Analytic Solution of Euler's Equations of Motion for an Asymmetric Rigid Body

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The problem of the time evolution of the angular velocity of a spinning rigid body, subject to torques about three axes, is considered. An analytic solution is derived that remains valid when no symmetry assumption can be made. The solution is expressed as a first-order correction to a previous solution, which required a symmetry or near-symmetry assumption. Another advantage of the new solution (over the former) is that it remains valid for large initial conditions of the transverse angular velocities.

1 Introduction

In recent years a considerable amount of effort has been devoted to the development of a comprehensive theory that will allow a better understanding of the complex dynamic behavior associated with the motion of rotating bodies. A cornerstone in this effort is the development of analytic solutions that can describe — at least qualitatively — the problem dynamics. The system of the associated equations, the celebrated Euler's equations of motion for a rigid body, consists of three nonlinear, coupled differential equations, the complete, general, solution of which is still unknown. Special cases for which solutions have been found include the torque-free rigid body and the forced symmetric case. Solutions for these two particular cases were known for some time and have been reported in the literature (Golubev, 1953; Leimanis, 1965; Greenwood, 1988). The discovery of complete solutions for those and other special cases, initially gave rise to optimism that a general solution was in sight; however, since then progress has been remarkably slow. The conjecture that studying several special cases would eventually lead to a comprehensive theory of the problem proved to be false. In fact, a complete characterization of the motion of a rotating solid body quickly turned out to be a formidable task, eluding the wit of some of the most prominent mathematicians of our time; see for example Leimanis (1965) and Golubev (1953) and the references therein. Even today, it is still not clear that a complete solution even exists. (It is well known, however, that for the closely related problem of a heavy rigid body rotating about a fixed point, integrability is possible for only four special cases (Golubev, 1953).)

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Most attempts to generalize the previous results were confined to some kind of perturbation approach of the known and well understood integrable, torque-free, and/or symmetry cases (Kraige and Junkins, 1976; van der Ha, 1984; Kane and Levinson, 1987; Or, 1992). Recently, significant results made it possible to extend the existing theory to include the attitude motion of a *near-symmetric spinning* rigid body under the influence of *constant* (Longuski, 1991; Tsiotras and Longuski, 1991) and *time-varying torques* (Tsiotras and Longuski, 1991,1993; Longuski and Tsiotras, 1993). The purpose of the present work is to extend these results to a spinning body with *large asymmetries*, subject to large initial angular velocities.

2 Equations and Assumptions

We are mainly interested in the problem of spin-up maneuvers of a non-symmetric spinning body in space, subject to constant torques and nonzero initial conditions. To this end, let M_1, M_2 and M_3 denote the torques (in the body-fixed frame) acting on a rigid body, and let ω_1, ω_2 and ω_3 denote the angular velocity components in the same frame. Then Euler's equations of motion for a rotating rigid body with principal axes at the center of mass are written as:

$$\dot{\omega}_1 = \frac{M_1}{I_1} + \frac{I_2 - I_3}{I_1}\omega_2\omega_3 \tag{1a}$$

$$\dot{\omega}_2 = \frac{M_2}{I_2} + \frac{I_3 - I_1}{I_2}\omega_3\omega_1 \tag{1b}$$

$$\dot{\omega}_3 = \frac{M_3}{I_3} + \frac{I_1 - I_2}{I_3}\omega_1\omega_2 \tag{1c}$$

These equations describe the evolution in time of the angular velocity components ω_1, ω_2, ω_3 in the body-fixed frame. For consistency we will assume that the spin axis is the 3-axis, corresponding to the maximum moment of inertia, and also that the ordering of the other principal moments of inertia is given by the inequalities $I_3 > I_1 \geq I_2$.

We henceforth define the *spin-up problem* of a rigid body rotating about its 3-axis, when the following conditions are satisfied:

$$M_1^2 + M_2^2 \leq M_3^2 \quad \text{and} \quad I_1^2\omega_1^2(0) + I_2^2\omega_2^2(0) \leq I_3^2\omega_3^2(0) \tag{2}$$

along with the condition that $sgn(M_3) = sgn(\omega_3(0))$. (Here sgn denotes the signum function defined as usual by $sgn(x) = +1$ for $x > 0$ and $sgn(x) = -1$ for $x < 0$.) This last condition simply states the requirement for spin-up, whereas the inequalities in (2) restrict the angles of the torque vector and the angular momentum vector at time $t = 0$ to be less than or equal to 45 deg from the body 3-axis. This, according to the previous discussion, implies that the transverse torques M_1, M_2, as well as the initial conditions $\omega_1(0), \omega_2(0)$, are considered as undesired deviations or perturbations from the *pure spin case*, namely when $M_1 = M_2 = \omega_1 = \omega_2 \equiv 0$. In practical problems these unwanted deviations tend to remain indeed small throughout the maneuver.

One more parameter needs to be introduced in order to describe the "relative effect" of the two inequalities (2) in the solution. This parameter, defined by

$$\rho_0 \triangleq \frac{\sqrt{M_1^2 + M_2^2}}{I_3\omega_3^2(0)}$$

describes the angle of departure of the angular momentum vector from its initial state (the angular momentum vector bias). During a spin-up maneuver (Longuski *et. al*, 1989), the angular momentum vector traces out a spiral path about a line in inertial space having an angle ρ_0 from the inertial 3-axis (see Fig. 1). The angle ρ_0 is small for cases where the transverse torques are "small" compared with the quantity $I_3\omega_3^2(0)$. The formula for ρ_0 applies even for asymmetric bodies as long as the angle of departure is small and the body is spinning about a stable principal axis. Throughout this work we assume that ρ_0 is relatively small, an assumption that is usually true for most satellite applications.

3 Analytic Solution

3.1 Assumptions

If we assume a near-symmetric (or symmetric) *spinning* rigid body with the spin axis being its axis of near-symmetry (or symmetry), then the near-symmetry assumption ($I_1 \approx I_2$) allows one to neglect the second term on the right-hand side of (1c) and therefore safely assume that the solution of ω_3 is approximated very closely by

$$\omega_3^0(t) = (M_3/I_3)\,t + \omega_3(0) \tag{3}$$

This allows the decoupling and complete integration of equations(1). The use of complex notation facilitates the analysis (Tsiotras and Longuski, 1991,1992,1993; Longuski and Tsiotras, 1993). Also introducing, for convenience, the new independent variable $\tau \triangleq \omega_3^0(t)$, one then writes the differential equation for the transverse angular velocities ω_1 and ω_2 as

$$\Omega' + i\,\rho\,\tau\Omega = F \tag{4}$$

where prime denotes differentiation with respect to τ, $i = \sqrt{-1}$ and where (Tsiotras and Longuski, 1993)

$$\Omega \triangleq \omega_1\sqrt{k_2} + i\,\omega_2\sqrt{k_1} \tag{5a}$$

$$F \triangleq (M_1/I_1)(I_3/M_3)\sqrt{k_2} + i\,(M_2/I_2)(I_3/M_3)\sqrt{k_1} \tag{5b}$$

$$\rho \triangleq k\,(I_3/M_3), \quad k_1 \triangleq (I_3 - I_2)/I_1, \quad k_2 \triangleq (I_3 - I_1)/I_2, \quad k \triangleq \sqrt{k_1 k_2} \tag{5c}$$

Integrating (4) one obtains the solution for ω_1 and ω_2 from

$$\begin{aligned}\Omega(\tau) &= \Omega_0 \exp(i\tfrac{\rho}{2}\tau^2) + \exp(i\tfrac{\rho}{2}\tau^2)\,F\int_{\tau_0}^{\tau}\exp(-i\tfrac{\rho}{2}u^2)du \\ &= \Omega_0 \exp(i\tfrac{\rho}{2}\tau^2) + \exp(i\tfrac{\rho}{2}\tau^2)\,F\sqrt{\frac{\pi}{\rho}}\{sgn(\tau)E(\sigma) - sgn(\tau_0)E(\sigma_0)\}\end{aligned} \tag{6}$$

where $\tau_0 = \omega_3^0(0)$ and $\Omega_0 \triangleq \Omega(\tau_0)\exp(-i\frac{\rho}{2}\tau_0^2)$ and where $\Omega(\tau_0)$ is the initial condition at $\tau = \tau_0$ ($t = 0$). The function $E(\cdot)$ in (6) represents the complex Fresnel integral of the first kind (Abramowitz and Stegun, 1972; Tsiotras and Longuski, 1993), defined by

$$E(x) \triangleq \int_0^x \exp(-i\frac{\pi}{2}u^2)\;du$$

The parameter σ is defined by $\sigma \triangleq \tau\sqrt{\rho/\pi}$. (Here we obviously assume $M_3 > 0$, so that $\rho > 0$; the case when $\rho < 0$ can be treated similarly (Tsiotras and Longuski, 1993).) Equation (6) gives the complete solution for the transverse components of the angular velocity ω_1 and ω_2 in the body-fixed frame, and for the symmetric case it provides the *exact* solution. For the nonsymmetric case, the accuracy of solution (6) depends on the "smallness" of the product $\omega_1\omega_2$, which will be discussed next.

3.2 The Effect of Asymmetry

In order to have a measure of the body asymmetry, we introduce the following *asymmetry parameter*

$$e \triangleq \frac{I_1 - I_2}{I_3}$$

Because of the well-known relationship $I_2 + I_3 \geq I_1$ between the moments of inertia (Greenwood, 1988) — for the assumed ordering of the principal axes — the parameter e takes values in the range $0 \leq e \leq 1$. The case of $e = 0$ corresponds to complete symmetry (about the 3-axis), whereas the extreme case of $e = 1$ (not considered here) corresponds to complete asymmetry (about the 3-axis). For the latter case one has $I_3 = I_1$ and $I_2 = 0$, i.e. the body resembles a thin rod along the 2-axis. (In the current work when we discuss a non-symmetric problem we have in mind values of e greater than 0.1 and perhaps as high as about 0.7.)

We note in passing, that the validity of solution (6) is not confined to near-symmetry cases. To understand this point, notice that the neglected term

$$g(t) = \frac{I_1 - I_2}{I_3}\,\omega_1(t)\omega_2(t) \tag{7}$$

in equation (1c) is small not only for the near-symmetry case, i.e. when $I_1 \approx I_2$, but also when the transverse angular velocity components ω_1 and ω_2 are small. This is indeed the case, for example, for a spin-stabilized vehicle (spinning about its 3-axis), when ω_1 and ω_2 tend to remain small for all times. For the pure spin case of a symmetric body we have of course that $\omega_1 = \omega_2 \equiv 0$. This fact justifies the often used terminology in the spacecraft dynamics literature which refers to ω_1 and ω_2 as the *angular velocity error components*. The previous assumption about the smallness of the term in equation (7) however does not incorporate the case where the initial conditions $\omega_1(0)$ and $\omega_2(0)$ are large (compared to the initial spin rate $\omega_3(0)$). As can be easily verified in such cases, the initial error

$$g(0) = \frac{I_1 - I_2}{I_3}\,\omega_1(0)\omega_2(0)$$

propagates quickly and renders the analytic solution inaccurate after a short time interval. On the other hand, as can also be easily verified through numerical simulations, analytic solutions based on the near-symmetry assumption remain insensitive to large inertia differences, as long as the initial conditions for ω_1 and ω_2 are zero. Therefore, the intent of this paper is to extend the analytic solutions for a near-symmetric rigid body subject to constant torques (Tsiotras and Longuski, 1991), when *both* large asymmetries and nonzero initial conditions for the transverse angular velocities are considered at the same time. In such a case, the neglected term (7) may not be negligible and the exact solution for ω_3 may depart significantly from the linear solution (3) for ω_3.

3.3 General Theory

A first correction to the linear zero-order solution $\omega_3^0(\cdot)$ is obtained as follows. Using solution (6), the differential equation for ω_3 can be approximated by

$$\dot{\omega}_3 = M_3/I_3 + e\,\omega_1^0\omega_2^0 \tag{8}$$

where the superscript zero denotes the zero-order solution of (1) (i.e. the solution with the term (7) in (1c) neglected). From (6) we can equivalently replace equation (8) with

$$\omega_3' = 1 + \epsilon\; Im[(\Omega^0)^2] \tag{9}$$

where $\epsilon \triangleq (I_1 - I_2)/2M_3k$ and $\Omega^0 = \omega_1^0\sqrt{k_2} + i\;\omega_2^0\sqrt{k_1}$, prime again denotes differentiation with respect to the independent variable $\tau = \omega_3^0$ and $Im(\cdot)$ denotes the imaginary part of a complex number. Under these assumptions and integrating (9) with respect to τ, one gets for the first-order correction for ω_3:

$$\omega_3(\tau) = \tau + \epsilon\; Im\int_{\tau_0}^{\tau}[\Omega^0(u)]^2 du \tag{10}$$

The first-order solution for ω_1 and ω_2 is then given by the solution of the differential equation

$$\Omega' + i\,\rho\,\omega_3(\tau)\,\Omega = F \tag{11}$$

Integrating, one obtains

$$\begin{aligned}\Omega(\tau) &= \Omega(\tau_0)\exp[i\,\rho\int_{\tau_0}^{\tau}\omega_3(u)\;du] \\ &+ \exp[i\,\rho\int_{\tau_0}^{\tau}\omega_3(u)\;du]\,F\int_{\tau_0}^{\tau}\exp[-i\,\rho\int_{\tau_0}^{u}\omega_3(v)\;dv]\;du\end{aligned} \tag{12}$$

Notice that this expression provides the general exact solution for $\Omega(\cdot)$ if knowledge of the time history of ω_3 is available *a priori*. Of course, this is not possible, in general, because of the coupled character of equations (1). However, we will assume that equation (10) gives a very accurate approximation of the exact ω_3, which can be used in (12).

The zero-order solution $\Omega^0(\cdot)$ required in (10) is given in (6). From the asymptotic expansion of the complex Fresnel integral one has that (Abramowitz and Stegun, 1972)

$$E(x) = \frac{1-i}{2} - \frac{\exp(-i\pi x^2/2)}{i\pi x}\{1 - \frac{1}{i\pi x^2} + \frac{1\cdot 3}{(i\pi x^2)^2} - \cdots\} \tag{13}$$

Thus, the Fresnel integral appearing in (6) can be approximated by

$$\int_{\tau_0}^{\tau}\exp(-i\,\tfrac{\rho}{2}u^2)\,du \approx \frac{i}{\rho}[\frac{\exp(-i\,\frac{\rho}{2}\tau^2)}{\tau} - \frac{\exp(-i\,\frac{\rho}{2}\tau_0^2)}{\tau_0}]$$

Substituting this expression in (6) and carrying out the algebraic manipulations, one approximates $[\Omega^0(\cdot)]^2$ by

$$[\Omega^0(\tau)]^2 = r_0\exp(i\rho\tau^2) + \frac{r_1}{\tau^2} + r_2\frac{\exp(i\,\frac{\rho}{2}\tau^2)}{\tau}$$

where r_j $(j = 0, 1, 2)$ are complex constants given by

$$r_0 \triangleq \left[\Omega_0 - i\,\frac{F}{\rho\tau_0}\exp(-i\,\tfrac{\rho}{2}\tau_0^2)\right]^2$$

$$r_1 \triangleq -\frac{F^2}{\rho^2}$$

$$r_2 \triangleq 2\,i\frac{F}{\rho}\left[\Omega_0 - i\,\frac{F}{\rho\tau_0}\exp(-i\,\tfrac{\rho}{2}\tau_0^2)\right]$$

The integral of $[\Omega^0(\cdot)]^2$ is then given by

$$\int_{\tau_0}^{\tau}\left[\Omega^0(u)\right]^2 du = r_0 h_0(\tau_0,\tau;\rho) + r_1 h_1(\tau_0,\tau) + r_2 h_2(\tau_0,\tau;\rho)$$

where

$$h_0(\tau_0,\tau;\rho) \triangleq \int_{\tau_0}^{\tau}\exp(i\rho u^2)\,du = \sqrt{\frac{\pi}{2\rho}}[sgn(\tau)\bar{E}(\tau\sqrt{2\rho/\pi}) - sgn(\tau_0)\bar{E}(\tau_0\sqrt{2\rho/\pi})]$$

$$h_1(\tau_0,\tau) \triangleq \int_{\tau_0}^{\tau}\frac{du}{u^2} = \frac{1}{\tau_0} - \frac{1}{\tau}$$

$$h_2(\tau_0,\tau;\rho) \triangleq \int_{\tau_0}^{\tau}\frac{\exp(i\frac{\rho}{2}u^2)}{u}\,du = \frac{1}{2}[Ei(\tfrac{\rho}{2}\tau_0^2) - Ei(\tfrac{\rho}{2}\tau^2)]$$

where bar denotes the complex conjugate and where

$$Ei(x) \triangleq \int_x^{\infty}\frac{e^{iu}}{u}\,du$$

is called the exponential integral (Abramowitz and Stegun, 1972). The integrals of h_j $(j = 0, 1, 2)$ can be then computed as follows

$$H_0(\tau_0,\tau;\rho) \triangleq \int_{\tau_0}^{\tau} h_0(\tau_0,u;\rho)\,du = -\sqrt{\frac{\pi}{2\rho}}sgn(\tau_0)\bar{E}(\tau_0\sqrt{2\rho/\pi})\,(\tau - \tau_0) + \sqrt{\frac{\pi}{2\rho}}sgn(\tau_0)\int_{\tau_0}^{\tau}\bar{E}(u\sqrt{2\rho/\pi})\,du \tag{14}$$

where the last integral is given by

$$\int \bar{E}(\tau\sqrt{2\rho/\pi})\,d\tau = \tau\bar{E}(\tau\sqrt{2\rho/\pi}) + \frac{i}{\sqrt{2\rho\pi}}\exp(i\rho\tau^2) \tag{15}$$

Similarly,

$$H_1(\tau_0,\tau) \triangleq \int_{\tau_0}^{\tau} h_1(\tau_0,u)\,du = \frac{\tau - \tau_0}{\tau_0} - \ln\left(\frac{\tau}{\tau_0}\right)$$

and

$$H_2(\tau_0,\tau;\rho) \triangleq \int_{\tau_0}^{\tau} h_2(\tau_0,u;\rho)\,du = \frac{1}{2}Ei(\tfrac{\rho}{2}\tau_0^2)\,(\tau - \tau_0) - \frac{1}{2}\int_{\tau_0}^{\tau} Ei(\tfrac{\rho}{2}u^2)\,du \tag{16}$$

where the last integral can be evaluated using

$$\int Ei(\tfrac{\rho}{2}\tau^2)\,d\tau = \tau Ei(\tfrac{\rho}{2}\tau^2) + 2\int \exp(i\tfrac{\rho}{2}u^2)\,du \tag{17}$$

We therefore have that the integral of $\omega_3(\cdot)$ required in (12) is given by

$$\int_{\tau_0}^{\tau} \omega_3(u)\, du = \frac{\tau^2}{2} - \frac{\tau_0^2}{2} + \epsilon\, Im(\sum_{j=0}^{2} r_j H_j) \tag{18}$$

Equation (18) gives the final expression for the integral of $\omega_3(\cdot)$ required in (12).

In order to proceed with our analysis, we need to calculate the last integral in (12). Any attempt to evaluate this integral by direct substitution of (18) into

$$I_\omega(\tau_0, \tau; \rho) = \int_{\tau_0}^{\tau} \exp[-i\,\rho \int_{\tau_0}^{u} \omega_3(v)\; dv]\; du \tag{19}$$

is futile. Notice however, that because of the oscillatory behavior of the kernel of the integral (19) one needs to know only the secular behavior of (18) in order to capture the essential contribution of (19). Thus, we next compute the secular effect due to the integrals $H_0(\tau_0, \tau; \rho)$ and $H_2(\tau_0, \tau; \rho)$. The integral $H_1(\tau_0, \tau)$ already has the required form.

From (14) and (15) and the asymptotic approximation of the Fresnel integral (13) one can immediately verify that, within a first order approximation, the integral $H_0(\tau_0, \tau; \rho)$ behaves as

$$H_0(\tau_0, \tau; \rho) \sim A_0^0 + A_0^1\, \tau \tag{20}$$

where

$$A_0^0 \triangleq -\frac{i}{2\rho}\exp(i\rho\tau_0^2), \qquad A_0^1 \triangleq \sqrt{\frac{\pi}{2\rho}}\left[\frac{1+i}{2} - sgn(\tau_0)\bar{E}(\tau_0\sqrt{2\rho/\pi})\right]$$

Similarly, using (16) and (17) and the fact that $\lim_{x\to\infty} Ei(x) = 0$, one can show that the integral $H_2(\tau_0, \tau; \rho)$ behaves, to a first order approximation, as

$$H_2(\tau_0, \tau; \rho) \sim A_2^0 + A_2^1\, \tau$$

where

$$A_2^0 \triangleq -\sqrt{\frac{\pi}{\rho}}\left[\frac{1+i}{2} - sgn(\tau_0)\bar{E}(\tau_0\sqrt{\rho/\pi})\right], \qquad A_2^1 \triangleq \frac{1}{2}Ei(\tfrac{\rho}{2}\tau_0^2)$$

Also writing the integral $H_1(\tau_0, \tau)$ in the form

$$H_1(\tau_0, \tau) = A_1^0 + A_1^1\tau - \ln(\tau) \tag{21}$$

where

$$A_1^0 \triangleq \ln(\tau_0) - 1, \qquad A_1^1 \triangleq \frac{1}{\tau_0}$$

we have for the secular part of (18)

$$\int_{\tau_0}^{\tau} \omega_3(u)\, du = \frac{\tau^2}{2} - \frac{\tau_0^2}{2} + b_0 + b_1\tau + b_2\ln(\tau) \tag{22}$$

where

$$\begin{aligned} b_0 &\triangleq \epsilon\, Im(r_0 A_0^0 + r_1 A_1^0 + r_2 A_2^0) \\ b_1 &\triangleq \epsilon\, Im(r_0 A_0^1 + r_1 A_1^1 + r_2 A_2^1) \\ b_2 &\triangleq -\epsilon\, Im(r_1) \end{aligned}$$

Unfortunately, the logarithmic term in (22) leads to an intractable form when substituted into (19) and we therefore approximate the former expression by

$$\int_{\tau_0}^{\tau} \omega_3(u)\, du \approx \frac{\tau^2}{2} - \frac{\tau_0^2}{2} + b_3 + b_1\tau \tag{23}$$

where $b_3 = \epsilon\, Im(r_0 A_0^0 - r_1 + r_2 A_2^0)$. This approximation amounts to the assumption that $ln(\tau/\tau_0) \approx 0$ in equation (21). Since the logarithmic function is dominated everywhere by any polynomial, we expect the error committed in passing from (22) to (23) to be relatively small, at least as $\tau \to \infty$. Using (23) in (19) we can finally write

$$\begin{aligned}
\int_{\tau_0}^{\tau} \exp[-i\,\rho \int_{\tau_0}^{u} \omega_3(v)\; dv]\; du \;&\approx\; \exp(i\frac{\rho}{2}\gamma_0) \int_{\tau_0}^{\tau} \exp[-i\frac{\rho}{2}(u+b_1)^2]\, du \\
&=\; \exp(i\frac{\rho}{2}\gamma_0)\sqrt{\frac{\pi}{\rho}}[sgn(\tilde{\tau})E(\tilde{\sigma}) - sgn(\tilde{\tau}_0)E(\tilde{\sigma}_0)]
\end{aligned}$$

where $\gamma_0 \triangleq \tau_0^2 + b_1^2 - 2b_3$, $\tilde{\tau} = \tau + b_1$ and $\tilde{\sigma} = \tilde{\tau}\sqrt{\rho/\pi}$.

3.4 Simplified Analysis

The analysis of the previous subsection allows for a direct calculation of the solution $\Omega(\cdot)$ from (12). In most cases encountered in practice, however, a simplified version of the previous procedure is often adequate. For example, for the case when $\rho_0 \ll 1$ (see Fig. 1) the initial conditions have a more profound effect than the acting torques in solution (6), and we can take just the asymptotic contribution of the non-homogeneous part of (6) to approximate the zero-order solution $\Omega^0(\cdot)$. Writing

$$\Omega^0(\tau) \approx \left\{\Omega_0^0 + F\sqrt{\frac{\pi}{\rho}} sgn(\tau_0)[(1-i)/2 - E(\sigma_0)]\right\} \exp(i\frac{\rho}{2}\tau^2) \triangleq B_0 \exp(i\frac{\rho}{2}\tau^2),$$

substituting this expression into (10), and approximating $E(\cdot)$ by its asymptotic limit $E(\infty) = (1-i)/2$, as $x \to \infty$, we get for $\omega_3(\cdot)$ that

$$\omega_3(\tau) = \tau + \alpha_0$$

where α_0 is the constant

$$\alpha_0 \triangleq \epsilon\sqrt{\pi/2\rho}\; sgn(\tau_0)\; Im\left\{B_0^2[\frac{1}{2}(1+i) - \bar{E}(\tau_0\sqrt{2\rho/\pi})]\right\}$$

We can therefore write for the first-order solution (11) of the transverse angular velocities

$$\Omega(\tau) = \hat{\Omega}_0^0 \exp[i\rho h(\tau)] + \exp[i\rho h(\tau)] F \int_{\tau_0}^{\tau} \exp[-i\rho h(u)]\; du \tag{24}$$

where

$$h(\tau) \triangleq \frac{\tau^2}{2} + \alpha_0\tau$$

and $\hat{\Omega}_0^0 \triangleq \Omega(\tau_0)\exp[-i\rho h(\tau_0)]$. From equations (6),(12) and (24) it is seen that the first-order solution for the transverse angular velocities ω_1 and ω_2 may be obtained in the same form as the zero-order solution; the initial condition of τ, however, has to be modified to include α_0. In other words, (24) can also be written in the more explicit form

$$\Omega(\tau) = \tilde{\Omega}_0^0 \exp(i\frac{\rho}{2}\tilde{\tau}^2) + \exp(i\frac{\rho}{2}\tilde{\tau}^2) F\sqrt{\frac{\pi}{\rho}}\{sgn(\tilde{\tau})E(\tilde{\sigma}) - sgn(\tilde{\tau}_0)E(\tilde{\sigma}_0)\} \tag{25}$$

where now $\tilde{\Omega}_0^0 \triangleq \Omega(\tau_0)\exp(-i\frac{\rho}{2}\tilde{\tau}_0^2)$, $\tilde{\tau} = \tau + \alpha_0$ and $\tilde{\sigma} \triangleq \tilde{\tau}\sqrt{\rho/\pi}$. It is interesting to compare equation (25) with (6). We see that the two equations have exactly the same form, but that equation (25) has a frequency shift which depends directly on ϵ.

4 A Formula for the Error

In this section we derive an error formula for the zero*th* order solution derived in (6), that is, we seek an expression for the difference between the exact solution and the approximate solution for the angular velocities, obtained by omitting the term $(I_1 - I_2)\omega_1\omega_2/I_3$ in equation (1c). Throughout this section, for notational convenience, we rewrite equations (1) in the form

$$\dot{x}_1 = a_1 x_2 x_3 + u_1 \tag{26a}$$
$$\dot{x}_2 = a_2 x_3 x_1 + u_2 \tag{26b}$$
$$\dot{x}_3 = a_3 x_1 x_2 + u_3 \tag{26c}$$

where a_j, x_j and u_j $(j = 1, 2, 3)$ are defined by

$$x_1 \triangleq I_1\omega_1, \qquad x_2 \triangleq I_2\omega_2, \qquad x_3 \triangleq I_3\omega_3 \tag{27a}$$
$$u_1 \triangleq M_1, \qquad u_2 \triangleq M_2, \qquad u_3 \triangleq M_3 \tag{27b}$$
$$a_1 \triangleq \frac{I_2 - I_3}{I_2 I_3}, \qquad a_2 \triangleq \frac{I_3 - I_1}{I_3 I_1}, \qquad a_3 \triangleq \frac{I_1 - I_2}{I_1 I_2} \tag{27c}$$

We also rewrite the equations that describe the reduced (zero*th* order) system in the form

$$\dot{x}_1^0 = a_1 x_2^0 x_3^0 + u_1 \tag{28a}$$
$$\dot{x}_2^0 = a_2 x_3^0 x_1^0 + u_2 \tag{28b}$$
$$\dot{x}_3^0 = u_3 \tag{28c}$$

Given any positive number $T \in [0, \infty)$, our aim is to compute the error between the solutions of (26) and (28) over the time interval $0 \le t \le T$. We can rewrite equations (26) and (28) in the compact form

$$\dot{x} = f(x) + g(x) \tag{29}$$
$$\dot{x}^0 = f(x^0) \tag{30}$$

where $x = (x_1, x_2, x_3), x^0 = (x_1^0, x_2^0, x_3^0)$ and $f : \mathbb{R}^3 \to \mathbb{R}^3$ and $g : \mathbb{R}^3 \to \mathbb{R}^3$ are the functions defined by

$$f(x) \triangleq \begin{bmatrix} a_1 x_2 x_3 + u_1 \\ a_2 x_3 x_1 + u_2 \\ u_3 \end{bmatrix}, \qquad g(x) \triangleq \begin{bmatrix} 0 \\ 0 \\ a_3 x_1 x_2 \end{bmatrix} \tag{31}$$

We also assume that (29) and (30) are subject to the same initial conditions, that is, $x(0) = x^0(0)$. Throughout the following discussion $||\cdot||$ will denote the usual Euclidean norm (or 2-norm) on $\mathbb{R}^3$, namely, $||x|| \triangleq (x_1^2 + x_2^2 + x_3^2)^{1/2}$.

Lemma 4.1 *The solution of the exact system (26), satisfies the inequality*

$$||x(t)|| \leq ||u||\, T + ||x(0)|| \triangleq B$$

for all $0 \leq t \leq T$*, where* $u = (u_1, u_2, u_3)$*.*

Proof. Multiplying equation (26a) by x_1, equation (26b) by x_2 and equation (26c) by x_3 and adding, and since $a_1 + a_2 + a_3 = 0$, one gets that

$$\dot{x}_1 x_1 + \dot{x}_2 x_2 + \dot{x}_3 x_3 = u_1 x_1 + u_2 x_2 + u_3 x_3$$

In other words,

$$\frac{1}{2}\frac{d}{dt}||x||^2 = < u, x > \tag{32}$$

where $< .,. >$ denotes the usual inner product on $\mathbb{R}^3$, namely $< x, y > \triangleq \sum_{j=1}^{3} x_j y_j$. Using the Cauchy-Schwarz inequality (32) gives

$$\frac{1}{2}\frac{d}{dt}||x||^2 \leq ||u|| \cdot ||x|| \tag{33}$$

The 2-norm $||\cdot||$ is a differentiable function on $\mathbb{R}^3$, so the differential inequality (33) can be solved for $||x(\cdot)||$ (here u is constant) to obtain

$$||x(t)|| \leq ||u||\, t + ||x(0)||, \qquad 0 \leq t \leq T \tag{34}$$

In particular, $||x(t)|| \leq \sup_{0 \leq t \leq T} ||u||\, t + ||x(0)|| = B$, as claimed. □

This result should not be surprising. If one looks carefully, ones sees that the vector x defined in equation (27a) is the angular momentum vector $\vec{\mathbf{H}}$, which obeys the equation $d\vec{\mathbf{H}}/dt = \vec{\mathbf{M}}$. This differential equation for $\vec{\mathbf{H}}$ requires that both $\vec{\mathbf{H}}$ and $\vec{\mathbf{M}}$ be expressed in the same coordinate system and that differentiation be carried out with respect to an inertial reference frame. In general, given the components M_1, M_2, M_3 of $\vec{\mathbf{M}}$ in the body-fixed system, does not provide any immediate information about the components of $\vec{\mathbf{M}}$ with respect to another (inertial) coordinate system. However the *magnitude* of $\vec{\mathbf{M}}$ is independent of the coordinate system. Equation (34) simply states the relationship between the magnitude of the acting torques and the time history of the magnitude of the angular momentum vector $\vec{\mathbf{H}}$. With this observation in mind, one can easily re-derive (34) starting from Euler's equation $d\vec{\mathbf{H}}/dt = \vec{\mathbf{M}}$.

Lemma 4.2 *Given a fixed positive number* T*, there exist positive constants* M *and* L*, such that the following conditions hold for all* $0 \leq t \leq T$*.*

$$||g(x(t))|| \leq M \tag{35a}$$

$$||f(x(t)) - f(x^0(t))|| \leq L\, ||x(t) - x^0(t)|| \tag{35b}$$

Proof. From Lemma 4.1 we have that for $t \in [0, T]$ all solutions of (26) satisfy $||x(t)|| \leq B$. In particular, $|x_j(t)| \leq B$, $j = 1, 2, 3$, for all $t \in [0, T]$, where $|\cdot|$ denotes absolute value. Clearly,

$$||g(x(t))|| = |a_3|\,|x_1(t)|\,|x_2(t)| \leq |a_3|\,B^2 \triangleq M$$

Now let $B_1 \triangleq \max_{0 \leq t \leq T}\{|x_1^0(t)|, |x_2^0(t)|, |x_3^0(t)|\}$. This number can be computed immediately, since the solution $x^0(\cdot)$ of the system (28) is known. If we define $B_0 \triangleq \max\{B, B_1\}$, then we have that all solutions of (29) and (30) are confined inside the region $\{x \in \mathbb{R}^3 : ||x|| \leq B_0\}$ for all $0 \leq t \leq T$. The partial derivatives of f are then bounded by

$$|\partial f_i / \partial x_j| \leq R, \qquad 1 \leq i, j \leq 3, \quad 0 \leq t \leq T, \quad ||x|| \leq B_0$$

where $R \triangleq \max\{|a_1|, |a_2|\}\,B_0$ and by the Mean Value Theorem (Boothby, 1986), we have

$$||f(x(t)) - f(x^0(t))|| \leq 3\,R\,||x(t) - x^0(t)||$$

for all $0 \leq t \leq T$, and therefore (35b) is satisfied with $L \triangleq 3\,R$. This completes the proof. □

Lemma 4.2 implies that over the time interval $0 \leq t \leq T$ the function g is bounded by M and the function f is Lipschitz continuous with Lipschitz constant L. These two results allow us, as the next theorem states, to find an *explicit* bound for the error of the approximate solution.

Theorem 4.1 *Let T be a given positive number and let M, L as in Lemma 4.2. Then, for $x(0) = x^0(0)$, the error between the solutions $x(\cdot)$ and $x^0(\cdot)$ over the time interval $0 \leq t \leq T$ is given by*

$$||x(t) - x^0(t)|| \leq \frac{M}{L} e^{Lt}, \qquad 0 \leq t \leq T$$

Proof. Subtract (30) from (29) to obtain

$$\dot{x} - \dot{x}^0 = f(x) - f(x^0) + g(x) \tag{36}$$

By integrating (36) and considering norms, we obtain the following estimate

$$||x(t) - x^0(t)|| \leq \int_0^t ||f(x(s)) - f(x^0(s))||\,ds + \int_0^t ||g(x(s))||\,ds$$

Now, use of Lemma 4.2 implies that

$$||x(t) - x^0(t)|| \leq L \int_0^t ||x(s) - x^0(s)||\,ds + Mt \tag{37}$$

Now, applying Gronwall's Lemma (Hille, 1969) to (37) gives finally that

$$||x(t) - x^0(t)|| \leq \frac{M}{L} e^{Lt} \tag{38}$$

This completes the proof. □

This error formula, involves only known quantities of the problem (time duration T of the maneuver, inertias I_1, I_2, I_3, acting torques M_1, M_2, M_3 and initial conditions $x_1(0), x_2(0)$ and $x_3(0)$) and can be computed immediately once these data are given. For most of the applications encountered in spacecraft problems it turns out however, that (38) provides a very conservative estimate of the true error, but usually this is the most one can expect, without resorting to the numerical solution of (1).

Having established an error formula for the angular momentum, it is an easy exercise to find a corresponding error formula for the angular velocity vector, using the simple relation between the two. Thus, the following corollary holds.

Corollary 4.1 *Let* $K \stackrel{\triangle}{=} \max\{1/I_1, 1/I_2, 1/I_3\}$. *The error between the exact and the zeroth order solutions of the angular velocities over the time interval* $0 \leq t \leq T$ *is given by*

$$||\omega(t) - \omega^0(t)|| \leq \frac{K\,M}{L} e^{Lt} \tag{39}$$

Proof. It follows immediately from the fact that

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} 1/I_1 & 0 & 0 \\ 0 & 1/I_2 & 0 \\ 0 & 0 & 1/I_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

and therefore that

$$||\omega(t)|| \leq \max\{1/I_1, 1/I_2, 1/I_3\}\, ||x(t)|| = K\, ||x(t)||$$

for all $0 \leq t \leq T$. □

5 Numerical Example

The analytic solution of Euler's equations of motion for an asymmetric rigid body is applied to a numerical example. The mass properties of the spinning body are chosen as $I_1 = 3500\ kg \cdot m^2$, $I_2 = 1000\ kg \cdot m^2$ and $I_3 = 4200\ kg \cdot m^2$. The constant torques are assumed to be $M_1 = -1.2\ N \cdot m$, $M_2 = 1.5\ N \cdot m$, $M_3 = 13.5\ N \cdot m$ and the initial conditions are set to $\omega_1(0) = 0.1\ r/s$, $\omega_2(0) = -0.2\ r/s$ and $\omega_3(0) = 0.33\ r/s$. Figure 2 shows the zero-order solution versus the exact solution for ω_1. Figure 3 shows the first-order solution versus the exact solution for ω_1. Notice the dramatic improvement of the first-order solution over the zero-order solution for this problem, where the asymmetry parameter, e, is 60%. The results for the ω_2 component of the angular velocity are similar. Finally, Fig. 4 presents the zero-order and the first-order solutions (given by (3) and (10) respectively) versus the exact solution for ω_3. Note the bias between the zero-order and the first-order secular terms (which is responsible for the frequency shift between Fig. 2 and Fig. 3). We mention at this point, although not demonstrated here, that the solution also remains valid for *spin-down* maneuvers, as long as the initial conditions $\omega_1(0)$ and $\omega_2(0)$ are small and as long as the spin rate ω_3 does not pass through zero. These observations are in agreement with the previous results of Tsiotras and Longuski (1991).

6 Conclusions

Analytic solutions are derived for the angular velocity of a non-symmetric spinning body subject to external torques about three axes. The solution is developed as a first-order correction to previously reported solutions for a near-symmetric rigid body. The near-symmetric solution provides accurate results even when the asymmetry is large, provided the initial condition for the transverse angular velocity is near zero. The problem of the asymmetry becomes apparent when the initial transverse angular velocities are not small. It is shown that the first-order solution for the angular velocity takes a simple form and is very accurate, at least for the cases when the effect of the transverse torques is not too large compared with the effect of the initial conditions. The formulation of the problem therefore allows for nonzero initial conditions in the transverse angular velocities, in conjunction with large asymmetries. Finally, an explicit formula for the bound of the error of the approximate solution is derived and a numerical example demonstrates the accuracy of the proposed analytic solution.

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Figure 1: Angular momentum behavior during spin-up (Longuski *et. al*, 1989)

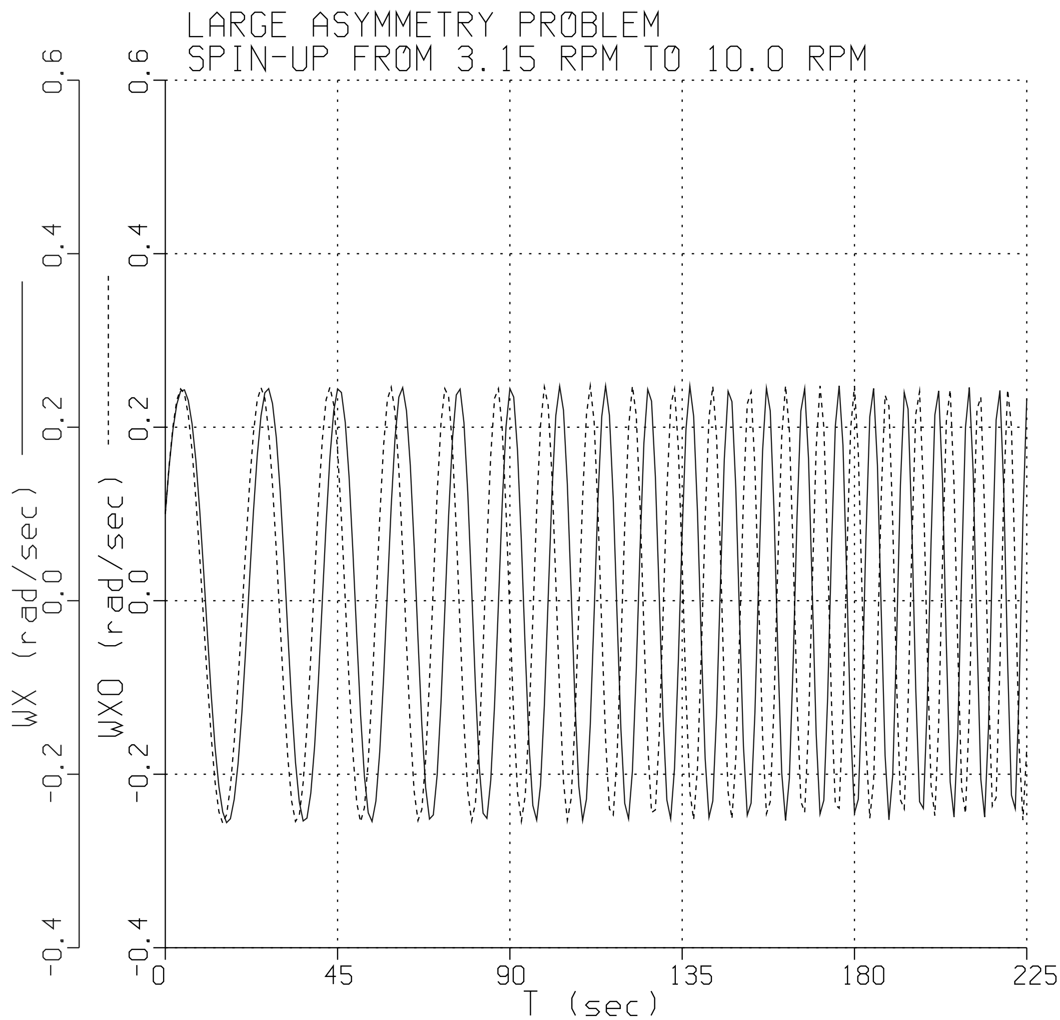

Figure 2: Zero-order vs. exact solutions for ω_1

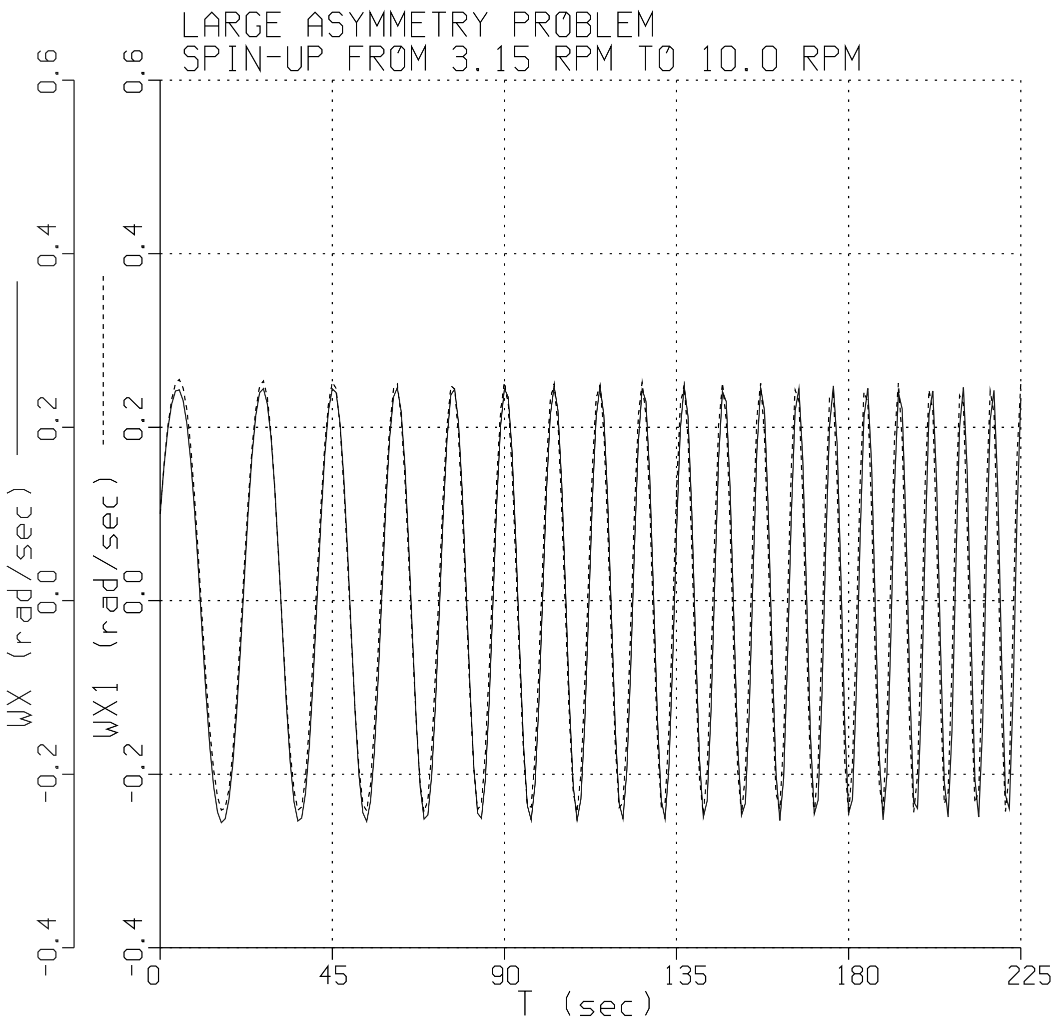

Figure 3: First-order vs. exact solutions for ω_1

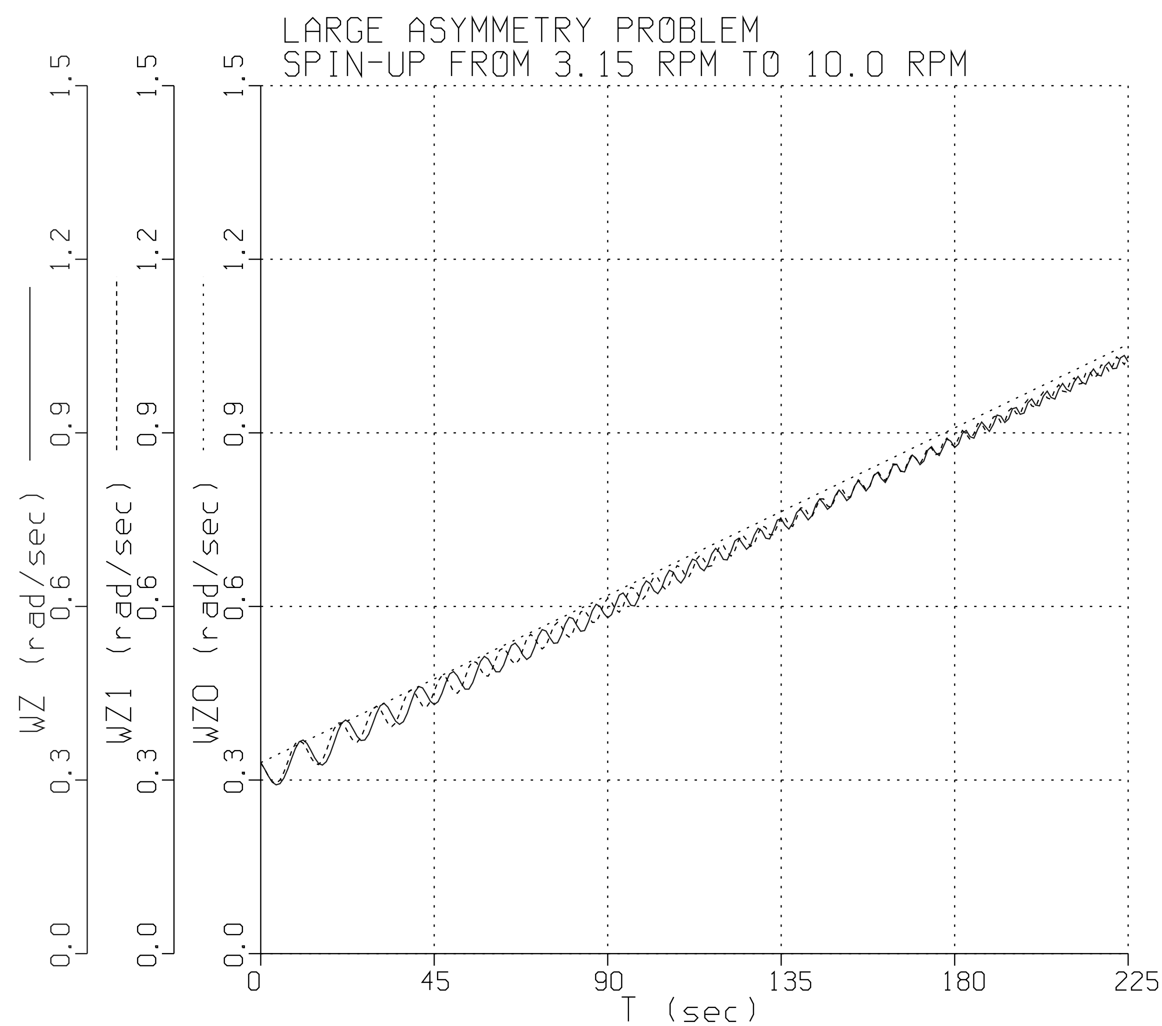

Figure 4: Zero-order and first-order vs. exact solutions for ω_3