_{0\nu}}\Delta V_{jk_{r_\star}} \tag{57}$$

We therefore conclude that the total fuel expenditure is minimized at the location where a weighted sum of ΔV is minimized, the weights being a ratio of mass and specific impulse for each satellite. If this ratio is the same for the two satellites, that is, $m_{s\mu}/c_{0\mu} = m_{s\nu}/c_{0\nu}$, then the minimum fuel expenditure during cooperative rendezvous is equivalent to minimizing the total ΔV. Furthermore, if the satellites have the same engine characteristics and nearly the same mass, minimizing fuel is the same as minimizing total ΔV.

Note that for a cooperative rendezvous on an orbit of radius r, both satellites s_μ and s_ν must have enough fuel to complete the rendezvous at an orbital slot ϕ_{k_r} on the orbit r. Next, we determine the necessary conditions for the feasibility of a cooperative rendezvous at a slot ϕ_{k_r} on the orbit r.

*This assumption is justified because a typical value of $c_0 = 2943$ m/s and the ΔV requirement for the transfers would be much smaller (of the order of 10 m/s).

For satellite s_μ to be able to complete the rendezvous, we must have

$$p^{\mu}_{ik_r} \leq f^-_{\mu}, \tag{58}$$

which, under the assumption $\Delta V/c_0 \ll 1$, implies

$$\frac{\left(m_{\mathrm{s}\mu} + f^-_{\mu}\right)}{c_{0\mu}} \Delta V_{ik_r} \leq f^-_{\mu}. \tag{59}$$

Similarly, for the satellite s_ν to be able to complete the rendezvous, we must have

$$p^{\nu}_{jk_r} \leq f^-_{\nu}, \tag{60}$$

which, under the assumption $\Delta V/c_0 \ll 1$, yields

$$\frac{\left(m_{\mathrm{s}\nu} + f^-_{\nu}\right)}{c_{0\nu}} \Delta V_{jk_r} \leq f^-_{\nu}. \tag{61}$$

Equations (59) and (61) imply that the radius r of the orbit for the cooperative rendezvous to take place is bounded above and below by $r_\ell \leq r \leq r_u$. Hence, minimizing the total fuel is equivalent to minimizing the weighted sum of ΔV over all locations of all orbits of radius r such that $r_\ell \leq r \leq r_u$.

Assuming the orbits r_i and r_o are close enough, that is $r_o - r_i \ll r_i$, we can derive explicit expressions for r_ℓ and r_u. To this end, let us consider the transfer of s_μ from the orbit r_i to some orbit $r > r_i$. For a given amount of fuel, the highest orbit the satellite s_μ can transfer to is the one given by a Hohmann transfer that would consume all its fuel. The velocity change for a Hohmann transfer from orbit r_i to r is given by

$$\Delta V_{ik_r} = \Delta r_u \sqrt{\frac{\mu}{r_i^3}}, \tag{62}$$

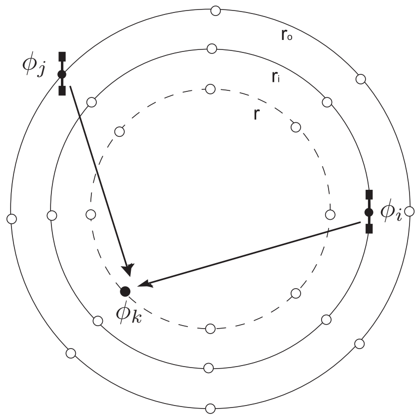
where $\Delta r_u = r - r_i$ and where we have assumed that $\Delta r_u / r_i \ll 1$. Using (62), we obtain from (59),

$$\Delta r_u \leq \frac{f^-_{\mu}}{\left(m_{\mathrm{s}\mu} + f^-_{\mu}\right)} c_{0\mu} \sqrt{\frac{r_i^3}{\mu}}. \tag{63}$$

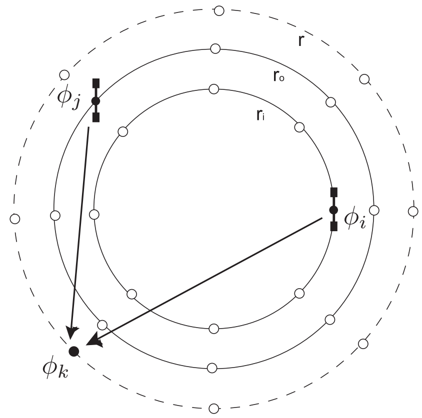
This expression yields the upper bound r_u as follows

$$r_u = r_i + \frac{f^-_{\mu}}{\left(m_{\mathrm{s}\mu} + f^-_{\mu}\right)} c_{0\mu} \sqrt{\frac{r_i^3}{\mu}}. \tag{64}$$

Let us now consider the transfer of s_ν from the orbit of radius r_o to the orbit $r < r_o$. For a given amount of fuel,



(a) $r < r_i < r_o$



(b) $r_i < r_o < r$

Figure 8. Cooperative rendezvous.

the lowest orbit the satellite s_ν can transfer to is the one given by a Hohmann transfer that would consume all its fuel. Letting $\Delta r_\ell = r - r_o$, and assuming again that $\Delta r_\ell / r_o \ll 1$, we have,

$$\Delta V_{jk_r} = \frac{1}{2}\Delta r_\ell \sqrt{\frac{\mu}{r_o^3}}. \tag{65}$$

Using the above expression, we obtain from (61),

$$\Delta r_\ell \leq 2\frac{f_\nu^-}{\left(m_{s\nu} + f_\nu^-\right)} c_{0\nu} \sqrt{\frac{r_o^3}{\mu}}, \tag{66}$$

which yields the following expression for the lower bound r_ℓ as follows

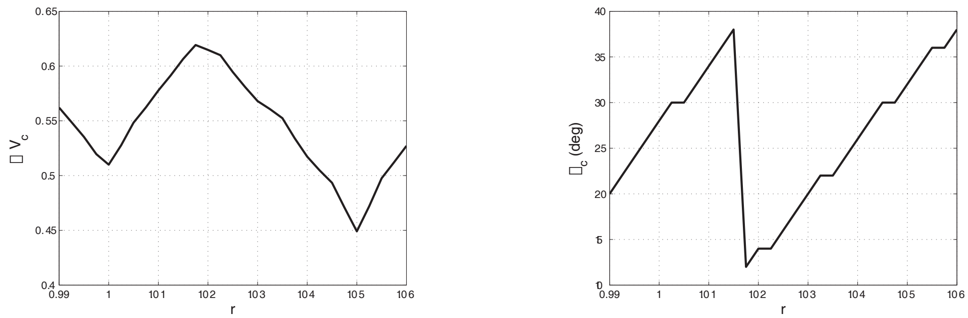
$$r_\ell = r_o - 2\frac{f_\nu^-}{\left(m_{s\nu} + f_\nu^-\right)} c_{0\nu} \sqrt{\frac{r_o^3}{\mu}}. \tag{67}$$

In summary, a cooperative rendezvous is feasible at an orbit of radius r if and only if $r_\ell \leq r \leq r_u$. If $r_\ell > r_i$ and $r_u > r_o$, none of the non-cooperative rendezvous are feasible and the rendezvous has to be cooperative.

From the above analysis (recall also (57)), we find that the fuel expenditure is minimized when the weighted sum

$$\frac{\left(m_{s\mu} + f_\mu^-\right)}{c_{0\mu}} \Delta V_{ik_r} + \frac{\left(m_{s\nu} + f_\nu^-\right)}{c_{0\nu}} \Delta V_{jk_r} \tag{68}$$

is minimized for all $r_\ell \leq r \leq r_u$. Assume now that the satellites s_μ and s_ν perform a HHCM rendezvous at an orbit of radius r. For $r_i \leq r \leq r_o$, the total ΔV required for the HHCM rendezvous remains roughly constant, say, ΔV_0.



(a) Cooperative rendezvous cost

(b) Optimal location for rendezvous

Figure 9. Case study ($r_i = 1, r_o = 1.05, \theta = 60\,\text{deg}, T = 1.5$).

For similar satellites, that is, $m_{\text{s}\mu} = m_{\text{s}\nu} = m_{\text{s}}$ and $c_{0\mu} = c_{0\nu}$, we have for the expression in (68)

$$\frac{m_{\text{s}} + f_\nu^-}{c_0}\Delta V_0 + \frac{f_\mu^- - f_\nu^-}{c_0}\Delta V_{ik_r} = \frac{m_{\text{s}} + f_\mu^-}{c_0}\Delta V_0 + \frac{f_\nu^- - f_\mu^-}{c_0}\Delta V_{jk_r}. \tag{69}$$

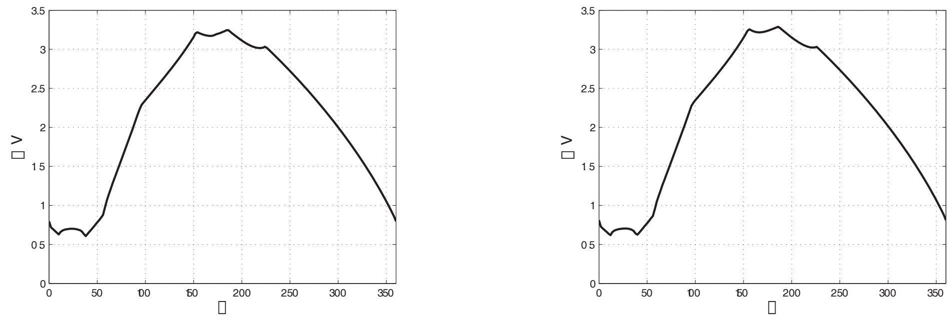
If $f_\mu^- < f_\nu^-$, the above expression is minimized when ΔV_{ik_r} is maximized, which occurs at $r = r_u$. Similarly, if $f_\nu^- < f_\mu^-$, the above expression is minimized when ΔV_{jk_r} is maximized, which occurs at $r = r_\ell$. In either case, the fuel-deficient satellite moves as close to the fuel-sufficient satellite as possible. This is a particularly important case for the refueling problem because refueling typically takes place towards the end of fuel life-time of the satellite. Hence, it is likely that the fuel-deficient satellites would be almost depleted of fuel. In such a case, even if enough time is permitted for Hohmann transfers to take place, the optimal rendezvous has to be cooperative, at an orbit of radius $r = r_\nu$ or $r = r_\mu$.

NUMERICAL EXAMPLES

In this section, we first consider an example of a cooperative rendezvous between two satellites in two different circular orbits. According to the previous developments, the terminal orbit of the satellites at the end of the cooperative maneuver is assumed to be circular as well. With the help of this example, we illustrate that the optimal rendezvous that minimizes the total ΔV, is either a non-cooperative Hohmann transfer or a cooperative maneuver that is comprised of a Hohmann transfer and a phasing maneuver, provided there is sufficient time to perform a phase-free Hohmann transfer between the orbits r_i and r_o. Note that the distance and time have both been non-dimensionalized, with unit distance equaling the radius r_i, and unit time equaling the time period of the satellite in radius r_i.

Example 1 *Cooperative rendezvous between two satellites in different circular orbits.*

Let $r_i = 1$, $r_o = 1.05$ and $\theta_0 = 60\,\text{deg}$. First we determine the optimal cooperative rendezvous for a time-of-flight less than the one necessary for a Hohmann transfer for either one of the non-cooperative maneuvers. For this example,



(a) Cooperative ΔV ($r = 1.0150$)

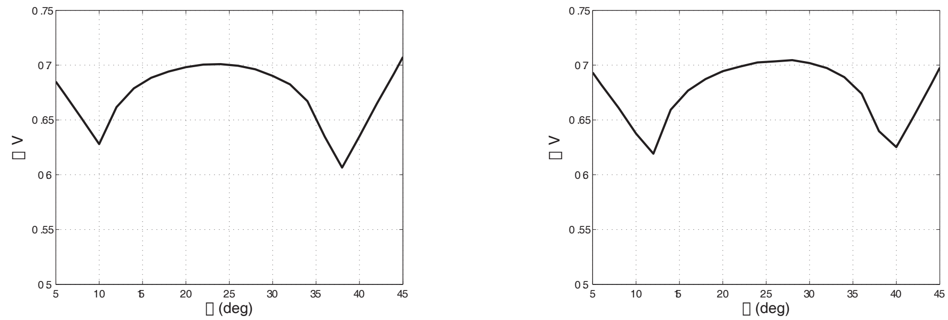
(b) Cooperative ΔV ($r = 1.0175$)

Figure 10. Explaining the discontinuity: Competing local minima.

a non-cooperative Hohmann transfer for which s_μ is the active satellite becomes possible when $t = 2.6290$. The other non-cooperative Hohmann transfer, in which s_ν is the active satellite, becomes possible at $t = 3.1479$. In other words, if $t < 2.6290$, non-cooperative Hohmann transfers are not feasible between the two satellites. We therefore consider the time for rendezvous to be $t = 1.50$. We determine the total cost (ΔV) of a cooperative rendezvous for all possible slots and compute the minimum. We consider cooperative rendezvous to occur in orbits of radius r, where $0.98 \leq r \leq 1.07$. This allows us to consider all three cases of cooperative rendezvous, namely (i) $r < r_i < r_o$, (ii) $r_i < r < r_o$ and (iii) $r_i < r_o < r$ (refer to Figs. 2 and 8). Figure 9(a) shows the variation of cooperative ΔV with the radius r of the orbit. On each orbit of radius r where the cooperative rendezvous takes place, there is an optimal location that yields the minimum ΔV for that particular orbit. Figure 9(b) depicts the variation of the optimal position of cooperative rendezvous $\phi_c(r)$ with r. The plot shows a discontinuity in the optimal rendezvous position as the value of r changes from $r = 1.0150$ to $r = 1.0175$. To investigate the reason for this discontinuity, let us consider the variation of $\Delta V_{ij}^c|_{k_r}$ over various slots Φ_r of a given intermediate orbit. Figure 10(a) shows such a variation for the orbit with radius $r = 1.0150$, while Fig. 10(b) shows the same for the orbit with radius $r = 1.0175$. A detailed view of the two competing local minima is shown in Fig. 11.

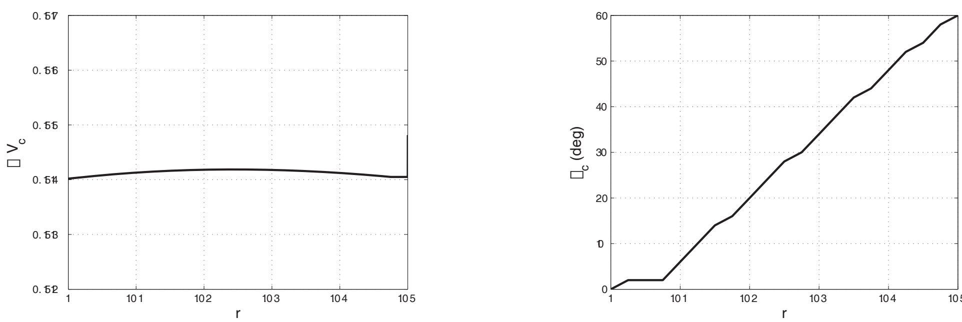
Both these plots show two local minima that compete with each other for the cheapest solution for cooperative rendezvous on that particular orbit r. Each of these local minima corresponds to a cooperative maneuver in which one of the orbital transfers is a Hohmann transfer. As r changes from $r = 1.0150$ to $r = 1.0175$, there is a change of optimal cooperative rendezvous from one local minimum to the other. This shift of the optimal position appears as a discontinuity in the plot of ϕ. Naturally, there is no discontinuity in the variation of ΔV.

Referring back to Fig. 9(a), we see that the minimum ΔV for cooperative rendezvous occurs at one of the orbits $r = r_i$ or $r = r_o$. The optimal cooperative rendezvous on orbit r_i occurs at the slot $\phi = 28\,\text{deg}$, while the optimal cooperative rendezvous on orbit r_o occurs at the slot $\phi = 32\,\text{deg}$. Calculations of the feasible slots for Hohmann transfers indicate that Hohmann transfers are possible for slots $\phi = 28.28\,\text{deg}$ to $\phi = 53.21\,\text{deg}$ on orbit r_i, while



(a) Cooperative ΔV ($r = 1.0150$) (b) Cooperative ΔV ($r = 1.0175$)

Figure 11. Explaining the discontinuity: Detail of the two competing local minima.

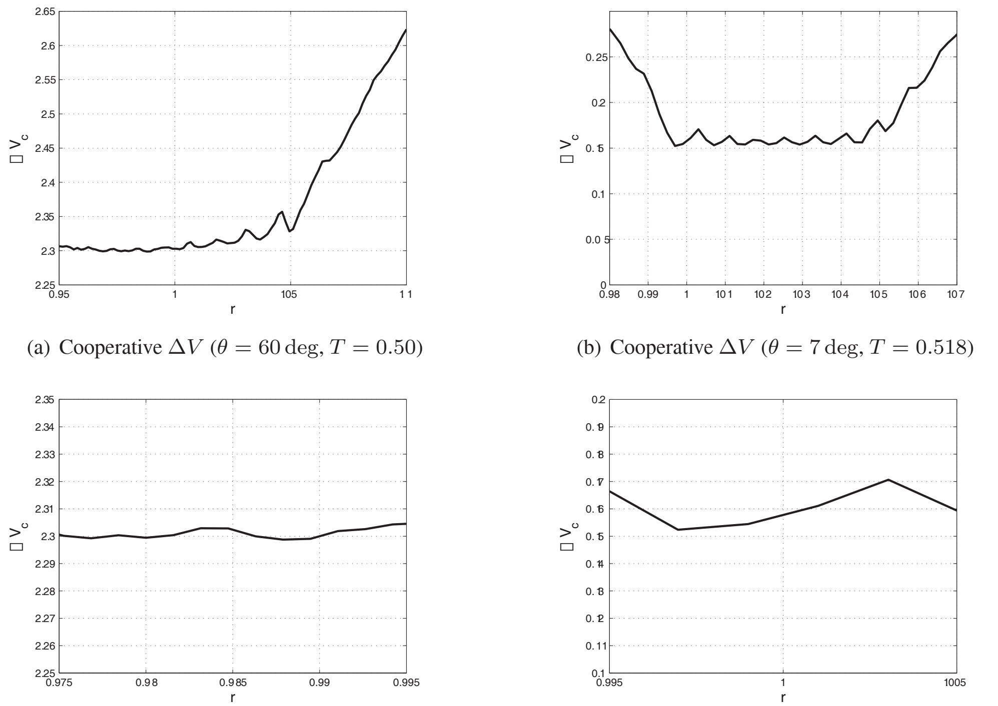


(a) Cooperative rendezvous cost (b) Optimal location for rendezvous

Figure 12. Case study ($r_i = 1, r_o = 1.05, \theta = 60\,\text{deg}, T = 3.0$).

Hohmann transfers are possible for slots $\phi = 6.39\,\text{deg}$ to $\phi = 31.32\,\text{deg}$ on orbit r_o. These are obtained by calculating the lead angles necessary for a Hohmann transfer, as given by equation (12). Because of the discretization used for our calculation of slots at intervals of $2\,\text{deg}$, we obtain the optimal rendezvous locations at $\phi = 28\,\text{deg}$ (instead of $\phi = 28.28\,\text{deg}$) on orbit r_i and at $\phi = 32\,\text{deg}$ (instead of $\phi = 31.32\,\text{deg}$) on orbit r_o. The results indicate that the optimal rendezvous locations on orbits r_i and r_o occur near the slots where Hohmann transfers are possible, indicating that the optimal cooperative rendezvous is indeed a Hohmann-phasing cooperative maneuver. Hence, when the time of rendezvous does not allow for a Hohmann non-cooperative transfer, the best possible cooperative maneuver is found to be comprised of a Hohmann transfer and a phasing maneuver.

Next, we investigate the optimal cooperative rendezvous between two satellites for a time of maneuver $T = 3.0$ that allows for non-cooperative rendezvous using Hohmann transfers. Figure 12(a) shows the variation of cooperative ΔV with the radius r of the orbit where the cooperative rendezvous takes place. Figure 12(b) depicts the variation of the optimal position of the cooperative rendezvous $\phi_c(r)$ with r. It is found that the non-cooperative Hohmann transfers provide the best ΔV for the rendezvous of the two satellites. In summary, this numerical example shows that the optimal rendezvous between two satellites in different orbits is either non-cooperative Hohmann ($T = 3.0$,



(a) Cooperative ΔV ($\theta = 60\,\text{deg}, T = 0.50$)

(b) Cooperative ΔV ($\theta = 7\,\text{deg}, T = 0.518$)

(c) Cooperative ΔV ($\theta = 60\,\text{deg}, T = 0.50$) (detail)

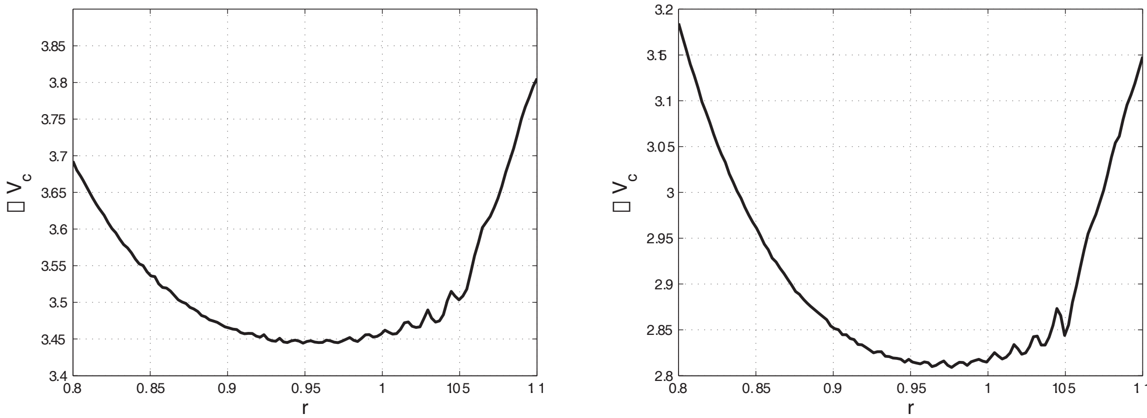
(d) Cooperative ΔV ($\theta = 7\,\text{deg}, T = 0.518$) (detail)

Figure 13. Case study for small time of flight ($r_i = 1, r_o = 1.05$).

$\Delta V = 0.15$) or it is a cooperative maneuver comprised of a Hohmann and a phasing maneuver ($T = 1.5$, $\Delta V = 0.45$).

Let us now consider a time for the rendezvous T, so that a phase-free Hohmann transfer between orbits r_i and r_o is not possible. For our example, if $T < 0.519$ a phase-free Hohmann transfer between r_i and r_o is not possible, so we let $T = 0.50$. In this case $r_\nu = 0.95$, so that HHCM maneuvers are not possible at any orbit of radius $r > 0.95$. The optimal rendezvous is cooperative and occurs at the orbit of radius $r = 0.9879$ (not a HHCM rendezvous). However, there are cases when the optimal solution is a HHCM rendezvous. For instance, when $r_i = 1$, $r_o = 1.05$, $\theta_0 = 7\,\text{deg}$ and $T = 0.518$, we have $r_\nu = 0.9975$ and the optimal maneuver is a HHCM rendezvous. The optimum occurs at the orbit of radius $r = 0.9975$, when the HHCM maneuver just becomes feasible. Figures 13(a) and 13(b) show the variation of ΔV for both of these cases. The optimal rendezvous in either case occurs at an orbit other than r_i or r_o. Figures 13(c) and 13(d) show the detail of the region where the minimum occurs. Note that the function is relatively flat in this region. We may use this result to compute (analytically) HHCM maneuvers that are only slightly suboptimal.

In order to confirm the occurrence of cooperative (but not HHCM) rendezvous when the time to rendezvous is very short, we repeated the analysis for the cases $T = 0.40$ and $T = 0.45$. The results are shown in Fig. 14. In both cases the HHCM maneuver is sub-optimal. In the first case the optimum occurs at $r = 0.9487$ whereas $r_\nu = 0.6735$. In the second case the optimum occurs at $r = 0.9765$ whereas $r_\nu = 0.8143$. Note that the optimal cooperative maneuver in

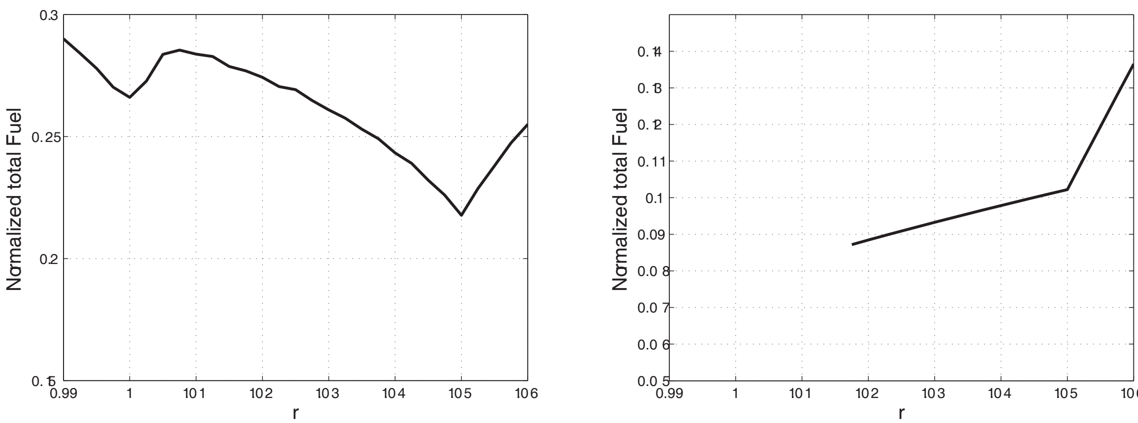


(a) $r_i = 1.00, r_o = 1.05, \theta_0 = 60\,\text{deg}, T = 0.40$

(b) $r_i = 1.00, r_o = 1.05, \theta_0 = 60\,\text{deg}, T = 0.45$

Figure 14. Optimal cooperative (but non-HHCM) rendezvous for short time of flight.

either case is substantially cheaper than the corresponding HHCM solution.

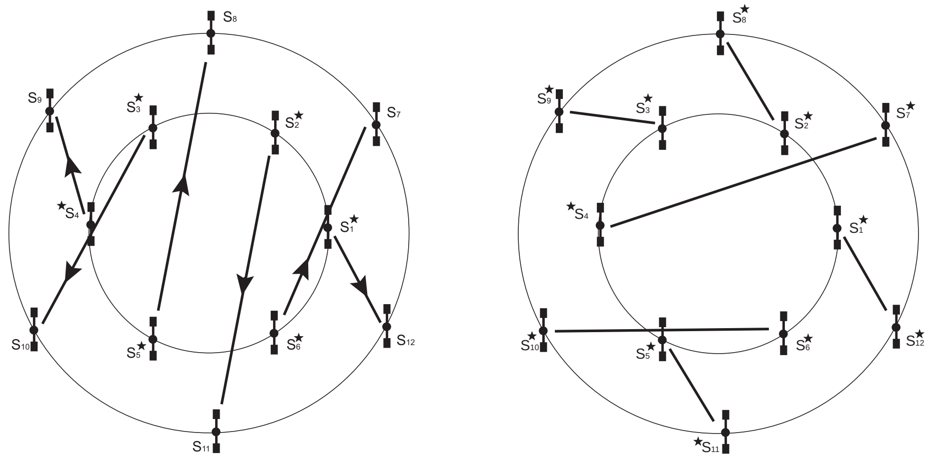


(a) Sufficient Fuel Case

(b) Insufficient Fuel Case

Figure 15. Variation of normalized fuel expenditure with r.

Thus far we have only considered the minimization of ΔV. Let us now consider the fuel expenditure during the cooperative rendezvous between the satellites in orbits $r_i = 1$ and $r_o = 1.05$ and with angle of separation $\theta_0 = 60$ deg. Fig. 15 shows the variation of fuel expenditure with r. The fuel expenditure has been normalized by dividing the total fuel expenditure by the maximum of the fuel capacities of the satellite. In the first case, the satellites have 25 and 5 units of fuel and the time for rendezvous is $T = 1.5$. The fuel-deficient satellite has enough fuel to complete the non-cooperative rendezvous. The plot shows that there are indeed two local minima at $r = 1$ and $r = 1.05$, that is, at the end orbits. In this case, the fuel is minimized at the same location as the total ΔV, and the optimal rendezvous is a HPCM maneuver. In the second case, the time of rendezvous is $T = 3.0$ and the fuel content of the satellites are 25



(a) Non-cooperative Strategy (b) Cooperative Strategy

Figure 16. Optimal assignments for P2P refueling.

and 1.3 units respectively. Had the fuel-deficient satellite had enough fuel to complete a non-cooperative Hohmann transfer, the optimal rendezvous would have taken place at $r = 1.00$. However, the 1.3 units of fuel is not sufficient for the fuel-deficient satellite to complete a non-cooperative Hohmann transfer. Consequently, the optimal rendezvous takes place at $r = 1.0175$. The fuel-deficient satellite uses all of its fuel in order to transfer to an orbit that is as close as possible to the fuel-sufficient satellite. The optimal rendezvous is HHCM.

The next and final example demonstrates the benefits of cooperative rendezvous for P2P refueling for a large number of satellites in two different coplanar circular orbits.

Example 2 *P2P refueling for a constellation of* 12 *satellites in two circular orbits, each orbit having* 6 *satellites.*

Consider a satellite constellation of two circular orbits, one at an altitude of 1000 Km and the other at an altitude of 1075 Km. The lower orbit is populated with 6 satellites $s_1, s_2, \ldots, s_6$, all of which are fuel-sufficient. The upper orbit has 6 fuel-deficient satellites $s_7, s_8, \ldots, s_{12}$. The orbital slots of the satellites in the lower orbit are given by $\Phi_1 = \{0, 60, 120, 180, 210, 270, 330\}\,\mathrm{deg}$. The fuel content of these satellites are 27, 29, 30, 29.5, 28.5 and 28 units respectively. Similarly, the orbital slots of the satellites in the upper orbit are given by $\Phi_2 = \{30, 90, 150, 210, 270, 330\}\,\mathrm{deg}$. The fuel content of these satellites are 0.75, 0.70, 0.80, 0.60 and 0.65 units respectively. Each satellite has a minimum fuel requirement of $\underline{f}_i = 12$ units, where $i = 1, \ldots, 12$, while the maximum amount of fuel is $\bar{f}_i = 30$ units. Each satellite has a permanent structure of $m_{si} = 70$ units, and a characteristic constant of $c_{0i} = 2943$ m/s. If all satellites are restricted to engage in non-cooperative rendezvous, then the optimal P2P assignments are $s_1 \leftrightarrow s_{12}$, $s_2 \leftrightarrow s_{11}$, $s_3 \leftrightarrow s_{10}$, $s_4 \leftrightarrow s_9$, $s_5 \leftrightarrow s_8$, and $s_6 \leftrightarrow s_7$. The corresponding total fuel expenditure during the refueling process is 12.80 units. The optimal solution is depicted in Fig. 16(a). In all these maneuvers, the fuel-sufficient satellites initiate a non-cooperative rendezvous with their corresponding fuel-deficient satellites and deliver fuel to them.

If the satellites are allowed to engage in cooperative rendezvous, the optimal C-P2P assignments are $s_1 \leftrightarrow s_{12}$, $s_2 \leftrightarrow s_8$, $s_3 \leftrightarrow s_9$, $s_4 \leftrightarrow s_7$, $s_5 \leftrightarrow s_{11}$, and $s_6 \leftrightarrow s_{10}$. All of these refueling transactions involve cooperative rendezvous at an intermediate orbit. The total fuel expenditure is given by 11.38 units, implying a reduction of fuel consumption by 11% when we allow for cooperative rendezvous between the satellites. Figure 16(b) shows the optimal C-P2P assignments. For instance, satellites s_1 and s_2 meet on the orbit of radius $r = 1.0042$ after both preforming Hohmann transfers. In fact, for all of the refueling transactions, the satellites engage in HHCM rendezvous. In each case, although the allotted time is enough for a non-cooperative Hohmann transfer between the participating satellites, the fuel-deficient satellites do not have enough fuel to complete the non-cooperative Hohmann transfer. Instead, they move as close as possible to the orbit of a fuel-sufficient satellite by expending all their fuel.

CONCLUSIONS

In this paper we have studied the problem of cooperative impulsive rendezvous between two satellites in circular orbits. We assume that the terminal orbit for each rendezvous maneuver is circular. We have specifically looked at cooperative maneuvers that are comprised of two Hohmann transfers (HHCM), or a Hohmann transfer and a phasing maneuver (HPCM). We derive bounds on the phasing angle that makes the HPCM rendezvous the optimal solution. We illustrate via an example that when the time to rendezvous is not sufficient for a non-cooperative Hohmann transfer between the satellites, the optimal rendezvous that yields the minimum ΔV is the Hohmann-phasing cooperative maneuver. However, if the time for the maneuver allows for a non-cooperative Hohmann transfer, the optimal solution is non-cooperative. In both these cases, we assume that the time of transfer is sufficient for a (phase-free) Hohmann transfer to take place between the orbits of the satellites. In other words, if the time of transfer is sufficient to allow for a phase-free Hohmann transfer between the orbits, the optimal solution is either a non-cooperative Hohmann transfer or is a cooperative maneuver that is comprised of a Hohmann transfer and a phasing maneuver. If the time to complete the rendezvous is too short, a cooperative Hohmann rendezvous in a lower orbit is the most likely optimal candidate. In all cases the optimal solution and associated cost can be calculated explicitly. Finally, we demonstrated the benefits of HHCM and HPCM rendezvous for cooperative peer-to-peer refueling.

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