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Analytic Solutions for a Spinning Rigid Body Subject to Time-Varying Body-Fixed Torques, Part II: Time-Varying Axial Torque

In this paper we extend the methodology developed in Part I in order to accommodate the case of an axial time-varying torque (in addition to the two transverse time-varying torques) acting on a rotating rigid body. The analytic solutions thus derived describe the general attitude motion of a near-symmetric rigid body subject to time-varying torques about all three body-fixed axes.

1 Introduction

The problem of attitude motion of a rotating rigid body is a very difficult one, and no general solution is known to exist. This comes as no surprise since the equations of motion consist of a system of six nonlinear, coupled differential equations and one cannot expect to find solutions to these equations without some simplifying assumptions. Typically such assumptions include axial symmetry of the body (see the survey by Janssens, 1980), a torque-free body (Junkins et al., 1973; Cochran and Shu, 1983; Kane and Levinson, 1987), or a weakly perturbed torqued body (Kraige and Junkins, 1976). For the case of a *self-excited* rigid body (Leimanis, 1965; Longuski, 1984), the general problem of forced motion is somewhat alleviated because the acting torques do not depend on the body orientation. Considerable progress has been achieved in recent years in developing analytic and semi-analytic solutions for this particular problem. Up to now, only rigid bodies subject to *constant* torques have been considered (Longuski, 1980; Van der Ha, 1984; Tsiotras and Longuski, 1991a). Solutions like these can help us better understand the underlying behavior of the dynamics of rotating rigid bodies under the influence of external torques, but they do not address the more complicated problem when the external torques vary with time. The purpose of this paper is to extend previous solutions to include cases when the torques are no longer required to be constants. Part I examined the case when the transverse torques were modeled as *polynomial* functions of time. Part II extends these results to include a time-varying axial torque, as well. This study is complementary to the study by Tsiotras and Longuski (1991b), where the transverse torques are modeled as *periodic* functions or can be expressed in terms of Fourier series. In such models, virtually all physically realizable torques can be accommodated.

We outline the general analytic procedure as follows. In the case of an axial time-varying torque, with the two transverse time-varying torques, the same simplifying assumptions made in Part I are used to reduce the solution of the problem to quadratures. The integrals appearing, however, are much more complicated than those in the constant axial-torque case, and there is no a priori guarantee that one can evaluate these integrals in closed form; the success of this task depends on each particular problem at hand. In many cases, a judicious choice of a new independent variable allows one to avoid much of the work involved in the evaluation of these integrals. When such a choice is not possible or difficult, a general method that allows approximate evaluation of a very large class of integrals is provided that can accommodate most of the cases of practical interest. The main difficulty of this method relies upon the construction of an explicit representation of the inverse transformation between the new independent variable and the original independent variable, i.e. time. Sufficient conditions when this is possible are discussed and a method to find this inverse transformation in terms of series expansions is also presented. The method of analytic continuation is used to extend the solution for all times outside the initial interval of the convergence of the series. Numerical simulations provide an indication of the accuracy and the validity of the solutions for both the angular velocity vector and the Eulerian angles.

2 Solution for the Angular Velocities and Eulerian Angles

The analysis in Part I was based on the assumption that the axial torque M_z is constant. The case when M_z is a function of the independent variable is much more complicated and the ability to find closed-form expressions for the solution in such a case is not assured. To be more specific, using the near-symmetry assumption $I_x \approx I_y$ in the differential equation for the spin velocity ω_z, one can readily find ω_z by simple integration as

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$$\omega_z(t) \approx \int_0^t \frac{M_z(u)}{I_z}\, du + \omega_z(0). \quad (1)$$

In general, M_z in the above expression is not constant, but it is given as a prescribed function of time. According to the analysis presented in Part I, the transverse components of the angular velocity ω_x and ω_y can be found by solving the following linear differential equation:

$$\frac{d\Omega}{dt} - ik\omega_z\Omega = M(t) \quad (2)$$

where the variables in the above expression have been introduced in Part I, and the forcing function $M(t)$ is given by $M(t) \triangleq (M_x(t)/I_x)\sqrt{k_y} + i(M_y(t)/I_y)\sqrt{k_x}$. One then can easily give the solution for the transverse angular velocities from (2) as

$$\Omega(t) = \Omega_0 \exp\left[ik\int_0^t \omega_z(u)du\right] + \exp\left[ik\int_0^t \omega_z(u)du\right]\int_0^t M(u)\exp\left[-ik\int_0^u \omega_z(v)dv\right]du. \quad (3)$$

It is clear from (3), that although the calculation of Ω and therefore for ω_x and ω_y has been reduced to quadratures, the practical applicability of the solution is limited by one's ability, ingenuity, or even luck to calculate these integrals in closed form. The same statement holds for the analytic solution of the Eulerian angles, as well. As shown in Part I, a small angle assumption reduces the problem of the kinematics to the solution of the two linear differential equations $\dot{\phi} = \omega_z$ and $\dot{\phi} + i\omega_z\phi = \omega$, the solution of which can be given in terms of quadratures by

$$\phi_z(t) = \int_0^t \omega_z(u)du + \phi_z(0) \quad (4)$$

$$\phi(t) = \phi_0 \exp\left[-i\int_0^t \omega_z(u)du\right] + \exp\left[-i\int_0^t \omega_z(u)du\right]\int_0^t \omega(u)\exp\left[i\int_0^u \omega_z(v)dv\right]du \quad (5)$$

where $\phi \triangleq \phi_x + i\phi_y$ and $\omega \triangleq \omega_x + i\omega_y$. For the special case when all three acting torques are constant, it can be shown that the evaluation of (3) can be performed exactly in terms of Fresnel integrals; the evaluation of (5) is more complicated, but very accurate approximations have been found by Tsiotras and Longuski (1991a). No general statements can be made however about the form of the solutions, when the acting external torques are arbitrary functions of time. Nevertheless, when ω_z does not become zero, one can apply the methodology developed in Part I for the constant M_z case, with a few modifications, to develop solutions that approximate the angular velocity vector solution for the time-varying M_z case as accurately as one may wish. The case of a nonvanishing spin rate ω_z could be, for example, the case of a spin-up maneuver ($M_z > 0$) from a positive value of initial spin rate ($\omega_z(0) > 0$), or the case of a spin-down maneuver ($M_z < 0$) from a negative value of initial spin rate ($\omega_z(0) < 0$). We proceed now to the development of the analysis for such a case.

2.1 Transformation of Independent Variable. Assuming hereafter that $\omega_z(t) \neq 0$ (for all t), we introduce the new independent variable $\tau = \tau(t)$ by

$$\tau \triangleq \int_0^t \omega_z(u)du. \quad (6)$$

In most instances we will suppress the explicit dependence of τ on t, but one should always keep in mind Eq. (6). When for reasons of clarity we need to stress this dependence, we will do so. The previous expression implies in particular that $d\tau/dt = \omega_z$. Therefore, one can write the differential Eq. (2) that governs the behavior for Ω, in terms of the new independent variable τ as follows:

$$\frac{d\Omega}{d\tau} - ik\Omega = F(\tau) \quad (7)$$

where now, however, the forcing function F has to be considered in terms of the new independent variable, that is,

$$F(\tau) \triangleq \frac{M(t(\tau))}{\omega_z(t(\tau))}. \quad (8)$$

In such a case, the solution for $\Omega(\tau)$ is given by

$$\Omega(\tau) = \Omega_0 \exp(ik\tau) + \exp(ik\tau)\int_0^\tau F(u)\exp(-iku)\, du. \quad (9)$$

It should be mentioned that the integral in (9) is equivalent to the integral in (3); however, the integration, now performed with respect to the new independent variable τ, has simplified the form of the integrand significantly. The difficulty of the aforementioned methodology relies, however, upon the explicit evaluation of the forcing term $F(\tau)$ in Eq. (7), or equivalently, upon our ability to express M/ω_z as a function of the new independent variable τ as in Eq. (8). This is, in general, not a trivial task and it is not assured that one can find such an analytic representation for F. There are cases, however, that such an expression is readily available by the nature of the problem. Hence, one can evaluate the complete solution of Ω without the need to express t as a function of τ, but one can use his intuition, or the physics of the problem in order to introduce, where possible, new variables that will simplify the problem considerably. For example, in a number of problems, the underlying behavior of the external torques is based on ϕ_z, so the choice of ϕ_z as our new independent variable can reduce the problem to the evaluation of elementary functions. In fact, as was shown by Bois (1986), in many cases involving spinning satellites the external torque depends upon the rotation angle of the satellite ϕ_z, or one can expand the dominating torque in terms of ϕ_z alone. For all these cases, the choice of ϕ_z as the new independent variable looks very promising. In general, however, it is not possible to express the acting torques as a function of the new independent variable directly, so a more general procedure is required that will encompass all the cases of practical interest. This will involve solving for t in terms of τ from Eq. (6). Sufficient conditions that allow one to obtain such an expression are given by the Inverse Function Theorem, and are discussed later on.

2.2 Angular Velocities. At this point, we will make the assumption that the external axial torques M_x, M_y, and M_z are *analytic* functions of time, that is, have convergent power series expansions about each point over their domain of definition. In other words, we assume that M_x, M_y, and M_z can be represented by the power series expansions

$$M_x(t) = \sum_{n=0}^{\infty} M_{x,n}t^n, \quad M_y(t) = \sum_{n=0}^{\infty} M_{y,n}t^n, \quad M_z(t) = \sum_{n=0}^{\infty} M_{z,n}t^n. \quad (10)$$

In these expressions the variable t should be considered as a variable in the *complex* plane even though, physically, t represents the real variable time. This is necessary, because the singularities that determine the convergence properties of power series lie in the complex plane and thus, power series have to be introduced having a complex argument. From analytic function theory (Knopp, 1956) it is known that the convergence of the above series is *absolute* and *uniform*. The importance of

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the absolute convergence is the fact, that one can operate with absolutely convergent series for the most part, as with ordinary sums. That is, one can add, subtract, and take (Cauchy) products, and rearrange the terms of absolutely convergent series without worrying about convergence, i.e., completely *formally*. On the other hand, uniform convergence of the series will imply, among other things, that one is allowed to integrate the series termwise, and the sum of the integrals will converge to the integral of the function that this series represents.

Using Eqs. (1), (6), and (10) we have that

$$\omega_z(t)=\sum_{n=0}^{\infty}\omega_{z,n}t^n, \quad \text{with } \omega_{z,n}=\frac{M_{z,n-1}}{I_z n} \tag{11}$$

for $n=1,2,3,\ldots$ and $\omega_{z,0}\triangleq\omega_z(0)$ and

$$\tau(t)=\sum_{n=0}^{\infty}b_n t^n, \quad \text{with } b_n=\frac{\omega_{z,n-1}}{n} \tag{12}$$

for $n=1,2,3,\ldots$ and $b_0\triangleq 0$.

Of course, for the special case when M_z is a polynomial in t, the foregoing series terminate after a finite number of terms. In order to write F as a function of the new independent variable τ, one has to invert Eq. (6) to write t as a function of τ and then compose this function into F. From the previous discussion, and Eqs. (8) and (10), it follows that one can represent the function M/ω_z with a power series

$$\frac{M(t)}{\omega_z(t)}=F(t)=\sum_{n=0}^{\infty}\bar{f}_n t^n \quad \text{for } |t|<r \tag{13}$$

where r is the radius of convergence of the series. Inside the disk of radius r (t considered complex), the series converges absolutely and uniformly. It is clear that, in general, F will have a finite radius of convergence, since any zeros of ω_z will be singular points of F in the complex plane. In fact, the radius of convergence equals the minimum distance to the origin of any zero of ω_z. It is clear that because of the assumption that $\omega_z(t)\neq 0$, none of the singularities of F can lie on the real axis. This assumption is essential, because later we will allow the use of analytic continuation in order to extend the solution to arbitrary time intervals. Nevertheless, there are singularities on the complex plane and for the case when M_z is a polynomial of order n, there are exactly $n+1$ singularities of F in the complex plane, all of which are *poles*. In general, the inverse function of (6) does not exist globally, unless the function $\tau=\tau(t)$ is bijective (one-to-one and onto). The Inverse Function Theorem states, however, that one is always able to invert a function in the vicinity of points where the derivative does not vanish. For our problem a *sufficient* condition for this to be true is that ω_z does not change sign inside the domain of its definition. This implies that $\tau(t)$ is monotonic, and thus (in fact globally) invertible. Therefore, inverting Eq. (12) we get

$$t(\tau)=\sum_{n=1}^{\infty}c_n\tau^n \tag{14}$$

where the coefficients of the inverted series can be calculated recursively, from the relationships (Knopp, 1956)

$$\begin{aligned}
&c_1=1/b_1, \quad c_2=-b_2/b_1^3, \quad c_3=(2b_2^2-b_1b_3)/b_1^5\\
&c_4=(5b_1b_2b_3-b_1^2b_4-5b_2^3)/b_1^7\\
&c_5=(6b_1^2b_2b_4+3b_1^2b_3^2+14b_2^4-b_1^3b_5-21b_1b_2^2b_3)/b_1^9\\
&c_6=(7b_1^3b_2b_5+7b_1^3b_3b_4+84b_1b_2^3b_3\\
&\quad -b_1^4b_6-28b_1^2b_2^2b_4-28b_1^2b_2b_3^2-42b_2^5)/b_1^{11}\\
&\vdots
\end{aligned} \tag{15}$$

Notice that the choice of $b_0=0$, which is arbitrary at this point, results in $c_0=0$, a condition that will be explained in detail later. Substitution now of Eq. (14) into Eq. (13) results in the following power series of F in terms of the new independent variable τ

$$F(t(\tau))=\sum_{n=0}^{\infty}f_n\tau^n \text{ for } |\tau|<R \tag{16}$$

where R is the radius of convergence of the composed series, completely determined by r.

A necessary and sufficient condition for the composition of $t(\tau)$ into F in (16) is that $|c_0|<r$, where as mentioned earlier r is the radius of convergence of (13). We therefore see that the choice of $c_0=0$ in (14) was more than a simple convenience, but rather it was imposed so that the method is well defined for every r. For a more complete analysis on the inversion and composition of power series as well as in general analytic function theory, see Knopp (1956) and Hille (1973).

Using Eq. (16) one can compute the required integral in (9) by

$$\int_0^\tau F(u)\exp(-iku)du=\sum_{n=0}^{\infty}f_n\int_0^\tau u^n\exp(-iku)\,du. \tag{17}$$

Integrals of this form can be evaluated using the recurrence formula

$$\int u^n\exp(-iu)du=iu^n\exp(-iu)-in\int u^{n-1}\exp(-iu)du \qquad n=1,2,3,\ldots. \tag{18}$$

Therefore, the complete solution for Ω is given by

$$\Omega(\tau)=\Omega_0\exp(ik\tau)+\exp(ik\tau)\sum_{n=0}^{\infty}f_nI_n(\tau;k) \tag{19}$$

where

$$I_n(\tau;k)\triangleq\int_0^\tau u^n\exp(-iku)\,du, \quad n=0,1,2,\ldots. \tag{20}$$

2.3 Eulerian Angles. Using the transformation introduced in (6) and the relationship between ω and Ω established in Part I, one writes the solution for the Eulerian angles (5) in the form

$$\begin{aligned}
\phi(\tau)&=\phi_0\exp(-i\tau)+\exp(-i\tau)\int_0^\tau\frac{\omega(u)}{\omega_z(t(u))}\exp(iu)du\\
&=\phi_0\exp(-i\tau)+\exp(-i\tau)\left[k_1\int_0^\tau\frac{\Omega(u)}{\omega_z(t(u))}\exp(iu)du\right.\\
&\qquad\left.+k_2\int_0^\tau\frac{\Omega^*(u)}{\omega_z(t(u))}\exp(iu)du\right]
\end{aligned} \tag{21}$$

where

$$k_1\triangleq\left(\sqrt{k_x}+\sqrt{k_y}\right)/2k \text{ and } k_2\triangleq\left(\sqrt{k_x}-\sqrt{k_y}\right)/2k.$$

Thus, the following two integrals need to be evaluated,

$$\int_0^\tau\frac{\Omega(u)}{\omega_z(t(u))}\exp(iu)du, \quad \int_0^\tau\frac{\Omega^*(u)}{\omega_z(t(u))}\exp(iu)du, \tag{22}$$

where the solution for Ω has been found from (19). Recall now, that according to the discussion for the solution for the angular velocities, $1/\omega_z(t)$ is analytic along the real line and using Eq. (14) we can write its expansion in terms of τ in the form of a power series

$$\frac{1}{\omega_z(t(\tau))}=\sum_{m=0}^{\infty}a_m\tau^m \quad \text{for } |\tau|<R. \tag{23}$$

If we substitute the above expression into (21) then after some

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straightforward algebraic manipulation, we may write for the first integral in (22)

$$\int_0^\tau \frac{\Omega(u)}{\omega_z(t(u))} \exp(iu)\,du = \Omega_0 \sum_{m=0}^{\infty} a_m I_m^*(\tau;k+1) + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} F_n a_m J_n^m(\tau;k) \quad (24)$$

where

$$J_n^m(\tau;k) \triangleq \int_0^\tau u^m \exp[i(k+1)u] I_n(u;k)\,du \quad (25)$$

and $I_m^*(\tau;k+1)$ can be computed by Eqs. (18) and (20). Using also the recurrence formula for $I_n(\tau;k)$ given in (18) one gets for fixed m, a recurrence formula for $J_n^m(\tau;k)$ as follows:

$$\begin{aligned} J_n^m(\tau;k) &= \frac{i}{k}\int_0^\tau u^{n+m}\exp(iu)\,du \\ &\quad - i\frac{n}{k}\int_0^\tau u^m \exp[i(k+1)u] I_{n-1}(u;k)\,du \\ &= \frac{i}{k} I_{n+m}^*(\tau;1) - i\frac{n}{k} J_{n-1}^m(\tau;k) \end{aligned} \quad (26)$$

for $n = 1,2,3, \ldots$ and $m = 0,1,2, \ldots$.

For $n=0$ and for some fixed integer m, the first term of the sequence of the integrals $J_n^m(\tau;k)$ is computed by

$$\begin{aligned} J_0^m(\tau;k) &= \int_0^\tau u^m \exp[i(k+1)u] I_0(u;k)\,du \\ &= \frac{i}{k}[I_m^*(\tau;1) - I_m^*(\tau;k+1)], \quad m = 0,1,2, \ldots \end{aligned} \quad (27)$$

because from (20) we have

$$I_0(\tau;k) = \int_0^\tau \exp(-iku)\,du = \frac{i}{k}[\exp(-ik\tau) - 1]. \quad (28)$$

It is not difficult to show that the second integral in (22), required for the solution of the Eulerian angles, is given similarly by

$$\int_0^\tau \frac{\Omega^*(u)}{\omega_z(t(u))} \exp(iu)\,du = \Omega_0^* \sum_{m=0}^{\infty} a_m I_m^*(\tau;k-1) + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} F_n^* a_m J_n^m(\tau;-k). \quad (29)$$

Equation (21) along with Eqs. (22), (24), and (29) give the solution to the Eulerian angles. In practice, one has to approximate the solution by truncating the infinite series in (24) and (29); however, because of the established absolute convergence of these series, the approximation can be made as accurately as one wishes. For most practical applications this will suffice.

3 Extension of the Solution

As mentioned at the beginning of the present analysis, the series representation of F in (13) has in general a finite radius of convergence r, which is determined by the zero of ω_z closest to the origin in the complex plane. It is therefore clear that one cannot expect the previous methodology to still be valid for time $t \geq r$. In fact, as we approach this value $t = r$ the rate of convergence of the series (16) becomes slower and slower, and therefore more and more terms are needed in order to achieve the same accuracy. One way to circumvent this difficulty is to use the principle of *analytic continuation* for F in order to find $F(t)$ for all t.

In short, the methodology goes as follows. Since, by assumption, ω_z has no zero on the real axis, one can choose a time $t_1 < r$ and expand F about $t = t_1$. The resulting series has a radius of convergence r_1 and therefore we can choose time t_2 such that $t_1 < t_2 < t_1 + r_1$ and expand F about $t = t_2$, etc. In general, given an expansion of F about a point t_n with radius of convergence r_n, we can find a point t_{n+1} such that $t_n < t_{n+1} < t_n + r_n$ and such that if we expand F in a power series about this point, the series will have a radius of convergence r_{n+1} which will be determined by the closest zero to t_{n+1} of ω_z in the complex plane.

Formalizing the method, we assume that given a time T, we seek the value of the solution (2) for all $t \in [0, T]$. It is obvious from (3) and (9) that the solution merely consists of evaluating the integral in (17)

$$R(\tau) \triangleq \int_0^\tau F(u)\exp(-iku)\,du \quad (30)$$

where $\tau = \tau(t)$. As a first step, we partition the interval $[0, T]$ into N subintervals $[t_n, t_{n+1}]$, $n = 0, 1, 2, \ldots, N-1$, such that $t_0 = 0$ and $t_N = T$. The time points $t_1, t_2, \ldots, t_{N-1}$ have been chosen by the procedure explained in the previous paragraph. Now given $t \in [0, T]$, let $t \in [t_n, t_{n+1}]$ for some n. Then we can write $R(\tau)$ as follows:

$$R(\tau) = R(\tau_n) + \int_{\tau_n}^\tau F(u)\exp(-iku)\,du \quad (31)$$

where $\tau_n = \tau(t_n)$. If we now let $\tau = \tau_n + \Delta\tau$, we can write

$$\begin{aligned} R(\tau) &= R(\tau_n) + \int_0^{\Delta\tau} F(\tau_n + u)\exp[-ik(\tau_n + u)]\,du \\ &= R(\tau_n) + \exp(-ik\tau_n)\int_0^{\Delta\tau} F(\tau_n + u)\exp(-iku)\,du. \end{aligned} \quad (32)$$

It is easy to show that $\Delta\tau$ is related to $\Delta t = t - t_n$ by

$$\Delta\tau = \int_0^{\Delta t} \omega_z(t_n + u)\,du. \quad (33)$$

Therefore, if we expand $M(t)/\omega_z(t)$ about $t = t_n$ and use the series inversion formula for (33) we can write $F(\tau_n + \Delta\tau)$ in terms of $\Delta\tau$, that is, we obtain an expansion of $F(\tau)$ about τ_n. Substitution of this expansion in (9) will give for the solution of Ω

$$\begin{aligned} \Omega(\tau) &= [\Omega_0 + R(\tau_n)]\exp(ik\tau) \\ &\quad + \exp(ik\Delta\tau)\int_0^{\Delta\tau} F_n(u)\exp(-iku)\,du \end{aligned} \quad (34)$$

where the explicit dependence of τ on t has been suppressed and F_n represents the series expansion of $F(\tau)$ about τ_n, that is, $F(\tau_n + \Delta\tau) = F_n(\Delta\tau) = \sum_{m=0}^{\infty} F_{n,m}(\Delta\tau)^m$. It is important to keep in mind that for $t_n \leq t \leq t_{n+1}$, or equivalently for $0 < \Delta\tau \leq \tau_{n+1} - \tau_n$ the independent variable in (34) is $\Delta\tau$. In order to stress this point we can rewrite (34) in the equivalent form

$$\begin{aligned} \Omega(\tau_n + \Delta\tau) &= \Omega_{0,n}\exp(ik\Delta\tau) + \exp(ik\Delta\tau)\int_0^{\Delta\tau} F_n(u)\exp(-iku)\,du \\ &= \Omega_{0,n}\exp(ik\Delta\tau) + \exp(ik\Delta\tau)\sum_{m=0}^{\infty} F_{n,m} I_m(\Delta\tau;k) \end{aligned} \quad (35)$$

where $\Omega_{0,n} \triangleq [\Omega_0 + R(\tau_n)]\exp(ik\tau_n)$ is constant for $t \in [t_n, t_{n+1}]$. The last equation is of the same form as Eq. (19). The explicit representation of the solution for Ω in terms of $\Delta\tau$ has an advantage that will become clear shortly, when we seek analytic solutions for the Eulerian angles.

In accordance with the previous discussion, it is necessary to extend also the solution for the Eulerian angles for all t using analytic continuation, since the expansion (23) is limited

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by the singularities of $1/\omega_z$ in the complex plane. To be more precise, given a time T, we are interested in the evaluation of the integral (36) for all $t \in [0, T[$:

$$Q(\tau) \triangleq \int_0^{\tau} \frac{\Omega(u)}{\omega_z(t(u))} \exp(iu) du. \quad (36)$$

The methodology to compute $Q(\tau)$ is along the same lines used for the evaluation of $R(\tau)$. Since the singularities of F in (13) are completely determined by the singularities of $1/\omega_z$, it will suffice to use the *same partition* of the interval $[0, T]$ into N subintervals $[t_n, t_{n+1}]$, $n = 0, 1, 2, \ldots, N-1$, as before. Now given $t \in [t_n, t_{n+1}]$, for some n, one writes $Q(\tau)$ as follows:

$$Q(\tau) = Q(\tau_n) + \exp(i\tau_n) \int_0^{\Delta\tau} \frac{\Omega(\tau_n + u)}{\omega_z(t(\tau_n + u))} \exp(iu) du. \quad (37)$$

By expanding ω_z about $t = t_n$ and using (33) we get the following expansion of $1/\omega_z$ about τ_n

$$\frac{1}{\omega_z(t(\tau_n + \Delta\tau))} = \sum_{m=0}^{\infty} a_{n,m} (\Delta\tau)^m \quad (38)$$

where the subscript n denotes expansion about τ_n. Using the previous equation along with Eqs. (37) and (35) we get that

$$\int_0^{\Delta\tau} \frac{\Omega(\tau_n + u)}{\omega_z(t(\tau_n + u))} \exp(iu) du = \Omega_{0,n} \sum_{m=0}^{\infty} a_{n,m} I_m^*(\Delta\tau; k+1) + \sum_{p=0}^{\infty} \sum_{m=0}^{\infty} F_{n,p} a_{n,m} J_p^m(\Delta\tau; k). \quad (39)$$

This expression is of the same form as the one in (24) and can be computed using the recursive formulas (18) and (26). We mention in passing that for the second integral in (22) one has, similarly,

$$\int_0^{\Delta\tau} \frac{\Omega^*(\tau_n + u)}{\omega_z(t(\tau_n + u))} \exp(iu) du = \Omega_{0,n}^* \sum_{m=0}^{\infty} a_{n,m} I_m^*(\Delta\tau; k-1) + \sum_{p=0}^{\infty} \sum_{m=0}^{\infty} F_{n,p}^* a_{n,m} J_p^m(\Delta\tau; -k). \quad (40)$$

4 Criticism of the Analytic Solutions

We have described a methodology that gives the solution for the angular velocity components in a body-fixed frame, and the solution for the Eulerian angles, for the case when arbitrary transverse and axial torques act simultaneously on the body. The sole hypothesis made is that the rotating rigid body be nearly symmetric about its spin axis; an additional mild assumption is that the acting torques are analytic functions of time. In practice, one usually approximates the acting torques with polynomials, or equivalently, by truncating the power series expansions until a desired degree of accuracy is achieved. Therefore, the model of polynomial torques for the acting torques encompasses a large class of applications. The methodology consists of introducing a new independent variable in order to simplify the arising integrals. In many cases, a judicious choice of the new independent variable will eliminate most of the tedious calculations, however, in general one will require the use of the inverse transformation (14) to express the forcing function in (7) in terms of the new independent variable. For the case when the spin rate ω_z does not vanish, one can introduce ω_z as the new independent variable. Then the inverse transformation is given in the form of an infinite power series, and this is true even when the actual torques are exact polynomials. Because in practice one has to truncate the infinite series (14) an error is always present in the solution. Even worse, since the series representation of the function F has a finite radius of convergence, one cannot expect to obtain a globally valid solution for all t. However, the theory of analytic continuation can be used to circumvent this difficulty and extend the previous analytic solution for all t, as demonstrated above. Moreover, because the series in (35), (39), and (40) converge absolutely and uniformly, the general procedure allows one to approximate the solution as accurately as one desires.

Next we shall demonstrate with a numerical example the accuracy of the derived analytic solutions, for a typical practical application involving a spinning spacecraft subject to external disturbing torques about all three axes.

5 Numerical Example

We consider the simplest case when the axial torque M_z is *linearly increasing with time* from $M_z(0) = 0$ to $M_z(t_f) = 28$ N−m for $t_f = 200$ s. That is, it is assumed that

$$M_z(t) = 0.1273\ t\ \text{N} - \text{m}. \quad (41)$$

The transverse torques M_x and M_y, are assumed to be the same as for the constant axial torque example, and are given in Part I. The rigid-body mass properties, as well as the initial conditions for the angular velocity vector, remain the same as for the constant axial torque case. Under these assumptions, one can compute $\tau(t)$ by Eq. (12), and $t(\tau)$ and $F(\tau)$ from (14) and (16). (In other words, we begin with $M_{x,n}$, $M_{y,n}$ and $M_{z,n}$ from Eqs. (10), we find b_n from Eqs. (11) and (12), $\bar{f}_n$ from Eq. (13), c_n from Eqs. (14) and (15) and ultimately f_n from Eq. (16).) For the given values, ω_z is a quadratic polynomial which has two conjugate imaginary roots at approximately $\pm i\,147$. Therefore, the method of analytic continuation has been used to extend the solution beyond the bound of 147 s, as described earlier in the paper. Three points have been chosen in the interval $0 \le t \le 200$; at $t_1 = 60$, $t_2 = 110$, and $t_3 = 160$ seconds, respectively. The operations with the power series were performed using the symbolic language manipulation software package MAPLE (Char et al., 1991). The results of the numerical simulations are depicted in Figs. 1 and 2. Only the ω_x (Fig. 1) and ϕ_x (Fig. 2) solutions are shown here, since ω_y and ϕ_y exhibit similar behavior. It is clear that upon the construction of the analytic solutions, a compromise must be made between the number of the terms kept in the power series in (16) and (23) and in the number of partition intervals for the analytical continuation of the solution. It is an unfortunate fact that decreasing the number of intervals required for con-

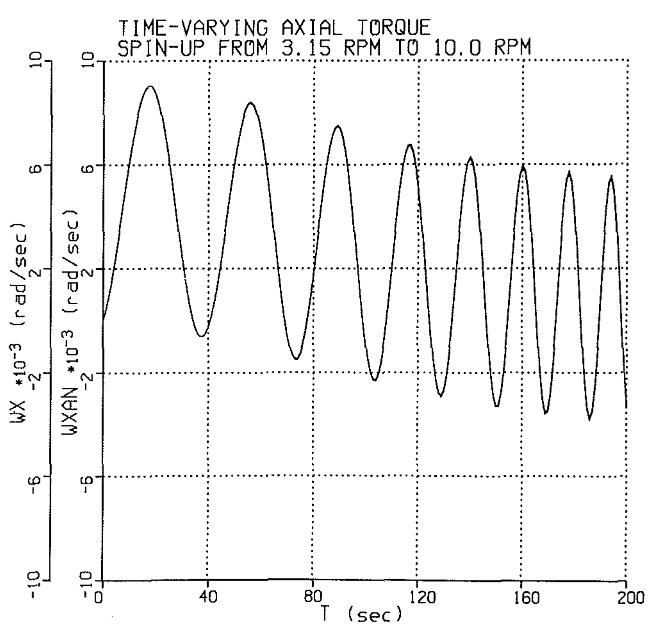

Fig. 1 Exact and analytic solutions for $\omega_x(t)$

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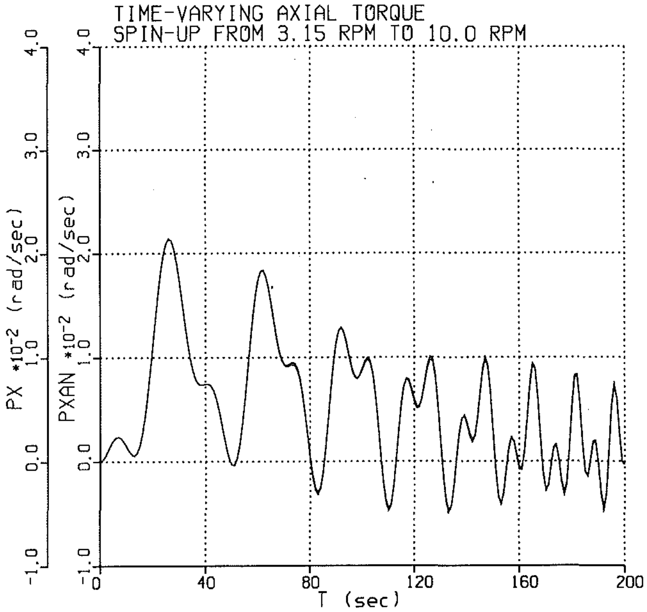

Fig. 2 Exact and analytic solutions for $\phi_x(t)$

tinuation of the solution beyond the bound determined by the singularities of $1/\omega_z$ comes at the expense of increasing the number of terms in the series expansion of F and $1/\omega_z$. In the example presented here, the first six terms were kept in the expansions of F and $1/\omega_z$ about each t_n, for $n=0, 1, 2, 3$.

6 Conclusions

Approximate analytic solutions have been derived for the attitude motion of a near-symmetric spinning rigid body, under the influence of transverse and axial time-varying torques, expressed as polynomial functions of time. In essence, the theory of Part I has been extended to include cases when all three components of the external torque vector vary with time. The difficulty of the method relies on expressing the external torques as functions of a new independent variable. Since the external torques are known functions of time, this involves the construction of the inverse transformation between the two independent variables. This is achieved using a series inversion lemma. As a consequence, the evaluation of the required integrals is always approximate, but because of the established convergence of the series, the approximation can be made arbitrarily accurate.

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